Superselection Theory for Subsystems

Roberto Conti
Dipartimento di Matematica, Università di Roma "Tor Vergata,"
I-00133 Rome, Italy

Sergio Doplicher
Dipartimento di Matematica, Università di Roma "La Sapienza,"
I-00185 Rome, Italy

John E. Roberts
Dipartimento di Matematica, Università di Roma "Tor Vergata,"
I-00133 Rome, Italy

Abstract

An inclusion of observable nets satisfying duality induces an inclusion of canonical field nets. Any Bose net intermediate between the observable net and the field net and satisfying duality is the fixed–point net of the field net under a compact group. This compact group is its canonical gauge group if the occurrence of sectors with infinite statistics can be ruled out for the observable net and its vacuum Hilbert space is separable.

Research supported by MURST, CNR–GNAFA and EU.

1 Introduction

In this paper, we take the view that a physical system is described by its observable net, a net $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ of von Neumann algebras over double cones in Minkowski space in its vacuum representation. Our goal is to analyze subsystems. Of course, physical intuition would lead us to believe that in physically realistic situations there will be no proper subsystems since any putative subsystem loses its identity through interaction with the ambient system. A result in this direction would be interesting as it would allow one to pinpoint many natural sets of generators. For example, one could, modulo technicalities, claim that the observable net is generated by the energy–momentum tensor density [6],[3]. Unfortunately, we have no such result. Instead, we turn to consider what might prove to be the exceptional case where there are proper subsystems. For example, if the original system admits symmetries, i.e. if there is a nontrivial group of local automorphisms of the net leaving the vacuum state invariant, then the fixed–point net provides an example of a subsystem and one may even wonder whether every subsystem arises in this manner.

In fact, the theory of superselection sectors provides a natural mechanism for giving examples of subsystems. Each observable net $\mathfrak{A}$ is contained in an associated canonical field net $\mathfrak{F}$ as the fixed–point net under a compact group G of gauge automorphisms. The fixed–point net $\mathfrak{B}$ under a proper subgroup G, containing the element changing the sign of Fermi fields, if present, can be treated as the observable net of some other physical system and $\mathfrak{A}$ will then be a subsystem of $\mathfrak{B}$.

Here are some of the questions we would like to answer. Can one classify subsystems? What is the relation between the superselection structure of a system and that of its subsystems and how are their canonical field nets related? Earlier partial results on the classification problem can be found in [17] and [5].

We have only been able to make sensible progress on classifying subsystems by restricting ourselves to systems satisfying duality in the vacuum representation. Since there are good reasons for believing that $\mathfrak{A}$ satisfies essential duality, this amounts to replacing $\mathfrak{A}$ by its dual net, thus ignoring, for example, the possibility of spontaneously broken gauge symmetries. This partial solution does have the merit of reducing the classification problem to that of finding all observable nets with a given dual net.

In Section 2 we study inclusions of observable nets and their functorial properties. Thus if $\mathfrak{A} \subset \mathfrak{B}$, we get an inclusion of the corresponding categories of 1–cocycles, $Z^1(\mathfrak{A}) \subset Z^1(\mathfrak{B})$ and hence of the corresponding categories of transportable morphisms, $\mathcal{T}_t(\mathfrak{A}) \subset \mathcal{T}_t(\mathfrak{B})$, and finally restricting to finite statistics gives $\mathcal{T}_f(\mathfrak{A}) \subset \mathcal{T}_f(\mathfrak{B})$. We interpret this latter inclusion in terms of the associated homomorphism from the gauge group of $\mathfrak{B}$ to that of $\mathfrak{A}$.

In Section 3, we study inclusions of field nets giving conditions for the existence of an associated conditional expectation. Conditional expectations also play a decisive role in proving that an inclusion of observable algebras satisfying duality give rise to an inclusion of the corresponding complete normal field nets.

In Section 4, we study intermediate nets, that is nets contained between the observable net and its canonical field net, showing that such nets are the fixed–points of $\mathfrak{F}$ under a closed subgroup L of the field net. After this, we prove a result showing that the sectors of an intermediate observable net correspond, as one would expect, to the equivalence classes of irreducible representations of L. This was the principal objective of this paper and is worth comparing with previous results. There are two known sets of structural hypotheses [13], [22] allowing one to conclude that a Bosonic net has no sectors. Our result would be a consequence whenever $\mathfrak{F}$ were Bosonic. However, in the absence of evidence that a Bosonic canonical field net satisfies the structural hypotheses, these results of [13] and [22] have been most useful in proving the absence of sectors in examples such as free field theories. For our result, we need to exclude infinite statistics for the observable net, a weaker hypothesis with little known about its validity. In addition we have to assume that the vacuum Hilbert space of $\mathfrak{A}$ is separable, as indeed it is, in practice.

The paper concludes with an appendix giving results on the harmonic analysis of the action of compact groups on von Neumann and C^*–algebras and on conditional expectations needed in the course of this paper.

2 Inclusions of Observable Nets

In this section, we will be considering an inclusion $\mathfrak{A} \subset \mathfrak{B}$ of observable (i.e. local) nets, with a view to seeing what can be said about the relation between the corresponding superselection sectors. Each observable net will be considered as acting irreducibly on its own vacuum Hilbert space (denoted in the sequel by $\mathcal{H}_{\mathfrak{A}}$, $\mathcal{H}_{\mathfrak{B}}$). Of course, if $\mathfrak{A} \subset \mathfrak{B}$, $\mathcal{H}_{\mathfrak{A}}$ is naturally identified with a subspace of $\mathcal{H}_{\mathfrak{B}}$.

However, we start by recalling some well known facts about superselection structure, cf. [14], [16], [7], [8]. This will allow us to introduce our notation and give a few definitions and useful results.

Throughout this paper, the superselection sectors for $\mathfrak{A}$ are understood to be the unitary equivalence classes of irreducible representations π satisfying the Selection Criterion (SC)

$$\pi|_{\mathfrak{A}(\mathcal{O}')} \cong \pi_0|_{\mathfrak{A}(\mathcal{O}')}, \ \mathcal{O} \in \mathcal{K}$$

with respect to a reference vacuum representation π_0 ($= \pi_0^{\mathfrak{A}}$). Here $\mathcal{K}$ denotes as usual the set of double cones in Minkowski space ordered under inclusion.

The representations satisfying the selection criterion are the objects of a W*-category $S(\pi_0)$ whose arrows are the intertwining operators.

Recall that $\mathfrak{A}^d$, the dual net of $\mathfrak{A}$, is defined by $\mathfrak{A}^d(\mathcal{O}) := \mathfrak{A}(\mathcal{O}')'$, the commutants being taken on $\mathcal{H}_{\mathfrak{A}}$. If $\mathfrak{A}$ is a local net, i.e. if $\mathfrak{A} \subset \mathfrak{A}^d$, we have inclusions $\mathfrak{A} \subset \mathfrak{A}^{dd} := (\mathfrak{A}^d)^d \subset \mathfrak{A}^d = \mathfrak{A}^{ddd}$.

If $\mathfrak{A}$ satisfies Haag duality, i.e. if $\mathfrak{A} = \mathfrak{A}^d$, we find that a π as above is equivalent to a representation of the form $\pi_0 \circ \rho$, where ρ is an endomorphism of $\mathfrak{A}$ with the following properties: it is localized in a double cone $\mathcal{O}$, i.e. $\rho \restriction_{\mathfrak{A}(\mathcal{O}')} = \mathrm{id}$, and is transportable, i.e. (inner) equivalent to an endomorphism localized in any other double cone. The category of these endomorphisms and their intertwiners is equivalent to $S(\pi_0)$ and hence may be equally used to describe the superselection structure. However, it has the added advantage of being a tensor W^*–category, denoted by $\mathcal{T}_t$, and reveals the latent tensor structure of superselection sectors. The objects of $\mathcal{T}_t$, the set of localized and transportable endomorphisms of $\mathfrak{A}$, will be denoted by $\Delta_t(\mathfrak{A})$.

If we wish to relate the superselection sectors of $\mathfrak{A}$ and $\mathfrak{B}$ using endomorphisms we run into two evident problems. On one hand, the restriction of an endomorphism ρ of $\mathfrak{B}$ to $\mathfrak{A}$ is not an endomorphism of $\mathfrak{A}$, unless $\rho(\mathfrak{A}) \subset \mathfrak{A}$, although it could still be regarded as a representation. Furthermore, it is not at first sight clear how a given localized, transportable endomorphism of $\mathfrak{A}$ can be extended to a similar endomorphism of $\mathfrak{B}$. However, this problem has a canonical solution indicated by an alternative approach to superselection sectors using net cohomology.

In the version in [23], net cohomology is conceived as a cohomology of partially ordered sets with coefficients in nets over the partially ordered set

$\mathcal{K}$. The formal description of this cohomology may be found in [23] and we restrict ourselves here to a pedestrian account of those concepts needed in this paper.

A 0–simplex a is just a double cone $\mathcal{O}$, i.e. an element of $\mathcal{K}$. A 1–simplex b is an ordered set $(\mathcal{O}, \mathcal{O}_0, \mathcal{O}_1)$ of double cones with $\mathcal{O}_0 \cup \mathcal{O}_1 \subset \mathcal{O}$. Its faces are the 0–simplices $\partial_o b = \mathcal{O}_0$, $\partial_1 b = \mathcal{O}_1$ and its support is $|b| = \mathcal{O}$. A 2–simplex c is an ordered set of double cones $(\mathcal{O}, \mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_{01}, \mathcal{O}_{02}, \mathcal{O}_{12})$ such that $\mathcal{O} \supset \mathcal{O}_0 \cup \mathcal{O}_1 \cup \mathcal{O}_2$ and such that its faces $\partial_0 c = (\mathcal{O}_0, \mathcal{O}_{01}, \mathcal{O}_{02})$, $\partial_1 c = (\mathcal{O}_1, \mathcal{O}_{01}, \mathcal{O}_{12})$ and $\partial_2 c = (\mathcal{O}_2, \mathcal{O}_{02}, \mathcal{O}_{12})$ are 1–simplices. Its support is $|c| = \mathcal{O}$. The set of n–simplices is denoted by Σ_n.

Definition *A 0–cocycle of $\mathfrak{A}$, a net of C^*–algebras over $\mathcal{K}$, is a map $z : \Sigma_0 \to \mathfrak{A}$ such that $z(\partial_0 b) = z(\partial_1 b)$, $b \in \Sigma_1$, and $z(a) \in \mathfrak{A}(a)$, $a \in \Sigma_0$.*

Hence the set $Z^0(\mathfrak{A})$ of 0–cocycles is $\cap_{\mathcal{O}} \mathfrak{A}(\mathcal{O})$.

Definition *A 1–cocycle of $\mathfrak{A}$ is a map $z : \Sigma_1 \to \mathcal{U}(\mathfrak{A})$ such that $z(\partial_0 c) z(\partial_2 c) = z(\partial_1 c)$, $c \in \Sigma_2$, and $z(b) \in \mathfrak{A}(|b|)$, $b \in \Sigma_1$.*

The 1–cocycles are considered as the objects of a C^*–category $Z^1(\mathfrak{A})$. An arrow between 1–cocycles, $w \in (z, z')$ is a mapping $w : \Sigma_0 \to \mathfrak{A}$ such that $z(b) w(\partial_1 b) = w(\partial_0 b) z'(b)$, $b \in \Sigma_1$, $w(a) \in \mathfrak{A}(a)$, $a \in \Sigma_0$. Note that if 1 denotes the trivial 1-cocycle $1(b) = I, b \in \Sigma_1$, then the elements of $(1, 1)$ are just the 0–cocycles. Two objects z, z' of $Z^1(\mathfrak{A})$ are cohomologous if (z, z') contains a unitary arrow and z is a 1–coboundary if it is cohomologous to 1.

Here is an example of 1–cocycle illustrating at the same time the relation with the theory of superselection sectors. Given a representation π of $\mathfrak{A}$ satisfying the selection criterion, pick for each $a \in \Sigma_0$ a unitary operator V_a such that

$$V_a \pi(A) = \pi_0(A) V_a, \quad A \in \mathfrak{A}(\mathcal{O}),\ a \subset \mathcal{O}',$$

and set

$$z(b) := V_{\partial_0 b} V^*_{\partial_1 b}, \quad b \in \Sigma_1,$$

then $z \in Z^1(\mathfrak{A}^d)$.

Conversely, given a 1–cocycle with values in a net $\mathfrak{A}$, define for $a \in \Sigma_0$,

$$\pi_a(A) := z(b) A z(b)^*, \quad \text{provided } A \in \mathfrak{A}^d(\mathcal{O})\ , b \in \Sigma_1,\ \partial_0 b = a,\ \partial_1 b \subset \mathcal{O}'.$$

One checks that π_a gives a well defined representation of $\mathfrak{A}^d$ and that $z(b) \in (\pi_{\partial_1 b}, \pi_{\partial_o b})$. Thus a 1–cocycle gives rise to a field $a \mapsto \pi_a$ of equivalent representations. Furthermore, π_a is localized in a in the sense that

$$\pi_a(A) = A, \quad A \in \mathfrak{A}^d(\mathcal{O}),\ \mathcal{O} \subset a'.$$

Details may be found in [23], §3.4.6, Theorem 1, Corollary 2, where it is also proved that $S(\pi_0)$ and $Z^1(\mathfrak{A}^d)$ are equivalent W^*–categories. It follows that $S(\pi_0)$ and $S(\pi_0^{dd})$ are equivalent as W^*–categories, where π_0^{dd} denotes the vacuum representation of the double dual net $\mathfrak{A}^{dd}$.

It should be noted that the above results on superselection sectors do not require any form of duality or even locality. But we are not able to define the tensor structure without a further hypothesis. We see, however, that essential duality, $\mathfrak{A}^d = \mathfrak{A}^{dd}$, will suffice for this purpose.

A variant of the above construction relates cocycles and endomorphisms. It is based on assuming relative duality

$$\mathfrak{A}(\mathcal{O}) = \mathfrak{A}^d(\mathcal{O}) \cap \mathfrak{A}, \quad \mathcal{O} \in \mathcal{K},$$

a weaker version of the more familiar assumption of duality. This is the C^*–version of duality and is defined without reference to the vacuum representation. In this context, it is natural to use nets of C^*–algebras. As a consequence of relative duality, an endomorphism localized in $\mathcal{O}$ satisfies $\rho(\mathfrak{A}(\mathcal{O}_1)) \subset \mathfrak{A}(\mathcal{O}_1)$ whenever $\mathcal{O} \subset \mathcal{O}_1$. Furthermore, an intertwiner between endomorphisms localized in $\mathcal{O}$ is automatically in $\mathfrak{A}(\mathcal{O})$. Relative duality suffices for a theory of transportable endomorphisms but to pass from superselection sectors to transportable endomorphisms we need duality or, at least, essential duality.

Consider an endomorphism ρ of $\mathfrak{A}$ such that, given $a \in \Sigma_0$, there is a unitary $\psi(a) \in \mathfrak{A}$ with

$$\psi(a)\rho(A) = A\psi(a) \quad A \in \mathfrak{A}(\mathcal{O}),\ \mathcal{O} \subset a'.$$

This is the analogue for endomorphisms of the selection criteria for representations. Our 1–cocycle $z(b) := \psi(\partial_o b)\psi(\partial_1 b)^*$, $b \in \Sigma_1$ now takes values in $\mathfrak{A}$. Any such 1–cocycle z now defines a field of endomorphisms:

$$\rho_a(A) := z(b)Az(b)^*, \quad \text{provided } A \in \mathfrak{A}(\mathcal{O})\ , b \in \Sigma_1,\ \partial_0 b = a,\ \partial_1 b \subset \mathcal{O}'.$$

ρ_a is localized in a in the sense that

$$\rho_a(A) = A, \quad A \in \mathfrak{A}(\mathcal{O}),\ \mathcal{O} \subset a'.$$

Since $z(b) \in (\rho_{\partial_1 b}, \rho_{\partial_0 b})$, we have a field $a \mapsto \rho_a$ of endomorphisms in $\mathcal{T}_t$, each of which is equivalent to the ρ we started from. Note that if we start with a cocycle of the form $z(b) := \psi(\partial_o b)\psi(\partial_1 b)^*$, as above, then $\rho_a = \mathrm{Ad}\psi_a\rho$. In particular, if ρ is localized in a we may take $\psi_a = I$ and hence arrange that $\rho_a = \rho$. We may regard our construction as leading to an equivalence of tensor C^*–categories between $Z^1(\mathcal{T}_t)$ and $\mathcal{T}_t$, cf. [23], §3.4.7, Theorem 5.

As in the theory of superselection sectors, the tensor C^*–category $\mathcal{T}_t(\mathfrak{A})$ admits a canonical permutation symmetry ε (in more than two spacetime dimensions).

We can now begin to examine an inclusion $\mathfrak{A} \subset \mathfrak{B}$ of nets satisfying relative duality. Such an inclusion obviously induces an inclusion functor $Z^1(\mathfrak{A}) \to Z^1(\mathfrak{B})$. Thus a 1–cocycle in a local net $\mathfrak{A}$ not only gives rise to a field $a \mapsto \rho_a$ of endomorphisms of $\mathfrak{A}$ but to a field $a \mapsto \tilde{\rho}_a$ of endomorphisms of any relatively local net $\mathfrak{B}$ extending the original field. We have seen that any element of $\Delta_t(\mathfrak{A})$ arises as a value of such a field and hence admits an extension to an element of $\Delta_t(\mathfrak{B})$. As the cocycle is not uniquely determined by the endomorphism, a little argument is needed to show that the extension is uniquely detemined, cf. Lemma 3 of §3.4.7 in [23].

Lemma 2.1 *Let z and z' be two 1–cocycles of a net $\mathfrak{A}$ satisfying relative duality and suppose that, for some $a \in \Sigma_0$, $\rho_a = \rho'_a$. Then, if $\mathfrak{A} \subset \mathfrak{B}$ is an inclusion of nets and $\mathfrak{A}$ and $\mathfrak{B}$ are relatively local, the endomorphisms $\tilde{\rho}_a$ and $\tilde{\rho}'_a$ of $\mathfrak{B}$ induced by z and z' agree.*

Proof. Let $b \in \Sigma_1$ with $\partial_0 b = a$ then $z(b)^* z'(b) \in (\rho'_{\partial_1 b}, \rho_{\partial_1 b})$ is an intertwiner of endomorphisms localized in $\partial_1 b$. Since $\mathfrak{A}$ satisfies relative duality, $z(b)^* z'(b) \in \mathfrak{A}(\partial_1 b)$ and the result follows since $\mathfrak{B}$ and $\mathfrak{A}$ are relatively local.

We now come to the main result of this section.

Theorem 2.2 *Let $\mathfrak{A} \subset \mathfrak{B}$ be an inclusion of nets satisfying relative duality. Then there is an induced structure preserving inclusion of $\mathcal{T}_t(\mathfrak{A})$ in $\mathcal{T}_t(\mathfrak{B})$ which corresponds to the above extension on endomorphisms and to the given inclusion on intertwiners.*

Proof. In view of the relation between cocycles and endomorphisms and Lemma 2.1, the only point which is not yet obvious is that the tensor structure is preserved by the inclusion. However, if z and z' are 1–cocycles in $\mathfrak{A}$ and $a \mapsto \rho_a$ and $a \mapsto \rho'_a$ are the corresponding fields of endomorphisms, then

$$z \otimes z'(b) := z(b)\rho_{\partial_1 b}(z'(b))$$

defines a 1–cocycle over $\mathfrak{A}$ whose associated field of endomorphisms is $a \mapsto \rho_a \rho'_a$, and the result now follows.

The extension of endomorphisms is also discussed in [2] under the name α–induction in the context of nets of subfactors [18].

The inclusion of Theorem 2.2 will of course map the unitary operator $\varepsilon(\rho, \rho')$ in $\mathcal{T}_t(\mathfrak{A})$ onto the corresponding operator for the extended endomorphisms. Furthermore, as is obvious from the cohomological description, we shall have

$$(\rho, \rho')_{\mathfrak{A}} = (\rho, \rho')_{\mathfrak{B}} \cap \mathfrak{A},$$

with an obvious notation.

In particular, whenever $\mathfrak{A}$ and $\mathfrak{B}$ satisfy duality, this result is at the same time a result about superselection structure and relates the superselection structure of $\mathfrak{A}$ to that of $\mathfrak{B}$. The extension of an endomorphism ρ with finite statistics, $\rho \in \Delta_f(\mathfrak{A})$, will again have finite statistics and we have an induced tensor *–functor from $\mathcal{T}_f(\mathfrak{A})$ to $\mathcal{T}_f(\mathfrak{B})$. In fact this also holds in the more general context provided we understand $\mathcal{T}_f$ to be the full subcategory of $\mathcal{T}_t$ having conjugates. Now $\mathcal{T}_f(\mathfrak{A})$ and $\mathcal{T}_f(\mathfrak{B})$ are equivalent to the tensor W^*–categories of finite dimensional continuous unitary representations of compact groups so that tensor *–functors correspond contravariantly to continuous homomorphisms between the groups in question[11]. In the context of superselection structure, the compact groups are the gauge groups. A gauge group appears as the group of automorphisms of a field net leaving the observable subnet pointwise fixed.

We would like to make this homomorphism explicit and therefore consider the following situation. We consider a commuting square of inclusions of nets $\mathfrak{A}_1 \subset \mathfrak{A}_2 \subset \mathfrak{F}_2$ and $\mathfrak{A}_1 \subset \mathfrak{F}_1 \subset \mathfrak{F}_2$. The $\mathfrak{A}_i$ are to be considered as observable nets, the $\mathfrak{F}_i$ as field nets, cf. [12], Definition 3.1 and Theorem 3.6. Thus we suppose $\mathfrak{A}_i$ to have trivial relative commutant in $\mathfrak{F}_i$ and $\mathfrak{F}_i$ to be local relative to $\mathfrak{A}_i$. We consider the subcategories $\mathcal{T}_i$ of $\mathcal{T}_f(\mathfrak{A}_i)$ induced by Hilbert spaces in $\mathfrak{F}_i$. The Hilbert spaces in question are unique and are supposed to generate $\mathfrak{F}_i$. We let G_i be the group of automorphisms of $\mathfrak{F}_i$ leaving $\mathfrak{A}_i$ pointwise invariant. These automorphisms leave the Hilbert spaces stable and we suppose that G_i is a compact group equipped with the topology of pointwise norm convergence on these Hilbert spaces. Finally, we suppose that each irreducible representation of G_i is realized on some such Hilbert space and that $\mathfrak{F}_i^{G_i} = \mathfrak{A}_i$. These last conditions ensure that the inclusion $\mathfrak{A}_i \subset \mathfrak{F}_i$ realizes $\mathcal{T}_i$ in a canonical way as a dual of G_i. Without wishing to get involved in further technicalities, we might say that the essence of these conditions is that the net $\mathfrak{F}_i$ is a crossed product of the net $\mathfrak{A}_i$ by the action of a group dual, where these terms are to be understood as adaptions to nets of von Neumann algebras of the corresponding concepts in [10].

Theorem 2.3 *Under the above conditions on a commuting square of inclusions, $\mathfrak{F}_1$ is stable under the action of G_2 and the restriction of G_2 to $\mathfrak{F}_1$ defines a homomorphism h from G_2 to G_1. The inclusion $\mathfrak{A}_1 \subset \mathfrak{A}_2$ induces an inclusion functor from $\mathcal{T}_1$ to $\mathcal{T}_2$ and this inclusion functor is precisely that induced by h. If N and K denote the kernel and image of h, respectively, then*

$$\mathfrak{F}_1 \vee \mathfrak{A}_2 = \mathfrak{F}_2^N,$$
$$\mathfrak{F}_1 \cap \mathfrak{A}_2 = \mathfrak{F}_1^K.$$

Proof. Note first that a Hilbert space $H(\rho)$ in $\mathfrak{F}_1$ inducing an object ρ of $\mathcal{T}_1$ must, when considered as a Hilbert space in $\mathfrak{F}_2$, induce the canonical extension of ρ to an object of $\mathcal{T}_f(\mathfrak{A}_2)$ by relative locality. This canonical extension is thus an object of $\mathcal{T}_2$ so that we do have an induced inclusion functor from $\mathcal{T}_1$ to $\mathcal{T}_2$. It also follows that $H(\rho)$ is stable under the action of G_2. But such Hilbert spaces generate $\mathfrak{F}_1$ so $\mathfrak{F}_1$ is stable under the action of G_2. The restriction of an element of G_2 to $\mathfrak{F}_1$ defines an automorphism of $\mathfrak{F}_1$ leaving $\mathfrak{A}_1$ pointwise invariant and is therefore an element of G_1. Thus restriction defines the required homomorphism h. Since the representation of G_2 on $H(\rho)$ arises by composing that of G_1 with h, h induces the above inclusion functor. Now N, being the kernel of h, obviously acts trivially on $\mathfrak{F}_1$ and $\mathfrak{A}_2$. Now $\mathfrak{F}_2^N$ is generated by the Hilbert spaces $H(\rho)$ in $\mathfrak{F}_2$ inducing objects of $\mathcal{T}_2$ and carrying irreducible representations of G_2 that are trivial in restriction to N. Regarding these as representations of K and inducing up to a representation of G_1, bearing in mind that every irreducible representation of G_1 is realized within $\mathfrak{F}_1$, we conclude that there is an isometry in $\mathfrak{A}_2$ mapping $H(\rho)$ into $\mathfrak{F}_1$. Thus $\mathfrak{F}_2^N$ is generated by $\mathfrak{F}_1$ and $\mathfrak{A}_2$. Next, note that the K–invariant part of a Hilbert space of $\mathfrak{F}_1$ inducing an object of $\mathcal{T}_1$ is G_2–invariant and hence lies in $\mathfrak{A}_2$. These Hilbert spaces generate $\mathfrak{F}_1^K$ and, as any element of $\mathfrak{F}_1 \cap \mathfrak{A}_2$ is K–invariant, we have $\mathfrak{F}_1 \cap \mathfrak{A}_2 = \mathfrak{F}_1^K$, completing the proof.

3 Inclusions of Field Nets

In the last section, we have treated inclusions of observable nets. However, observable nets are frequently defined by starting with a net $\mathfrak{F}$ of fields with

Bose–Fermi commutation relations. From a mathematical point of view, these are simply the $\mathbb{Z}_2$–graded version of an observable net. Hence to have a basic formalism which is sufficiently flexible, we need to consider inclusions of $\mathbb{Z}_2$–graded nets.

We define a (concrete) $\mathbb{Z}_2$–graded net $\mathfrak{F}$ to be a net of von Neumann algebras over $\mathcal{K}$, represented on its (vacuum) Hilbert space $\mathcal{H}_{\mathfrak{F}}$, together with an involutive unitary operator k inducing a net automorphism α_k of $\mathfrak{F}$. The even (Bose) part $\mathfrak{F}_+$ of $\mathfrak{F}$ is the fixed–point net under α_k, the odd (Fermi) part $\mathfrak{F}_-$ changes sign under α_k. The twisted net $\mathfrak{F}^t$ is defined as $\mathfrak{F}_+ + ik\mathfrak{F}_-$ and is, in an obvious way, itself a $\mathbb{Z}_2$–graded net. The $\mathbb{Z}_2$–graded or twisted dual net of $\mathfrak{F}$ is defined by

$$\mathfrak{F}^d(\mathcal{O}) := \cap_{\mathcal{O}_1 \subset \mathcal{O}'} \mathfrak{F}^t(\mathcal{O}_1)'.$$

It is understood to act on the same Hilbert space with the same unitary k. $\mathfrak{F}$ satisfies twisted duality if it coincides with its twisted dual net $\mathfrak{F}^d$. $\mathfrak{F}$ is said to have Bose–Fermi commutation relations if

$$\mathfrak{F}(\mathcal{O}_1) \subset \mathfrak{F}^t(\mathcal{O}_2)', \quad \mathcal{O}_1 \subset \mathcal{O}_2',$$

or, equivalently, if $\mathfrak{F} \subset \mathfrak{F}^d$. If, in addition, $\mathfrak{F}$ is irreducibly represented on $\mathcal{H}_{\mathfrak{F}}$, we refer to the triple $\mathfrak{F}, k, \mathcal{H}_{\mathfrak{F}}$ as being a field net.

By an inclusion of $\mathbb{Z}_2$–graded nets we mean compatible (normal) inclusions $\mathfrak{B}(\mathcal{O}) \subset \mathfrak{F}(\mathcal{O})$ of von Neumann algebras together with an inclusion of Hilbert spaces $\mathcal{H}_{\mathfrak{B}} \subset \mathcal{H}_{\mathfrak{F}}$ compatible with the inclusion of nets and such that $k_{\mathfrak{B}}$ is the restriction of $k_{\mathfrak{F}}$ to $\mathcal{H}_{\mathfrak{B}}$. We further require that $\mathcal{H}_{\mathfrak{B}}$ be cyclic and separating for each $\mathfrak{F}(\mathcal{O})$.

Typically, an observable net may be defined from a field net acted on by a compact group G of net automorphisms with $\alpha_k \in G$ by taking $\mathfrak{A}$ to be the fixed–point net under G. Under these circumstances, $\mathfrak{A}$ satisfies duality if $\mathfrak{F}$ satisfies twisted duality (except in one space dimension) and there is a normal conditional expectation m of nets from $\mathfrak{F}$ onto $\mathfrak{A}$ obtained by averaging over the group. However, it has been known since the beginnings of the theory of superselection sectors that the existence of such a normal conditional expectation follows simply from the hypothesis that $\mathfrak{A}$ satisfies duality, without any reference to a compact group G. We present here some related results.

Let $\mathfrak{F}, k, \mathcal{H}_{\mathfrak{F}}$ be a $\mathbb{Z}_2$–graded net and let E be a projection on $\mathcal{H}_{\mathfrak{F}}$, commuting with k and cyclic and separating for $\mathfrak{F}$, i.e. for each $\mathfrak{F}(\mathcal{O})$. Let

$\mathfrak{F}^E(\mathcal{O}) := \mathfrak{F}(\mathcal{O}) \cap \{E\}'$ and let $\mathfrak{F}_E$ and k_E denote the restriction of $\mathfrak{F}^E$ and k to the subspace $E\mathcal{H}_{\mathfrak{F}}$. Then the triple $\mathfrak{F}_E, k_E, E\mathcal{H}_{\mathfrak{F}}$ is itself a $\mathbb{Z}_2$–graded net. If we started with a field net, we would only get a field net if we knew that $\mathfrak{F}_E$ acts irreducibly on $E\mathcal{H}_{\mathfrak{F}}$.

We now ask whether $\mathfrak{F}$ admits a conditional expectation m of nets such that

$$m(F)E = EFE, \quad F \in \mathfrak{F}.$$

In this case, m would project onto the subnet $\mathfrak{F}^E$ and be locally normal, see Lemma A.7 of the Appendix. In particular, in the case of a field net $\mathfrak{F}_E$ would act irreducibly on $E\mathcal{H}_{\mathfrak{F}}$.

Lemma 3.1 *If $\mathfrak{F}$ is a field net and $\mathfrak{F}_E$ satisfies twisted duality, then there is a conditional expectation of $\mathfrak{F}$ such that*

$$m(F)E = EFE, \quad F \in \mathfrak{F}.$$

Proof By Corollary A.8b of the Appendix, we must show that

$$[EFE, EF'E] = 0, \quad F \in \mathfrak{F}(\mathcal{O}), \quad F' \in \mathfrak{F}(\mathcal{O})'.$$

Now if $\mathcal{O}_1 \subset \mathcal{O}'$ and $B \in \mathfrak{F}^t(\mathcal{O}_1)_E$ then $[EFE, B] = 0$. Hence $E\mathfrak{F}(\mathcal{O})E \upharpoonright E\mathcal{H}_{\mathfrak{F}} \subset (\mathfrak{F}_E)^d(\mathcal{O}) = \mathfrak{F}_E(\mathcal{O})$. Thus there is a $G \in \mathfrak{F}(\mathcal{O})$ with $GE = EG = EFE$, and $[GE, EF'E] = 0$, as required.

To have a more systematic approach, we begin by proving an analogue of Lemma A.7 of the Appendix for $\mathbb{Z}_2$–graded nets.

Lemma 3.2 *Let $\mathfrak{F}, k$ be a $\mathbb{Z}_2$–graded net on a Hilbert space $\mathcal{H}$. Let E be a k–invariant projection cyclic for $\mathfrak{F}$ and $\mathfrak{F}^d$. Let ${}^E\mathfrak{F}$ be the net defined by:*

$${}^E\mathfrak{F}(\mathcal{O}) := \{F \in \mathfrak{F}(\mathcal{O}) : EFE \in (E\mathfrak{F}^d E)^d(\mathcal{O})\}.$$

Then ${}^E\mathfrak{F} \supset \mathfrak{F}^E$ and is weak–operator closed. This makes ${}^E\mathfrak{F}$ into a $\mathfrak{F}^E$–bimodule. Given $F \in \mathfrak{F}(\mathcal{O})$, there is a $m(F) \in \mathfrak{F}^{ddE}(\mathcal{O})$ such that

$$m(F)E = EFE$$

if and only if $F \in {}^E\mathfrak{F}(\mathcal{O})$.

Proof If $F \in {}^E\mathfrak{F}(\mathcal{O})$, then

$$EFE \in (E\mathfrak{F}^{dt}E)(\mathcal{O}_1)', \quad \mathcal{O}_1 \subset \mathcal{O}'.$$

Pick $G \in \mathfrak{F}^{dt}(\mathcal{O}_1)$, then

$$EF^*EG^*GEFE \leq \|EFE\|^2 EG^*GE,$$

hence there exists $m(F) \in \mathfrak{F}^{dt}(\mathcal{O}_1)'$, such that $m(F)E = EFE$. Since E is separating for each $\mathfrak{F}^{dt}(\mathcal{O}_1)'$ and $\mathcal{O}'$ is path–connected, $m(F)$ is independent of the choice of $\mathcal{O}_1 \subset \mathcal{O}'$. Hence $m(F) \in \cap_{\mathcal{O}_1 \subset \mathcal{O}'} \mathfrak{F}^{dt}(\mathcal{O}_1)' = \mathfrak{F}^{dd}(\mathcal{O}_1)$. If $F \in \mathfrak{F}(\mathcal{O})$ and there is an $m(F) \in \mathfrak{F}^{ddE}(\mathcal{O})$, such that $m(F)E = EFE$ then $F \in {}^E\mathfrak{F}(\mathcal{O})$. The remaining assertions are evident.

Remark To have a closer analogy with Lemma A.7 of the Appendix, we should require that $\mathfrak{F} = \mathfrak{F}^{dd}$. Since, after all, we require $\mathcal{M} = \mathcal{M}''$ in the Appendix.

Corollary 3.3 *Let $\mathfrak{F} = \mathfrak{F}^{dd}, k, \mathcal{H}$ be a $\mathbb{Z}_2$–graded net and E a projection cyclic for $\mathfrak{F}$ and $\mathfrak{F}^d$. Then the following conditions are equivalent.*

- a) *There is a conditional expectation m on $\mathfrak{F}$ such that $m(F)E = EFE$, $F \in \mathfrak{F}$.*
- a′) *There is a conditional expectation m^d on $\mathfrak{F}^d$ such that $m^d(F^d)E = EF^dE$, $F^d \in \mathfrak{F}^d$.*
- b) *$E\mathfrak{F}E \subset (E\mathfrak{F}^dE)^d$.*
- b′) *$E\mathfrak{F}^dE \subset (E\mathfrak{F}E)^d$.*
- c) *$({}_E\mathfrak{F})^d = {}_E(\mathfrak{F}^d)$.*
- c′) *$({}_E(\mathfrak{F}^d))^d = {}_E\mathfrak{F}$.*

Here ${}_E\mathfrak{F}$, for example, denotes the restriction of $E\mathfrak{F}E$ to $E\mathcal{H}$.

Proof Suppose b) holds, then ${}^E\mathfrak{F} = \mathfrak{F}$ and, by Lemma 3.2, m becomes a conditional expectation onto $\mathfrak{F}_E$, since it is idempotent and of norm 1, giving a). Similarly, b′) implies a′). Taking duals, we see that b) and b′) are equivalent. It is clear that a) implies b) and that a′) implies b′). Now

$$({}_E\mathfrak{F})^d(\mathcal{O}) = \cap_{\mathcal{O}_1 \subset \mathcal{O}'}({}_E\mathfrak{F}^t)(\mathcal{O}_1)' = \cap_{\mathcal{O}_1 \subset \mathcal{O}'}(E\mathfrak{F}^t(\mathcal{O}_1)E \upharpoonright E\mathcal{H})',$$

so if a) holds then by Corollary A.8c of the Appendix,

$$({}_E\mathfrak{F})^d(\mathcal{O}) = \cap_{\mathcal{O}_1 \subset \mathcal{O}'}(E\mathfrak{F}^t(\mathcal{O}_1)'E) \upharpoonright E\mathcal{H} = E \cap_{\mathcal{O} \subset \mathcal{O}'} \mathfrak{F}^t(\mathcal{O}_1)'E \upharpoonright E\mathcal{H},$$

where we have used the fact that E is separating for each $\mathfrak{F}^t(\mathcal{O}_1)'$ and that $\mathcal{O}'$ is path–connected. Thus a) implies c). The implication a′) implies c′) follows by exchanging the role of $\mathfrak{F}$ and $\mathfrak{F}^d$. Now, trivially, c) implies b′) and, again, c′) implies b) follows.

Of course, a direct application of Corollary A.8 of the Appendix shows that the above conditions are also equivalent to

$$({}_E\mathfrak{F})(\mathcal{O})' = {}_E(\mathfrak{F}(\mathcal{O})'), \quad \mathcal{O} \in \mathcal{K}.$$

However, in view of the superficial similarities with c), it is worth stressing that equivalence depends on being in more than two spacetime dimensions.

What becomes clear from the above discussion is that the problem of studying the subsystems of a given system can be divided up in a natural way. We can begin with the simple class of subsystems characterized by cyclic projections E and the existence of a conditional expectation as above. Let us call such subsystems *full* since they are the largest subsystems on their Hilbert spaces and are uniquely determined by their Hilbert spaces. If $\mathfrak{A}$ is a full subsystem of $\mathfrak{B}$ and $\mathfrak{B}$ is itself a full subsystem of $\mathfrak{F}$, then $\mathfrak{A}$ is a full subsystem of $\mathfrak{F}$. We see from Lemma 3.1 that a subsystem satisfying twisted duality is full. Furthermore, if $\mathfrak{F}$ satisfies twisted duality, then, by Corollary 3.3, a subsystem is full if and only if it satisfies twisted duality. A second step might then be to analyze subsystems having the same Hilbert space.

In the following result, we give an analogue of Corollary A.9 of the Appendix and look at full subsystems from the point of view of the subsystem.

Lemma 3.4 *Let $\mathfrak{B} \subset \mathfrak{F}$ be an inclusion of $\mathbb{Z}_2$–graded nets and E the associated projection from $\mathcal{H}_{\mathfrak{F}}$ to $\mathcal{H}_{\mathfrak{B}}$, then the following conditions are equivalent.*

a) There is a (necessarily unique, injective and $\mathbb{Z}_2$–graded) net morphism $\nu : \mathfrak{B}^d \to \mathfrak{F}^d$ such that

$$\nu(B)\Phi = B\Phi, \quad B \in \mathfrak{B}^d, \quad \Phi \in \mathcal{H}_{\mathfrak{B}}.$$

b) $\mathfrak{B}^d = {\mathfrak{F}^d}_E$.

If the conditions are fulfilled, there is a unique normal conditional expectation m of $\mathfrak{F}^d$ onto $\nu(\mathfrak{B}^d)$ such that

$$m(F)E = EFE, \quad F \in \mathfrak{F}^d.$$

Proof ν is obviously unique, hence $\mathbb{Z}_2$–graded, since $\mathcal{H}_{\mathfrak{B}}$ is cyclic for each $\mathfrak{F}^t(\mathcal{O})$, hence separating for each $\mathfrak{F}^d(\mathcal{O})$. Given a), we note that $E\nu(B)E = \nu(B)E$ and replacing B by B^*, we see that $\nu(B) \in \mathfrak{F}^{dE}$. Hence $B \in \mathfrak{F}^d{}_E$, yielding b). Conversely, if b) is satisfied, given $B \in \mathfrak{B}^d(\mathcal{O})$, there is an $F \in \mathfrak{F}^d(\mathcal{O})$ with $FE = EF$ and $F\Phi = B\Phi$, $\Phi \in \mathcal{H}_{\mathcal{B}}$. Hence, we may pick $\nu(B) = F$ to give a map $\nu : \mathfrak{B}^d \to \mathfrak{F}^d$ and it follows from uniqueness that ν is a net morphism. Now suppose the conditions are satisfied and that $F \in \mathfrak{F}^d(\mathcal{O})$ and $B \in \mathfrak{B}^t(\mathcal{O}_1)$ with $\mathcal{O}_1 \subset \mathcal{O}'$. Then

$$EFEB = EFBE = EBFE = BEFE.$$

Hence the restriction of EFE to $\mathcal{H}_{\mathfrak{B}}$ lies in $\mathfrak{B}^d(\mathcal{O}) = \mathfrak{F}^d{}_E(\mathcal{O})$ by b). The result now follows by Lemma A.7 of the Appendix.

Remarks For an inclusion of field nets, $\mathfrak{B} \subset \mathfrak{F}^d{}_E \subset \mathfrak{B}^d$, b) is trivially fulfilled if $\mathfrak{B}$ satisfies twisted duality. Now suppose that $\mathfrak{B}$ satisfies twisted duality for wedges then

$$\mathfrak{B}^d(\mathcal{O}) = \cap_{\mathcal{W}\supset\mathcal{O}}\mathfrak{R}(\mathcal{W}),$$

where $\mathfrak{R}(\mathcal{W})$ denotes the von Neumann algebra associated with the wedge $\mathcal{W}$. Now given spacelike double cones, $\mathcal{O}$ and $\mathcal{O}_1$, there is a wedge $\mathcal{W}$ such that $\mathcal{O} \subset \mathcal{W} \subset \mathcal{O}_1'$. Hence

$$\mathfrak{B}^d(\mathcal{O}) \subset \mathfrak{R}(\mathcal{W}) \subset \mathfrak{F}^t(\mathcal{O}_1)'{}_E,$$

and, taking the intersection over $\mathcal{O}_1$, we see that b) is again satisfied. If $\mathfrak{B}$ satisfies essential twisted duality, i.e. if $\mathfrak{B}^d = \mathfrak{B}^{dd}$, then we cannot conclude from the above that b) is satisfied since we do not know that we have an inclusion $\mathfrak{B}^{dd} \subset \mathfrak{F}^{dd}$. If $\mathfrak{B} = \mathfrak{B}^{dd}$, $\mathfrak{F} = \mathfrak{F}^{dd}$ and E is also cyclic for each $\mathfrak{F}^d(\mathcal{O})$ in Lemma 3.4, then we may deduce from Corollary 3.3 that, under the equivalent conditions of Lemma 3.4, $\mathfrak{B} = \mathfrak{F}_E$.

We now consider an inclusion $\mathfrak{A} \subset \mathfrak{B}$ of nets of local von Neumann algebras over double cones each satisfying duality in their respective Hilbert spaces $\mathcal{H}_{\mathfrak{A}}$ and $\mathcal{H}_{\mathfrak{B}}$. Let E denote the projection of $\mathcal{H}_{\mathfrak{B}}$ onto $\mathcal{H}_{\mathfrak{A}}$. Then, as follows e.g. from Lemma 3.4, there is a conditional expectation of nets of von Neumann algebras m of $\mathfrak{B}$ onto $\mathfrak{A}$ such that

$$EBE = m(B)E, \quad B \in \mathfrak{B}.$$

Furthermore, the intertwiners spaces between transportable localized morphisms of $\mathfrak{A}$ and their extensions to $\mathfrak{B}$ are related by

$$m(\rho, \rho')_{\mathfrak{B}} = (\rho, \rho')_{\mathfrak{A}},$$

see the remarks following Theorem 2.2.

We now introduce the canonical field net $\mathfrak{F}$ of $\mathcal{B}$ and let $m_{\mathfrak{B}}$ be the associated conditional expectation from $\mathfrak{F}$ onto $\mathfrak{B}$. We recall[12] that the canonical field net is defined for observable nets satisfying duality and Property B. Let $\mathfrak{E}$ denote the net of C^*–algebras generated by the Hilbert spaces in $\mathfrak{F}$ implementing the transportable localized morphisms of $\mathfrak{A}$. Then by Lemma A.2, the restriction of $m \circ m_{\mathcal{B}}$ to $\mathfrak{E}$ is the unique conditional expectation n onto the subnet $\mathfrak{A}$. By [10], this shows that $\mathfrak{E}$ is the C^*–cross product of $\mathfrak{A}$ by the action of $\mathcal{T}_{\mathfrak{A}}$.

Now let α denote the canonical action of the gauge group G of $\mathfrak{A}$ on the net $\mathfrak{E}$. Then we have

$$\int \alpha(F)\, d\mu(g) = n(F), \quad F \in \mathfrak{E},$$

where μ denotes Haar measure on G. Let $\mathcal{H}_{\mathfrak{F}}$ denote the canonical Hilbert space of $\mathfrak{F}$ and $\mathcal{H}$ the Hilbert subspace generated by $\mathcal{H}_{\mathfrak{A}}$ and $\mathfrak{E}$. Since $\omega \circ m \circ m_{\mathfrak{B}} = \omega$,

$$\omega \circ n = \omega$$

where ω is a state defined by a vector of $\mathcal{H}_{\mathfrak{A}}$. It follows that states of $\mathfrak{E}$ defined by vectors in $\mathcal{H}_{\mathfrak{A}}$ are gauge invariant. Hence

$$(\Phi, F\Psi) = (\Phi, n(F)\Psi), \quad F \in \mathfrak{E}, \quad \Phi, \Psi \in \mathcal{H}_{\mathfrak{A}}.$$

Given $F_i \in \mathfrak{E}$ and $\Phi_i \in \mathcal{H}_{\mathfrak{A}}$, $i = 1, 2 \ldots, n$ define

$$U_g \sum_i F_i \Phi_i := \sum_i \alpha_g(F_i)\Phi_i,$$

$$\| \sum_i \alpha_g(F_i)\Phi_i \|^2 = \sum_{i,j} (\Phi_i, \alpha_g(F_i^* F_j)\Phi_j) = \sum_{i,j} (\Phi_i, F_i^* F_j \Phi_j).$$

Thus we get a unitary action of G on $\mathcal{H}$.

We next remark that $\mathcal{H}_{\mathfrak{A}}$ is the space of G–invariant vectors in $\mathcal{H}$. In fact, if $U_g\Phi = \Phi$, $g \in G$ and Φ is orthogonal to $\mathcal{H}_{\mathfrak{A}}$. then

$$(\Phi, F\Psi) = (\Phi, \alpha_g(F)\Psi) = (\Phi, n(F)\Psi) = 0, \quad \Psi \in \mathcal{H}_{\mathfrak{A}}, \quad \mathcal{F} \in \mathfrak{E}.$$

Thus $\Phi = 0$.

We can now check easily, that our data consisting of a representation of $\mathfrak{A}$ on $\mathcal{H}$ restricting to the vacuum representation on $\mathcal{H}_{\mathfrak{A}}$, a unitary action of G on $\mathcal{H}$ and a homomorphism $\rho \mapsto H_\rho$ from the semigroup of objects of $\mathcal{T}_{\mathfrak{A}}$ has all the properties needed to generate the canonical field net of $\mathfrak{A}$ [12], p.66. Since $\mathcal{H}_{\mathfrak{A}}$ is cyclic for $\mathfrak{E}(\mathcal{O})'$ hence separating for $\mathfrak{E}(\mathcal{O})''$, the canonical field net is canonically isomorphic to $\mathcal{O} \mapsto \mathfrak{E}(\mathcal{O})$. Thus we have shown the following result.

Theorem 3.5 *Let $\mathfrak{A} \subset \mathfrak{B}$ be an inclusion of nets of observable algebras satisfying duality and Property B, then there is a canonical inclusion of the corresponding canonical field nets.*

4 Sector Structure of Intermediate Nets

In this section, we consider an inclusion of nets $\mathfrak{A} \subset \mathfrak{B}$ and examine in more detail the relation between the sectors of $\mathfrak{A}$ and those of $\mathfrak{B}$. As we have little to say in general, we restrict our attention to the case that $\mathfrak{A} \subset \mathfrak{B} \subset \mathfrak{F}(\mathfrak{A})$. We first show that under these circumstances, $\mathfrak{B}$ is the fixed–point net of $\mathfrak{F}(\mathfrak{A})$ under a closed subgroup of the gauge group G of $\mathfrak{A}$. To this end, we denote by H_ρ the Hilbert space in $\mathfrak{F} := \mathfrak{F}(\mathfrak{A})$ inducing $\rho \in \Delta_f$. Set $K_\rho := H_\rho \cap \mathfrak{B}$. Then K_ρ is a Hilbert space in $\mathfrak{B}$. We claim

a) $K_\rho K_\sigma \subset K_{\rho\sigma}$,

b) $TK_\rho \subset K_\sigma$, if $T \in (\rho, \sigma)$,

c) $K_{\bar\rho} = JK_\rho$, where J is an antiunitary from H_ρ to $H_{\bar\rho}$ intertwining the actions of the gauge group.

Indeed a) is obvious whilst b) follows from the fact that $T \in \mathfrak{A}$. Finally, c) follows from the fact that we may define such an antiunitary J by $J\psi = \psi^* \bar R$, with $\bar R \in (\iota, \rho\bar\rho)$ as in the definition of conjugate endomorphisms, cf. Theorem 3.3 of [8, II], or a standard solution of the conjugate equations, cf. [19]. It follows[21] that there is a unique closed subgroup L of the gauge group G such that each K_ρ is precisely the fixed-points of the action of L on H_ρ.

We now make use of the fact that when $\mathfrak{B}$ satisfies duality, there is a locally normal conditional expectation m from $\mathfrak{F}$ onto $\mathfrak{B}$. Let $\psi, \psi' \in H_\rho$, then

$$m(\psi)B = \rho(B)m(\psi), \quad B \in \mathfrak{B}.$$

Hence

$$\psi^* m(\psi')B = \psi^* \rho(B)m(\psi') = B\psi^* m(\psi'),$$

and since $\mathfrak{B}' \cap \mathfrak{F}(\mathfrak{A}) = \mathbb{C}I$, $\psi^* m(\psi') \in \mathbb{C}I$ and $m(\psi') \in H_\rho$. Since $\mathfrak{F}$ is generated as a net of linear spaces closed in say the s–topology by the elements of the Hilbert spaces H_ρ, $\mathfrak{B}$ is generated in the same way by K_ρ. Thus $\mathfrak{B}$ is the fixed–point net under the action of L. Thus we have proved the following result.

Theorem 4.1 *Let $\mathfrak{F}$ be the canonical field net of the observable net $\mathfrak{A}$ and $\mathfrak{B}$ an intermediate net, $\mathfrak{A} \subset \mathfrak{B} \subset \mathfrak{F}$, satisfying duality, then there is a closed subgroup L of the gauge group G of $\mathfrak{A}$ such that $\mathfrak{B} = \mathfrak{F}^L$.*

Related results in the context of inclusions of von Neumann algebras can be found in [15].

Lemma 4.2 *The following are equivalent:*

a) L is a normal subgroup of G,

b) $\alpha_g(\mathfrak{B}) \subset \mathfrak{B}$, $g \in G$,

c) $\mathfrak{B}$ is generated by Hilbert spaces inducing endomorphisms in $\Delta_f(\mathfrak{A})$.

Proof. a) $\Rightarrow$ b) is obvious. Hilbert spaces inducing endomorphisms in $\Delta_f(\mathfrak{A})$ are G-invariant so c) $\Rightarrow$ b). If b) holds then given $g \in G$ and $k \in L$, $B \in \mathfrak{B}$

$$\alpha_{gkg^{-1}}(B) = \alpha_g \alpha_k \alpha_{g^{-1}}(B) = \alpha_g \alpha_{g^{-1}}(B) = B$$

since $\alpha_{g^{-1}}(B) \in \mathfrak{B}$. Thus $\alpha_{gkg^{-1}}$ is an automorphism of $\mathfrak{F}$ leaving $\mathfrak{B}$ pointwise fixed. Thus $gkg^{-1} \in L$ and L is a normal subgroup, giving b) $\Rightarrow$ a). Suppose a) then consider the set of Hilbert spaces in $\mathfrak{B}$ inducing endomorphisms in $\Delta_f(\mathfrak{A})$. These must be L-invariant and each thus carry a canonical representation of G/L and $\mathfrak{B}^{G/L} = \mathfrak{F}^G = \mathfrak{A}$. We know that $\mathfrak{B}$ is generated by $H \cap \mathfrak{B}$ where H is a Hilbert space inducing an element of $\Delta_f(\mathfrak{A})$. This Hilbert space may not have support I but it is an invariant subspace for the action of G and is hence in the algebra generated by Hilbert spaces above. Obviously, c) implies b), completing the proof.

We next discuss a situation where two members of an inclusion of observable nets $\mathfrak{A} \subset \mathfrak{B}$ have coinciding canonical field nets. We start with a net $\mathfrak{B}$

and suppose that its canonical field net $\mathfrak{F}$ has a compact gauge group K of internal symmetries with $K \supset G$, where G is the gauge group of $\mathfrak{B}$ and then define $\mathfrak{A}$ to be the fixed–point net $\mathfrak{F}^K$.

We recall that if K is spontaneously broken then $\mathfrak{A}$ does not satisfy duality[21]. Its dual net $\mathfrak{A}^d$ is the fixed-point net of $\mathfrak{F}$ under the closed subgroup of unbroken symmetries and does satisfy duality. Furthermore, $\mathfrak{A}$ and $\mathfrak{A}^d$ have the same superselection structure. Hence in line with our strategy of considering only nets satisfying duality, we may restrict ourselves to the case that K is unbroken. We recall that, if $\mathfrak{F}$ has the split property, then the group K^{max} of all unitaries leaving Ω invariant and inducing net automorphisms of $\mathfrak{F}$ is automatically compact in the strong operator topology[9].

In the above situation $\{\mathfrak{F}, \mathfrak{A}, K, \mathcal{H}_{\mathfrak{A}}\}$ is a field system with gauge symmetry for $\mathfrak{A}$. Furthermore, $\rho \in \Delta_f(\mathfrak{A})$ is induced by a finite-dimensional Hilbert space H in $\mathfrak{F}$ since this is true of its extension to an element of $\Delta_f(\mathfrak{B})$. But this means that every sector of $\mathfrak{A}$ is realized on the vacuum Hilbert space of $\mathfrak{F}$ so that $\mathfrak{F}$ is the canonical field net of $\mathfrak{A}$ and K is the gauge group. We have thus proved the following result

Proposition 4.3 *Let $\mathfrak{B}$ be an observable net with canonical field net $\mathfrak{F}$ and gauge group G. Suppose $\mathfrak{F}$ has an unbroken compact group K of internal symmetries. Then the fixed–point net $\mathfrak{F}^K$ has $\mathfrak{F}$ as canonical field net.*

Finally, we consider the sector structure of an intermediate observable net $\mathfrak{A} \subset \mathfrak{B} \subset \mathfrak{F}(\mathfrak{A})$ satisfying duality. As we know from Theorem 4.1, $\mathfrak{B}$ is the fixed–points of $\mathfrak{F}(\mathfrak{A})$ under the action of a closed subgroup L of the gauge group G. We shall suppose that the vacuum Hilbert space of $\mathfrak{A}$ is separable, that Property B of Borchers holds for $\mathfrak{A}^d$ and that *each representation of $\mathfrak{A}$ satisfying the selection criterion is a direct sum of irreducibles with finite statistics.*

We now pick a representation $\hat{\pi}$ of $\mathfrak{B}$ satisfying the selection criterion for $\mathfrak{B}$. To analyse this representation, we choose an associated 1–cocycle z as in §2. Since $\mathfrak{B}$ satisfies duality, $z(b) \in \mathfrak{B}(|b|) \subset \mathfrak{F}(|b|)$. If we consider z as a 1–cocycle of $\mathfrak{F}$, it can be used, as discussed in §2, to define representations $\tilde{\pi}_a$ of $\mathfrak{F}$ where

$$\tilde{\pi}_a(F) := z(b)Fz(b)^*, \quad F \in \mathfrak{F}(\mathcal{O}),\, b \in \Sigma_1,\, \partial_0 b = a,\, \partial_1 b \subset \mathcal{O}'.$$

Note that $z(b)$ is a Bosonic operator in $\mathfrak{F}$. Restricting $\tilde{\pi}_a$ first to $\mathfrak{B}$ and then to the vacuum Hilbert space of $\mathfrak{B}$ gives the representations $\hat{\pi}_a$ associated with z considered as a cocycle of $\mathfrak{B}$. Thus $\hat{\pi}_a$ is equivalent to $\hat{\pi}$.

We now let π denote the restriction of some fixed $\tilde{\pi}_a$ to $\mathfrak{A}$. Now if the vacuum Hilbert space of $\mathfrak{F}$ is non–separable, then π cannot satisfy the selection criterion as its restriction to each $\mathfrak{A}(\mathcal{O}')$ is equivalent to a direct sum of uncountably many copies of the identity representation of $\mathfrak{A}(\mathcal{O}')$. However, we shall see that π is just a direct sum of representations satisfying the selection criterion. It suffices to show that any cyclic subrepresentation satisfies the selection criterion. Such a cyclic representation is, like π, locally normal and hence acts on a separable Hilbert space as a consequence of the following well known result.

Lemma 4.4 *Let $\mathfrak{A}$ be an observable net acting on a separable vacuum Hilbert space and ω be a locally normal state. Then the GNS representation π_ω of $\mathfrak{A}$ is separable.*

A proof may be found for example in §5.2 of [1].

Lemma 4.5 *Every cyclic subrepresentation of π satisfies the selection criterion.*

Proof We turn the equivalence of representations in restriction to $\mathfrak{A}(\mathcal{O}')$ into a question of the equivalence of two projections E_0 and F_0 in the representation of $\mathfrak{A}(\mathcal{O}')$ obtained by restricting the vacuum representation $\hat{\pi}_0$ of $\mathfrak{F}$ to $\mathfrak{A}(\mathcal{O}')$. E_0 is the projection onto the subspace given by the vacuum sector of $\mathfrak{A}$. F_0 is determined as follows. To be able to exploit the Borchers property, we choose a double cone $\mathcal{O}_0$ with $\mathcal{O}_0^- \subset \mathcal{O}$, and a unitary U such that

$$U\pi(A) = \hat{\pi}_0(A)U, \;\; A \in \mathfrak{A}(\mathcal{O}_0'),$$

and set $F_0 := UFU^*$, where F corresponds to the (cyclic) subrepresentation of π, $F \in \pi(\mathfrak{A})'$, with separable range.

Let $\sigma, \hat{\sigma}_0$ and $\tau, \hat{\tau}_0$ denote the restrictions of $\pi, \hat{\pi}_0$ to $\mathfrak{A}(\mathcal{O}')$ and $\mathfrak{A}(\mathcal{O}_0')$, respectively. Then $U \in (\tau, \hat{\tau}_0) \subset (\sigma, \hat{\sigma}_0)$ and $F_0 \in (\hat{\tau}_0, \hat{\tau}_0) \subset (\hat{\sigma}_0, \hat{\sigma}_0)$. Since $\hat{\pi}_0$ is, in restriction to $\mathfrak{A}$, a direct sum of representations satisfying the selection criterion, E_0 has central support I in both $(\hat{\tau}_0, \hat{\tau}_0)$ and $(\hat{\sigma}_0, \hat{\sigma}_0)$.

Thus there are projections e_0 and f_0 with $e_0 \prec E_0$, $f_0 \prec F_0$ and $e_0 \simeq f_0$ in $(\hat{\tau}_0, \hat{\tau}_0)$. Moreover, by Property B for $\mathfrak{A}^d$, $e_0 \simeq E_0$ in $(\hat{\sigma}_0, \hat{\sigma}_0)$ Thus E_0 is equivalent to the subprojection f_0 of F_0 in $(\hat{\sigma}_0, \hat{\sigma}_0)$. Since F_0 is separable and E_0 has infinite multiplicity by Property B, we have

$$E_0 \prec F_0 \prec \infty E_0 \simeq E_0.$$

Thus E_0 and F_0 are equivalent, completing the proof.

Corollary 4.6 *π is normal on $\mathfrak{A}$, $\tilde{\pi}_a$ is normal on $\mathfrak{F}$ and $\hat{\pi}$ is normal on $\mathfrak{B}$, where the term normal refers to the vacuum representation of $\mathfrak{F}$.*

Proof. The first statement follows at once from Lemma 4.5, the second by invoking Theorem A.6 of the Appendix and the third is obvious since, as we have seen, $\hat{\pi}$ is equivalent to a subrepresentation of the restriction $\tilde{\pi}_a$ to $\mathfrak{B}$.

Now any normal representation of $\mathfrak{B}$ is just a direct sum of subrepresentations of the defining representation so we have proved the following result.

Theorem 4.7 *Let $\mathfrak{A}$ be an observable net on a separable Hilbert space whose dual net satisfies Property B and suppose that every representation of $\mathfrak{A}$ satisfying the selection criterion is a direct sum of irreducible representations with finite statistics. Let $\mathfrak{B}$ be an intermediate observable net satisfying duality, i.e. $\mathfrak{A} \subset \mathfrak{B} \subset \mathfrak{F}(\mathfrak{A})$ and L the associated compact group as in Theorem 3.1. Then every representation of $\mathfrak{B}$ satisfying the selection criterion is a direct sum of sectors with finite statistics and these are labelled by the equivalence classes of irreducible representations of L.*

As a particular case of this, we note that when $\mathfrak{F}$ contains Fermi elements, then the Bose part of $\mathfrak{F}$ is the fixed–point algebra of $\mathfrak{F}$ under $\mathbb{Z}_2$ and has precisely two sectors. In the case where $\mathfrak{A}$ has only a finite number of superselection sectors, the above result is already known, cf. [4],[20]. Theorem 4.7 has an immediate corollary.

Corollary 4.8 *Under the hypothesis of Theorem 4.7, the field nets of $\mathfrak{A}$ and $\mathfrak{B}$ coincide, $\mathfrak{F}(\mathfrak{A}) = \mathfrak{F}(\mathfrak{B})$.*

5 Appendix

In this appendix we collect together various results needed in the course of this paper. They have in common that they do not involve the net structure but typically the harmonic analysis of the action of compact groups on von Neumann algebras and C^*–algebras and conditional expectations. The results are looked at in terms of the structure of the category of finite-dimensional continuous, unitary representations of the group rather than the group itself. Consequently, the results transcend group theory. This degree of generality is not needed in this paper.

Lemma A.1 *Let m be a conditional expectation from the *–algebra $\mathcal{B}$ onto the *–subalgebra $\mathcal{A}$ and H a Hilbert space in $\mathcal{B}$ such that $m(H) \subset H$, then m restricted to H is the orthogonal projection onto the closed subspace $\mathcal{A} \cap H$.*

Proof If $\psi, \psi' \in H$ then $\psi^* m(\psi)$ is a scalar. Thus $\psi^* m(\psi') = m(\psi^* m(\psi')) = m(\psi)^* m(\psi')$ so $m(\psi)^*\psi' = \psi^* m(\psi')$. Hence m restricted to H is selfadjoint and as it is anyway involutive, it is the orthogonal projection onto $m(H) = \mathcal{A} \cap H$, as required.

There are some obvious corollaries of this result. Suppose that $\mathcal{B}$ is generated by $\mathcal{A}$ and a collection $\mathcal{H}$ of Hilbert space in B then there is at most one conditional expectation m of $\mathcal{B}$ onto $\mathcal{A}$ such that $m(H) \subset H$ for each $H \in \mathcal{H}$. If we suppose that $\mathcal{B}$ is a C^*–algebra then it suffices if $\mathcal{A}$ and $\mathcal{H}$ generate $\mathcal{B}$ as a C^*–algebra. If $\mathcal{B}$ is a von Neumann algebra and m is normal then it suffices if $\mathcal{A}$ and $\mathcal{H}$ generate $\mathcal{B}$ as a von Neumann algebra. These results apply in particular to the case where $\mathcal{B}$ is the cross product of $\mathcal{A}$ by the action of a dual object of a compact group. Note, too, that the hypothesis $m(H) \subset H$ is redundant if the canonical endomorphism of H maps $\mathcal{A}$ into itself and if $\mathcal{A}' \cap \mathcal{B} = \mathbb{C}$. Thus 'minimal' or perhaps better irreducible cross products have a unique mean.

Lemma A.2 *Let $\mathcal{A} \subset \mathcal{B} \subset \mathcal{F}$ be inclusions of C^*–algebras and $m_{\mathcal{B}}$ a conditional expectation of $\mathcal{F}$ onto $\mathcal{B}$. Let $\mathcal{H}$ denote a category of Hilbert spaces in $\mathcal{F}$ each normalizing $\mathcal{B}$ and $\mathcal{A}$ and such that $m_{\mathcal{B}}(H) \subset H$ for each object H of $\mathcal{H}$. Let m a conditional expectation of $\mathcal{B}$ onto $\mathcal{A}$. Suppose that, whenever H is an object of $\mathcal{H}$ and σ_H the corresponding endomorphism, then*

$$\psi \in \mathcal{A}, \quad \psi A = \sigma_H(A)\psi, \quad A \in \mathcal{A},$$

implies $\psi \in H$. Let $\mathcal{E}$ denote the C^*–subalgebra of $\mathcal{F}$ generated by $\mathcal{A}$ and the objects H of $\mathcal{H}$ then $m \circ m_{\mathcal{B}}$ restricted to $\mathcal{E}$ is the unique conditional expectation n of $\mathcal{E}$ onto $\mathcal{A}$ with $n(H) \subset H$ for all objects H of $\mathcal{H}$.

Proof The uniqueness of n holds since $\mathcal{E}$ is generated by $\mathcal{A}$ and the objects of $\mathcal{H}$ and since $n(H) \subset H$ for each such object H. Now taking n to be the restriction of $m \circ m_{\mathcal{B}}$ to $\mathcal{E}$, n is trivially a conditional expectation onto $\mathcal{A}$. If $\psi \in H$, then

$$n(\psi)A = n(\psi A) = \sigma_H(A)n(\psi), \quad A \in \mathcal{A},$$

since H normalizes $\mathcal{A}$. Hence $n(\psi) \in H$ by hypothesis, completing the proof.

By a partition of the identity on a Hilbert space $\mathcal{H}$ we mean a set E_i, $i \in I$ of (self-adjoint) projections with sum the identity operator. Each element $X \in \mathcal{B}(\mathcal{H})$ can then be written $X = \sum_{i,j} E_i X E_j$ with convergence in say the s–topology. The set of elements for which this sum is finite forms a *–subalgebra $\mathcal{B}(\mathcal{H})_I$ of $\mathcal{B}(\mathcal{H})$ which is a direct sum of the subspaces $E_i\mathcal{B}(\mathcal{H})E_j$.

We let s_f denote the topology on the *–subalgebra which is the direct sum of the s–topologies on these subspaces.

Lemma A.3 *Let E_i, $i \in I$ be a partition of the unit on a Hilbert space $\mathcal{H}$ and π a representation of $\mathcal{B}(\mathcal{H})_I$ on $\mathcal{H}_\pi$, continuous in the s_f–topology when $\mathcal{B}(\mathcal{H}_\pi)$ is given the s–topology. Then if $\pi(E_i)$, $i \in I$, is a partition of the identity, π extends uniquely to an s–continuous representation of $\mathcal{B}(\mathcal{H})$.*

Proof If π extends to an s–continuous representation, again denoted by π, we must have

$$\pi(X) = \sum_{i,j} \pi(E_i X E_j), \quad X \in \mathcal{B}(\mathcal{H}),$$

so any extension is unique. On the other hand, this expression for $\pi(X)$ is obviously defined on the dense subspace spanned by the subspaces $\pi(E_i)\mathcal{H}_\pi$, $i \in I$. Hence, it suffices to show that $\pi(X)$ is bounded there. Let J be a finite subset of I and $E_J := \sum_{j \in J} E_j$. Then the von Neumann algebra $E_J \mathcal{B}(\mathcal{H}) E_J$ is a *–subalgebra of $\mathcal{B}(\mathcal{H})_I$ so that

$$\|\pi(E_J X E_J)\| \leq \|E_J X E_J\| \leq \|X\|, \quad X \in \mathcal{B}(\mathcal{H})$$

and $\pi(X)$ is bounded. Computing matrix elements from the dense subspace, we see that we have a representation of $\mathcal{B}(\mathcal{H})$. To see that it is normal, it suffices to show that its restriction to the compact operators is non–degenerate. However, its restriction to the compact operators on each $E_i\mathcal{H}$ is non-degenerate on $\pi(E_i)\mathcal{H}_\pi$. But $\pi(E_i)$ is a partition of the identity, so the result follows.

Remark Another way of looking at the above result is that $\mathcal{B}(\mathcal{H})$ is the inductive limit of the von Neumann algebras $\mathcal{B}(E_J\mathcal{H})$ as J runs over the set of finite subsets of I, ordered under inclusion. The inductive limit is here understood in the category of von Neumann algebras with normal, but not necessarily unit–preserving *–homomorphisms.

We now consider a von Neumann algebra $\mathcal{M}$ and a faithful, normal conditional expectation m onto a von Neumann subalgebra $\mathcal{A}$. Consider $\mathcal{M}$ as a left $\mathcal{A}$–module with the $\mathcal{A}$–valued scalar product $m(XY^*)$ derived from m.

Lemma A.4 *A representation π of $\mathcal{M}$, s–continuous in restriction to $\mathcal{A}$ is also s–continuous in restriction to any submodule $\mathcal{N}$ of finite rank.*

Proof When $\mathcal{N}$ has finite rank, we can find a finite orthonormal basis ψ_i using the Gram–Schmidt orthogonalization process. Thus for each $X \in \mathcal{N}$, we have

$$X = \sum_i m(X\psi_i^*)\psi_i.$$

Suppose $X_n \to X$ in the s–topology on $\mathcal{N}$. Then $\pi(m(X_n\psi_i^*)) \to \pi(m(X\psi_i^*))$ and hence $\pi(X_n) \to \pi(X)$ as required.

We will need some variant of this result where $\mathcal{M}$ is just a C^*–algebra. We could assume that $\mathcal{N}$ has a finite orthonormal basis or say assume that it is a finite–rank projective module where the coefficients can be chosen continuous in the s–topology.

To make a bridge between Lemmas A.3 and A.4, we need another structure related to the notion of hypergroup. We consider a set Σ and a mapping $(\sigma, \tau) \mapsto \sigma \otimes \tau$ from $\Sigma \times \Sigma$ into the set of finite subsets of Σ. We suppose further that Σ is equipped with an involution (conjugation) $\sigma \mapsto \bar{\sigma}$ with the property that $\rho \in \sigma \otimes \tau$ if and only if $\tau \in \bar{\sigma} \otimes \rho$ and if and only if $\sigma \in \rho \otimes \bar{\tau}$. Furthermore there is a distinguished element $\iota \in \Sigma$ such that $\iota \otimes \sigma$ and $\sigma \otimes \iota$ both consist of the single point σ for each $\sigma \in \Sigma$.

If Σ is as above then a C^*–algebra $\mathcal{B}$ will be said to be Σ–graded if there are norm–closed linear subspaces $\mathcal{B}_\sigma$, $\sigma \in \Sigma$, spanning $\mathcal{B}$ such that $\mathcal{B}_\sigma^* = \mathcal{B}_{\bar{\sigma}}$ and if $\mathcal{B}_\sigma \mathcal{B}_\tau \subset \mathcal{B}_{\sigma\otimes\tau}$. Here $\mathcal{B}_{\sigma\otimes\tau}$ denotes the norm–closed subspace spanned by the $\mathcal{B}_\rho$ as ρ runs over the elements of $\sigma\otimes\tau$. Note that $\mathcal{B}_\iota$ is a C^*–subalgebra of $\mathcal{B}$ and that each $\mathcal{B}_\sigma$ is a $\mathcal{B}_\iota$–bimodule.

A representation π of a Σ–graded C^*–algebra $\mathcal{B}$ is a representation of $\mathcal{B}$ on a Hilbert space $\mathcal{H}$ which is a direct sum of closed linear subspaces $\mathcal{H}_\sigma$ such that

$$\pi(\mathcal{B}_\sigma)\mathcal{H}_\tau \subset \mathcal{H}_{\sigma\otimes\tau},$$

where $\mathcal{H}_{\sigma\otimes\tau}$ is defined in the obvious manner.

Lemma A.5 *Let π be a representation of a Σ–graded C^*–algebra and E_σ the projection on $\mathcal{H}_\sigma$ then*

$$E_\sigma \pi(\mathcal{B}) E_\tau \subset E_\sigma \pi(\mathcal{B}_{\sigma\otimes\bar{\tau}}) E_\tau.$$

Proof $\pi(\mathcal{B}_\rho)\mathcal{H}_\tau \subset \mathcal{H}_{\rho\otimes\tau}$. Thus $E_\sigma\pi(\mathcal{B}_\rho)E_\tau = 0$ unless $\sigma \in \rho \otimes \tau$, i.e. unless $\rho \in \sigma \otimes \bar{\tau}$.

The obvious example of the above structure is to consider a compact group G acting on a C^*–algebra $\mathcal{B}$ and to take Σ to be the set of equivalence classes of irreducible, continuous unitary representations of G. We now set $\mathcal{B}_\sigma := m_\sigma(\mathcal{B})$, where

$$m_\sigma(B) := \int_G \alpha_g(B)\overline{\chi_\sigma(g)}, \quad B \in \mathcal{B},$$

and χ_σ denotes the normalized trace of σ.

In the same way, if (π, U) is a covariant representation of $\{\mathcal{B}, \alpha\}$, we get a representation of the Σ–graded C^*–algebra $\mathcal{B}$ by using

$$E_\sigma := \int_G \overline{\chi_\sigma(g)}U(g)$$

to define the closed linear subspace $\mathcal{H}_\sigma$.

We now put the above results together in the form of a theorem needed in the body of the text.

Theorem A.6 *Given a C^*–algebra $\mathcal{B}$ acting irreducibly on a Hilbert space $\mathcal{H}$ and a continuous unitary representation U of a compact group G inducing an action $\alpha : G \to Aut(\mathcal{B})$ on $\mathcal{B}$ with full Hilbert spectrum, then every representation of $\mathcal{B}$ normal on $\mathcal{B}^G$ is normal on $\mathcal{B}$.*

Proof Let $\mathcal{A}$ denote the fixed point algebra and let E_σ be as above. Since $\mathcal{B}$ is irreducible, E_σ is in the weak closure of $\mathcal{A}$, so that extending π to this weak closure by normality, we have a partition $\pi(E_\sigma)$ of the unit in the representation space of π. Then by Lemma A.5 above, $E_\sigma\mathcal{B}E_\tau$ is finite–dimensional as a left $\mathcal{A}$–module. Since the action has full Hilbert spectrum, i.e. every irreducible representation of G is realized on some Hilbert space in $\mathcal{B}$, the argument of Lemma A.4 applies and shows that a representation π of $\mathcal{B}$ normal on $\mathcal{A}$ is normal on each $E_\sigma\mathcal{B}E_\tau$. The result now follows from Lemma A.3.

We come now to a result on the existence of normal conditional expectations, beginning with a simple lemma of interest in its own right.

Lemma A.7 *Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and E a cyclic and separating projection for $\mathcal{M}$. Let $\mathcal{M}^E := \mathcal{M} \cap \{E\}'$ and ${}^E\mathcal{M} := \{M \in \mathcal{M} : EME \in (E\mathcal{M}'E)'\}$, then ${}^E\mathcal{M}$ is a weak-operator closed $\mathcal{M}^E$–bimodule containing $\mathcal{M}^E$ as a subbimodule. Given $M \in \mathcal{M}$ there is a $\mu(M) \in \mathcal{M}^E$ such that*

$$\mu(M)E = EME$$

if and only if $M \in {}^E\mathcal{M}$.

Proof Given $M \in {}^E\mathcal{M}$ and $M' \in \mathcal{M}'$, then

$$EM^*EM'^*M'EME \leq ||EME\|^2\, EM'^*M'E,$$

since EME and $EM'^*M'E$ commute. Thus E being cyclic for $\mathcal{M}'$, there exists a unique bounded operator $\mu(M)$ such that

$$\mu(M)M'E = M'EME.$$

Obviously, $\mu(M) \in \mathcal{M}$ and a computation shows that $\mu(M^*) = \mu(M)^*$. Setting $M' = I$, it now follows that $\mu(M)$ commutes with E. On the other hand, if $\mu(M)E = EME$ for some $M \in \mathcal{M}$ then $M \in {}^E\mathcal{M}$. The remaining assertions are evident.

Specializing to the case that ${}^E\mathcal{M} = \mathcal{M}$ gives the following result.

Corollary A.8 *Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and E a cyclic and separating projection for $\mathcal{M}$. Then the following conditions are equivalent.*

- *a) There is a conditional expectation μ on $\mathcal{M}$ such that $\mu(M)E = EME, \quad M \in \mathcal{M}$.*
- *a') There is a conditional expectation μ' on $\mathcal{M}'$ such that $\mu'(M')E = EM'E, \quad M' \in \mathcal{M}'$.*
- *b) $[E\mathcal{M}E, E\mathcal{M}'E] = 0$.*
- *c) $({}_E(\mathcal{M}'))' = {}_E\mathcal{M}$.*

Here ${}_E\mathcal{M}$, for example, denotes the restriction of $E\mathcal{M}E$ to $E\mathcal{H}$. The conditional expectations μ and μ' are automatically normal.

Proof Suppose b) holds then ${}^E\mathcal{M} = \mathcal{M}$ and by Lemma A.7, μ becomes a normal conditional expectation onto $\mathcal{M}^E$ since it is idempotent and of norm 1. We have therefore deduced a) and by symmetry a$'$). Now suppose a) holds, then $\mu(\mathcal{M})$ is just $\mathcal{M}^E$ and $\mathcal{M} = {}^E\mathcal{M}$, proving b). Furthermore, its restriction to $E\mathcal{H}$ is $\mu(\mathcal{M})_E$. Thus $({}_E\mathcal{M})' = \mu(\mathcal{M})'_E$. Since $\mu(\mathcal{M}) = \mathcal{M} \cap (E)'$, elements of the form $M'_1 + M'_2EM'_3$ with $M'_i \in \mathcal{M}'$ form an s–dense *–subalgebra in its commutant and restricting this to $E\mathcal{H}$, we have proved c). Trivially, c) implies b), so the conditions of the corollary are equivalent.

Remark If σ is an (inner) automorphism of $\mathcal{B}(\mathcal{H})$, the above conditions are satisfied by $\sigma(\mathcal{M})$ and $\sigma(E)$ and the corresponding conditional expectation is $\sigma\mu\sigma^{-1}$. In particular, if $\sigma\mathcal{M} = \mathcal{M}$ and $\sigma(E) = E$, then $\mu\sigma = \sigma\mu$.

Corollary A.9 *Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of von Neumann algebras on Hilbert spaces $\mathcal{K}$ and $\mathcal{H}$, respectively. Let E, the projection from $\mathcal{H}$ onto $\mathcal{K}$, be cyclic and separating for $\mathcal{M}$, then the following conditions are equivalent.*

a) There is a (necessarily unique and injective) morphism $\nu : \mathcal{N}' \to \mathcal{M}'$ such that

$$\nu(N')\Phi = N'\Phi, \quad N' \in \mathcal{N}', \quad \Phi \in \mathcal{K}.$$

b) $\mathcal{N}' = \mathcal{M}'_E$.

c) There is a conditional expectation m of $\mathcal{M}$ onto $\mathcal{N}$ such that

$$m(M)E = EME, \quad M \in \mathcal{M}.$$

Here $\mathcal{M}'_E$ denotes the restriction of $\mathcal{M}'^E$ to $E\mathcal{H} = \mathcal{K}$.

Proof ν is obviously unique since $\mathcal{K}$ is cyclic for each $\mathcal{M}$. Given a), we note that $E\nu(N')E = \nu(N')E$ and replacing N' by N'^*, we see that $\nu(N') \in \mathcal{M}'^E$. Hence $N' \in \mathcal{M}'_E$, yielding b). Conversely, if b) is satisfied, given $N' \in \mathcal{N}'$, there is an $M' \in \mathcal{M}'$ with $M'E = EM'$ and $M'\Phi = N'\Phi$. Hence, we may pick $\nu(N') = M'$ to give a map $\nu : \mathcal{N}' \to \mathcal{M}'$ and it follows from uniqueness that ν is a morphism. Now if $M \in \mathcal{M}$ and $N' \in \mathcal{N}'$, then by a), N' commutes with the restriction of EME to $\mathcal{K}$. Thus $EME \in \mathcal{N} \subset \mathcal{M}^E$ and c) follows from Lemma A.7. Conversely, if c) holds then $\mathcal{N} = \mathcal{M}_E$ and b) follows by calculating commutants.

Remark Of course, when the conditions of Corollary A.9 are satisfied, there is also a conditional expection m' of $\mathcal{M}'$ onto $\nu(\mathcal{N}')$ such that

$$m'(M')E = EM'E, \quad M' \in \mathcal{M}'.$$

This follows from Corollary A.8.

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