EMBEDDING TANGLES IN LINKS

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ABSTRACT

Extending and reproving a recent result of D. Krebes, we give obstructions to the embedding of a tangle in a link.

Keywords: Tangle, link, determinant, branched covering

1. Introduction

A recent paper [2] of David Krebes poses the following interesting question: given a tangle $\mathcal{T}$, and a link $\mathcal{L}$, when can $\mathcal{T}$ sit inside $\mathcal{L}$? A *tangle* is a 1-manifold with 4 boundary components, properly embedded in a 3-ball; this is somewhat more general than the usual definition. An embedding of $\mathcal{T}$ in $\mathcal{L}$ is determined by a ball in S^3, whose intersection with $\mathcal{L}$ is the given tangle; we will indicate an embedding by $\mathcal{T} \hookrightarrow \mathcal{L}$. Krebes gives the following simple criterion for when $\mathcal{T}$ embeds in $\mathcal{L}$. Complete the tangle to a link in either of the two obvious ways, giving rise to the *numerator* $n(\mathcal{T})$ and *denominator* $d(\mathcal{T})$. For any oriented link $\mathcal{L}$, its determinant $\det(\mathcal{L})$ is defined to be $\det(V+V')$, where V is a Seifert matrix for $\mathcal{L}$. Krebes shows

Theorem 1. *If $\mathcal{T} \hookrightarrow \mathcal{L}$, then*

$$\gcd(\det(n(\mathcal{T})), \det(d(\mathcal{T}))) \,\Big|\, \det(\mathcal{L}) \tag{1.1}$$

The proof of Theorem 1 given in [2] is essentially combinatorial, and uses an interpretation of the determinant in terms of link diagrams and the Kauffman bracket. On the other hand, the determinant has a homological interpretation; it is essentially the order of the homology of the 2-fold branched cover S^3, branched along the link. In this paper, we will prove a simple fact about the homology of certain 3-manifolds, which readily implies Theorem 1. In essence, we replace the divisibility condition above with the conclusion that the homology of the 2-fold branched cover of the link must contain a subgroup of a certain size. Our approach has the advantage that it gives some stronger results on the embedding problem, which do not seem approachable via the combinatorial route. A slightly different argument yields a similar conclusion about other branched coverings as well. Some examples, and further remarks on embeddings, are given in the final section.

To state the result, let us use the notation $|M|$ for the order of the first homology group of M, where by convention the order is defined to be 0 if the homology is infinite. Also, if ∂M is a torus T^2, and $\alpha \subset T$ is a simple closed curve which does not bound a disc, then let $M(\alpha)$ denote the result of Dehn filling with slope α. (In other words, glue $S^1 \times D^2$ to M so that ∂D^2 is glued to α.)

Theorem 2. *Suppose that M is an orientable 3-manifold, and that α, β are simple closed curves on $T = \partial M$ which generate all of $H_1(T)$. Suppose that $M \subset N$, where N is a closed, orientable 3-manifold. Then*

$$\gcd(|M(\alpha)|, |M(\beta)|) \,\Big|\, |N|$$

It is worth remarking that as a consequence of Theorem 2, the quantity

$$f(M) = \gcd(|M(\alpha)|, |M(\beta)|)$$

is independent of the choice of pair α, β, and hence defines an invariant of M. For, given another such pair, say α', β', the theorem says that $\gcd(|M(\alpha)|, |M(\beta)|)$ divides both $|M(\alpha')|$ and $|M(\beta')|$, and so $\gcd(|M(\alpha)|, |M(\beta)|) \,\Big|\, \gcd(|M(\alpha')|, |M(\beta')|)$. The remark follows by reversing the roles of α, β and α', β'.

Some further results on the embedding problem, using invariant derived from the Kauffman bracket, can be found in the recent preprint [1].

2. Proof of Theorem 2

Before beginning the proof of the theorem, we remark that unless $H_1(N)$ is torsion, then the theorem has no content. So we can assume that N is a rational homology sphere for the remainder of this section. Writing

$$N = M \cup_T X$$

it follows from a standard Poincaré duality argument that both M and X have the rational homology of a circle. Hence we can write (non-canonically)

$$H_1(M) = \mathbf{Z} \oplus \mathbf{Z}/q_1 \oplus \ldots \mathbf{Z}/q_s.$$

In particular, the torsion subgroup $T_1(M) \subset H_1(M)$ has order $q_1 \cdots q_s$.

Under the map $j_* : H_1(T) \to H_1(M)$, the classes α and β go to

$$j_*(\alpha) = (a, a_1, \ldots, a_s) \quad \text{and} \quad j_*(\alpha) = (b, b_1, \ldots, b_s)$$

respectively. (The coefficients are with respect to generators of the summands of $H_1(M)$ in the splitting given above.)

Claim: $|M(\alpha)| = a\,|T_1(M)|$.

To see this, note that $H_1(M(\alpha)) \cong H_1(M)/ < \alpha >$. The given splitting of $H_1(M)$ corresponds to a presentation of that group by the $s \times (s+1)$ matrix

$$\begin{pmatrix} 0 & q_1 & 0 & \dots & 0 \\ 0 & 0 & q_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & q_s \end{pmatrix} \tag{2.2}$$

Killing α adds an additional row, to get the following presentation matrix for $H_1(M(\alpha))$:

$$\begin{pmatrix} a & a_1 & a_2 & \dots & a_s \\ 0 & q_1 & 0 & \dots & 0 \\ 0 & 0 & q_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & q_s \end{pmatrix} \tag{2.3}$$

which has determinant $aq_1 \cdots q_s = a\,|T_1(M)|$. But the order of $H_1(M)$ is the same as the determinant of any (square) presentation matrix for it.

By the same argument for homology of β we get that

$$\gcd(|M(\alpha)|, |M(\beta)|) = \gcd(a\,|T_1(M)|, b\,|T_1(M)|) = \gcd(a,b)|T_1(M)| \tag{2.4}$$

Now we turn to the situation at hand, and consider the homology of N, which we place into the long exact sequence of the pair (N, M).

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_2(N,M) & \xrightarrow{\partial} & H_1(M) & \longrightarrow & H_1(N) & \longrightarrow H_1(N,M) & \longrightarrow & 0 \\ & & \uparrow{\cong} & & \uparrow{j_*} & & & & & \\ & & H_2(X,T) & \xrightarrow{\partial} & H_1(T) & & & & & \end{array}$$

By exactness, $H_1(M)/\partial(H_2(N,M))$ injects into $H_1(N)$, and so the theorem will follow if we can show that $\gcd(a,b)|T_1(M)|$ divides the order of $H_1(M)/\partial(H_2(N,M))$. Again by duality, $H_2(X,T) \cong H^1(X) \cong \mathbf{Z}$ is generated by a relative 2-cycle C with boundary γ lying in T. By the commutativity of the diagram (the isomorphism is an excision) it follows that $H_1(M)/\partial(H_2(N,M))$ is just $H_1(M)/ < \gamma >$. As before, the image of γ in $H_1(M)$ may be written $(c, c_1, \dots, c_s)$, so that $|H_1(M)/ < \gamma > | = c\,|T_1(M)|$. Since α and β generate $H_1(T)$, we can write $\gamma = m\alpha + n\beta$ which implies that $c = ma + nb$. This means that $\gcd(a,b) \Big| c$, or in other words

$$\gcd(a,b)|T_1(M)| \Big| c\,|T_1(M)|.$$

Since $c\,|T_1(M)| \Big| |H_1(N)|$ the theorem follows. □

3. Complements and Examples

First, let us observe that Theorem 2 implies Theorem 1. The basic point is that the 2-fold cover of the ball, branched along a trivial tangle, is $S^1 \times D^2$. The different ways of completing a tangle $\mathcal{T}$ to a link give rise to different Dehn fillings of $M =$ the 2-fold cover of the ball, branched along $\mathcal{T}$. It is easy to see that the meridians of the solid tori corresponding to the numerator $n(\mathcal{T})$ and denominator $d(\mathcal{T})$ have intersection number ± 1 in $T = \partial M$ and thus generate $H_1(T)$. Now if $\mathcal{T} \hookrightarrow \mathcal{L}$ as in Theorem 1, there is an embedding $M \subset N$, where N is the 2-fold cover of the 3-sphere branched along $\mathcal{L}$. The 2-fold cover of the ball, branched along a trivial tangle, is a solid torus. Hence the hypotheses of Theorem 2 are satisfied, and we get that

$$\gcd(|n(\mathcal{T})|, |d(\mathcal{T})|) \Bigm| |\det(\mathcal{L})|$$

Second, the proof gives a somewhat stronger condition, involving the homology groups themselves, rather than their orders. We give the statement in terms of the homology of 3-manifolds, with the understanding that passing to the 2-fold cover of a link gives rise to a restriction on embeddings of tangles in links.

Corollary 1. *Suppose that $H_1(N)$ is torsion, and that $M \subset N$. Then there is an injection of $\mathbf{Z}/c \oplus T_1(M)$ into $H_1(N)$, where c has the same significance as in the proof of Theorem 2.*

In applying these results to specific tangles, the most useful part of the conclusion is the fact that the inclusion map of M into N induces an injection on $T_1(M)$. This is true in a more general setting, by an argument which is perhaps more conceptual than the calculation proving Theorem 2.

Theorem 3. *Suppose M is an orientable 3-manifold with connected boundary, and $i : M \subset N$, where N is an orientable 3-manifold with $H_1(N)$ torsion. Then the inclusion map i_* induces an injection of $T_1(M)$ into $H_1(N)$.*

Proof of Theorem 3: The linking pairing $\lambda : T_1(M) \times T_1(M, \partial M) \to \mathbf{Q}/\mathbf{Z}$ is non-degenerate, by Poincaré duality. So if $x \in T_1(M)$ is non-zero, there is an element $y \in T_1(M, \partial M)$ with $\lambda(x, y)$ non-zero in $\mathbf{Q}/\mathbf{Z}$; represent each of these by (absolute or relative) cycles with the same names. Since ∂M is connected, $H_1(M) \to H_1(M, \partial M)$ is surjective, so y lifts to $\bar{y} \in H_1(M)$. There is no good reason that $\bar{y}$ represents a torsion class in M, but by hypothesis it is a torsion class in N. Now $\lambda(x, i_*\bar{y})$ (as calculated in N) may be calculated as the intersection number of $\bar{y}$ with a 2-chain bounding $n \cdot x$, and C can be chosen to lie in M since x is a torsion class in M. Hence $\lambda(x, i_*\bar{y}) = \lambda(x, y) \neq 0$ in $\mathbf{Q}/\mathbf{Z}$, and therefore x is nontrivial in $H_1(N)$. □

Remark: It is not possible to deduce Theorem 2 (and hence Theorem 1) from Theorem 3. To do so would amount to proving that (in the notation of Theorem 2) $\gcd(|M(\alpha)|, |M(\beta)|) = |T_1(M)|$. However, this is not the case, as the following example indicates; the example has $T_1(M) = \mathbf{Z}/3$ but $\gcd(|M(\alpha)|, |M(\beta)|) = 9$.

Consider an oriented solid torus M_0 with a basis α, β for $H_1(T)$ chosen so that β generates $H_1(M_0)$ and α bounds a disk. Let $K \subset M_0$ be an oriented knot, representing 3 times β in $H_1(M_0)$. The meridian μ of K is determined by the orientation, and we choose a longitude λ by requiring that λ be homologous to 3β in $H_1(M_0 - K)$. (If M_0 were embedded in S^3 in a standard way, so that β bounds a disk in $S^3 - \mathrm{int}(M_0)$, then λ would be the longitude of K in S^3.) Now let M be the result of Dehn surgery on M_0, with coefficient 9/1. In other words, remove a neighborhood of K, and glue in a solid torus killing $9\mu + \lambda$. The homology of $M_0 - \nu(K)$ is generated by μ and β, and the surgery kills $9\mu + 3\beta$, so $H_1(M) = \mathbf{Z} + \mathbf{Z}/3$.

On the other hand, the homology of $M(\alpha)$ is gotten by killing α, which is homologous to 3μ and so $H_1(M(\alpha))$ is presented by the matrix

$$\begin{pmatrix} 9 & 3 \\ 3 & 0 \end{pmatrix} \tag{3.5}$$

yielding $H_1(M(\alpha)) = \mathbf{Z}/3 \oplus \mathbf{Z}/3$. Likewise, $H_1(M(\beta))$ is presented by the matrix

$$\begin{pmatrix} 9 & 3 \\ 0 & 1 \end{pmatrix} \tag{3.6}$$

yielding $H_1(M(\beta)) = \mathbf{Z}/9$.

This same example shows that the hypothesis in Theorem 3, that $H_1(N)$ be torsion, is necessary. For if N is obtained by filling M with slope $\alpha + \beta$, then $H_1(N) \cong \mathbf{Z}$, and so $T_1(M)$ doesn't inject.

Theorem 3 may be applied to branched covers of tangles, of degrees other than 2. In applying this remark, one must take care, because for $k > 2$ the k-fold covers of the ball, branched along $\mathcal{T}$ are not uniquely specified by the branch locus. These different covers may well have differing homology groups, so in practice, one might have to calculate the homology groups for all of the different possibilities.

Corollary 2. *Suppose that $\mathcal{T} \hookrightarrow \mathcal{L}$, and that N is a k-fold cover of S^3 branched along $\mathcal{L}$. Let M be the induced cover of B^3, branched along $\mathcal{T}$. If N is a rational homology sphere, then $T_1(M)$ is a subgroup of $T_1(N)$.*

Most of the results so far have concerned only the torsion part of the homology, but there are some things which can be said about the torsion-free part of the homology. One simple result is the following.

Theorem 4. *Suppose that M has boundary of genus g, and that $i : M \subset N$. Then $\dim(H_1(N; \mathbf{Q})) \geq \dim(H_1(M; \mathbf{Q})) - g$.*

The proof is straightforward; if the quantity $\dim(H_1(M; \mathbf{Q})) - g$ is greater than zero, then there is a subspace of that dimension in $H_1(M; \mathbf{Q})$ which pairs non-trivially with a subspace of $H_2(M; \mathbf{Q})$ of the same dimension. Hence both of these inject into the homology of N.

Example 1. It is not hard to give examples where the homological approach gives more information than can be deduced from the determinants alone. The simplest I can think of is the following. Let $\mathcal{T}$ be the tangle

$$(T_3)^* + (T_3)^* + (T_{-3})^*$$

pictured below in Figure 1.

According to [2], the numerator and denominator of this tangle may be calculated

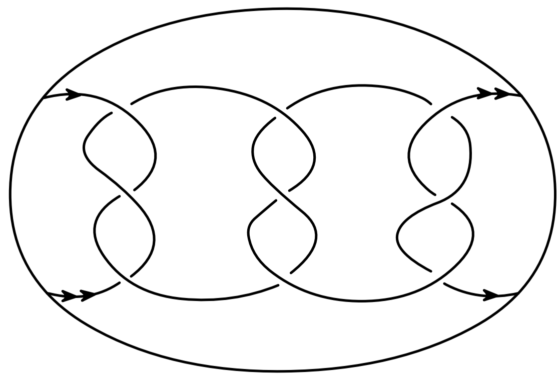

Figure 1

as the numerator and denominator of the fraction obtained by the grade school addition of fractions, without canceling common factors:

$$\frac{-1}{3}+\frac{-1}{3}+\frac{1}{3}=\frac{-6}{9}+\frac{1}{3}=\frac{-18+9}{27}=\frac{-9}{27}$$

and so $\gcd(|n(\mathcal{T})|,|d(\mathcal{T})|)=9$. This would allow, in principle, that $\mathcal{T}$ might embed in a 2-bridge knot (or link) corresponding to the fraction $9p/q$, for the determinant of such a knot is $9p$. But we calculate below that for $M=$ the 2-fold cover of B^3 branched along $\mathcal{T}$, we have $H_1(M)=\mathbf{Z}\oplus\mathbf{Z}/3\oplus\mathbf{Z}/3$, whereas the 2-fold cover of a 2-bridge knot or link has cyclic homology.

The calculation of $H_1(M)$ proceeds as in the usual calculation of 2-fold covers branched along knots, as described in §6 of [3]. If surgery is done on the middle crossings in each of the three $\pm1/3$ tangles which make up $\mathcal{T}$, then $\mathcal{T}$ becomes trivial. Straightening it out (for the purpose of drawing the branched cover, it's legal to move the endpoints around) gives the surgery picture in Figure 2.

Passing to the double branched cover give a picture of M as surgery on a 6-component link in $S^1\times D^2$, whose homology may be readily calculated to give the result quoted above.

4. An obstruction to embedding in a trivial link

An obstruction to embedding tangles in the trivial link, of a somewhat different sort, may be derived from the invariants $I^n(\mathcal{T})$ defined in [4]. To explain this, we recall that for a 2-component oriented link $\mathcal{L}=(L_x,L_y)$ of linking number $\lambda=0$, Cochran [5] defined 'higher linking numbers' $\beta^n_x(\mathcal{L})$ and $\beta^n_y(\mathcal{L})$. For $n=1$ these are both equal to the Sato-Levine invariant [6] while for $n>1$ they depend on the ordering of the components. For a tangle $\mathcal{T}$ it is possible to choose a tangle sum with a trivial tangle (a *closure* of $\mathcal{T}$ in the terminology of [4]) to get a link $\mathcal{L}$ with

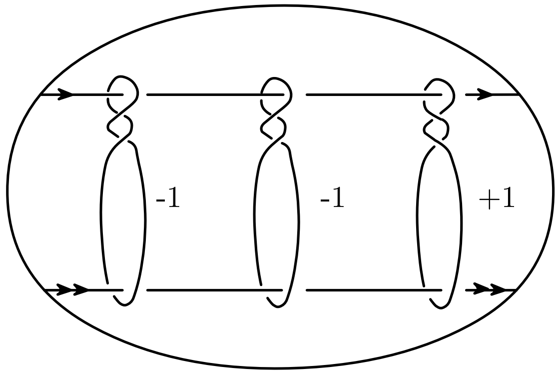

Figure 2

$\lambda = \beta^1 = 0$. There is no canonical choice for $\mathcal{L}$, but we showed that

$$|I^n(\mathcal{T})| = |\beta_x^n(\mathcal{L}) - \beta_y^n(\mathcal{L})|$$

independent of $\mathcal{L}$ (and of the order of the components because of the absolute value signs).

Theorem 5. *Suppose that $\mathcal{T}$ is a tangle with no loops such that $\mathcal{T} \hookrightarrow \mathcal{J}$ = the trivial 2-component link. Then $I^n(\mathcal{T}) = 0$ for all $n \geq 2$.*

Proof: Consider first a 2-string tangle $\mathcal{K}$ with no loops, and the 4-punctured sphere Σ in the boundary of its exterior. Then Σ is compressible if and only if $\mathcal{K}$ is *split*, where $\mathcal{K}$ is split if the ball can be split into two sub-balls by a properly embedded disk with the two strings of $\mathcal{K}$ lying in different sub-balls. This follows directly from the loop theorem and Dehn's lemma. The structure of a split tangle is very simple: it is a trivial tangle possibly with knots tied in each string. The tangle is trivial if and only if there are no knots. In particular, a split tangle has $I^n = 0$ for all n, because it has a completion which is a split link, which in turn has all of its $\beta^n = 0$.

Now consider a tangle $\mathcal{T}$ with $I^n \neq 0$, and suppose that $\mathcal{T} \hookrightarrow \mathcal{J}$, or in other words that $\mathcal{J}$ splits as a sum $\mathcal{T} + \mathcal{T}'$. From the preceding paragraph $I^n(\mathcal{T}) \neq 0$ implies that the surface Σ is incompressible in the exterior of $\mathcal{T}$. But then Σ must be compressible in the exterior of $\mathcal{T}'$. For if it weren't then Σ would be an incompressible 4-punctured sphere in the exterior of the unlink. Applying the preceding paragraph once more, it follows that $\mathcal{T}'$ is split. If there is a knot in one of the strands of $\mathcal{T}'$, then that would give a knot in the corresponding component of the unlink, which cannot be. It follows from all of this that $\mathcal{T}'$ must in fact be a trivial tangle, so that the the unlink $\mathcal{T} + \mathcal{T}'$ may be used to calculate $I^n(\mathcal{T})$ and show that it is zero. This contradicts our assumption that $I^n(\mathcal{T}) \neq 0$. □

The tangles cited in [4] give rise to examples of tangles which cannot be embedded in a trivial link.

5. Generalizations of Tangles

We close with a few remarks on some of the questions raised at the end of [2]. One question concerned the existence of a family of completions of any tangle with $2t$ strands, which would play the role of the numerator and denominator of a 2-string tangle. Following the proof of Theorems 2 and 3 we see how to construct such a family. Note that the 2-fold branched cover of a trivial $2t$-tangle is a handlebody of genus $2t-1$. By twisting the strings around, one can vary the attaching of this complementary handlebody so as to kill of the homology of ∂M in various fillings. The gcd of the orders of the homology of all these fillings then would have to divide the determinant of any link in which the tangle was embedded.

Another generalization of a tangle pointed out in §14 of [2] is an embedded arc (or more generally a 1-manifold) in a more complicated submanifold of S^3, such as a solid torus. In particular, Krebes asks whether a particular arc $\mathcal{A}$ in a $S^1 \times D^2$ (Figure 17 of [2]) can sit inside an unknot. Our approach, especially Corollary 2 gives (in principle) an obstruction, if there were torsion in the homology of some cover of $S^1 \times D^2$ branched along $\mathcal{A}$. Unfortunately, it seems that the homology of all of the cylic covers of the solid torus, branched along this arc, is torsion-free. Hence we cannot apply Theorem 3 to deduce anything about embeddings of this pair in a link. Likewise, it does not seem possible to use Theorem 4, because the rational homology of each of the cyclic branched covers is the same as for a trivial arc.

6. Acknowledgements

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