

ARTINIAN ALGEBRAS AND DIFFERENTIAL FORMS

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0. INTRODUCTION

This article concerns commutative algebras over a field k of characteristic zero which are finite dimensional as vectorspaces, and particularly those of such algebras which are graded. Here the term graded is applied to non-negatively graded algebras A with A_0 reduced and finite dimensional. Thus the trivial grading $A = A_0$ is only allowed if A is a product of finite field extensions of k . It was conjectured in [2] that for all finite dimensional algebras A which are not principal ideal algebras (i.e. have at least one nonprincipal ideal), the following submodule of the Kähler differentials is nonzero:

$$\tau(A) = \bigcap \ker(\Omega_A \rightarrow \Omega_B)$$

Here the intersection is taken over all principal ideal algebras B and all homomorphisms $A \rightarrow B$. In this paper we prove that the conjecture holds for both Gorenstein graded and standard graded algebras. The conjecture, called Artinian Berger Conjecture (ABC) in [2], was introduced in connection with the classical Berger Conjecture (BC) of [1]. In fact it is proven [2, Theorem 0.1] that $\text{ABC} \implies \text{BC}$. (Both the BC and the result of [2] are recalled in 1.2 below.) However we shall show here that the implication is not as straightforward as one could expect. For instance our theorem that ABC holds for standard graded algebras (2.1) implies BC for some graded and some ungradable algebras of dimension one (2.2). Our theorem that 0-dimensional Gorenstein graded algebras satisfy ABC (3.2) proves no case of the BC (3.3); in particular it does not prove the latter for 1-dimensional Gorenstein graded algebras.

The ABC is connected to other interesting questions such as when is an artinian algebra embeddable in a principal ideal algebra. For instance we show that non-principal Gorenstein algebras are not embeddable (3.1) and we classify standard graded algebras in terms of their degree of embeddability (2.5).

The rest of this paper is organized as follows. Some basic properties from [2] are recalled in section 1. Sections 2 and 3 are devoted respectively to graded standard and Gorenstein algebras. Theorem 2.1 and Proposition 2.4 were obtained jointly with P. Solernó. Theorem 3.1 was proven independently –and almost simultaneously– by S. Geller and C. Weibel.

1. PRELIMINARIES

The two main results of this paper are stated for local algebras over an arbitrary field of characteristic zero. The following lemma allows us to prove them in the algebraically closed case only. By part ii) of the lemma below we can rephrase

the main results of this paper (2.1 and 3.1) into one; namely that ABC holds for products of Gorenstein graded and of standard graded algebras.

Lemma 1.1. *Let A, A_1 and A_2 be k -algebras; write \bar{k} for the algebraic closure of k . Then:*

- i) If ABC holds for $A \otimes_k \bar{k}$ then it holds for A .*
- ii) If ABC holds for A_1 and A_2 then it holds for the product $A_1 \times A_2$.*

Proof. It is immediate from [2, 1.1.1 and 2.0] ■

Artinian Berger Conjecture - Berger Conjecture: The main result of [2] states that if ABC holds for every 0-Krull dimensional k -algebra, then BC holds for every 1-Krull dimensional reduced algebra. Actually the proof given in [2] proves the following more precise statement:

Lemma 1.2. *Suppose a pull back diagram:*

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array} \quad (1)$$

is given where $A \hookrightarrow B$ is an inclusion of 0-Krull dimensional algebras, the vertical maps are surjective, A is not a principal ideal algebra, S is reduced normal and $\text{Krull-dim} S = 1$. Then R is reduced singular, $\text{Krull-dim} R = 1$ and S is its normalization. If $\Omega_A \rightarrow \Omega_B$ is not injective then $\Omega_R \rightarrow \Omega_S$ is not injective either, i.e. BC holds for R .

Proof. That R satisfies the hypothesis of BC is not hard to see. If $\Omega_A \rightarrow \Omega_B$ is not injective then $\Omega_R \rightarrow \Omega_S$ cannot be injective, by [2, 1.1.1] ■

Remark 1.3. Note that if A occurs in a diagram such as (1) then it is necessarily embeddable. It follows from the proof of [2, 2.3] that the canonical map:

$$H_{dR}^0(A) \rightarrow H_{dR}^0(A_{red}) \quad (2)$$

is an isomorphism if A is embeddable. On the other hand if A is not embeddable and (2) is an isomorphism, then it satisfies ABC. To see this, first use 1.1 ii) to restrict to the local case. Next let (A, M) be local and unembeddable, and let $0 \neq m$ such that any map $A \rightarrow B$ into a principal ideal algebra maps it to zero. Then its differential $dm \in \tau(A)$ is a non-zero element by (2). It should be noted that the map (2) need not be an isomorphism in general, even under our standing assumption that $\text{char}(k) = 0$, as shown by the example below. What our assumption that $\text{char}(k) = 0$ does imply is that (2) holds when A is graded –in the sense of this paper; see §0. Indeed in this case the derivation $D(a) = \text{deg}(a)a$ is injective on A_+ , whence so must be $d : A \rightarrow \Omega_A$.

Example 1.4. Consider the ideal $I := \langle X^3Y, X^5, XY^3 + 2X^3, 3X^2Y^2 + 5Y^4 \rangle \subset k[X, Y]$; note I contains both partial derivatives of $F = X^4 + X^2Y^3 + Y^5$. Hence the image f of F in $A = k[X, Y]/I$ is in $K = \ker(H_{dR}^0(A) \rightarrow H_{dR}^0(A_{red}) = k)$. One checks that $\{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^4, x^2y^2\}$ is a basis of the vectorspace A and that $f = \frac{x^4}{5}$, whence it is a nonzero element of K . As per the remark above, this implies that A is unembeddable. To see that ABC actually holds for A , proceed as follows. First one checks that the basis element $w = x^2y^2$

is mapped to zero by any map into a principal ideal algebra. Hence $dw \in \tau(A)$. Further, $dw \neq 0$, since w goes to an element of positive degree in the graded algebra $A/\langle x^3 \rangle$.

In order to prove the main results of this paper we shall use the following artinian version of valuation theory.

Truncated Valuations: Let $\alpha : A \rightarrow B = k[t]/\langle t^{N+1} \rangle$ be a map. Consider the map $\nu = \nu_\alpha : A \rightarrow \{0, 1, \dots, N, \infty\}$ given by $\nu(a) = e$ when $\alpha(a) = t^e u$ for some invertible element $u \in B$, and $\nu(a) = \infty$ if $\alpha(a) = 0$. We call ν the *truncated valuation* associated to α . Note that truncated valuations are not valuations in the classical sense of [5, Ch. VI, §8, page 32]. However our truncated valuations do retain some of the properties of classical valuations. For instance we have $\nu(xy) = \nu(x) + \nu(y)$ and $\nu(x+y) \leq \min\{\nu(x), \nu(y)\}$. Thus ν is a morphism from the non-cancellative multiplicative monoid (A, \cdot) to the additive monoid $\{0, 1, \dots, N, \infty\}$, where $n + \infty = \infty + n = \infty$. The order in $\{0, 1, \dots, N, \infty\}$ is the obvious one. Another trivial, but useful property of truncated valuations is the following. Let $a_1, \dots, a_n \in A$ be a k -linearly independent set, $S = \sum ka_i$, and $d = n - \dim \ker \alpha \cap S$. Assume $d \neq 0$, i.e. assume $S \not\subseteq \ker \alpha$. Then there exist elements $x_1, \dots, x_n \in A$ such that:

$$\nu(x_1) < \nu(x_2) \cdots < \nu(x_d), \quad \nu(x_i) = \infty \quad (i > d) \quad (3)$$

and such that still $\sum kx_i = S$. This property follows from the fact that the $n \times N$ matrix having as rows the coefficients of $\alpha(a_1), \dots, \alpha(a_n)$ can be made into an upper triangular matrix through row operations.

2. STANDARD GRADED ALGEBRAS

Let $A = A_0 \oplus A_1 \oplus \cdots \oplus A_n$ be a finite dimensional graded algebra. We say that A is *standard* if the ideal $M = A_1 \oplus \cdots \oplus A_n$ is maximal and is generated by degree one elements. Thus (A, M) is a local ring with residue field A_0 . Hence any standard graded algebra may be written as $A = F[X_1, \dots, X_m]/I$ where F is a finite field extension of the ground field k , and I is homogeneous for the standard grading of the polynomial ring.

The main theorem of this section is the following:

Theorem 2.1. *Standard graded algebras satisfy ABC.*

The next example shows how ungraded cases of BC can be deduced from the theorem above.

Example 2.2. Consider the smooth curve of genus one $S = k[X, Y]/\langle X(X^2 - 1) + Y^3 \rangle$ and the standard graded algebra $A = k[X, Y]/\langle X, Y \rangle^4$. Form the pull-back:

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

Here $B = k[t]/\langle t^{16} \rangle = S/\langle X, Y \rangle^{16}$ and $A \rightarrow B$ is the inclusion $X \mapsto t^4, Y \mapsto t^5$. It is not hard to see –using [4]– that S is not gradable whence R cannot be gradable either. Thus by lemma 1.2 and the theorem above, BC holds for the ungradable ring R .

The proof of theorem 2.1 will take us the rest of this section. Since standard grading is preserved by change of ground field, the proof of the theorem immediately reduces to the case when k is algebraically closed. For the rest of this section we shall assume k is algebraically closed.

It will be essential for our proof to measure the degree of embeddability of standard graded algebras in truncated polynomial rings. The critical degree defined in 2.4 below will be a useful invariant for this purpose.

Lemma 2.3. *Let (A, M) be a standard graded algebra. If A is not a principal ideal algebra then the following is a finite nonempty set:*

$$\mathcal{E} = \{i \in \mathbb{N} / \exists B \text{ truncated polynomial algebra and } \alpha : A \rightarrow B \text{ s.t. } \dim \alpha(A_i) \geq 2\}$$

Proof. Let $n \geq 1$ be such that $M^{n+1} = 0$. Then $\mathcal{E} \subset \{1, \dots, n\}$. Further, $1 \in \mathcal{E}$ because there exists a map $A/M^2 \rightarrow k[t]/\langle t^4 \rangle$ sending M/M^2 onto $\langle t^2, t^3 \rangle$ since $\dim M/M^2 \geq 2$. ■

Definition 2.4. In the situation of the lemma above, we define the *critical degree* of A as the integer $\text{crit. deg.}(A) = \max(\mathcal{E})$.

Proposition 2.5. *Let A be a standard graded algebra. Let r be its critical degree. Then there exists a surjective homomorphism:*

$$A \twoheadrightarrow Q(r) = k[X, Y] / \langle X^{r+1}, X^r Y, Y^2 \rangle$$

Proof. Let $\alpha : A \rightarrow B$ be a map with values in a truncated polynomial algebra such that $\dim \alpha(A_r) \geq 2$. Let $\{x, y, z_3, \dots, z_m\}$ be a basis of A_1 such that $e = \nu(x) < f = \nu(y)$ are the least two valuations in A_1 , and such that $\nu(z_i) > f$, ($i = 3, \dots, m$). Consider the ideal $I = \langle x^{r+1}, x^r y, y^2, z_3, \dots, z_m \rangle \subset A$. We have a surjective map $\phi : Q(r) \twoheadrightarrow A/I, X \mapsto x, Y \mapsto y$. A valuation argument shows that for each $1 \leq i \leq r$ the elements $x^i, x^{i-1}y$ are linearly independent in $(A/I)_i$. It follows that ϕ is an isomorphism. ■

Remark 2.6. It is not hard to show –using valuations– that $\text{crit. deg. } Q(r) = \lfloor \frac{r}{2} \rfloor + 1$ (here $\lfloor x \rfloor$ is the integer part of x). Hence the fact that an epimorphism $A \twoheadrightarrow Q(r)$ exists does not imply any relation between the critical degree of A and r . In other words, $\text{crit. deg.}(A) \leq \max\{r : \exists A \twoheadrightarrow Q(r)\}$, but the inequality may be strict.

Proof of Theorem 2.1. Assume A is not a principal ideal algebra and let r be its critical degree. We shall show that for all $x, y \in A_1$ the element:

$$\omega(x, y) = x^{r-1}(xdy - ydx) \in \tau(A) \tag{4}$$

On the other hand it is not hard to show that if x, y are as in the proof of 2.5 above then composite $A \twoheadrightarrow A/I \xrightarrow{\phi^{-1}} Q(r)$ maps ω to a non-zero element. Thus it suffices to prove (4). Let $\beta : A \rightarrow k[t]/\langle t^{N+1} \rangle$ be any map. We have to show that $\beta(\omega) = 0$. Write e and f for the valuations of x and y , and $e' = \min\{\nu(a_1) : a_1 \in A_1\}$. We have that $\nu(\omega) \geq re + f$. If $re + f = \infty$ we are done. Otherwise $0 \neq \beta(x^r y) \in \beta(A_{r+1})$. Then $\beta(A_{r+1}) = k\beta(x^r y)$ and therefore $re + f = (r+1)e'$, whence $e = f = e'$. Choose a basis $\{x, z_2, \dots, z_m\}$ of A_1 with $\nu(z_i) > \nu(x)$. Then $y = \lambda x + p$ where $\lambda \in k$ and $p = \sum \lambda_i z_i, \lambda_i \in k$. As

$\omega(x, -)$ is linear, $\omega(x, y) = \lambda\omega(x, x) + \omega(x, p)$. But now $\nu(p) > \nu(x)$, whence $\beta(\omega(x, y) = \beta(\omega(x, p)) = 0$ by the argument above. ■

3. GORENSTEIN ALGEBRAS

Recall a finite dimensional local algebra (A, M) is called *Gorenstein* (or self-injective) if for the socle we have $\dim_k \text{soc}(A) = 1$.

Theorem 3.1. *Let A be a finite dimensional Gorenstein local algebra over a field k . Assume A is not a principal ideal algebra. Then every homomorphism from A into a principal ideal algebra maps the socle of A to zero. In particular A is unembeddable.*

Proof. We may assume k is algebraically closed (by 1.1). Let M be the maximal ideal of A and let n be a positive integer such that $M^n \neq 0$ and $M^{n+1} = 0$. As A is Gorenstein, $M^n = \text{soc } A$. If a homomorphism to a product of truncated polynomial algebras which is not zero on M^n exists, then there must also exist a homomorphism $f : A \rightarrow k[t]/(t^{N+1})$ such that $f(M^n) \neq 0$. Because $\dim M^n = 1$, f must be injective on M^n . Let $\{x_1, \dots, x_m\} \subset M$ a minimal generating set of M with ascending valuations as in (3) above. Then $f(x_1^n) \neq 0$ and any other monomial of degree n in the x_i is in $\ker f$. As f is injective on M^n we have $M^n = kx_1^n$, and any other monomial in the x_i is zero. We have proved the following statement for $j = 0$: if $n - j \geq 1$ then $M^{n-j} = kx_1^{n-j} + \dots + kx_1^n$. Assume by induction that this assertion is true for j . If $n - j = 1$ we are done. Otherwise $n - (j+1) \geq 1$. Let $\alpha \in M^{n-(j+1)}$; we must prove that $\alpha \in kx_1^{n-(j+1)} + \dots + kx_1^n$. Since $x_1\alpha \in M^{n-j}$, then $x_1\alpha = \sum_{i=0}^{n-(j+1)} \lambda_i x_1^{n-i}$, by induction. Write $\beta = \alpha - \sum_{i=0}^{n-j} \lambda_i x_1^{n-i-1}$. We have $x_1\beta = 0$. Hence $\infty = \nu(x_1\beta) = \nu(x_1) + \nu(\beta) \leq \nu(x_i) + \nu(\beta) = \infty$ for $1 < i \leq m$. We claim $x_i\beta = 0$ in A . To see this note first that $x_i\beta = \sum_{l=0}^j \mu_{il} x_1^{n-l}$. Next, apply f to both sides of this identity to get that $\mu_{il} = 0$ for all l . Hence $x_i\beta = 0$ as claimed. It follows that $\beta \in M^n$ whence $\beta = \lambda x_1^n$ for some λ . Then $\alpha \in kx_1^{n-j-1} + \dots + kx_1^n$ as we wanted to prove. ■

Corollary 3.2. *Let $A = A_0 \oplus A_1 \oplus A_2 \oplus \dots$ be a finite dimensional graded algebra. Assume (A, A_+) is a local Gorenstein ring. Then A satisfies ABC.*

Proof. Immediate from the theorem and from remark 1.3. ■

Remark 3.3. The result above does not give any information on the one dimensional BC. Indeed by theorem 3.1 above, no zero dimensional Gorenstein algebra A can occur in the pullback diagram (1). Also note that the hypothesis that A be graded can be replaced by the hypothesis that (2) be an isomorphism, or even that $d(\text{soc}(A)) \neq 0$. The latter condition is exactly what is needed to make the proof above work, and is not automatic. Indeed one checks that the algebra A of example 1.4 has $\text{soc}(A) = kx^4$; it was shown already that $d(x^4) = 0$, whence A is an ungradable Gorenstein algebra for which the proof above does not work. On the other hand ABC does hold for A , as shown in 1.4 above.

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attention. The latter, due to Weibel, is a variation of the example appearing in page 241 of [3]. Thanks also to L. Avramov from whom we learned of Saito's paper [4].

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