ON THE CYCLIC HOMOLOGY OF COMMUTATIVE ALGEBRAS OVER ARBITRARY GROUND RINGS

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0. Introduction.

We consider commutative algebras and chain algebras over a fixed commutative ground ring k as in the title. We are concerned with the problem of computing the cyclic (and Hochschild) homology of such algebras via free DG-resolutions $\Lambda V \tilde{\twoheadrightarrow} A$. We find spectral sequences:

(1)
$$E^2_{p,q} = H_p(\Lambda V \otimes \Gamma^q(dV)) \Rightarrow HH_{p+q}(\Lambda V)$$
$$E'^2_{p,q} = H_p(\Lambda V \otimes \Gamma^{\leq q}(dV)) \Rightarrow HC_{p+q}(\Lambda V)$$

The algebra $\Lambda V \otimes \Gamma(dV)$ is a divided power version of the de Rham algebra; in the particular case when k is a field of characteristic zero, the spectral sequences above agree with those found in [BuV], where it is shown they degenerate at the E^2 term. For arbitrary ground rings we prove here (theorem 2.3) that if $V_n = 0$ for $n \geq 2$ then $E^2 = E^\infty$. From this we derive a formula for the Hochschild homology of flat complete intersections in terms of a filtration of the complex for crystalline cohomology, and find a description of E'^2 also in terms of crystalline cohomology (theorem 3.0). The latter spectral sequence degenerates for complete intersections of embedding dimension ≤ 2 (Corollary 3.1). Without flatness assumptions, our results can be viewed as the computation Shukla (cyclic) homology (as defined in 1.4 below–see also [PW], [S]). Particular cases of theorem 3.0 and corollary 3.1 have been obtained in [GG],[L] and [LL]. To our knowledge this is the first paper to give a unified proof for all of these.

The remainder of this paper is organized as follows. Section 1 contains the definitions, results and notations used in the paper. Section 2 is devoted to computing the homology of free algebras. The spectral sequences (1) are obtained in 2.1-2.2; the degeneracy result is proved in 2.3.; the fact that the same degeneracy result is not valid for general free algebras ΛV with $V_2 \neq 0$ –unless $k \supset \mathbb{Q}$– is proved in 2.4. In Section 3 we apply the results of section 2 to complete intersections (Theorem 3.0 and Corollary 3.1).

1. Definitions and notations.

1.0 Chain algebras. We consider algebras over a fixed *ground ring* k which is assumed unital and commutative. Algebras are unital and come equipped with a non-negative gradation $A = \oplus_{n\geq 0} A_n$ and a differential $\partial : A_n \longrightarrow A_{n-1}$ with $\partial^2 = 0$ and $\partial(ab) = \partial(a)b + (-1)^{|a|}a\partial b$, and are strictly commutative in the graded sense. Thus not only is $ba = (-1)^{|a||b|}ab$ but also $a^2 = 0$ if $|a|$ is odd. Algebras with this added structure and strict commutativity condition are called *chain algebras* or simply algebras. Usual commutative algebras are considered as chain algebras concentrated in degree zero. Two algebra maps $f, g : A \longrightarrow B$ are *homotopic* –and we write $f \approx g$– if they are homotopic as chain maps, and are *specially homotopic* if the homotopy $s : A_* \longrightarrow B_{*+1}$ is special, i.e. if it verifies $s(ab) = s(a)b+(-1)^{|a|}as(b)$. Algebra maps which induce an isomorphism in homology are called *quisms* or quasi-isomorphisms and are denoted $\xrightarrow{\sim}$. Maps which are surjective in every degree are denoted $\twoheadrightarrow$. We abuse language and call a chain algebra *free* if it is free as a strictly commutative graded algebra. This means that $A = \Lambda V$ is the symmetric algebra –in the graded sense– of some free graded module V; no condition is imposed on the differential $\partial : A_* \longrightarrow A_{*-1}$.

The following theorem is well-known to specialists. The particular case of 1.1-ii) when k is a field was proved in [BuV, 1.1].

Theorem 1.1. *(Folklore)*

i) *Let ΛV be a free algebra, and let $f{:}A \overset{\sim}{\twoheadrightarrow} B$ be a surjective quism. Then the induced map*

$$f_*{:}\mathrm{Hom}(\Lambda V, A) \twoheadrightarrow \mathrm{Hom}(\Lambda V, B)$$

is surjective. Moreover, if $f_*(g^1) \approx f_*(g^2)$ are homotopic via a special homotopy $s{:}\Lambda V_* \longrightarrow B_{*+1}$, then $g^1 \approx g^2$ via a special homotopy $\hat{s}{:}\Lambda V_* \longrightarrow A_{*+1}$ such that $f \circ \hat{s} = s$.

ii) *For every $A \in$ <u>ChCGA</u> there exists a free algebra ΛV and a surjective quism*

$$\Lambda V \overset{\sim}{\twoheadrightarrow} A$$

Such a surjective quism will be called a model of A. If furthermore A has finite type and k is noetherian, then ΛV can be chosen of finite type.

We omit the proof of the theorem above for lack of space; it is available upon request. For the proof of 2.4 below we shall need the following explicit version of 1.1-ii), which in addition gives an idea of the proof of the theorem above.

1.1.1 Addendum. *Suppose the following data is given:*

i) *A free algebra $Q^n = \Lambda V^n$, with $V^n = V_0 \oplus \cdots \oplus V_n$.*

ii) *A morphism $f^n{:}Q^n \longrightarrow A$ inducing an isomorphism $H_m(Q^n) \xrightarrow{\cong} H_m(A)$ for $m < n$ and a surjection $Z_n(Q^n) = \ker(\partial : Q^n_n \to Q^n_{n-1}) \twoheadrightarrow Z_n(A)$.*
Then there exist:

i') *A free chain algebra $Q = \Lambda V$ with $V_i = V_i^n$ for $i = 0 \ldots n$.*

ii") *A surjective quism $f{:}Q \overset{\sim}{\twoheadrightarrow} A$ with $f\big|_{Q^n} = f^n$.*

Corollary 1.2. *Let* F:<u>*ChCGA*</u> $\longrightarrow \mathcal{D}$ *be a functor from the category of chain algebras and with values in any category* $\mathcal{D}$*. Assume* F *maps quisms and specially homotopic maps between free algebras to isomorphisms and to equal maps. Then the following construction is functorial. For each* $A \in$ <u>*ChCGA*</u> *choose a model* $\epsilon_A : \Lambda V(A) \tilde{\twoheadrightarrow} A$ *–if* A *is free already, choose* $\epsilon_A = id_A$*– and set* $\hat{F}(A) = F(\Lambda V(A))$*. For each map* $A \longrightarrow B \in$ <u>*ChCGA*</u> *choose a lifting* $f' : \Lambda V(A) \longrightarrow \Lambda V(B)$ *of* f *with* $\epsilon_B f' = f\epsilon_A$*, and set* $\hat{F}(f) = F(f')$*. Furthermore the functor* $\hat{F}$ *maps all quisms into isomorphisms, the map* $F(\epsilon_A) : \hat{F}(A) \longrightarrow F(A)$ *is a natural transformation, and is universal (final) among all natural transformations whose source is a functor mapping quisms into isomorphisms.*

Proof. Straightforward from Theorem 1.1. □

1.3 Double Mixed complexes. By a *double mixed complex* we shall understand a chain complex in the category of (single) mixed complexes in the sense of Kassel. Thus a double mixed complex is a bigraded module $(p, q) \mapsto M_{p,q}$ equipped with three k-linear maps of degree ± 1: D which lowers the q index and fixes p; ∂ which fixes q and lowers p, and B which increases q and fixes p. These maps satisfy $0 = D^2 = B^2 = \partial^2 = D\partial + \partial D = B\partial + \partial B = DB + BD$. The Hochschild homology of a double mixed complex M as above is the homology of the double complex $(M_{*,*}, D, \partial)$, and its cyclic homology is the homology of the triple complex $({}_BM_{*,*,*}, D, \partial, B)$ where for each fixed value of r, ${}_BM_{*,*,r}$ is the usual triangular double complex for cyclic homology ([K]). The homology of a triple complex is defined by taking the usual Tot of double complexes twice. One can do this in different ways which of course yield the same complex but which suggest different filtrations and thus different spectral sequences. We like to think of $\mathrm{Tot}_B\, M_{*,*}$ as the Tot of the double complex $M'_{p,q} = \bigoplus_{i=0}^{q} M_{p-i,q-i}$ with D as vertical boundary and $B + \partial$ as row boundary. Our choice of spectral sequences is coherent with this. We use the column filtration of $M_{*,*}$ to obtain the spectral sequence:

$$E^1_{p,q} := H_q(M_{p,*}, D) \Rightarrow HH_{p+q}(M) \tag{2}$$

Note that $E^1 = (E^1_{*,*}, 0, \partial, B)$ is a double mixed complex. We write ${E'}^1$ for the corresponding double complex as above. Thus ${E'}^1$ is the first term of the spectral sequence for the column filtration of the double complex M'. For the second term in the spectral sequence E we have to take homology with respect to ∂, while for ${E'}^2$ we take homology with respect to $\partial + B$. Hence if we put:

$$HH^p_n(M) := E^\infty_{n-p,p} \quad \text{and} \quad HC^p_n(M) := {E'}^\infty_{n-p,p} \tag{3}$$

we obtain:

(4)

$$E^2_{p,q} = HH^q_{p+q}(E^1) \Rightarrow HH_{p+q}(M) \quad \text{and} \quad {E'}^2_{p,q} = HC^q_{p+q}(E^1) \Rightarrow HC_{p+q}(M)$$

We shall be especially concerned with two double mixed complexes. One is the cyclic mixed complex $(C_{*,*}(A), b, B, \partial)$ with $C_{p,q}(A) = (A \otimes (A/k)^{\otimes q})_p$, and with the usual boundary maps, see e.g [CGG, 1.6]. Another is the mixed complex of Γ-differential forms, which we shall define in 1.6 below.

Lemma-Definition 1.4. *(Shukla cyclic homology–compare [PW], [S]) For each $n \geq 0$, the functors $A \mapsto HH_n(A), HC_n(A)$, going from chain algebras to filtered modules equipped with the filtration of 1.3 above, satisfy the hypothesis of Corollary 1.2. In particular $\widehat{HH}_n(A)$ and $\widehat{HC}_n(A)$ are defined; we call them* Shukla homology *and* Shukla cyclic homology*; the filtration they carry is what we shall call the* Hodge filtration. *The p-th layers of each filtration are $\widehat{HH}^p_n(A) := HH^p_n(C(\Lambda V))$ and $\widehat{HC}^p_n(A) := HC^p_n(C(\Lambda V))$.*

Proof. Immediate from the fact that free algebras are flat as modules, and the fact that a special homotopy s between maps $f, g : A \longrightarrow B$ of not necessarily free algebras A and B induces a homotopy $s : (C_{*,q}(A), \partial) \longrightarrow (C_{*,q}(B), \partial)$ between the maps f and g induce on each column of $C_{*,*}$. $\square$

1.5 Divided powers.

Recall from [B,I.1.1] that a system of *divided power operations* for a pair (A, I) consisting of a plain algebra A and an ideal $I \subset A$ is a family of maps $\gamma_n : I \longrightarrow A$ $(n \geq 0)$, with $\gamma_0(x) = 1, \quad \gamma_1(x) = x, \quad \gamma^p(x) \in I \quad (p \geq 1)$ and satisfying a number of identities ([B I.1.1.1-1.1.6]) which make them formally analogue to $x \mapsto \frac{x^n}{n!}$. It is shown in [B, I.2.3.1], that the forgetful functor going from the category of triples (A, I, γ) to the category of pairs (A, I) has a left adjoint. We write $D(A, I) = (\bar{A}, \bar{I}, \gamma)$ for this adjoint functor, which we call the *divided power envelope* of (A, I). Oftentimes we shall abuse notation and write $D(A, I)$ –or simply D– for $\bar{A}$. The ideals $D \supset \mathcal{F}_n :=< \{\gamma^{q_1}(x_1) \dots \gamma^{q_r}(x_r) : r \geq 1, \sum_{i=1}^r q_i \geq n\} >$ define a descending filtration. We call $\mathcal{F}$ the γ*-filtration*. By the *crystalline complex* of (A, I) we mean the largest quotient $\bar{\Omega}$ of the de Rham cochain algebra of $\bar{A}$ for which the induced derivation $\bar{d}$ is a γ*-derivation*, i.e. maps $\gamma_n(x) \mapsto \gamma_{n-1}(x)dx$; explicitly $\bar{\Omega} = \Omega / < d\gamma_n(x) - \gamma_{n-1}(x)dx >$. There is a natural isomorphism $\bar{\Omega} \cong D \otimes_A \Omega$ of cochain algebras with γ-derivation (cf. [I ,Ch.0, 3.1.6]). Our choice of name for $\bar{\Omega}$ comes from the fact that, for example if k is the field with p elements and A is smooth, essentially of finite type over k, it computes the crystalline cohomology $H^*_{crys}(A/I)$ of A/I over k (cf. [I,Ch.0, 3.2.3-4.]).

We shall be especially interested in the particular case of the divided power envelope of pairs of the form $(\Lambda V, < V >)$ where V is a k- module and ΛV is the symmetric algebra. In this case we write $\Gamma(V)$ for $D(\Lambda V, < V >)$. If V is a graded k-module, we put $\Gamma(V) := (\Lambda V_{odd}) \otimes \Gamma(V_{even})$. If V happens to be free on a homogeneous basis $\{v_i : i \in I\}$, then $\Gamma(V)$ is free on the homogeneous basis $v_I \gamma^Q(v_J) := v_{i_1} \dots v_{i_r} \gamma^{q_1}(v_{j_1}) \dots \gamma^{q_s}(v_{j_s})$, where $I = (i_1, \dots i_r)$, $J = (j_1, \dots j_s)$, $Q = (q_1 \dots q_s)$, the v_i are of odd degree and the v_j are of even degree. We write $\Gamma^q(V)$ for the k-submodule generated by all the $v_I \gamma^Q(v_J)$ with $q = |Q| := \sum_{i=0}^s q_i$, and $\Gamma^{\leq q}(V) = \oplus_{p=0}^q \Gamma^p(V)$.

1.6. The mixed complex of Γ*-forms.* Let ΛV be a free chain algebra, and let ∂ be its differential. Write dV for the graded module V shifted by one: $dV_n = V_{n-1}$. Form the graded algebra $\Lambda V \otimes \Gamma(dV)$ and equip it firstly with the degree-increasing γ-derivation d extending $v \mapsto dv$ and secondly with the degree-decreasing γ-derivation δ with prescriptions $\delta(a) = \partial(a)$ $(a \in \Lambda V)$ and $\delta dv = -d\partial(v)$. The (double) *mixed complex of* Γ*-differential forms* is $M(\Lambda V)_{p,q} := (\Lambda V \otimes \Gamma^q(V))_p$ with 0 as the D boundary, δ as the ∂ boundary and d as the B boundary. Note that the spectral sequences (4) for this complex degenerate at the second term. We shall show in 2.1 below that $M(\Lambda V)$ is isomorphic to the spectral mixed complex E^1 of $C(\Lambda V)$.

2. The homology of free algebras.

Lemma 2.0. *Let $\Lambda V = (\Lambda V, 0)$, $\Lambda W = (\Lambda W, 0)$ be free algebras, with zero boundary maps. For each $p \geq 0$, let $K^p = \Lambda V \otimes \Lambda W \otimes \Gamma^p(dW)$, and put:*

$$K = \oplus_{p \geq 0} K^p \simeq \Lambda V \otimes \Lambda W \otimes \Gamma(dW)$$

Equip K with the γ-derivation $D{:}K^p_* \longrightarrow K^{p-1}_*$ determined by $D(dw) = w$ $(w \in W)$, $D^2 = 0$. Then:

i) *Consider the natural projection $\pi : \Lambda V \otimes \Lambda W \longrightarrow \Lambda V$; then $\ker \pi = < W > = D(K^1)$ is both the ideal generated by W and the image of D. We write I for this ideal.*

ii) *The ideal $K' = \mathrm{Ker}(\pi \otimes Id) \subset K$ is contractible. Precisely, there is a ΛV-linear map $h{:}K'^p_* \longrightarrow K'^{p+1}_{*+1}$ with $hD + Dh = Id$, $h(w) = dw$, $h^2 = 0$ $(w \in W)$ and $h(I^r K) \subset I^{r-1} K$ $(r \geq 1)$. In particular $\pi \otimes 1{:}K \tilde{\twoheadrightarrow} \Lambda V$ is a free resolution of ΛV as a $\Lambda V \otimes \Lambda W$-module.*

Proof. Part i) is trivial. Next observe that it suffices to prove ii) in the case $V = 0$, for if $h = h(0) : \Lambda W \otimes \Gamma^*(dW) \mapsto \Lambda W \otimes \Gamma^{*+1}(dW)$ is a homotopy satisfying the prescriptions of the lemma for $V = 0$, then $h(V) = 1 \otimes h : \Lambda V \otimes \Lambda W \otimes \Gamma^*(dW) \longrightarrow \Lambda V \otimes \Lambda W \otimes \Gamma^{*+1}(dW)$ satisfies the prescriptions for V. Assume $V = 0$; if $W = 0$ there is nothing to prove. Otherwise choose a well-ordered, nonempty basis $(B, <)$ of W. For each $w \in B$, define $h : \Lambda(kw) \otimes \Gamma^*(kdw) \longrightarrow \Lambda(kw) \otimes \Gamma^{*+1}(kdw)$ as follows. Put $h(w^n dw) = 0$ and $h(w^{n+1}) = w^n dw$ if $|w|$ is even, $h(w\gamma_p(dw)) = \gamma_{p+1}(dw)$ and $h(\gamma_p(dw)) = 0$ if $|w|$ is odd, and $h(1) = 0$ in either case. Given any strictly increasing sequence $w_1 < \cdots < w_r$, extend h to $A(w_1, \ldots, w_r) := \Lambda(kw_1) \otimes \Gamma^{*+1}(kdw_1) \otimes \cdots \otimes \Lambda(kw_r) \otimes \Gamma^{*+1}(kdw_r)$ by $h(x_1 \otimes \cdots \otimes x_r) = (-1)^{|x_1| + \ldots |x_{r-1}|} x_1 \otimes \ldots x_{r-1} \otimes h(x_r)$. One checks immediately that these prescriptions give a well defined linear map $h : \Lambda W \otimes \Gamma^*(\Lambda W) = \bigcup_{r \geq 1} \bigcup_{w_1 < \cdots < w_r \in B} A(w_1, \ldots, w_r) \longrightarrow \Lambda W \otimes \Gamma^{*+1}(\Lambda W)$ and that this map is a homotopy satisfying the requirements of the lemma. $\square$

Proposition 2.1. *Let ΛV be a free chain algebra. Consider the spectral sequence E^1 of (2) associated to the standard mixed complex $C(\Lambda V)$. Then the double mixed complex $(E^1_{*,*}, 0, \partial, B)$ is isomorphic to the double mixed complex of Γ-differential forms defined in 1.6 above.*

Proof. The plan of the proof is as follows. Firstly we construct a map $f : \Lambda V \otimes \Gamma(dV) \longrightarrow E^1$ and show it commutes with the relevant boundary maps. Secondly we find an isomorphism $\bar{F} : \Lambda V \otimes \Gamma(dV) \cong E^1$. Thirdly we prove that $f = \bar{F}$. For the definition of f proceed as follows. It is not hard to see that $d : \Lambda V \longrightarrow \Lambda V \otimes \Gamma^1(dV)$ is the universal derivation, so that $\Lambda V \otimes \Gamma^1(dV)$ is the graded ΛV-module of Kähler differentials of ΛV. Hence $\Lambda V \otimes \Gamma^1(dV) \cong E^1_{*,1}$ as graded modules, and the isomorphism maps da to the class of Ba $(a \in \Lambda V)$. Hence $\partial(da) = -d\partial(a)$ goes to the class of $-B(\partial(a)) = \partial(B(a))$, whence the isomorphism

$\Lambda V \otimes \Gamma^1(dV) \cong E^1_{*,1}$ is a chain module map, because it is a module map which commutes with the boundary on the generators da. Next, the shuffle product and divided power operations of Cartan ([Ca, Exp.4, Th.4; Exp.7, Th.1]) –actually their analogue for the cyclic bar construction– make E^1 into a DG algebra with divided operations for the ideal generated by $E^1_{*,1}$. By the universal property of $\Lambda V \otimes \Gamma(dV)$, we have a map $f : \Lambda V \otimes \Gamma(dV) \to E^1$ commuting with ∂. Next we must check that $fd = Bf$. Note that the image of f is generated, as a graded k-module, by classes of monomials of the form $M = f(a\gamma_{p_1}(dy_1)\dots\gamma_{p_r}(dy_r)) = [a * (1 \otimes y_1^{\otimes p_1}) * \cdots * (1 \otimes y_r^{\otimes p_r})]$, where $[\,]$ denotes homology class, $*$ is the shuffle product, $a \in \Lambda V$, $y_i \in V$, $r \geq 1$, $p_i \geq 0$ and $p_i \leq 1$ if $|y_i|$ is even. Thus we must show that $B(M) = [Ba * (1 \otimes y_1^{\otimes p_1}) * \cdots * (1 \otimes y_r^{\otimes p_r})]$. The same proof as in the ungraded case [LQ, Lemma 3.1] proves that the formula $B(x * By) = Bx * By$ holds for chain algebras. Using this formula and induction, we see it suffices to show that, for $y \in \Lambda V_{even}$, $x \in C(\Lambda V)$ and $p \geq 2$, we have

$$B(x * (1 \otimes y^{\otimes p})) = B(x) * (1 \otimes y^{\otimes p}) \tag{5}$$

Whenever p is not a zero divisor in k, (5) follows from the formula of [LQ, 3.1] and the fact that $1 \otimes y^{\otimes p} = \frac{1}{p}B(y^{\otimes p})$ over $k[1/p]$. In particular (5) holds for $\mathbb{Z}$, whence it holds for arbitrary ground rings, by naturality. This finishes the first part of the proof. Next apply the lemma above with $1 \otimes V$ as V and $TV = \{v \otimes 1 - 1 \otimes v : v \in V\}$ as W, to obtain a free resolution $(K, 0) = (\Lambda V \otimes \Lambda V \otimes \Gamma(dV), D) \tilde{\twoheadrightarrow} (\Lambda V, 0)$. Write C' for the Hochschild acyclic resolution. By universal property, the inclusion map $F{:}\Lambda V \otimes \Lambda V \otimes \Gamma^1(dTV) \hookrightarrow C'^1(\Lambda V)$, $dTv \longmapsto 1 \otimes v \otimes 1$ extends to an algebra homomorphism $F{:}K \longrightarrow C'(\Lambda V)$ with $F(K^p) \subset C'^p(\Lambda V)$ and $F(\gamma^p(dTv)) = \gamma^p F(dTv)$, $p \geq 0$, $v \in V$. Because $F(DdTv) = Tv = b'F(dTv)$, F is a chain map. Since both K and C' are resolutions, the induced map $\bar{F}$: $\bar{K} = (\Lambda V \otimes \Gamma(dV), 0) \tilde{\longrightarrow} C(\Lambda V), b)$ is a quism. Thus, $\bar{F}$ induces an isomorphism of divided power algebras $\Gamma^p_q(\Lambda V) \simeq E^1_{p,q}$. But $\bar{F}(dv) = f(dv)$ for all $v \in V$. Thus, $\bar{F}$ and f are the same map. $\square$

Corollary 2.2. *Let A be an algebra, $\Lambda V \tilde{\twoheadrightarrow} A$ be a free model, $M(\Lambda V)$ the mixed complex of γ-forms of 1.6 above. Then there are spectral sequences:*

$$E^2_{p,q} = HH^p_{p+q}(M(\Lambda V)) \Rightarrow \widehat{HH}_{p+q}(A), \quad {E'}^2_{p,q} = HC^p_{p+q}(M(\Lambda V)) \Rightarrow \widehat{HC}_{p+q}(A)$$

Furthermore, for the Hodge filtration, we have $E^\infty_{p,q} = \widehat{HH}^p_{p+q}(A)$ and ${E'}^\infty_{p,q} = \widehat{HC}^p_{p+q}(A)$.

Proof. Immediate from lemma 1.4. $\square$

Theorem 2.3. *In the situation of the proposition above, assume further that $V_n = 0$ for $n \geq 2$. Then the spectral sequence for Hochschild homology degenerates, and there is an isomorphism of graded algebras*

$$HH_*(\Lambda V) \cong H_*(\Lambda V \otimes \Gamma(dV))$$

Proof. We use the notations of the proof of the proposition above. The plan of the proof is as follows. Firstly, we equip the algebra K with a γ-derivation $\Delta{:}K \longrightarrow K$

Secondly, we define a chain homomorphism $F'{:}(K,\Delta) \longrightarrow (C', b'+\partial')$. Thirdly, we show that the induced map $\bar{F}'{:}(\bar{K},\bar{\Delta}) \longrightarrow (C, b+\partial)$ is a quism. Finally, we prove that $(\bar{K},\bar{\Delta}) \simeq (\Lambda V \otimes \Gamma(dV), \delta)$. Let δ' be the γ-derivation determined by $\delta'(x) = \partial' x$, $\delta'(dTv_0) = 0$, $\delta'(dTv_1) = -h(\partial v_1)$, $x \in \Lambda V \otimes \Lambda V$, $v_i \in V_i$, $i = 0, 1$. By definition of δ', we have $\delta'^2 = 0$ and $(D+\delta')^2 = 0$. Thus $\Delta := D+\delta'{:}K_* \longrightarrow K_{*-1}$ is a boundary derivation, and (K,Δ) is a chain algebra. Let $F{:}K \longrightarrow C'$ be as in the proof of the proposition above; then F comutes with D and b' but not necessarily with Δ. Define a morphism $F'{:}K \longrightarrow C'$ as follows. Firstly, let $F'(w) = F(w)$, $w \in L = \Lambda V \otimes \Lambda V \otimes \Gamma(dTV_0) \subset K$. Secondly, choose a basis $\mathcal{B}$ of V_1 and put $F'(dTv) = F(dTv) + 1 \otimes ((F\delta' - \partial' F)dT(v)))$ $(v \in \mathcal{B})$. Because $x \longmapsto 1 \otimes x$ is a chain homotopy, we have:

$$F'(\Delta dTv) = (b'+\partial')F'(dTv) \tag{6}$$

We remark that $G(dTv) = (F' - F)(dTv) \in \Lambda V_0 \otimes \Lambda V_0 \otimes \Gamma^2(TV_0)$. In particular:

$$(G(dTv))^2 = 0 \qquad (v \in \mathcal{B}) \tag{7}$$

For each $p \geq 1$, put $F'(\gamma^p(dTv)) = \gamma^p F(dTv) + \gamma^{p-1} F(dTv) G(dTv)$ $(v \in \mathcal{B})$. Fix $v \in \mathcal{B}$, and write $\eta = dTv$. Then:

$$\begin{aligned}
&F'(\gamma^p(\eta))F'(\gamma^q(\eta)) = \\
&\qquad (\text{by } (7)) = \gamma^p F(\eta)\gamma^q F(\eta) + (\gamma^{p-1}F(\eta)\gamma^q F(\eta) + \gamma^p F(\eta)\gamma^{q-1}F(\eta))G(\eta) \\
&\qquad = \binom{p+q}{p}\gamma^{p+q}F(\eta) + \left[\binom{p+q-1}{p-1} + \right. \\
&\qquad\quad \left.\binom{p+q-1}{p}\right]\gamma^{p+q-1}(F(\eta))G(\eta) \\
&\qquad = \binom{p+q}{p} F(\gamma^{p+q}(\eta))
\end{aligned}$$

Thus, the maps $\eta \longmapsto F(\gamma^p(\eta))$, $p \geq 1$, extend to divided power operations $\theta^p{:}dT(V) \longrightarrow C'$, $v \longmapsto F(\gamma^p(dTv))$ $(v \in \mathcal{B})$ satisfying all of the conditions of [B I.1.1]. By (6), $b' + \partial'$ is a θ-derivation. Thus, F' extends uniquely to a chain homomorphism $F'{:}K \longrightarrow C'$ with $F' \circ \gamma^p(\eta) = \theta^p \circ F'(\eta)$, $p \geq 1$. Let $\bar{F}{:}(K,\Delta) \otimes_{\Lambda V \otimes \Lambda V} \Lambda V \simeq (\Lambda V \otimes \Gamma(dV), \bar{\delta}') \longrightarrow (C, b+\partial)$ be the induced map. Consider the subalgebra $L \subset K$ defined above. By definition, $\bar{F}' = \bar{F}$ both on L and on K/L. Since $\bar{F}$ is a quism, so is $\bar{F}'$. To finish the proof, it suffices to show that $\bar{\delta}' = \delta{:}\Lambda V \otimes \Gamma(dV)_* \longrightarrow \Lambda V \otimes \Gamma(dV)_{*-1}$ is the usual derivation. Let $\mathcal{B}_* \subset V_*$ be the ordered basis used in the definition of the homotopy $h{:}K \longrightarrow K$. Because both $\bar{\delta}'$ and δ are γ-derivations and $\bar{\delta}' \equiv \delta$ on $L \subset K$, it is enough to show that:

$$\bar{\delta}'(dv) = -\mu(hT(\partial v)) = -d\partial v = \delta(dv) \qquad (v \in \mathcal{B}_1) \tag{8}$$

Here $\mu{:}C' \twoheadrightarrow C$ is the projection map and $T{:}\Lambda V \longrightarrow J = \ker(b' : \Lambda V \otimes \Lambda V \longrightarrow \Lambda V)$, $T(a) = a \otimes 1 - 1 \otimes a$ is the universal non commutative derivation of degree zero. We have $\operatorname{Ker} \mu = JC'$ and:

$$T(ab) = (a \otimes 1)T(b) + (b \otimes 1)T(a) \qquad (\mathrm{mod}\ J^2) \tag{9}$$

Now, for each $v \in B_1$, ∂v is a sum of monomials of the form: $\lambda v_1 \dots v_r,\ \ \lambda \in k,\ \ v_1 < \dots < v_r,\ \ v_i \in V_0$. Thus $d\partial v$ is the sum of the monomials $\lambda v_1 \dots \overset{v_i}{\vee} \dots v_r dv_i$. On the other hand, by (9), $T\partial v$ is the sum of the monomials $\lambda v_1 \dots \overset{v_i}{\vee} \dots v_r T v_i$ plus an element in J^2. But by Lemma 2.0-ii), we have $h(J^2) \subset JK^1$. Thus $h(T\partial V)$ is congruent modulo JK^1 to the sum of the monomials $\lambda v_1 \dots \overset{v_i}{\vee} \dots v_r dTv_i$. It follows that $\mu(T\partial v) = d(\partial v)$. We have established the identity (8); this concludes the proof. $\square$

Proposition 2.4. *Let k be any ground ring. Suppose that for every model $\Lambda V \tilde{\twoheadrightarrow} k$ (V arbitrary) the spectral sequence E of Corollary 2.2 degenerates at the second term. Then $k \supset \mathbb{Q}$.*

Proof. Suppose that $k \not\supset \mathbb{Q}$; then there is an integer $p \geq 2$ that is not invertible in k. We are going to exhibit a free algebra ΛV that is quasi-isomorphic to $k = \Lambda(0)$, but for which $H_{2p}(\Lambda V \otimes \Gamma(dV)) \neq 0 = H_{2p}(\Lambda(0) \otimes \Gamma(0))$. Consider the graded k- module V' defined by $V'_0 = V'_n = 0$ if $n \neq 1, 2$ and $V'_1 = ky$, $V'_2 = kz$. Equip $\Lambda V'$ with the derivation $\partial' y = 0$, $\partial' z = y$. Let $f' : \Lambda V' \longrightarrow k$ be the natural projection. Thus, $H_0(\Lambda V') = k$ and $H_i(\Lambda V') = 0$ for $i = 1, 2$. Hence by 1.1.1, f' extends to a quism $f : \Lambda V \tilde{\twoheadrightarrow} k$ where $V_i = V'_i$ for $i \leq 2$ and where $\partial(a') = \partial'(a')$ for $a' \in \Lambda V'$. Consider the element $\gamma^p(dy) \in (\Lambda V \otimes \Gamma^p(dV))_{2p}$; we have $\delta(\gamma^p(dy)) = \gamma^{p-1}(dy)(-d\partial y) = 0$. I claim that $\gamma^p(dy)$ is not a boundary. Suppose otherwise that there exists $\alpha \in (\Lambda V \otimes \Gamma^p(dV))_{2p+1}$ with $\delta\alpha = \gamma^p(dy)$. Extend $\{y, z\}$ to a basis of V. Then $(\Lambda V \otimes \Gamma^p(dV))_{2p+1}$ is the free ΛV-module on the monomials $v_I \gamma^Q(v_J)$ of 1.6 above. Because $V_2 = kz$, it follows that $\beta = \gamma^{p-1}(dy)dz$ is the only basis element of $\Gamma^p_{2p+1}(\Lambda V)$ whose boundary is a multiple of $\gamma^p(dy)$. Thus $\delta(\alpha)$ must be a multiple of $\delta(\beta)$. But $\delta(\beta) = \delta(\gamma^{p-1}(dy)dz) = -\gamma^{p-1}(dy)dy = -p\gamma^p(dy)$; since p is not invertible in k, it follows that the element α does not exist. $\square$

3. Complete Intersections.

Theorem 3.0. *(Complete Intersections) Let I be an ideal in the polynomial ring $R = k[x_1, \dots x_n]$, and set $A = R/I$. Assume I is generated by an R-sequence, i.e. assume there is a model $\Lambda V \tilde{\twoheadrightarrow} A$ with $V_0 =< x_1, \dots, x_n >$ and $V_n = 0$ for $n \geq 2$. Then, with the notations of 1.5 above:*

(1) *Consider the complex:*

$$L^p: \quad \bar{\Omega}^p/\mathcal{F}_1\bar{\Omega}^p \overset{\bar{d}}{\longleftarrow} \mathcal{F}_1\bar{\Omega}^{p-1}/\mathcal{F}_2\bar{\Omega}^{p-1} \overset{\bar{d}}{\longleftarrow} \dots \overset{\bar{d}}{\longleftarrow} \mathcal{F}_p\bar{\Omega}^0/\mathcal{F}_{p+1}$$

Then $\widehat{HH}^p_n(A) = H_{n-p}(L^p)$ and $\widehat{HH}_n(A) \cong \oplus_{p=0}^n \widehat{HH}^p_n(A)$.

(2) *Consider the complex:*

$$L'^p: \quad \bar{\Omega}^p/\mathcal{F}_1\bar{\Omega}^p \overset{\bar{d}}{\longleftarrow} \bar{\Omega}^{p-1}/\mathcal{F}_2\bar{\Omega}^{p-1} \overset{\bar{d}}{\longleftarrow} \dots \overset{\bar{d}}{\longleftarrow} \bar{\Omega}^0/\mathcal{F}_{p+1}$$

Then the spectral sequence of Corollary 2.2 has ${E'}^2_{p,q} = H_q(L'^p)$.

Proof. In view of corollary 2.2 and Theorem 2.3, it suffices to show that there are quasi-isomorphisms $(\Lambda V \otimes \Gamma^p(V), \delta) \tilde{\twoheadrightarrow} L^p$ and $(\Lambda V \otimes \Gamma^{\leq p}(dV), d + \delta) \tilde{\twoheadrightarrow} L'^p$. By virtue of [B, 3.4.4 and 3.4.9], the argument given in [CGG, proof of Th. 3.3 on page 229] to prove the case $k \supset \mathbb{Q}$ works here also. $\square$

Corollary 3.1. *If in the theorem above the number of variables is $r \leq 2$ then*

$$\widehat{HC}'^{p}_{n}(A) = H_{n-p}(L'^{p}) \qquad \text{for all} \quad n \geq p \geq 0 \quad \square$$

Remark 3.2. It is not hard to see that our Hodgewise complexes L^p of a complete intersection are quasi-isomorphic to those found in [GG]. For $r = 1, 2$ the sum of the Hodgewise complexes L'^p of the corollary above give the complexes of [LL, 2.5.3] and [L, 2.10]. All of these are generalizations of the Feigin-Tsygan complexes ([FT], [CGG]).

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