

INFINITESIMAL K -THEORY

BY GUILLERMO CORTIÑAS

Departamento de Matemática
Facultad de Cs. Exactas
Universidad de La Plata
Argentina

0. Introduction.

In this paper we study the fiber F of the rational Jones-Goodwillie character

$$F := \text{hofiber}(ch : K^{\mathbb{Q}}(A) \rightarrow HN^{\mathbb{Q}}(A))$$

going from K -theory to negative cyclic homology of associative rings. We describe this fiber F in terms of sheaf cohomology. We prove that, for $n \geq 1$, there is an isomorphism (Th. 6.2):

$$(0) \quad \pi_n(F) \cong H_{inf}^{-n}(A, K^{\mathbb{Q}})$$

between the homotopy of the fiber and the hypercohomology groups of $K^{\mathbb{Q}}$ on a non-commutative version of Grothendieck's infinitesimal site ([Dix]). We fall short of expressing F as the K -theory of a category. However we construct a natural map (7.4-7.6):

$$(1) \quad K_*^{\mathbb{Q}}(Free^{inf}) \rightarrow H_{inf}^*(A, K^{\mathbb{Q}}(Free)) \cong \pi_*(F)$$

between the K -theory of locally free modules on the non-commutative infinitesimal site and hypercohomology K -theory of free modules. These K -theory results are obtained as a very particular case of a general construction. The input of this construction consists of a category C equipped with a suitable notion of infinitesimal deformation, (e.g. C =rings, deformation=surjection with nilpotent kernel) and a functor $X : C \rightarrow \text{Spaces}$. The output is a natural map:

$$H_{inf}(-, X) \rightarrow X$$

where the source is generalized sheaf cohomology and maps deformations into weak equivalences. Delooping the fiber of this map we obtain a character $c^{\tau} : X \rightarrow \tau X$ which induces an isomorphism of relative groups:

$$(2) \quad c^{\tau} : \pi_* X(f) \cong \pi_* \tau X(f)$$

for any given deformation f . We show that, for $C =$ associative rings and $X = K^{\mathbb{Q}}$, the map $c_n^{\tau} : K_n^{\mathbb{Q}}(A) \rightarrow \tau_n K(A)$ is the Jones-Goodwillie character ($n \geq 1$), and (2) is Goodwillie's isomorphism [G]. For general C and $KM =$ group completion of a permutative category M , we obtain a map $KM^{inf} \rightarrow H_{inf}(-, KM)$ of which (1) is a particular case. The hypercohomology construction is also interesting for other functors. For instance if C is the category of associative algebras over a field of characteristic zero, then for the Cuntz-Quillen de Rham supercomplex:

$$(3) \quad H_{inf}(A, X_{dR}) \cong HP_*(A)$$

For $C =$ commutative algebras and $X =$ the commutative de Rham complex Ω :

$$(4) \quad H_{inf}^*(A, \Omega) = H_{inf}(Spec A, O)$$

The right hand side of (4) is the infinitesimal cohomology of the structure sheaf in the sense of Grothendieck [Dix]. For some functors X such as the de Rham complexes (3) and (4), there is another interpretation for $H_{inf}(-, X)$. This is the derived functor analogy of Cuntz-Quillen ([CQ2], [C1]). For associative algebras over a field it consists of the following. Given an algebra A choose a presentation $A = R/I$ as a quotient of a quasi-free algebra, and put

$$(5) \quad LX(A) = \operatorname{holim}_{\mathbb{N}^{op}}(n \mapsto X(R/I^n))$$

We formalize and generalize this construction. First we show that a category with deformations C admits a structure akin to that of a closed model category which permits calculation of the localization $C[Def^{-1}]$ as the homotopy category of cofibrant objects. (For associative algebras over a field, cofibrant=quasi-free; for commutative algebras, cofibrant=smooth). Then we show that, under certain conditions, hypercohomology can be computed as the left derived functor with respect to the localization above (Prop. 5.5):

$$(6) \quad H_{inf}(A, X) \cong LX$$

For categories of rings the isomorphism above occurs whenever X satisfies a power series version of the Poincaré lemma (Th. 3.5). Such is notably the case of the de Rham complexes (3), (4) in their respective categories of definition (Th. 4.0). We also show that $X = BE^+ \otimes \mathbb{Q}$, the rational plus construction of the elementary group satisfies (6) (the general linear group does not, as there are problems with K_1). Computationwise, this means that, for $n \geq 2$, $H_{inf}^{-n}(A, K^{\mathbb{Q}})$ can be computed as in (5) (Ths. 4.3, 6.2).

This paper continues the line of research I started in [C1]. The main results of [C1] reappear here either in a more general form (sections 2-4) or with shorter, simpler proofs (section 4).

The rest of this paper is organized as follows. In section 1 we develop the language of categories with deformations. In section 2 we construct the localization $C \rightarrow C[Def^{-1}]$ for a category with deformations. A general criterion for the existence of derived functors with respect to this localization is established in section 3. In section 4 we show that both non-commutative de Rham cohomology and the rational plus construction of the elementary group meet this criterion and compute

their derived functors. The sheaf theoretic approach is developed in section 5 where the character c^τ mentioned above is constructed. The isomorphism (0) is proved in section 6. Also in this section we conjecture that $H^n(A, K^\mathbb{Q}) = 0$ for positive n , and show that this conjecture is related to finding a non commutative analogue of Grothendieck's isomorphism (4). The map (1) is constructed in section 7; a more concrete interpretation of $Free^{inf}$ as a category of integrable connections with as maps the gauge transformations is discussed.

1. Categories with Deformations.

Shrinking functors in categories of interest 1.0. By a *category of interest* we mean a pointed category C which is closed under finite limits and finite colimits. In particular kernels and cokernels exist in C . We further require that a weak version of the first Noether isomorphism holds in C . We shall presently make this condition precise. First recall that a map in a category of interest is called *normal* if it is a kernel of its cokernel, and is a quotient map (or is conormal) if it is a cokernel of its kernel. We require that if $i : I \hookrightarrow A$ and $j : J \hookrightarrow A$ are normal and i factors as jk then the following sequence is exact:

$$(NIT) \quad 0 \rightarrow J/I \rightarrow A/I \rightarrow A/J \rightarrow 0$$

i.e. the first nonzero map is a kernel of the second, which is a cokernel of the first. A category of interest has products and co-products. We use Cuntz' notations for co-products; we write $QA = A * A$ for the coproduct of A with itself, $\mu : QA \rightarrow A$ for the folding map, $qA := \ker \mu$ and ∂_i for the natural inclusions ($i = 0, 1$). Recall a subobject $N \subset A$ of an object $A \in C$ is an equivalence class of monomorphisms. The fact that a subobject is normal is indicated by $N \triangleleft A$. Consider the category $C \triangleright C$ with as objects the pairs (A, N) consisting of an object $A \in C$ and a normal subobject $N \triangleleft A$, and as maps $(A, N) \rightarrow (A', N')$ the maps $A \rightarrow A'$ in C sending N into N' . Note $C \triangleright C$ is equipped with a natural projection $\pi : C \triangleright C \rightarrow C$. A *shrinking* functor in a category of interest is a functor $C \triangleright C \rightarrow C \triangleright C$ preserving π , $(A, N) \mapsto (A, s(A, N))$, $f \mapsto f$, which satisfies the following:

$$s1 \quad s(A, N) \subset N$$

$$s2 \quad s\left(\frac{A}{s(A, N)}, \frac{N}{s(A, N)}\right) = 0$$

Notice that the inclusion in $s1$ is necessarily normal, because $s(A, N) \triangleleft A$ is. We shall be especially concerned with maps $f : A \rightarrow B$ such that $s(A, \ker f) = 0$. We remark that, in view of (NIT), condition $s2$ says that $A/s(A, N) \rightarrow A/N$ is such a map. We shall use the following notation for the powers of the functor s ; we shall write $s(A, N)^n$ to mean $s(A, s(A, s(\dots, s(A, N) \dots)))$ (n times).

Remark 1.1. One can equivalently define shrinking functors as functors on the category of normal monomorphisms. To see the equivalence, proceed as follows. Start with a shrinking functor s in the sense of the definition above. Given any concrete normal mono (or monic) $\alpha : N \rightarrow A$, choose a representative $s(A, N) \rightarrow A$ of $s(A, \text{class of } \alpha)$; if $N = 0$ is the fixed 0 object (C is pointed), choose $s(A, 0) = 0$. The construction is functorial because if $\alpha : N \rightarrow A$ and $\beta : M \rightarrow B$ are monos and $f : A \rightarrow B$ is a map such that $f \circ \alpha$ factors through β then the factorization is unique.

Conversely, if we start with a shrinking functor defined on concrete monos α , then the class of $s(\alpha)$ depends only on the class of α , as is easily checked. Further these constructions are inverse to each other. Precisely starting with an s on $C \triangleright C$, then going to the associated functor on monos and coming back induces the identity, and the reverse composition is naturally equivalent to the identity. Hence we may –and do– identify both notions. Notice also that any shrinking functor defined for normal monos can be extended to a functor from the arrow category to the category of normal monos, by $f : B \rightarrow A \mapsto s(A, \ker(\text{coker } f))$.

Main Examples 1.2. Note that $s(A, N)$ does not have to depend on N ; we may always set $s(\cdot) = 0$. More interestingly, if C is the category of (not necessarily unital) associative rings, then the square two sided ideal $s(A, N) := N^2$ is a shrinking functor. A shrinking functor which depends on both variables is $s(A, N) = \langle [A, N] \rangle$, the ideal generated by all commutators. Another is $s(A, N) = \langle [A, N] \rangle + N^2$. Lie rings also admit at least two interesting shrinking functors; we may either set $s(A, N) = [N, N]$ or $s(A, N) = [A, N]$. The same is true of groups; simply substitute Lie brackets by commutators. The name of shrinking is meant to convey the idea that as we iterate the functor, the result is smaller. Of course this need not happen in particular cases, as we may have $s(A, N)^n = N$ for all n , e.g. if $N = A$ is a unital associative ring and $s(A, N) = N^2$. However the property that $s(A, N)^n = 0$ for some n , i.e. the ‘shrinkability’ of N in A is interesting, as it gives, depending on the choice of $s(\cdot)$, the notions of nilpotency, solvability and filtered commutativity. All these examples are particular cases of the following.

General Example 1.3. Let C be a category of interest, and let s be a shrinking functor. Then $r(A, N) = (A/s(A, N), N/s(A, N))$ is a functor, and is equipped with a natural map $\epsilon : 1 \rightarrow r$. Note that the map $\epsilon(A, N)$ has as kernel the ‘diagonal’ subobject $\Delta s(A, N) = (s(A, N), s(A, N))$. The functor r maps $C \triangleright C$ into the full subcategory $E \subset C \triangleright C$ of those pairs (A, N) such that $s(A, N) = 0$, and ϵ is an isomorphism on objects of E . It is not hard to check that $C \triangleright C$ is a category of interest if C is one. Note also that E is closed under kernels and cokernels. Conversely, let C be a category of interest, and let $E \subset C \triangleright C$ be a full subcategory, closed under kernels and cokernels. Suppose a functor $r : C \triangleright C \rightarrow E$ is given, together with a natural map $\epsilon : 1 \rightarrow r$. Suppose further that for all $(A, N) \in C \triangleright C$ the map $\epsilon(A, N)$ is a quotient map, has a diagonal subobject as kernel and that if (A, N) happens to live in E , then $\epsilon(A, N)$ is an isomorphism. Then it is not hard to check that $(A, N) \mapsto (A, \Delta^{-1}(\ker \epsilon(A, N)))$ is a shrinking functor. Note further that these constructions are mutually inverse. Thus a shrinking functor is the same thing as the data (C, E, r, ϵ) . For instance in the Lie ring examples above, the choice $s(A, N) = [N, N]$ comes from choosing as E the category of pairs (A, N) with N abelian. The choice $s(A, N) = [A, N]$ comes from the subcategory $E' \subset E$ where in addition we require the action to be trivial. The group and associative ring examples are similarly obtained.

Pro-Example 1.4. Let C be a category; we write $\text{Pro-}C$ for the category of countably indexed pro-objects. Thus –by [CQ3]– every object of $\text{Pro-}C$ is isomorphic to an inverse system of the form $\{A_n : n \in \mathbb{N}\}$. We regard C as the full subcategory of constant pro-objects in $\text{Pro-}C$. By [AM], $\text{Pro-}C$ is a category of interest if C

is. If C is of interest and is equipped with a shrinking functor $s(A, N)$, then it is possible to extend $s(\cdot)$ to all of $\text{Pro-}C$ in such a way that if $\{A_n : n \in \mathbb{N}\}$ is an inverse system with structure maps $\sigma : A_* \rightarrow A_{*-1}$ and $N_* \triangleleft A_*$ is a collection of normal subobjects such that $\sigma N_n \subset N_{n-1}$ then

$$(7) \quad s(A, N) = \{s(A_n, N_n) : n \in \mathbb{N}\}$$

In fact since every representative of a normal subobject in $\text{Pro-}C$ is isomorphic—in the arrow category—to an inverse system of normal sub-objects as above, the formula (7) is almost the definition of a functor; there are however problems with the many choices involved. To get an unambiguous definition proceed as follows. First extend the definition of $s(A, N)$ from normal subobjects to arbitrary maps $f : B \rightarrow A$ as in 1.1. The pro-extension of $(A, f) \mapsto s(A, f)$ is a well-defined functor, and sends maps which are inverse systems of monomorphisms into inverse systems of monomorphisms, whence it sends every monomorphism to a normal monomorphism, because every monomorphism is isomorphic to one of such form in the arrow category. In particular we obtain a functor mapping pairs (A, N) of a pro-object A and a sub-pro-object N to a sub-pro-object $s(A, N) \subset N$; one checks that this functor satisfies s1 and s2 above. We write $s(A, N)^\infty := \lim_n s(A, N)^n$ for the inverse limit; this, as well as the limit of any inverse system indexed by the natural numbers exists in $\text{Pro-}C$ ([AM]). For example if A and N are as in (7) above, we have:

$$(8) \quad s(A, N)^\infty := \{s(A_n, N_n)^n : n \in \mathbb{N}\}$$

Note that $s(\cdot)^\infty$ is idempotent as a functor $\text{Pro-}C \triangleright \text{Pro-}C \rightarrow \text{Pro-}C \triangleright \text{Pro-}C$; one checks further it is again a shrinking functor.

Homotopy 1.5. Let C be a category with a shrinking functor and let $A \in C$. We say that two maps $f, g \in C(A, B)$ are *congruent*,—and write $f \equiv g$ —if $f * gs(QA, qA)^n = 0$ for some $n \geq 1$. Thus if $s(\cdot)$ is idempotent,—e.g. as in (8)—then $f \equiv g$ iff $f * g$ factors through $QA/s(QA, qA)$. In general, $f \equiv g$ iff $f * g$ factors through

$$\text{Cyl}A := QA/s(QA, qA)^\infty$$

in $\text{Pro-}C$. One checks that \equiv is a reflexive and symmetric relation, and that it is compatible with composition on both sides in the restricted sense that $f_0 \equiv f_1 \Rightarrow gf_0 \equiv gf_1$ and $f_0h \equiv f_1h$ (whenever composition makes sense). It follows that the equivalence relation \sim generated by \equiv is compatible with composition in the ample sense that $f_0 \sim f_1$ and $g_0 \sim g_1$ imply $f_0g_0 \sim f_1g_1$. We say that f and g are *homotopic* if $f \sim g$. In the particular case when s is idempotent, $\text{Cyl}A$ is already an object of C . Such is the case of $\text{Pro-}C$ with either of the shrinking functors (7) and (8). Both shrinking functors induce the same cylinder and therefore the same homotopy relation. Note also that, as coproducts in $\text{Pro-}C$ are indexwise—i.e. $Q\{A_i : i \in I\} = \{QA_i : i \in I\}$ —the pro-extension of the functor Cyl , $\text{Cyl}(\{A_i : i \in I\}) = \{QA_i/s(QA_i, qA_i)^n : (i, n) \in I \times \mathbb{N}\}$ is precisely the natural cylinder associated to the shrinking functor of $\text{Pro-}C$.

Lemma 1.5.1. *Let*

$$\begin{array}{ccc}
 & & B \\
 & \nearrow^{f_0} & \downarrow p \\
 R & \xrightarrow{f} & A
 \end{array}$$

be a commutative diagram in $\text{Pro-}C$. Suppose $s(B, \ker p)^\infty = 0$. Then $f_0 \equiv f_1$.

Proof. Consider the sum map $h = f_0 * f_1 : QR \rightarrow B$. We have $ph\partial_i = f$, ($i=0,1$); hence $ph = f * f = f\mu$. Thus h maps $qR = \ker \mu \subset \ker f\mu$ into $\ker p$, and $s(QR, qR)^\infty$ into $s(B, \ker p)^\infty = 0$. Therefore h induces a homotopy $CylR \rightarrow B$. \square

Corollary 1.5.2. *Let p be as in the lemma above. Assume further that p admits a right inverse; then p is a homotopy equivalence. Precisely if $ps = 1_A$ then $sp \equiv 1_B$.*

Proof. Apply the lemma with $R = B$, $f_0 = 1_B$, $f_1 = sp$ and $f = p$. \square

Fibrations 1.6. A class of *fibrations* in a category of interest is a class Fib of maps which satisfies the following variation of the dual of Waldhausen's axioms for a class of cofibrations [Wa p.320]:

- (1) Fib1 Fib contains all isomorphisms.
- (2) Fib2 Fib is closed under composition and base change by arbitrary maps.
- (3) Fib3 The folding map $A^{*n} \rightarrow A$ defined as the identity in each summand is a fibration ($A \in C$, $n \in \mathbb{N}$).

As opposed to Waldhausen's fibrations, ours do not necessarily include the map $A \rightarrow 0$. On the other hand (Fib3) is not required in [Wa]. Fibrations shall be denoted by a double headed arrow \twoheadrightarrow . If C and Fib are as above, then we say that an object $A \in C$ is *rel-projective* if it has the left lifting property of [Q1] with respect to Fib . We say that C (or rather (C, Fib)) has *enough rel-projectives* if every object $A \in C$ is the target of a fibration $P_A \twoheadrightarrow A$ with P_A rel-projective.

Underlying example 1.7. In most of the examples considered in this paper, fibrations are those maps which admit a right inverse in an *underlying category*. By the latter we understand a fixed category of interest S together with a faithful embedding $C \subset S$ which preserves inverse limits. Hence we can equip $\text{Pro-}C$ with the class of fibrations consisting of those maps having a right inverse in $\text{Pro-}S$. In all the examples 1.1 we may take $S = \text{Sets}_*$, the category of pointed sets; in the ring examples, we may also choose $S = \text{Abelian Groups}$. If instead of rings we look at algebras over some ground ring k , $S = k\text{-Mod}$ is another natural choice. If, as is the case in these examples, the embedding has a left adjoint, then C has sufficient relatively projectives. For if $\perp : C \rightarrow C$ is the associated cotriple, then the co-unit map $\perp A \twoheadrightarrow A$ is a fibration with rel-projective source. Note that all this structure is preserved by the pro-category. Indeed the pro-extension of a faithful functor is faithful because both filtrant direct and inverse limits preserve injections, and if L is left adjoint to the inclusion $C \subset S$ then its pro-extension if

left adjoint to $\text{Pro-}C \subset \text{Pro-}S$ (cf. [AM]). We remark that this class of fibrations has the following extra property:

$$\text{(Fib4)} \quad fg \in \text{Fib} \implies f \in \text{Fib}$$

Categories with deformations 1.8. By a *category with* (infinitesimal, iterative) *deformations* (or *thickenings*) we shall understand a category of interest C together with a shrinking functor and a class of fibrations. We require further that the following axiom be satisfied:

(Def) If $B \twoheadrightarrow A$ is a fibration with kernel I then $B/s(B, I) \twoheadrightarrow A$ as well as each of the maps $B/s(B, I)^{n+1} \twoheadrightarrow B/s(B, I)^n$ ($n \in \mathbb{N}$) is a fibration.

Note that if s happens to be idempotent only the first condition is relevant. For example if fibrations are as in 1.7, then $\text{Pro-}C$ satisfies (Def) (by (Fib4)); if further every quotient map in C is split in the underlying category, then C satisfies (Def) also. If C is any category with deformations, and $A, B \in C$ are objects, then by a *deformation* of A by B we shall mean a fibration $f : B \twoheadrightarrow A$ such that if $N = \ker f$ then $s(A, N)^n = 0$ for some $n \geq 1$. Thus if $s(\cdot, \cdot)$ happens to be idempotent, the latter condition simply means that $s(A, N) = 0$. In general we have $s(A, N)^\infty = 0$. For example, it follows from NIT, s2 and Def that the map $B/s(B, \ker f)^n \xrightarrow{\sim} A$ induced by a fibration $f : B \twoheadrightarrow A$ is always a deformation. In particular

$$(9) \quad QA/s(QA, qA)^n \xrightarrow{\sim} A$$

is a deformation by Fib3.

Cofibrancy 1.8.1. A map $A \twoheadrightarrow B \in C$ is called a *cofibration* if it has the left lifting property (in the sense of [Q1]) with respect to deformations; an object A is called *cofibrant* if $0 \twoheadrightarrow A$ is a cofibration. For example relatively projective objects are always cofibrant; if $s = 0$ they are all the cofibrants. Here are some concrete examples. We fix $\text{Fib} = \text{all surjections}$ in all cases, except as noted. If C is the category of commutative algebras over a field, and $s(A, I) = I^2$, then cofibrancy is the same as smoothness; for associative algebras, cofibrant = quasi-free in the sense of Cuntz-Quillen. For commutative algebras over a ring which is not a field, smooth = cofibrant + flat. If C is the category of groups, and $s(G, N) = [G, N]$ is the relative commutator subgroup, then G is cofibrant iff $H^2(G, M) = 0$ for every trivial G -module M . Equivalently, by the universal coefficient theorem, G is cofibrant iff G_{ab} is free abelian and $H_2(G, \mathbb{Z}) = 0$. In particular superperfect groups (i.e. universal central extensions of perfect groups) are cofibrant in this setting. Similar considerations can be made regarding the category of Lie algebras over a field with $s(L, N) = [L, N]$. As noted above, in the case of groups we may also take $s(G, N) = [N, N]$; here cofibrant groups are those for which $H^2(G, M) = 0$ for all G -modules M . Clearly free groups satisfy this; the converse is a theorem of Stallings and Swan (cf. [Co]). Thus the cofibrants are just the free groups in this case. One can take any of these examples and pass to the pro-category, extending s as in (8). Then one can either take as fibrations the maps which are split in Pro-Sets , or all effective epimorphisms. The latter choice gives more deformations and therefore less cofibrants (cf. [CQ3]).

The following elementary lemma shall be of use in what follows.

Lemma 1.9. *Let C be a category with deformations. Then:*

- 1) *The class of deformations contains all isomorphisms and is closed under composition and under base change by arbitrary maps.*
- 2) *The class of cofibrations contains all isomorphisms and is closed under composition and cobase change by arbitrary maps.*
- 3) *An object A is cofibrant iff every deformation $\pi : B \rightsquigarrow A$ is split; i.e. there exists $s : A \rightarrow B$ such that $\pi s = 1$.*

Proof. The first assertion of 1) follows from Fib1, the fact that isomorphisms have trivial kernel and the fact that $-$ by $s1-$ $s(A, 0) = 0$ ($A \in C$). Let $f : A \rightsquigarrow B$ and $g : B \rightsquigarrow C$ be deformations. Write $I = \ker f$ and $J = \ker g$, and choose n and m such that $s(A, I)^n = 0$ and $s(B, J)^m = 0$. Then gf is a fibration and $K = \ker gf$ is the pullback of $J \subset B$ along f . Hence $s(A, K)^m$ maps to zero in B and therefore is contained in I . Thus $s(A, K)^{m+n} = 0$, and gf is a deformation, proving the second assertion. Let $\pi : X \rightsquigarrow Y$ be a deformation and let $f : A \rightarrow Y$ be any map. Write P for the pullback and $\hat{\pi}$ and \hat{f} for the induced maps. Then $\hat{\pi}$ is a fibration by Fib2. Further, \hat{f} maps $K = \ker \hat{\pi}$ isomorphically onto $I = \ker \pi$ and $s(P, K)^n$ monomorphically into $s(X, I)^n$ which is zero for $n \gg 0$. Therefore $\hat{\pi}$ is a deformation. Thus 1) is proven. The dual statements of 2) are immediate from the definition of cofibration. The third assertion follows from the fact that deformations are closed under base change. \square

Corollary 1.10. (*Homotopy Extension*) *If R is a cofibrant object in C and $X \rightsquigarrow Y \in C$ is a deformation then the dotted arrow in the commutative diagram below exists in Pro- C :*

$$\begin{array}{ccc} R & \longrightarrow & X \\ \partial_0 \downarrow & \nearrow & \downarrow \\ Cyl R & \longrightarrow & Y \end{array}$$

Proof. From the lemma the map $\partial_0 : R \rightarrow QR$ is a cofibration. Hence there exists a map $h : QR \rightarrow X$ making the obvious diagram commute. As $X \rightsquigarrow Y$ is a deformation, the map h must kill some power $s(QR, qR)^n$; the induced map is a pro-map $Cyl R \rightarrow X$ with the desired property. \square

Cofibrant models 1.11. We say that C has *enough cofibrant objects* if for each object $A \in C$ there exists a fibration $RA \twoheadrightarrow A$ for some cofibrant object RA . Such is the case for example if C has enough rel-projectives. By a *model* of an object $A \in C$ we understand a deformation $R \rightsquigarrow A$ where R is cofibrant; we say that C has *enough models* if each object has a model. Such is the case for example if C has sufficient cofibrant objects and $s(\cdot)$ is idempotent. Indeed if $\pi^A : RA \twoheadrightarrow A$ is a fibration with RA cofibrant and $I := \ker \pi^A$, then the induced map $\pi'^A : R'A := RA/s(RA, I) \rightsquigarrow A$, which is a deformation by $s2$ and Def, is a model. To see this, we must show that the dotted arrow out of $R'A$ in the diagram below exists for every deformation $p : X \rightsquigarrow Y$:

$$(10) \quad \begin{array}{ccc} RA & \dashrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ R'A & \dashrightarrow & Y \end{array}$$

But the arrow out of RA exists by cofibrancy of the latter, and it maps $s(RA, I)$ into $\ker p$ and $s(RA, I) = s(RA, s(RA, I))$ into $s(X, \ker p) = 0$. Hence $RA \rightarrow X$ factors through $R'A$ as claimed. The same argument shows that if R is any cofibrant object and $I \triangleleft R$ is any normal subobject, then $R/s(R, I)$ is cofibrant. In the general case, i.e. when the shrinking is not idempotent, one needs to pass to the pro-category to find sufficient models.

The idea of cofibrant models is analogous to that of projective resolutions in abelian categories such as modules over a ring, and to simplicial resolutions in non-abelian ones such as Rings. As with resolutions, we can define derived functors from cofibrant models. To formalize this we need to have a derived category; this is constructed in the next section.

2. The derived category.

Throughout this section we work under the following:

STANDING ASSUMPTION: C is a category with deformations having sufficient cofibrant objects.

The main purpose of this section is to prove the theorem below.

Theorem 2.0. *Let C be a category with deformations; assume C has sufficient cofibrant objects. Then the localization $C[Def^{-1}]$ of C at the class of deformations exists, and is equivalent to the category whose objects are all fibrations $R \twoheadrightarrow A$ with R cofibrant, ($A \in C$), and where a map $p_1 : R \twoheadrightarrow A \rightarrow p_2 : S \twoheadrightarrow B$ is the homotopy class of a map of pro-objects $R/s(R, \ker p_1)^\infty \rightarrow S/s(S, \ker p_2)^\infty$ in $\text{Pro-}C$. If furthermore, $s(\cdot)$ is idempotent, then $C[Def^{-1}]$ is also equivalent to the homotopy category of cofibrant objects.*

The proof of the theorem requires two lemmas.

Lemma 2.1. *Let $f : A \rightarrow B$ be a map in C and let $\pi^A : RA \twoheadrightarrow A$ and $RB \twoheadrightarrow B$ be fibrations with RA and RB cofibrant. Write $R'A := RA/s(RA, \ker \pi^A)^\infty$, $R'B := RB/s(RB, \ker \pi^B)^\infty$ and $\pi'^A : R'A \xrightarrow{\sim} A$ and π'^B for the induced maps. Then:*

- (1) *There exists a map $\hat{f} : R'A \rightarrow R'B$ lifting f ; i.e., such that $\pi'^B \circ \hat{f} = f \circ \pi'^A$.*
- (2) *Any two liftings of f as in (1) are congruent.*

Proof. By cofibrancy of RA and Def , one can lift f to a map f' from the constant pro-object RA into $R'B$. This map sends $IA := \ker \pi^A$ to $\ker \pi'^B$ and $s(RA, IA)^\infty$ to zero. Thus f' factors through a map \hat{f} , proving (1). The second statement is immediate from 1.5.1. \square

Remark 2.1.1. Note that we do not affirm nor negate that $R'A$ is cofibrant. Such an assertion does not make sense as we have not equipped $\text{Pro-}C$ with any particular class of fibrations. However the same argument proves that the statement obtained by replacing $R'A \rightarrow A$ and $R'B \rightarrow B$ by arbitrary deformations with cofibrant source holds. The point is that, unless we are in the idempotent case (cf. 1.11), the standing assumption does not imply the existence of sufficient models in C .

Lemma 2.2. *Let $f : A \xrightarrow{\sim} B$ be a deformation. Then any lifting \hat{f} as in 2.1-1) above is a homotopy equivalence.*

Proof. We keep the notations of the proof of the previous lemma. In view of 2.1-2) it suffices to show that there is one lifting which is a homotopy equivalence. We shall construct such a lifting as a composite $R'A \rightarrow R'_*P_* \rightarrow R'B$, and show that each of the two components admits an inverse up to congruence. Write $R'_0A = R'_0P_0 = A$, $R'_0B = B$. Construct by induction a commutative diagram:

$$\begin{array}{ccccc} R'_{n+1}P_{n+1} & \longrightarrow & P_{n+1} & \longrightarrow & R'_{n+1}B \\ & & \downarrow & & \downarrow \\ & & R'_nP_n & \longrightarrow & R'_nB \end{array}$$

Here the inner diagram is a pullback, and the two maps out of P_{n+1} are deformations by 1.9. Write R'_*P_* for the inverse system $n \mapsto R'_nP_n$ just constructed. The pro-map $R'_*P_* \rightarrow R'B$ is the first component of the lifting as announced above. Next we construct a congruence inverse for this map. By cofibrancy of RB and induction, the fibration $RB \twoheadrightarrow B$ lifts to a family of maps $RB \rightarrow R'_nP_n$ making the following diagram commute:

$$\begin{array}{ccccc} R'_{n+1}P_{n+1} & \longrightarrow & P_{n+1} & \longrightarrow & R'_{n+1}B \\ & \swarrow \text{dotted} & \downarrow & \swarrow \text{dotted} & \downarrow \\ & & R'_nP_n & \longrightarrow & R'_nB \\ & \swarrow \text{dotted} & & \swarrow \text{dotted} & \downarrow \\ & & & & RB \end{array}$$

The map $RB \rightarrow P_{n+1}$ exists by cartesianity and the map $RB \rightarrow R'_{n+1}P_{n+1}$ is a lifting of the latter along the deformation $R'_{n+1}P_{n+1} \xrightarrow{\sim} P_{n+1}$. Write $l_n : RB \rightarrow R'_nP_n$ for the map just constructed. Then l_n maps $s(RB, IB)^n = \ker(RB \rightarrow R'_nB)$ into $J_n := \ker(R'_nP_n \xrightarrow{\sim} R'_nB)$, whence $l_n(s(RB, IB)^{nm}) \subset s(R'_nP_n, J_n)^m$ for all m . Since by construction $s(R'_nP_n, J_n)^m = 0$ for $m \gg 0$, we have that $l_n(s(RB, IB)^m) = 0$ for $m \geq \alpha_n$ where α_n is some integer sufficiently large which depends on n . Choose these numbers in such a way that $\alpha_n < \alpha_{n+1}$. Since by construction $l = \{l_n : RB \rightarrow R'_nP_n\}$ is a map of inverse systems $RB \rightarrow R'_*P_*$, the induced map $l' = \{R'_{\alpha_n}B \rightarrow R'_nP_n\}$ is a map of inverse systems also, going from the inverse system $R'_{\alpha_n}B : n \mapsto R'_{\alpha_n}B$ to the inverse system R'_*P_* . Because by construction the sequence $\{\alpha_n\}$ is strictly increasing, it is cofinal, whence l' is a map of pro-objects $R'B \rightarrow R'_*P_*$. Furthermore the composite $R'_{\alpha_n}B \rightarrow R'_*P_* \rightarrow R'_nB$ is the natural projection, which represents the identity of the pro-object $R'B$. Hence the map $l' : R'B \rightarrow R'_*P_*$ is a right inverse of the map $R'_*P_* \rightarrow R'B$ in Pro- C . Thus by 1.5.2, the maps l and $R'_*P_* \rightarrow R'B$ are homotopy inverse. We have thus constructed one of two announced maps and proved it is a homotopy equivalence. Next we construct the second map. Use the cofibrancy of RA to lift the fibration π^A along the deformation $R_1P_1 \xrightarrow{\sim} A$ first to a map $RA \rightarrow R_1P_1$ and then by induction to a pro-map $t : RA \rightarrow R'_*P_*$ covering the identity of

A . By a similar argument as above, it is not hard to see that t induces a map $t' : R'A \rightarrow R'_*P_*$ still covering the identity of A . It remains to show that t' has a homotopy inverse. By 1.5.1 it suffices to show that there is also a map in the opposite direction $u : R'_*P_* \rightarrow R'A$ covering the identity of A . The latter map shall be constructed as a composite $u : R'_*P_* \xrightarrow{\tau} R'_\beta P \xrightarrow{v} R'A$. We define the map v as follows. First use cofibrancy of RP_1 and the fact that $R'_1A \xrightarrow{\sim} A$ is a deformation to lift the composite $RP_1 \rightarrow R'_1P_1 \xrightarrow{\sim} A$ to a map $v'_1 : RP_1 \rightarrow R'_1A$. Then because v'_1 covers the identity of A , and because $R'_1A \xrightarrow{\sim} A$ is a deformation, the map v'_1 factors through a map $v_1 : R'_{\beta_1}P_1 \rightarrow R'_1A$ for some $\beta_1 > 1$. To construct v_2 , first lift $RP_2 \rightarrow R'_2P_2 \xrightarrow{\sim} R'_1P_1$ along the deformation $R'_{\beta_1}P_1 \xrightarrow{\sim} R'_1P_1$ to a map $RP_2 \rightarrow R'_{\beta_1}P_1$. Next lift the composite of the latter map followed by v_1 to a map $v'_2 : RP_2 \rightarrow R'_2A$, covering the identity of R'_1A . By the same argument as above, there is an integer $\beta_2 > \beta_1$ such that the map v'_2 factors through a map $v_2 : R'_{\beta_2}P_2 \rightarrow R'_2A$. Proceeding inductively, we get an inverse system $R_{\beta_*}P_*$ and a map of inverse systems $v : R'_{\beta_*}P_* \rightarrow R'A$ covering the identity of A , as claimed. Next we need to find a map $\tau : R'_*P_* \rightarrow R'_{\beta_*}P_*$ covering the identity of A . As a preliminary step, note that we already have a map in the opposite direction. Indeed by the construction of $R'_{\beta_*}P_*$, the projection maps $\theta_n : R'_{\beta_n}P_n \xrightarrow{\sim} R'_nP_n$ commute with structure maps, and therefore assemble into a map of inverse systems; furthermore each θ_n is a deformation, by (Def). Next we shall construct the map τ and in so doing we shall show it is an isomorphism inverse to the map θ just considered. Because θ covers the identity of A , it will follow that the same is true of τ . Now to the construction of τ . Consider RP_{β_1} ; by cofibrancy, there is a map from the latter to $R'_{\beta_1}P_1$ lifting the projection $RP_{\beta_1} \rightarrow R'_{\beta_1}P_{\beta_1} \xrightarrow{\sim} R_1P_1$ along θ_1 . By *NIT* and *s2*, the kernel K_1 of θ_1 satisfies $s(R'_{\beta_1}P_1, K_1)^{\beta_1} = 0$; hence the indicated lifting maps $s(RP_{\beta_1}, IP_{\beta_1})^{\beta_1}$ to zero and hence factors through $R'_{\beta_1}P_{\beta_1}$. Consider the map $\beta : \mathbb{N} \rightarrow \mathbb{N}$, $\beta(n) = \beta_n$; proceeding inductively, we get a map of inverse systems $\tau_n : R'_{\beta_{n+1}(1)}P_{\beta_{n+1}(1)} \rightarrow R'_{\beta_n(1)}P_{\beta_n(1)}$, which represents a pro-map $\tau : R'_*P_* \rightarrow R_{\beta_*}P_*$. One checks that both $\tau\theta$ and $\theta\tau$ are restriction maps, and therefore represent identity maps. \square

Remark 2.2.1. Note that, in the proof above, as $A \mapsto RA$ is not a functor—nor is $RA \rightarrow A$ a deformation—we do not really have a pro-object RP . This difficulty, as well as that remarked in 2.1.1, can be avoided if one assumes that fibrations in C are as in the underlying example 1.7, and that the forgetful functor has an adjoint. For then the cotriple $RA = \perp A$ is functorial and $R'A$ is cofibrant by (10).

Proof of Theorem 2.0. For each $A \in C$, choose a fibration $\pi^A : RA \rightarrow A$ with RA cofibrant; if A is cofibrant already, choose $\pi^A = 1$. Write $IA := \ker \pi^A$. Let C' be the category whose objects are those of C and where a map $f : A \rightarrow B$ is the homotopy class of a map of pro-objects $R'A := RA/s(RA, IA)^\infty \rightarrow R'B$. Define a functor $\gamma : C \rightarrow C'$ as follows. Set $\gamma A = A$ on objects, and for each map $f : A \rightarrow B$ choose a lifting $\hat{f} : R'A \rightarrow R'B$, and define $\gamma f := [\hat{f}]$ as the homotopy class of \hat{f} . Then γ is a functor by 2.1, and maps deformations into isomorphisms by 2.2. Next, we have to prove that any functor $F : C \rightarrow D$ which inverts deformations factors uniquely as $F = \hat{F}\gamma$. Define $\hat{F} : C' \rightarrow D$ by $\hat{F}A = FA$ on objects; to define \hat{F} on arrows, proceed as follows. Given a homotopy class

$\alpha \in C'(A, B)$ choose a representative $f \in \text{Pro-}C(R'A, R'B)$ of α , then choose a representative $f_{nm} : R'A_m := RA/s(RA, IA)^m \rightarrow R'B_n$ of f and finally set $\hat{F}\alpha = F(R'B_n \xrightarrow{\sim} B)Ff_{nm}F(R'A_m \xrightarrow{\sim} A)^{-1}$. One checks immediately –using (9)– that $\hat{F}\alpha$ is independent of the choices made in its definition and that it is functorial. Next we have to prove that \hat{F} is unique. Suppose $G : C' \rightarrow D$ is another functor such that $G\gamma = F$. Then $GA = FA = \hat{F}A$, hence \hat{F} and G agree on objects; similarly, if $\alpha \in C'(A, B)$ is the class of a map $g : R'A \rightarrow R'B$ which admits a representative $f : A \rightarrow B$, then $G\alpha = G\gamma f = \hat{F}\alpha$. In general, if $f : R'A_m \rightarrow R'B_n$ represents g , then the following diagram commutes in $\text{Pro-}C$:

$$\begin{array}{ccccccc} R'A & \xleftarrow{\hat{\pi}_m^A} & R'(R'A_m) & \xrightarrow{\hat{f}} & R'(R'B_n) & \xrightarrow{\hat{\pi}_n^B} & R'B \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A & \xleftarrow{\pi_m^A} & R'A_m & \xrightarrow{f} & R'B_n & \xrightarrow{\pi_n^B} & B \end{array}$$

Hence the following diagram commutes up to homotopy:

$$\begin{array}{ccc} R'(RA_m) & \xrightarrow{\hat{f}} & R'(R'B_n) \\ \hat{\pi}_m^A \downarrow & & \hat{\pi}_n^B \downarrow \\ R'A & \xrightarrow{g} & R'B \end{array}$$

Thus for $\alpha = [g]$ we have $G\alpha = G(\gamma(\pi_n^B)\gamma f\gamma(\pi_m^A)^{-1}) = \hat{F}\alpha$, and $G = \hat{F}$, as claimed. This finishes the proof of the first assertion of the theorem. Next, let C'' be the category of fibrations $R \twoheadrightarrow A$ as in the theorem. One checks that $A \mapsto \pi^A$, $[f] \mapsto [f]$ is a fully faithful functor $F : C' \rightarrow C''$. I claim further that every object in C'' is isomorphic to one in the image of F . Note the claim implies that F is a category equivalence as we have to prove. To prove the claim, let $p : R \twoheadrightarrow A \in C''$ be an object. Then by cofibrancy of RA the map π^A can be lifted to a map $\hat{p} : RA \rightarrow R' = R/s(R, \ker p)^\infty \in PC$, which passes to the quotient to give a map $\hat{p}' : R'A \rightarrow R' \in PC$, whose homotopy class is a map $\pi^A \rightarrow p \in C''$. Next observe that by 2.1-2), the map $[\hat{p}']$ is an isomorphism, and the claim is proved. To finish proof of the theorem it only remains to prove the last assertion; the proof is similar to that just given for the next to last statement of the theorem, so we shall just sketch it. Assume that s is idempotent; by its very definition, γ induces a fully faithful embedding γ' of the homotopy category of cofibrant objects in the localization of C . But by 1.11 and (Def) every object in the localization is isomorphic to one in the image of γ' . This completes the proof. \square

Remark 2.3. The proof of 2.0 does not require C to be of interest. For example the same proof as above applies for unital rings.

3. Derived Functors and Poincaré Lemma.

Throughout this section, C is a fixed category with deformations satisfying the Standing Assumption of §3. We recall the following definition from [Q1, Ch.I, §4.1].

Definition 3.0. Let C be a category with deformations, let $\gamma : C \rightarrow C[Def^{-1}]$ be the localization functor of 2.0 above, and let $F : C \rightarrow D$ be a functor. By the *total left derived functor* of F we mean a functor $\underline{\underline{LF}} : C[Def^{-1}] \rightarrow D$ together with a natural transformation $\alpha : \underline{\underline{LF}}\gamma \rightarrow F$ having the following universal property. Given any $G : C[Def^{-1}] \rightarrow D$ and natural transformation $\beta : G\gamma \rightarrow F$ there is a unique natural transformation $\theta : G \rightarrow \underline{\underline{LF}}$ such that the following diagram commutes:

$$(11) \quad \begin{array}{ccc} G\gamma & \xrightarrow{\beta} & F \\ \downarrow \theta\gamma & \nearrow \alpha & \\ \underline{\underline{LF}}\gamma & & \end{array}$$

The following lemma establishes a criterion for the recognition of derived functors.

Lemma 3.1. *With the notations of the definition above, Let C be a category with deformations having sufficient cofibrant objects. Assume the shrinking functor is idempotent. Let $F : C \rightarrow D$ be a functor. Then the following conditions are equivalent:*

- i) F carries deformations between cofibrant objects into isomorphisms.
- ii) The map $F(\mu : CylR \xrightarrow{\sim} R)$ is an isomorphism for every cofibrant object $R \in C$.
- iii) F carries homotopic maps between cofibrant objects into equal maps.
- iv) The following construction is independent –up to isomorphism– of the choices made in its definition. Given an object $A \in C$ choose a cofibrant model $RA \xrightarrow{\sim} A$, and set $LF A = FR$. Given a map $f : A \rightarrow B$ choose a lifting $\hat{f} : RA \rightarrow RB$ and set $LF(f) = F(\hat{f})$.
- v) The derived functor exists and $\underline{\underline{LF}}\gamma(A) = F(A)$ for all cofibrant objects A .

Under the equivalent conditions above, the construction LF of (iv) is functorial and $LF = \underline{\underline{LF}}\gamma$.

Proof. That i) \implies ii) \iff iii) is clear from the definition of homotopy; ii) \implies i) follows from 1.9-3) and 1.5.1.; iii) \implies iv) is immediate from 2.1.1. If i) does not hold, then there is a deformation $f : R \xrightarrow{\sim} S$ with both R and S cofibrant such that $F(f)$ is not an isomorphism. Hence the choices $RS = R$ and $RS = S$ lead to two distinct values of $LF(S)$. Thus iv) \implies i). The last assertion of the lemma implies that iv) \iff v). Note that, once LF is assumed to be well-defined, it is automatically functorial, and –by 2.2– maps deformations into isomorphisms. Hence –by 2.0– it factors uniquely as $LF = G\gamma$. The proof that $G = \underline{\underline{LF}}$ is essentially the same as the proof of [Q1, I.4, Prop. 1]; details are left to the reader. \square

Functors to Simplicial Sets 3.2. Suppose a functor $X : C \rightarrow \mathbb{S}Sets$ is given. Let $A \in C$ and let RA and IA be as in the previous section. Consider the homotopy limit:

$$(12) \quad LX(A) = \text{holim}(n \mapsto X(RA/s(RA, IA)^n))$$

Note that this construction is analogous to the one used to define the hyper(co) homology of a functor from some abelian category to its chain complexes. Simply think of $R'A \rightarrow A$ as a resolution and of holim as a total complex. More formally the homotopy type of LX is analogous to the composite of the derived functor in the sense of the definition above and the localization functor. Note that, for (12) to make sense, we need to have a true functor to SSets , as opposed to a homotopy type. However we at least want that if $X \xrightarrow{\sim} Y$ is a natural weak equivalence, then $LX \xrightarrow{\sim} LY$. For this we need X and Y to be fibrant (i.e. Kan) simplicial sets (cf.[BK, XI 5.6]). Now if X is fibrant then by the proof of the cofinality theorem of [BK,XI 9.2], the following assignment is a functor $\text{Pro-}C \rightarrow \text{HoSSets}$ extending (12):

$$(13) \quad \mathbf{X}\{A_i : i \in I\} = \text{homotopy type of } \text{holim}_I XA_i$$

Suppose that a deformation category structure is given in $\text{Pro-}C$, such that $R'A$ is cofibrant and $R'A \xrightarrow{\sim} A$ is a deformation; such is the case e.g. if fibrations are as in example 1.7. Then if the equivalent conditions of 3.1 hold for (13), the homotopy type of (12) is a functor, –by 3.1-iv– and is precisely the restriction of the derived functor of (13) to the subcategory $C \subset \text{Pro-}C$. We remark that this construction, however useful, is not really the derived functor of the homotopy type of $X : C \rightarrow \text{SSets}$, which need not exist. This is because by definition 3.0 the universal property (11) must hold for functors G with values in HoSSets , while the universal property of the holim in (12) requires functors to SSets . On the plus side, if we happen to have a functorial choice of $RA \rightarrow A$ then (12) yields not just a homotopy type but a true sset. Such is notably the case when fibrations are as in the underlying example 1.7 and the forgetful functor has a left adjoint. This example has the advantage that one does not need to check the conditions of 3.1 for all the cofibrant objects of $\text{Pro-}C$, as the next lemma shows.

Lemma 3.3. *Let C be a category of interest with a shrinking functor. Assume C is equipped with a faithful functor $C \rightarrow S$ into another category of interest S which has a left adjoint. Write \perp for the associated cotriple, $JA := \ker(\perp A \rightarrow A)$, and $UA = \perp A/s(\perp A, JA)^\infty$. Consider C and $\text{Pro-}C$ as categories with deformations, with fibrations defined as in 1.7 and shrinking functor as in (8). Let $X : C \rightarrow \text{Fibrant SSets}$ be a functor. Then the functor (13) satisfies the equivalent conditions of 3.1 iff the functor X satisfies the following:*

vi) For every object $A \in C$ the map $X(\mu(UA) : \text{Cyl}UA \rightarrow UA)$ is a weak equivalence.

Proof. Note that, by 1.5.2, *vi)* is logically weaker than 3.1-iii). Hence it suffices to show that *vi)* implies at least one –and then all– of the equivalent conditions of 3.1. Assume *vi)* holds; we shall prove that 3.1-i) holds also. It follows from *vi)* and the fact that holim preserves weak equivalences of fibrant ssets ([BK, XI5.6]) that if A is any pro-object then the map $\mathbf{X}(\mu(UA))$ is an isomorphism in HoSSets . Thus for $A, B \in \text{Pro-}C$, $\mathbf{X} : \text{Pro-}C(UB, A) \rightarrow \text{HoSSets}(\mathbf{X}(UB), \mathbf{X}(A))$ sends homotopic maps to equal maps. Thus \mathbf{X} sends the following maps into isomorphisms:

- all homotopy equivalences $UB \rightarrow UA$,
- those homotopy equivalences $UB \rightarrow A$ which admit a strict –not just homotopy– right inverse

Now let $p : B \xrightarrow{\sim} A \in \text{Pro-}C$ be a deformation of cofibrant objects; we must prove $\mathbf{X}(p)$ is an isomorphism. Consider the following diagram:

$$\begin{array}{ccc} UB & \xrightarrow{Up} & UA \\ \left(\begin{array}{c} \downarrow \pi^B \\ \downarrow \pi^A \end{array} \right) & & \\ B & \xrightarrow{p} & A \end{array}$$

Here the arrows going up are right inverses of those going down; they exist by cofibrancy of both A and B . By 1.5.2 and 2.2, each of π^B , π^A and Up is a homotopy equivalence, whence \mathbf{X} maps each of them –and therefore also the map p – to an isomorphism. \square

Associative Rings 3.4. Let C be the category of –not necessarily unital– associative rings, equipped with the shrinking functor $s(A, I) = I^2$ and with as fibrations the surjective maps. Then the hypothesis of the lemma above are satisfied for $S = \text{Sets}_*$, the category of pointed sets, and the forgetful functor $C \rightarrow S$. We shall show below (Theorem 3.5) that condition vi) of 3.3 is equivalent to a Poincaré lemma for power series. The key observation needed to prove theorem 3.5 is that, as we shall see presently, the pro-ring $CylUA$ is a power series pro-ring. To make this assertion precise, we need some notation. Given a pointed set S and a ring B , we write $B\{S\}$ for the ring of polynomials in the non commutative variables $S - \{*\}$ (we identify $*$ with 0), and $\langle S \rangle \subset B\{S\}$ for the two-sided ideal generated by the variables. We think of the power series on S as a pro-ring; for each (pro-) ring B and pointed set S , we put:

$$(14) \quad B\{\{S\}\} = B\{S\} / \langle S \rangle^\infty$$

The rest of this subsection will be devoted to proving that there is a natural isomorphism:

$$(15) \quad CylUA \cong UA\{\{A\}\}$$

Our proof uses Lemma 3.7, which is proved separately below. We need some more notation. Write V for the free abelian group on $A - \{0\}$, TV for the tensor algebra (over \mathbb{Z}), and let I be the kernel of the adjunction map $TV \rightarrow A$. Thus in the notation of Lemma 3.3, $TV = \perp A$, $I = JA$, and $UA = TV/I^\infty$. The left hand side of (15) is a quotient of QTV ; precisely, identifying TV with its image through the inclusion $\partial_1 : TV \rightarrow QTV = T(V \oplus V)$, $\partial_1(v) = (v, 0)$, $v \in V$, we have $Cyl(TV/I^\infty) = QTV/\mathcal{F}^\infty$, where \mathcal{F}^∞ is the pro-ideal $\mathcal{F}^1 = \langle I \rangle + \langle q(I) \rangle + \langle qTV \rangle \supset \dots \supset \mathcal{F}^n = \langle I^n \rangle + \langle q(I^n) \rangle + \langle qTV \rangle^n \supset \dots$. Similarly, we have $UA\{\{A\}\} = \{(TV/I^n)A / \langle A \rangle^n : n \in \mathbb{N}\} = QTV/\mathcal{G}^\infty$, where $\mathcal{G}^n := \langle I^n \rangle + \langle \partial_0(V) \rangle^n$, and $\partial_0(v) = (0, v)$. Consider the isomorphism $\alpha : QTV = T(V \oplus V) \rightarrow QTV$, $\alpha(v, w) := (v + w, -w)$, $(v, w) \in V \oplus V$. We shall show that α maps the filtration \mathcal{F} into a filtration equivalent to \mathcal{G} , and thus induces an isomorphism as in (15). For this purpose we consider four new filtrations by ideals of QTV . We put $\mathcal{T}^n = \langle I \rangle + \langle qTV \rangle^n$, $\mathcal{T}^n = \langle I^n \rangle + \langle qTV \rangle^n$, $\mathcal{G}^n = \langle I \rangle + \langle \partial_0(V) \rangle^n$

$\mathcal{G}''^n = \langle I^n \rangle + \langle \partial_0(V) \rangle^n$. One checks that the isomorphism α maps \mathcal{F}' isomorphically onto \mathcal{G}' . Hence it suffices to show that \mathcal{F} is equivalent to \mathcal{F}' and that \mathcal{G} is equivalent to \mathcal{G}' . But since $\mathcal{F}''^n \supset \mathcal{F}^n \supset \mathcal{F}'^n$ and $\mathcal{G}''^n \supset \mathcal{G}^n \supset \mathcal{G}'^n$ we are reduced to showing that, for N sufficiently large, we have $\mathcal{F}''^n \supset \mathcal{F}'^N$ and $\mathcal{G}''^n \supset \mathcal{G}'^N$; both these inclusions follow from Lemma 3.7 below.

Theorem 3.5. (*Poincaré Functors*) *Regard the categories of associative rings and pro-rings as deformation categories as in 3.4 above. Let $X : \text{Rings} \rightarrow \text{Fibrant Ssets}$ be a functor. Extend X to a functor $\mathbf{X} : \text{pro-Rings} \rightarrow \text{Ho-Ssets}$ as in (13). We have:*

- 1) *If any of the following holds, then $\underline{\mathbf{LX}}$ exists and $\underline{\mathbf{LX}}\gamma(A) = \mathbf{X}(A)$ for all quasi-free pro-rings A :*
 - i) (*Poincaré Lemma for pro-power series*) *For every ring A and every set S , the map $\mathbf{X}(A\{\{S\}\} \xrightarrow{\sim} A)$ associated to the pro-power series (14) is an isomorphism.*
 - ii) (*Poincaré Lemma for polynomials*) *$\mathbf{X}(A[t] \rightarrow A)$ is an isomorphism for all $A \in \text{Rings}$.*
 - iii) *Either of the above holds for all pro-rings of the form R/I^∞ where R is a quasi-free ring and $I \triangleleft R$ is an ideal.*
- 2) *If $\underline{\mathbf{LX}}$ exists and $\underline{\mathbf{LX}}\gamma(A) = \mathbf{X}(A)$ for all quasi-free pro-rings A , then i) above holds for all quasi-free pro-rings. In such case we say that X is a Poincaré functor.*

Proof. If 1-i) holds then $\underline{\mathbf{LX}}$ exists and has the desired property by 3.3 and 3.4. If 1-ii) holds and $A = A_0 \oplus A_1 \oplus A_2 \dots$ is a graded ring, then X maps the projection $A \rightarrow A_0$ to a weak equivalence. As $A\{\{S\}\}$ is pro-graded, it follows that 1-ii) \implies 1-i). If (iii) holds then so does the condition of 3.3, as $\text{Cyl}UA$ is of the indicated form, cf. (15). This proves 1). If $\underline{\mathbf{LX}}$ exists and is as in 2), then by 3.1-i), \mathbf{X} must map all deformations of quasi-free pro-rings into isomorphisms. One checks that $A\{\{S\}\}$ is quasi-free if A is, whence 2) follows. \square

Remark 3.6. By essentially the same arguments as above, it is not hard to see that a functor $F : \text{pro-Rings} \rightarrow \text{Any Category}$ satisfies 3.1 iff it satisfies 3.5-2).

Lemma 3.7. *Let $A \subset B$ be rings and let $\epsilon : B \rightarrow A$ be a homomorphism such that $\epsilon a = a, (a \in A)$. Set $I = \ker \epsilon$, and let $J \subset A$ be an ideal. Consider the following filtration in B :*

$$B \supset \mathcal{F}^n = \langle J^n \rangle + I^n$$

Then there is an isomorphism:

$$B/\mathcal{F}^\infty \cong B/(\langle J \rangle + I)^\infty$$

Proof. Let $\mathcal{G}^n = \langle J \rangle^n + I^n$. It is straightforward to check that $(\langle J \rangle + I)^{2n} \subset \mathcal{G}^n$, whence $B/(\langle J \rangle + I)^\infty \cong B/\mathcal{G}^\infty$. Thus we must prove that $B/\mathcal{G}^\infty \cong B/\mathcal{F}^\infty$. It is clear that $\mathcal{G}^n \supset \mathcal{F}^n$. I claim that for $N = n^2 + n - 1$, we also have $\mathcal{G}^N \subset \mathcal{F}^n$. To prove the claim –and the lemma– it suffices to show that $\langle J \rangle^N \subset \mathcal{F}^n$. Every element of $\langle J \rangle^N$ is a sum of products of the form:

If we distribute all parenthesis we get a sum in which all those terms not in I^n have at most $n-1$ factors in I and at least n^2 factors in J . We must show that at least n of the latter appear side by side, forming a string. Assume the contrary holds. Then every string of j 's must be broken off before the n -th step by an i . The minimum number of i 's that are necessary for this to happen occurs when each sequence of j 's is broken off exactly after the $n - 1$ -th j in a row. Since there are $n^2 = (n + 1)(n - 1) + 1$ j 's, the minimum number must be $n + 1$, which is a contradiction. \square

4. The derived functors of rational K -theory and Cyclic Homology.

In this section we give examples of derived functors in the sense of the previous section. The first one is that which motivated this paper.

Theorem 4.0. *Consider the Cuntz-Quillen supercomplex [CQ2 (1)] as a functor $X : \mathbb{Q}$ -Algebras \rightarrow Supercomplexes. Then X is Poincaré, and its derived functor is represented by the homotopy type of the periodic cyclic complex.*

Proof. Immediate from [CQ2,(9)], [CQ2, 8.1] and 3.5-6 above. \square

Remark 4.1. One can also derive the commutative de Rham complex in the category of commutative rational algebras. The resulting derived functor is the infinitesimal cohomology of Grothendieck [Dix], also called algebraic de Rham ([H]) and crystalline ([FT]).

4.2. The derived functor of rational K -theory. Recall Goodwillie's isomorphism:

$$(16) \quad K_*^{\mathbb{Q}}(A, I) \cong HN_*(A \otimes \mathbb{Q}, I \otimes \mathbb{Q})$$

between the relative rational K -group of a nilpotent ideal and its analogue in negative cyclic homology [G]. Using this isomorphism, we show in Theorem 4.3 below that $K^{\mathbb{Q}}$ is (almost) a Poincaré functor and that its derived functor is essentially the fiber of the Jones-Goodwillie character:

$$(17) \quad ch_* \otimes \mathbb{Q} : K_*^{\mathbb{Q}}(A) \rightarrow HN_*(A) \otimes \mathbb{Q}$$

In order to apply the framework developed in the previous section, we need some preliminaries. First of all we need a fibrant functorial model for K -theory. One such model is $K = \mathbb{Z}_{\infty}NGL$, the Bousfield-Kan completion of the nerve of the general linear group. As is well-known, this is just a functorial plus construction. To get rational K -theory we complete again: $K^{\mathbb{Q}} = \mathbb{Q}_{\infty}K = \mathbb{Q}_{\infty}NGL$. Actually $\mathbb{Q}_{\infty}X$ is fibrant for every sset X , so we can choose any other –not necessarily fibrant– functorial model for integral K -theory. Next note that the character (17) is induced by a natural map of fibrant ssets $ch : K^{\mathbb{Q}} \rightarrow SN^{\mathbb{Q}}$. For example in the plus construction approach, the simplicial map ch is constructed as follows. Start off with the Hurewicz map $NGL \rightarrow \mathbb{Z}NGL$. Next consider the Dold-Kan functor $S : \text{Chain Complexes of abelian groups} \rightarrow \text{S.abelian groups}$, and follow the Hurewicz map with the result of applying S to the chain map $\mathbb{Z}NGL \rightarrow \mathbb{Z}NGL$ of [G]. We

thus get a map $NGl \rightarrow SCN_{\geq 1}$. Finally \mathbb{Q} -complete on both sides to obtain a map $ch : K^{\mathbb{Q}} \rightarrow \mathbb{Q}_{\infty}SCN_{\geq 1} =: SN^{\mathbb{Q}}$. This is essentially construction of [We], modulo geometric realization. Note for example that the map $SCN_{\geq 1} \otimes \mathbb{Q} \xrightarrow{\sim} SN^{\mathbb{Q}}$ is an equivalence –by virtue of [BK, V 3.3]– whence $\pi_*SN^{\mathbb{Q}} \cong HN_* \otimes \mathbb{Q}$. Now the map $ch : K^{\mathbb{Q}} \rightarrow SN^{\mathbb{Q}}$ is defined for unital rings. We extend it to non-unital rings in the usual manner, i.e. by considering the unital ring $\tilde{A} = A \oplus \mathbb{Z}$ and taking the homotopy fiber of the simplicial map associated to the projection $\tilde{A} \rightarrow \mathbb{Z}$. One checks that if $I \triangleleft A$ is a nilpotent ideal in a not necessarily unital ring A , then $K_n^{\mathbb{Q}}(A, I) = K_n^{\mathbb{Q}}(\tilde{A}, I)$ and $HN_n^{\mathbb{Q}}(A, I) = HN_n^{\mathbb{Q}}(\tilde{A}, I)$ (as observed in [Wo, proof of Th. 1.1]) whence (16) holds for not necessarily unital rings. Next extend both $K^{\mathbb{Q}}$ and $SN^{\mathbb{Q}}$ to pro-rings as in (13). We observe that, as holim preserves homotopy fibration sequences of fibrant ssets, the isomorphism (16) holds for arbitrary deformations of pro-rings. In particular for the pro-power series (14) we have $K_n^{\mathbb{Q}}(A\{\{S\}\}, \langle S \rangle) \cong HN_n(A \otimes \mathbb{Q}\{\{S\}\}, \langle S \rangle \otimes \mathbb{Q})$. Now if A is quasi-free as a ring then $A \otimes \mathbb{Q}$ and $A \otimes \mathbb{Q}\{\{S\}\}$ are quasi-free as \mathbb{Q} -algebras. Hence $K_n^{\mathbb{Q}}(A\{\{S\}\}, \langle S \rangle) = 0$ for $n \geq 2$, as HN_n of a quasi-free algebra is zero in degrees ≥ 2 . However $K_1^{\mathbb{Q}}(A\{\{S\}\}, \langle S \rangle) = \ker(HH_1(A\{\{S\}\}) \rightarrow HH_1(A)) \neq 0$ in general, whence $K^{\mathbb{Q}}$ is not Poincaré. To get a Poincaré functor out of $K^{\mathbb{Q}}$ we just need to eliminate its first homotopy group. We do this by substituting the elementary group for Gl ; i.e. we consider

$$KE^{\mathbb{Q}}(A) := \mathbb{Q}_{\infty}NE(A)$$

Theorem 4.3. (The derived functor of K -theory)

The functor $A \mapsto K^{\mathbb{Q}}(A)$ is not Poincaré. However, the functor $A \mapsto KE^{\mathbb{Q}}(A)$ above is, and therefore it has a left derived functor $LKE^{\mathbb{Q}}$. Set $LK_n^{\mathbb{Q}}(A) := \pi_n LKE^{\mathbb{Q}}$; then:

i) There is an exact sequence:

$$\begin{aligned} \dots HN_{n+1}A \rightarrow LK_n^{\mathbb{Q}}(A) \rightarrow K_n^{\mathbb{Q}}(A) \rightarrow HN_n(A) \rightarrow \\ \dots HN_3(A) \rightarrow LK_2^{\mathbb{Q}}(A) \rightarrow K_2^{\mathbb{Q}}(A) \rightarrow HN_2(A) \end{aligned}$$

ii) If $A = R/I$ is a presentation of A where $R \otimes \mathbb{Q}$ has Hochschild dimension $\leq n-1$, –i.e. $HH^n(R \otimes \mathbb{Q}, -) = 0-$, then $LK_n^{\mathbb{Q}}(A) = \pi_n(\lim_m K^{\mathbb{Q}}(R/I^m))$.

Proof. The first two assertions follow from the discussion above. To prove i), consider the exact sequence of K -groups associated with the deformation $\pi^A : UA \xrightarrow{\sim} A$. Then $LK_n^{\mathbb{Q}}(A) = K_n^{\mathbb{Q}}(UA)$ ($n \geq 2$) and $K_n(\pi^A) \cong HN_n(\pi^A)$ ($n \geq 1$). On the other hand $HN_n(UA) = 0$ for $n \geq 2$, and therefore $HN_n(\pi^A) \cong HN_{n+1}(A)$, for $n \geq 2$. This proves that the sequence is exact at $LK_2^{\mathbb{Q}}(A)$ and to the left. By the same argument as above, the natural map $HN_2(A) \hookrightarrow HN_1(\pi^A)$ is injective, whence $K_2^{\mathbb{Q}}(A) \rightarrow K_1^{\mathbb{Q}}(\pi^A)$ factors through ch_2 . It follows that the sequence is exact also at $K_2^{\mathbb{Q}}(A)$, completing the proof of i). If $A = R/I$ is a presentation as in ii), then $HN_m(R/I^{\infty}) \otimes \mathbb{Q} = 0$ for $m \geq n$. Hence by i), $LK_n(A) = LK_n(R/I^{\infty}) = K_n(R/I^{\infty})$. \square

Remark 4.4. For commutative algebras over \mathbb{Q} part ii) of the theorem applies to presentations $A = R/I$ with R smooth of Krull dimension $\leq n$. Indeed the latter

coincides with Hochschild's for smooth algebras. In general, for any R as in ii), we have an exact sequence ([BK]):

$$0 \rightarrow \lim_m^1 K_n^{\mathbb{Q}}(R/I^m) \rightarrow LK_n^{\mathbb{Q}}(A) \rightarrow \lim_m K_n^{\mathbb{Q}}(R/I^m) \rightarrow 0$$

As the map $LK_n^{\mathbb{Q}}(A) = K_n^{\mathbb{Q}}(R/I^\infty) \rightarrow K_n^{\mathbb{Q}}(A)$ is induced by the projection $R/I^\infty \xrightarrow{\sim} A$, it factors through the \lim term above. Hence the \lim^1 term is an obstruction for the surjectivity of the character $ch : K_{n+1}^{\mathbb{Q}}(A) \rightarrow HN_{n+1}(A)$.

5. Sheaf Theoretic Approach.

In this section we go back to the general setting of sections 1-3; we work in a fixed category C with deformations. In the previous section we have seen the virtues and pitfalls of the derived functor construction of section 3. The pitfall being mainly that $\underline{L}F$ may not exist for a given functor F . In this section we produce another object; the infinitesimal hypercohomology of a functor to simplicial sets. This construction is closely related to the derived functor construction, and has the advantage of being defined without any hypothesis on the functor in question.

5.0 Infinitesimal cohomology. Given $A \in C$ consider the category $inf(C/A)$ with as objects the deformations $B \xrightarrow{\sim} A$ and as maps $(B_0 \xrightarrow{\sim} A) \rightarrow (B_1 \xrightarrow{\sim} A)$ the maps $B_0 \rightarrow B_1 \in C$ making the diagram commute. Assume C -whence also $inf(C/A)$ - is small. We regard $inf(C/A)^{op}$ as a site with the indiscrete topology; i.e. as coverings we take the families $\{B \cong B'\}$ consisting of a single isomorphism. Thus a sheaf on $inf(C/A)^{op}$ is just any covariant functor on $inf(C/A)$. Recall that if G is a sheaf of abelian groups, then its cohomology groups are defined as the right derived functors of its global sections $\lim_{inf(C/A)} G$. By analogy, if X is a sheaf of simplicial sets, we define its hypercohomology as the right derived functor of its holim. Precisely, for each A the category $SSets^{inf(C/A)}$ is a closed model category ([BK, proof of XI 8.1]), so we take the total right derived functor:

$$H_{inf}(A, X) = \mathbf{R} \operatorname{holim}_{= inf(C/A)} X$$

Although this definition makes sense in general, we shall apply it for ssets with the property that $\pi_n X$ is an abelian group for all n . If X is fibrant (i.e. if $X(B)$ is fibrant for all $B \xrightarrow{\sim} A \in inf(C/A)$) this homotopy type is calculated by $H_{inf}(A, X) \cong \operatorname{holim}_{inf(C/A)} X$. If X is any sheaf, then $H_{inf}(A, X) = \operatorname{holim}_{inf(C/A)} X'$ where X' is any fibrant sheaf with a cofibration and weak equivalence $X \xrightarrow{\sim} X'$. For functors $X : C \rightarrow SSets$, this construction has properties in common with the derived functor of section 3. For example $H_{inf}(A, X)$ always maps deformations into isomorphisms. This follows from the cofinality theorem for holim, once one observes that if $f : B \xrightarrow{\sim} A$ is a deformation then $f_* : inf(C/B) \rightarrow inf(C/A)$ is left cofinal in the sense of [BK, XI 9.1]. In turn, the cofinality of f_* is immediate from the fact that, for each $\pi : E \xrightarrow{\sim} A \in inf(C/A)$, the pullback $P \xrightarrow{\sim} A$ of π along f is a final object of the over category f/π . Another common feature between L and H_{inf} is that, if X is fibrant, then $H_{inf}(A, X) = \operatorname{holim}_{inf(C/A)} X \rightarrow \operatorname{holim}_{inf(C/A)} X(A) \rightarrow X(A)$ is a natural map, and is a weak equivalence iff X maps deformations into weak equivalences. A difference between $H_{inf}(A, X)$ and LX is that the first one exists

independently of any property of X , appart from the smallness of C . Another is simply that $H_{inf}(A, X)$ does not have the universal property of LX . Namely if $Y \rightarrow X$ is a map of fibrant sheaves, we have a commutative diagram:

$$\begin{array}{ccc} H_{inf}(A, Y) & \longrightarrow & H_{inf}(A, X) \\ \downarrow & & \downarrow \\ Y(A) & \longrightarrow & X(A) \end{array}$$

If Y maps deformations into weak equivalences then the first vertical map is a weak equivalence. This implies that it has a –perhaps not natural– homotopy inverse, which we can compose with the first horizontal map to get a map $Y A \rightarrow H_{inf}(A, X)$ making the diagram commute up to homotopy. To have a natural homotopy inverse of the first vertical map we need that Y be cofibrant as an object of SSets^C , which is a very rare property. In fact the latter is a model category, so one can replace any object by one that is cofibrant in this sense. This means that the lifting exists in the homotopy category of SSets^C . But we do not want an object of $\text{Ho}(\text{SSets}^C)$, we want a true functor, i.e. an object of HoSSets^C . So in general H_{inf} and L are distinct. A more precise comparison between these is given in 5.5 below. We use the following indexing:

$$H_{inf}^n(A, X) := \pi_{-n} H_{inf}(A, X)$$

Lemma 5.1. (*Čech pro-covering*) *Let C be a –not necessarily small– category with deformations, and let $f : R \rightarrow A$ be a fibration. Consider the n -fold sum map $f^{*n} : R^{*n} \rightarrow A$; write $q_n = \ker f^{*n}$. If R is cofibrant, then the functor $\check{f} : \Delta \times \mathbb{N}^{op} \rightarrow inf(C/A)$, $(n, m) \mapsto R^{*n+1}/s(R^{*n+1}, q_{n+1})^m$ is left cofinal.*

Proof. We have to show that, for every object $\pi : B \xrightarrow{\sim} A \in inf(C/A) -B$ for short– the category \check{f}/B is null homotopic. By definition an object of \check{f}/B is a pair $(n, m) \in \Delta \times \mathbb{N}^{op}$ together with a map $\alpha : R^{*n+1}/s(R^{*n+1}, q_{n+1})^m \rightarrow B \in inf(C/A)$. Let $I = \ker \pi$, and let $r \geq 1$ such that $s(B, I)^r = 0$. Let $g : \Delta \times \Delta^r \rightarrow inf(C/A)$ be the restriction of \check{f} . We have a functor $\theta : \check{f}/B \rightarrow g/B$ given by $(n, m, \alpha) \mapsto (n, \min(m, r), \hat{\alpha})$, where $\hat{\alpha}$ is the map induced by passage to the quotient by $s(R^{*n+1}, q_{n+1})^{\min(m, r)}$. There is also a natural faithful inclusion $\iota : g/B \subset \check{f}/B$. We have $\theta \iota = 1$; quotient by $s(-, -)^{\min(-, r)}$ gives a natural map $1 \rightarrow \iota \theta$. Hence \hat{f}/B is homotopy equivalent to g/B . Next consider $h : \Delta \rightarrow inf(C/A)$, $n \mapsto g(n, r)$. Then $\theta' : (n, m, \alpha) \mapsto (n, \frac{R^{*n+1}}{s(R^{*n+1}, q_{n+1})^r} \xrightarrow{\sim} \frac{R^{*n+1}}{s(R^{*n+1}, q_{n+1})^m} \xrightarrow{\alpha} B)$ is a left inverse of the natural inclusion $\iota' : h/B \subset g/B$, and is equipped with a natural map $\iota' \theta' \rightarrow 1$. Hence g/B is weakly equivalent to h/B . Now the deformation $R^{*n}/s(R^{*n}, q_n)^r \xrightarrow{\sim} A$ is clearly the n -fold coproduct of $R/s(R, q_1)^r \xrightarrow{\sim} A$ in the category of those deformations $p : C \xrightarrow{\sim} A$ which satisfy $s(C, \ker p)^r = 0$. From this latter fact, the definition of h/B , the definition of the Grothendieck construction and the fact that the latter is the homotopy colimit in CAT cf. [T1,1.2], it follows immediately that $h/B = \text{hocolim}_{\Delta^{op}} (n \mapsto S^{n+1})$. Here $S = \text{hom}_{inf(C/A)}(R/s(R, q_0)^r, B)$ –which is a nonempty set because R is cofibrant– is thought of as a discrete category. Hence taking nerves, $N(h/B) \approx \text{hocolim}_{\Delta^{op}} (n \mapsto NS^{n+1})$ (by [T1, 1.2]). But as S is discrete this last hocolim is nothing but the simplicial set $n \mapsto S^{n+1}$ which is null homotopic. \square

Corollary 5.2. *Let C be a category with deformations having sufficient cofibrant objects, and $X : C \rightarrow \mathbb{S}\text{Sets}$ a functor. Assume $X(A)$ is fibrant for all $A \in C$. Given $A \in C$ choose a fibration $f : RA \rightarrow A$ with RA cofibrant, and consider $H_{inf}(-, X) : A \mapsto \text{holim}_{\Delta \times \mathbb{N}^{op}} X\check{f}$. Then $H_{inf}(-, X)$ is a functor $C \rightarrow \text{Ho } \mathbb{S}\text{Sets}$, is independent of the choices made in its definition, maps deformations to isomorphisms and is equipped with a natural map $H_{inf}(-, X) \rightarrow X$. Further, if $A \in C' \subset C$ is a small subcategory of interest containing a fibration $R \rightarrow A$ with R cofibrant, then $H_{inf}(A, X)$ is the sheaf cohomology of X on the infinitesimal site $\text{inf}(C'/A)$ as defined in 5.0 above.*

Proof. Straightforward from the cofinality theorem for holim ([BK, XI 9.2]) and the discussion 5.0. \square

5.3 Spectral Sequence. Let C be as in 5.2 above, and let X be a sheaf of fibrant ssets. We assume $\pi_0 X, \pi_1 X$ are abelian groups. Fix a fibration $f : R \rightarrow A$ as in 5.2 and form the homotopy limit $\text{holim}_{\Delta \times \mathbb{N}^{op}} X\check{f}$. By [BK XI 4.3], the latter is isomorphic to $\text{holim}_{\Delta}(n \mapsto \text{holim}_{\mathbb{N}^{op}}(m \mapsto X(R^{*n+1}/s(R^{*n+1}, q_{n+1})^m))$. Thus by [BK, X 7.2] we have a spectral sequence

$$(18) \quad E_2^{r,s} = \pi^r \pi_{-s} \text{holim}_{\mathbb{N}^{op}}(m \mapsto X\check{f}(-, m))$$

for $0 \leq r \leq -s$, which, if $E_2^{r,-r} = 0$, converges conditionally to $H_{inf}^{r-s}(A, X)$. If X is a sheaf of fibrant spectra then the fringe constraints disappear, and the sequence is always conditionally convergent. A closely related spectral sequence is obtained as follows. Write $H_{inf}(A, X) = \text{holim}_{\text{inf}(C'/A)} X$ where $C' \subset C$ is any small subcategory as in 5.2 above. Then by [BK, XI 7.1], we also have a spectral sequence:

$$(18') \quad E_2'^{r,s} = H_{inf}^r(A, \pi_{-s} X)$$

with properties similar to those of (18). This may be regarded as a trivial kind of Atiyah-Hirzebruch-Brown-Gersten ([BO]) spectral sequence. The sequences (18) and (18') agree in some cases; this is discussed in 5.4.1 below.

Lemma 5.4. *The groups $E_2^{r,s}$ are independent of the choice of f .*

Proof. Let $f : R \rightarrow A$ be a fibration with R cofibrant. Let $f \in C' \subset C$ be a small subcategory with deformations. Consider the category $D(A)$ having as objects the fibrations $\rightarrow A \in C'$ and as maps $(\alpha_0 : B_0 \rightarrow A) \rightarrow (\alpha_1 : B_1 \rightarrow A)$ the pro-maps $B_0/s(B_0, \ker \alpha_0)^\infty \rightarrow B_1/s(B_1, \ker \alpha_1)^\infty$. Give $D(A)^{op}$ the indiscrete topology, and consider $\alpha \mapsto \pi_s \text{holim}_{\mathbb{N}^{op}} X(B/s(B, \ker \alpha)^\infty)$ as a sheaf π_s of abelian groups on $D(A)^{op}$. Then, by 2.1, we have $E_2^{0,s} = H^0(D(A), \pi_{-s})$. On the other hand the proof of [A, 3.1] shows that for $r \leq 1$ the groups $E_2^{r,*}$ vanish on injectives. Summing up, $E_2^{r,s} = H^{-r}(D(A), \pi_{-s})$, and the lemma follows. \square

Remark 5.4.1. If the shrinking functor is idempotent, then the argument of the proof of the lemma above shows that $E_2^{r,s} = E_2'^{r,s}$. In general if we have a deformation category structure in $\text{pro-}C$ extending that of C , (as in 1.7), then, again by the proof of the lemma $E_2^{r,s} = H^{-r}(D(A), \pi_{-s})$ the cohomology

of the sheaf $B \rightarrow \pi_s \operatorname{holim}_I X B_i$ on $\operatorname{inf}(\operatorname{Pro}\text{-}C/A)$. If in addition the functor $C \rightarrow \text{Abelian Groups}$, $A \mapsto \pi_{-s}X(A)$ maps deformations into surjections, then $H_{\operatorname{pro}\text{-}\operatorname{inf}}^*(A, \pi_{-s}X) \cong H_{\operatorname{inf}}^*(A, \pi_{-s}X)$, whence $E_{*,s}^2 = E'_{*,s}^2$. In fact in general if $G : C \rightarrow \text{Ab}$ is any functor and $f : R \rightarrow A$ is a fibration with cofibrant source, then the complex associated with the Čech pro-covering \check{f} :

$C(A, G) :$

$$\lim G(R/s(R, q_1)^n) \xrightarrow{\partial} \lim G(R^{*2}/s(R^{*2}, q_2)^n) \xrightarrow{\partial} \lim G(R^{*3}/s(R^{*3}, q_3)^n) \xrightarrow{\partial} \dots$$

computes $H_{\operatorname{inf}}^*(A, G)$. This can be seen e.g. by mimicking the proof that the Čech complex computes presheaf cohomology [A, 3.1]. The complex C can also be used to compute infinitesimal hypercohomology of some chain complexes. Indeed if $G : C \rightarrow ((\text{positively graded chain complexes of abelian groups}))$ is a functor which maps deformations into surjections, then we have a weak equivalence $M(H_{\operatorname{inf}}(A, SG)) \xrightarrow{\sim} \operatorname{Tot}_*(C(A, G))$. Here $C(A, G)$ is regarded as a (second quadrant) double complex, Tot is the total chain complex, $S : \text{Chain Complexes} \rightarrow \text{Abelian Groups}$ is the Dold-Kan functor considered in 4.2 above and M is its left adjoint, the (Moore) normalized chain complex. This fact follows from a long chain of homotopy equivalences which essentially use that S , having a left adjoint (namely M), preserves limits, that it maps surjections to fibrations, that $MS = 1$, and that $\operatorname{holim} = \operatorname{lim}$ for both towers of fibrations (by [BK, XI 4.1-v]) and cosimplicial abelian groups (by combining [BK, XI 4.4, X 4.9, 4.3, and 5.2-ii]).

Proposition 5.5. *Let C be a category with deformations with idempotent shrinking functor. Let $X : C \rightarrow \text{Fibrant SSets}$ be a functor. Assume the $\pi_n X$ are abelian groups for all n . Write $\check{X}(A) = X1_A$ for the composite of X with the Čech pro-covering 5.1 induced by the identity map. Then*

$$H_{\operatorname{inf}}(A, X) = \underline{\underline{\mathbf{L}\check{X}}}$$

as homotopy types. If in addition, $\underline{\underline{\mathbf{L}\mathbf{X}}}$ exists and $\underline{\underline{\mathbf{L}\mathbf{X}}}\gamma(R) = \mathbf{X}(R)$ for cofibrant objects R , then $H_{\operatorname{inf}}(A, X) = \underline{\underline{\mathbf{L}\mathbf{X}}}(A)$. The latter occurs iff the natural map $H_{\operatorname{inf}}^0(R, \pi_n X) \xrightarrow{\cong} \pi_n XR$ is an isomorphism for cofibrant R and $n \geq 0$. In such case the E_2 -term of the spectral sequence (18) vanishes for $r \neq 0$.

Proof. The first assertion follows from 5.2 and 3.1-iv). If $\underline{\underline{\mathbf{L}\mathbf{X}}}$ exists and agrees with \mathbf{X} on cofibrants, then the cosimplicial group $n \mapsto \pi_r(R^{*n+1}/s(R^{*n+1}, q_{n+1}))$ is constant for cofibrant R . Hence the spectral sequence vanishes outside the y -axis. It follows that the natural map $H_{\operatorname{inf}}(R, X) \rightarrow XR$ is a weak equivalence, and $H_{\operatorname{inf}}(A, X) \cong \underline{\underline{\mathbf{L}\mathbf{X}}}(A)$. The remaining assertion follows from the fact that, by 5.4.1, $E'_2 = E_2$ in this case. \square

Remark 5.6. (Stratifying site and homotopy) The proposition above suggests an interpretation of H_{inf} in terms of homotopy, as defined in section 1. Indeed we may think of \check{X} as a homotopization of X . The construction $X \mapsto \check{X}$ transforms any functor X into one which maps deformation retractions into weak equivalences. In making the construction \check{X} we are forcing X into a functor which satisfies the conditions of 2.1. Back to the sheaf theoretic approach, we may interpret

$\tilde{X}(A)$ as the sheaf hypercohomology on the stratifying site $\text{strat}(X/A)$, i.e. the site consisting of those deformations $\tilde{\rightarrow} A$ which are split. Note that A is cofibrant iff $\text{strat}(C/A) = \text{inf}(C/A)$. In terms of the stratifying site, the proposition above states that $H_{\text{inf}}(A, -) = H_{\text{strat}}(R/s(R, I)^\infty, -)$ for any given fibration $R \rightarrow A$ with cofibrant source having kernel I .

Application 5.7. (Tautological Character) Let C be a category with deformations having sufficient cofibrants. Assume for simplicity that there is a functorial choice of fibration $\pi^A : RA \rightarrow A$ with RA cofibrant, so that, given a functor $X : C \rightarrow \text{Fibrant Spectra}$, we can regard $H_{\text{inf}}(A, X) = \text{holim}_{\Delta \times \mathbb{N}^{\text{op}}} X \tilde{\pi}^A$ as a functor to fibrant spectra, rather than just homotopy types. Here by a fibrant spectrum we mean a sequence $X = \{X^n : n \geq 0\}$ of fibrant ssets and weak equivalences $X^n \xrightarrow{\sim} \Omega X^{n+1}$. Write $\tau X^n := (\text{hofiber}(H_{\text{inf}}(A, X^{n+1}) \rightarrow X^{n+1}))$. Then we have a map $c^\tau : X \rightarrow \tau X$; we call this the tautological character. By definition it induces a weak equivalence $X(B \tilde{\rightarrow} A) := \text{hofiber}(XB \rightarrow XA) \xrightarrow{\sim} \tau X(B \tilde{\rightarrow} A)$ of the relative spaces of a deformation. Hence c^τ satisfies a tautological Goodwillie theorem (2). Of course the identity $X = X$ has the same property; unlike the identity however, the map c^τ is universal in the following sense. If $c : X \rightarrow Y$ is another map (character) for which the Goodwillie theorem holds, then from the fact that holim preserves fibration sequences, it follows that the induced map $\tau X \xrightarrow{\sim} \tau Y$ is a weak equivalence. Hence c^τ factors through c . In other words the tautological character is a coarser invariant than any other character with a Goodwillie theorem.

6. Infinitesimal K -theory.

The purpose of this section is to apply the infinitesimal hypercohomology machine to rational K -theory. First of all, choose any connective functorial spectrum $K(A)$ with homotopy $\pi_n K(A) = K_n(A)$, $n \geq 0$. Then set $K^\mathbb{Q} = \mathbb{Q}_\infty K$. Most of the spectral sequence (18) can be computed immediately using 4.3 and 5.5; we have:

$$(19) \quad E_2^{rs} = \begin{cases} LK_{-s}^\mathbb{Q}(A) & r = 0, s \leq -2 \\ K_0^\mathbb{Q}(A) & r = s = 0 \\ 0 & r \neq 0, s \neq -1 \end{cases}$$

It follows that

$$H_{\text{inf}}^{-n}(A, K^\mathbb{Q}) = LK_n^\mathbb{Q}(A) \quad n \geq 2$$

Note also that, with the possible exception of the map $d_2 : K_0^\mathbb{Q}(A) \rightarrow E_2^{2,-1}$, all spectral differentials are zero. Hence, for a full computation it suffices to compute this map and the terms $E_2^{r, -1}$ $r \geq 0$. By 5.4.1, we have:

$$(20) \quad E_2^{r, -1} = H_{\text{inf}}^r(A, K_1^\mathbb{Q})$$

Next we use Goodwillie's theorem (16) to relate the groups (20) with the corresponding groups for negative cyclic homology. We have a commutative diagram with exact rows:

$$(22) \quad \begin{array}{ccccccc} 0 & \rightarrow & D(A, K_1^\mathbb{Q}) & \longrightarrow & C(A, K_1^\mathbb{Q}) & \longrightarrow & K_1^\mathbb{Q}(UA) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & D(A, \text{HN}^\mathbb{Q}) & \longrightarrow & C(A, \text{HN}^\mathbb{Q}) & \longrightarrow & \text{HN}^\mathbb{Q}(UA) \rightarrow 0 \end{array}$$

Here the terms on the right hand side are the cochain complexes associated to the constant cosimplicial objects, and the D -terms are the kernels of the projections. The vertical maps are induced by the character of Jones-Goodwillie. From 5.4.1, 5.6, and the long cohomology sequence of (22), we get the isomorphism:

$$(23) \quad H_{inf}^r(A, K_1^{\mathbb{Q}}) \cong H_{inf}^r(A, HN_1^{\mathbb{Q}}) \quad r \geq 2$$

Conjecture 6.0: $H_{inf}^n(A, HN_1^{\mathbb{Q}}) = 0$ for $n \geq 1$.

The conjecture is not stated for $r = 0$ because in this case we prove it below, so it is not a conjecture. We shall justify the conjecture by showing that several things one expects should happen depend on its validity. For instance note that the groups (23) determine the negative homotopy of both $H_{inf}(A, K^{\mathbb{Q}})$ and $H_{inf}(A, N^{\mathbb{Q}})$, where the latter is the hypercohomology of the connective spectrum associated by Dold-Kan to the rational negative cyclic complex truncated below zero. Hence the conjecture implies that this negative homotopy groups are zero. Moreover we have a commutative diagram:

$$(24) \quad \begin{array}{ccccc} H(A, K^{\mathbb{Q}}) & \longrightarrow & K^{\mathbb{Q}}(A) & \xrightarrow{c_\tau} & \tau K^{\mathbb{Q}}(A) \\ H(-, ch) \downarrow & & \downarrow ch & & \downarrow \tau ch \\ H(A, N^{\mathbb{Q}}) & \longrightarrow & N^{\mathbb{Q}}(A) & \longrightarrow & \tau N^{\mathbb{Q}}(A) \end{array}$$

Here c_τ is the tautological character, the columns are induced by the character of Jones-Goodwillie and the last of these is a weak equivalence by (16) and 5.7. Hence for ch to be the tautological character we need $H_{inf}^{*n}(A, HN_1) = 0$ for all n , i.e. we need the conjecture to hold for A . We prove next (Lemma 6.1) that $H_{inf}^0(A, HN_1) = 0$. In theorem 6.2 below we use this vanishing result to extend the long exact sequence of theorem 4.3 so as to include K_1 .

Lemma 6.1. $H_{inf}^0(A, HN_1^{\mathbb{Q}}) = 0$.

Proof. It suffices to prove the lemma for \mathbb{Q} -algebras. Consider the functor $\Omega_{\mathfrak{h}}^1$, $A \mapsto \Omega^1 A/[A, \Omega^1 A]$. I claim it suffices to show that $H_{strat}^0(A, \Omega_{\mathfrak{h}}^1) = 0$ for all \mathbb{Q} -algebras A . To prove the claim, proceed as follows. Note that, for a quasi-free pro-algebra, the pro-cyclic mixed complex is homotopy equivalent to the pro-de Rham mixed complex $MX = (\Omega^0 \oplus \Omega_{\mathfrak{h}}^1, b, d_{\mathfrak{h}})$. Hence $H_{inf}^0(A, HN_1^{\mathbb{Q}}) \cong \pi_1 H_{inf}(A, NX)$, the hypercohomology of the negative cyclic complex of MX . As $NX_1 = \Omega_{\mathfrak{h}}^1$ and is zero in higher degrees, it follows that $\pi_1 H_{inf}(A, NX) \subset H_{inf}^0(A, \Omega_{\mathfrak{h}}^1)$. Now by 3.7, 5.4.1, and 5.6, we have $H_{inf}^0(A, \Omega_{\mathfrak{h}}^1) \cong H_{strat}^0(TA/JA^\infty, \Omega_{\mathfrak{h}}^1)$. The claim follows from the fact that \lim preserves kernels. To prove $H^0(A, \Omega_{\mathfrak{h}}^1) = 0$, we must show that if $\omega \in \Omega^1 A$ is a 1-form such that the class of $(\partial_0 - \partial_1)\omega$ in $\Omega^1(QA/qA^n)_{\mathfrak{h}}$ is zero for all $n \geq 0$, then $\mathfrak{h}\omega = 0 \in \Omega^1 A_{\mathfrak{h}}$. We shall show something stronger, namely if $(\partial_0 - \partial_1)\omega$ is zero in $\Omega^1(QA/qA^2)_{\mathfrak{h}}$ then $w_{\mathfrak{h}} = 0$. We have $QA/qA^2 = A \oplus \Omega^1 A$ and $\Omega^1(A \oplus \Omega^1 A) = \Omega^1 A \oplus \hat{A} \otimes_{\mathbb{Q}} \Omega^1 A \oplus \Omega^1 \otimes_{\mathbb{Q}} A \oplus \Omega^1 A \otimes_{\mathbb{Q}} \Omega^1 A$. We get $\Omega^1(A \oplus \Omega^1 A)_{\mathfrak{h}} \cong \Omega^1 A_{\mathfrak{h}} \oplus \hat{A} \otimes_{\mathbb{Q}} \Omega^1 A_{\mathfrak{h}} \oplus \Omega^2 \Omega^1 A_{\mathfrak{h}}$. Here M is the subspace generated by the elements of

the form $1 \otimes b\omega a - b \otimes \omega a - a \otimes b\omega + ab \otimes \omega$ and of the form $1 \otimes [b, \omega]$. The relation which permits eliminating the factor $\Omega^1 A \otimes A$ is

$$(25) \quad \omega \otimes a = 1 \otimes aw - a \otimes \omega$$

It follows that the left multiplication map $\tilde{A} \otimes_{\mathbb{Q}} \Omega^1 A \rightarrow \Omega^1 A$ induces a projection $\Omega^1(A \oplus \Omega^1 A)_{\natural} \rightarrow \Omega^1 A_{\natural}$. Let $\Omega^1 A \ni \omega = \sum a_i db_i$ ($a_i \in \tilde{A}$, $b_i \in A$). Then $p(\partial_0 - \partial_1)\omega = p(\sum da_i \otimes b_i + a_i \otimes db_i + da_i \otimes db_i) = \omega_{\natural}$ by (25). \square

Theorem 6.2. *Let A be a ring. Then:*

- i) *The tautological character $c_n^{\tau} : K_n^{\mathbb{Q}}(A) \rightarrow \tau K_n^{\mathbb{Q}}(A)$ coincides with the Jones-Goodwillie character $ch_n : K_n^{\mathbb{Q}}(A) \rightarrow HN_n^{\mathbb{Q}}(A)$ for $n \geq 2$. For $n = 1$, the map c_1^{τ} factors as ch_1 followed by an injection.*
- ii) *There is a natural map of spectra $f : \text{hofiber}(ch : K^{\mathbb{Q}}(A) \rightarrow N^{\mathbb{Q}}(A)) \rightarrow H_{inf}(A, K^{\mathbb{Q}})$ such that the induced map $\pi_n(f)$ is an isomorphism for $n \geq 1$.*
- iii) *For $n \geq 2$ the infinitesimal hypercohomology groups agree with the derived functor groups of Theorem 4.3; we have $H_{inf}^{-n}(A, K^{\mathbb{Q}}) = LK_n^{\mathbb{Q}}(A)$. The long exact sequence 4.3-i) extends to the right as follows:*

$$K_2^{\mathbb{Q}}(A) \rightarrow HN_2(A) \rightarrow H_{inf}^0(A, K_1^{\mathbb{Q}}) \rightarrow K_1^{\mathbb{Q}}(A) \rightarrow HN_1(A) \rightarrow H^1(A, K_1^{\mathbb{Q}})$$

If furthermore, conjecture 6.0 holds for A , then the last map above is surjective, and $H_{inf}^n(A, K) = \tau_n K = 0$ for $n < 0$.

Proof. It follows from the lemma above and the exact homotopy sequence of the commutative diagram of fibrations (24). \square

Infinitesimal v. De Rham cohomology 6.2. A theorem of Grothendieck [Dix, Th. 4.1] establishes that for a smooth scheme of characteristic zero, its de Rham cohomology is the same as the infinitesimal cohomology of the structure sheaf:

$$(26) \quad H_{dR}^*(X) \cong H_{inf}^*(X, O_X)$$

We note that, even for a smooth affine scheme $\text{Spec} A / \text{Spec} \mathbb{Q}$, the infinitesimal site in the sense of Grothendieck is larger than the infinitesimal site of A as an object of the category with deformations of commutative \mathbb{Q} -algebras. However the resulting cohomologies are the same, cf. [Dix, isomorphism (*) on page 338]. Next we investigate a non commutative analogue of Grothendieck's theorem. The non-commutative analogue of de Rham cohomology we use is the cohomology of the complex:

$$(27) \quad DR : \quad HH_0(A) \xrightarrow{Bi} HH_1(A) \xrightarrow{Bi} HH_2(A) \xrightarrow{Bi} \dots$$

Theorem 6.3. (Compare (26)) *Let A be a quasi-free \mathbb{Q} -algebra. Then there is a natural map*

$$H_{DR}^n(A) \rightarrow H_{inf}^n(A, O/[O, O])$$

(Here we write $O/[O, O]$ for HH_0 to emphasize the relation of this statement with Grothendieck's theorem (26).) *This map is an isomorphism for $n = 0$. If conjecture 6.0 holds for A then it is an isomorphism for all n .*

Proof. Consider the hypercohomology of the complex (27). As A is quasi-free, this can be calculated from the Čech pro-covering associated with the identity $A = A$. Thus $H_{inf}(A, DR)$ is the cohomology of the double cochain complex with as n -th column the DR -complex of the quasi-free pro-algebra $P^n = A^{*n+1}/q_{n+1}^\infty$. Here $HH_r(P^n) = H_r(\lim_m C(P_m^n))$ is the homology of the limit of the Hochschild complexes. As P^n is quasi-free, we have $HH_r(P^n) = 0$ for $r \geq 2$. On the other hand, by the tubular neighborhood theorem ([CQ1, Th2]) $P^n \cong T_{\tilde{A}}(M_n)/\langle M_n \rangle^\infty$, where $M_n = \Omega^1 A \oplus \dots \oplus \Omega^1 A$ (n factors). It is not hard to see that in general, if $B = B_0 \oplus B_1 \oplus \dots$ is a graded algebra, then for each r the pro-vector space $\{HH_r(B/B_+^m) : m \in \mathbb{N}\}$ is isomorphic to the pro-vector space $\{\oplus_{i=0}^m HH_r^{deg=i}(B) : m \in \mathbb{N}\}$ which is clearly Mittag-Leffler. In our case, this implies that $HH_*(P^n) = \lim_m HH_*(P_m^n)$. Hence $H_{inf}^*(A, DR)$ is the cohomology of the double cochain complex:

$$(28) \quad \begin{array}{ccccc} HH_1(A) & \longrightarrow & \lim_m HH_1(P_m^1) & \longrightarrow & \lim_m HH_1(P_m^2) & \longrightarrow \\ d \uparrow & & d \uparrow & & d \uparrow & \\ HH_0(A) & \longrightarrow & \lim_m HH_0(P_m^1) & \longrightarrow & \lim_m HH_0(P_m^2) & \longrightarrow \end{array}$$

By 5.4.1, the rows of this complex compute $H_{inf}^*(A, HH_i)$. Hence we have a spectral sequence:

$$(29) \quad E_1^{r,s} = H_{inf}^r(A, HH_s) \implies H_{inf}^{r+s}(A, DR)$$

On the other hand if we first take cohomology of the columns, and then of the rows, we get $HP_i(A)$ in the $(0, i)$ entry and zero elsewhere. Hence $H_{inf}^n(A, DR) = HP_n(A)$ for $n = 0, 1$ and is zero if $n \geq 2$. The theorem now follows from the convergence of the spectral sequence (29), and the fact that $\lim_m HH_1(P_m^n) = HH_1(P^n) = HN_1(P^n) = \lim_m HN_1(P_m^n)$. \square

7. Categorical character.

Throughout this section C is a fixed small category with deformations having sufficient cofibrant objects. We consider functors from C to CAT , the large category of all small categories.

Homotopy limits and colimits in CAT 7.0. By the *homotopy colimit* of a functor $F : I \rightarrow \text{CAT}$ ($I \in \text{CAT}$) we mean the Grothendieck construction:

$$\text{hocolim } F := \int F$$

and by the *homotopy limit* we mean the pullback:

$$\begin{array}{ccc} \text{holim}_I F & \longrightarrow & 0 \\ \downarrow & & \downarrow id \\ \text{HOM}(I, \text{hocolim}_I F) & \xrightarrow{\pi_*} & \text{HOM}(I, I) \end{array}$$

Here HOM is the functor category, π_* is induced by the natural projection $\pi : \text{hocolim}_I F \rightarrow I$, 0 is the category with only one map and id maps the only object of 0 to the identity functor. In other words, $\text{holim}_I C$ is the category whose objects are the functors $s : I \rightarrow \text{hocolim}_I C$ such that $\pi s = 1$ and whose maps are the natural transformations which project to identity maps through π . Thomason ([T1]) showed that, upon taking nerves, the hocolim defined above has the same homotopy type as its simplicial counterpart, i.e. $N \text{hocolim}_I F \approx \text{hocolim}_I NF$. On the other hand it was observed in [L, p74] that there is an isomorphism of simplicial sets $N \text{holim}_I F \cong \text{holim}_I NF$. By definition a map of categories is a weak equivalence if its nerve is a weak equivalence of SSets . Hence by [BK, XII 4.2], hocolim preserves weak equivalences. Similarly holim preserves weak equivalences between categories having fibrant nerve, by [BK, XI 5.6]. Moreover it is proven in [L, Th. 1] that holim maps adjoint functors to adjoint functors, and thus to weak equivalences. Here is a description of both $\text{hocolim}_I F$ and $\text{holim}_I F$ in terms of objects and arrows. An object of $\text{hocolim}_I F$ is a pair $x_i := (i, x)$ where $i \in I$ and $x \in F_i := F(i)$. A map $x_i \rightarrow y_j$ is a pair (α, ρ) with $\alpha : i \rightarrow j \in I$ and $\rho : \alpha x \rightarrow y \in F_j$. Here we abbreviate $F(\alpha)(\rho)(\mu)$ as $\alpha(\rho)\mu$. Hereafter we shall omit F 's and $(,)$'s whenever no confusion is possible. Composition is defined as in a semidirect product: $(\alpha, \rho)(\beta, \mu) = (\alpha\beta, \rho\alpha(\mu))$. On the other hand, $\text{holim}_I C$ is the category of all pairs of families $(x, \rho) := (\{x_i\}_{i \in I}, \{\rho_\alpha\}_{\alpha \in I})$, indexed respectively by the objects and the maps of I , where $x_i \in F_i$, and if $\alpha : i \rightarrow j$ is a map in I , then $\rho_\alpha : \alpha x_i \rightarrow x_j$ is a map in F_j . The family of ρ 's is subject to the conditions:

$$\rho_1 = 1 \quad \rho_{\alpha\beta} = \rho_\alpha \alpha(\rho_\beta)$$

In the first identity, the 1 on the left is an identity map $1 : i \rightarrow i$ and the 1 on the right is the identity of x_i in F_i ; in the second identity, $i_0 \xleftarrow{\alpha} i_1 \xleftarrow{\beta} i_2$ are composable maps in I and the identity is of maps $\alpha\beta x_{i_2} \rightarrow x_{i_0}$ in F_{i_0} . A map $f : (x, \rho^x) \rightarrow (y, \rho^y)$ in $\text{holim}_I F$ is a family of maps $f_i : x_i \rightarrow y_i \in F_i$ indexed by the objects of I such that the following diagram commutes for every map $\alpha : i \rightarrow j \in I$:

$$\begin{array}{ccc} \alpha x_i & \xrightarrow{f_i} & \alpha y_i \\ \rho^x \downarrow & & \downarrow \rho^y \\ x_j & \xrightarrow{f_j} & y_j \end{array}$$

Sheaves of objects of a functor $F : C \rightarrow \text{CAT}$ 7.1. Let C be a small category with deformations and let $F : C \rightarrow \text{CAT}$ be a functor. Write:

$$F_{\text{inf}}(A) := \text{holim } F$$

We call $F_{inf}(A)$ the category of *sheaves of objects* of F on $inf(C/A)$. If F is constant, we recover the usual notion of sheaves as functors $inf(C/A) \rightarrow F(A)$. Because holim commutes with nerves, we have a map:

$$NF_{inf}(A) \rightarrow H_{inf}(A, NF)$$

which is natural up to homotopy. This map is a weak equivalence if NF is fibrant, i.e. if F takes values in the category of small groupoids=categories where every map is an isomorphism, but not in general. We note that as holim preserves adjointness, the functor F_{inf} maps deformations into weak equivalences. Moreover we have a natural evaluation map $\epsilon : F_{inf}(A) \rightarrow F(A)$, $(x, \rho^x) \mapsto x_A$, $\tau \mapsto \tau_A$. It is straightforward to check that $F_{inf inf} \cong F_{inf}$, whence ϵ is (uni)versal among natural maps $G \rightarrow F$ where $G_{inf} \cong G$. In other words the construction $F \mapsto F_{inf}$ has properties similar to those of $H_{inf}(-, -)$.

Example 7.2. (Stratified objects) Next we give an explicit description of $F_{inf}(A)$ for a groupoid functor F and cofibrant A , up to category equivalence. In the case when C is the category of commutative algebras of finite type over a field, and $F(A)$ is the isomorphism category of finitely generated modules we recover [BO, Prop. 2.11] (see also [Dix, 4.2]). Write $\partial_i^n : A \rightarrow P_n^1 = A * A/s(A, qA)^n$ for the i -th coface map ($i = 0, 1$). Given an object $(x, \rho^x) \in F_{inf}$, consider the family of isomorphisms:

$$\epsilon_n : \partial_0^n x \xrightarrow{\sim} \partial_1^n x \in F(P_n^1)$$

defined by $\epsilon_n = \rho_{\partial_1}^{-1} \rho_{\partial_0}$. Then:

- i) $\epsilon_0 = 1$
- ii) The family $\{\epsilon_n : n \geq 0\}$ is compatible with the maps $P_{n+1}^1 \xrightarrow{\sim} P_n^1$.
- iii) (Cocycle Condition) The following identity holds:

$$\partial_2(\epsilon_n) \partial_0(\epsilon_n) = \partial_1(\epsilon_n)$$

We call the family of ϵ a *stratification* on x . Conversely, an object $x \in FA$ with a stratification ϵ_* , yields an object of $F_{inf}(A)$ as follows. Given $B \xrightarrow{\sim} A \in inf(C/A) = strat(C/A)$ choose a section s_B and set $x_B = s_B x$. If $\alpha : B \rightarrow C \in inf(C/A)$ is a map, let $h : P_n^1 \rightarrow C$ be a homotopy $\alpha s_B \equiv s_C$, and set $\rho_\alpha = h(\epsilon_n) : \alpha x_B \rightarrow x_C$. It is straightforward to check that these assignments define mutually inverse equivalences between $F_{inf} A$ and the category of objects of FA equipped with a stratification. The same argument shows that, even if A is not cofibrant –but F is still a groupoid– then the category of stratified objects is equivalent to $F_{strat} = \text{holim}_{strat(C/A)} F$ (compare [Dix, 4.2]).

Categorical Character 7.3. Let M be a permutative monoidal category, and consider the simplicial category $M^+ : n \mapsto M^{n+2}$ of [T1, 4.3.1]. Form the fibrant spectrum $n \mapsto Sp_n M$ of simplicial sets associated to the topological spectrum constructed in [T1, 4.2.1]; the 0-th space of this fibrant spectrum has the weak homotopy type of the nerve of $\text{hocolim}_{\Delta^{op}} M^+$, which is a categorical model for the group completion of the realization $BM = |NM|$. Set:

$$KM : Sp_n M$$

Now suppose $f : M \rightarrow N$ is a functor of permutative monoidal categories, preserving products in the strong sense, i.e. up to natural isomorphism, and assume M is a groupoid. Then by [T2 5.2] there is another permutative category $P(f)$ and a functor $N \rightarrow P(f)$ such that the sequence of base spaces $Sp_0 M \rightarrow Sp_0 N \rightarrow Sp_0 P(f)$ is a fibration up to homotopy. Use this fibration sequence to define relative groups as follows. Let $A \mapsto MA$ be a functor going from a category C with deformations into the category of (small) permutative monoidal categories which are groupoids, with as maps the strong product preserving functors. Given $f : A \rightarrow A' \in C$, write:

$$KM(f) := KP(f) \quad \text{and} \quad KM^{rel} A = KP(M_{inf} A \rightarrow MA)$$

Tautology 7.4. *Let C be a category with deformations. Let $M : C \rightarrow \text{CAT}$ be a functor, mapping objects to groupoids which are permutative monoidal categories and maps to functors preserving products in the strong sense, i.e. up to natural isomorphism. Then there is a natural character $c : KM(A) \rightarrow KM^{rel}(A)$ which is induced by a map of categories, has $KM^{inf} A$ as fiber, and is such that any deformation $f : B \xrightarrow{\sim} A \in C$ induces an isomorphism of the relative groups $KM_n(f) \cong KM_n^{rel}(f)$ $n \geq 0$. This character fits into a homotopy commutative diagram with homotopy fibration rows:*

$$\begin{array}{ccccc} KM_{inf} A & \longrightarrow & KMA & \xrightarrow{c} & K(M^{rel} A) \\ \downarrow & & \downarrow 1 & & \downarrow \\ H_{inf}(A, KM) & \longrightarrow & KMA & \xrightarrow{c^\tau} & \tau KMA \end{array}$$

Here the bottom row is the homotopy sequence of the tautological character 5.7.

Remark 7.5. In the discussion above, we applied the *inf* construction *before* group completing; we considered $\text{hocolim}_{\Delta^{op}} M_{inf}^+$. The reader may wonder what happens if in place of the latter category we use $(\text{hocolim}_{\Delta^{op}} M^+)_{inf}$. The fact is that the two categories are homotopy equivalent. This is a problem of interchanging holim and hocolim , which in general is not possible, but which in the particular case where the index category of holim has a final object, yields homotopy equivalent spaces (cf. [C2]).

7.6 Case of rings: K -theory of connections. The tautology above applies notably in the case when C is a category of rings and $MA = \coprod Gl_n A$. For A cofibrant, $(\coprod Gl_n)_{inf} A$ can be described, upon the identification 7.2, as the category having as objects the pairs $(m, n \mapsto \epsilon_n)$ where the first coordinate is a non-negative integer and the second is a family of matrices $\epsilon_n \in Gl_m(P_n^1)$ satisfying i)-iii) above. There is no map $(m, \epsilon) \rightarrow (r, \theta)$ unless $r = n$; if $r = n$ a map is a matrix $\alpha \in Gl_m(A)$ with $\theta \partial_0(\alpha) = \partial_1(\alpha)\epsilon$. In the case when C is the category of commutative algebras essentially of finite type over a field of characteristic zero and $A \in C$ is smooth, a stratification on A is the same thing as a flat connection (cf. [BO]). Thus the category $(\coprod Gl_n)_{inf} A$ is identified with the category having as objects the pairs (n, ∇) where ∇ is a flat connection on A^n . There are no maps $(n, \nabla) \rightarrow (n', \nabla')$ if $n \neq n'$, and if $n = n'$ a map is a matrix $\alpha \in Gl_n(A)$ such that $\nabla' \alpha = \alpha \nabla$. We

remark that isomorphism classes of connections in this sense are the classical gauge equivalence classes [BL]. Next consider the case when C is some small category of associative but not necessarily commutative algebras over a characteristic zero field, and let A be quasi-free. As indicated above objects of $(\coprod Gl_n)_{inf} A$ are stratified free modules (m, ϵ) . Because ϵ_0 is to be the identity, the map $\epsilon_1 : P_1^1 = A \oplus \Omega^1 A \rightarrow A \oplus \Omega$ is necessarily of the form $\epsilon_1 = 1 + \nabla$ where ∇ is a right connection on A^m . I tried to prove that the cocycle condition is equivalent to this right connection being flat, but quit overwhelmed by the horrendous calculations. I got as far as proving that flatness is equivalent to extending ∇ to a cocycle ϵ_2 . The cocycle identity for ϵ_3 already takes several pages to write down. This seems to indicate one should use a deeper argument than just brute force. In the commutative case such deeper argument comes from the interpretation of stratified modules as D -modules ([BO 2.11-3])). I have not been able to find a good analogy of this interpretation in the non-commutative case.

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DEPARTAMENTO DE MATEMÁTICA, FACULTAD DE CIENCIAS EXACTAS, CALLE 50 Y 115 (1900)
LA PLATA, ARGENTINA.

E-mail address: `willie@mate.unlp.edu.ar`