On the inconsistency of the Camassa-Holm model with the shallow water theory

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Abstract

In our paper we show that the Camassa-Holm equation does not represent a long wave asymptotic due to a major inconsistency with the theory of shallow water waves. We state that any solution of the Camassa-Holm equation, which is not asymptotically close to a solution of the Korteweg–de Vries equation is an artefact of the model and irrelevant to the theory of shallow water waves.

Keywords: Camassa-Holm equation, peakon, long wave asymptotic expansion.

1 Introduction

A partial differential equation

$$2\,\omega\, U_y + U_\tau + 3\,U_y U - U_{\tau yy} - 2\,U_y U_{yy} - U U_{yyy} = 0 \tag{1}$$

known in the literature as Camassa-Holm equation is a fine example of an integrable system with many interesting and rather unusual properties. As an integrable equation it was discovered in Fokas & Fuchssteiner (1980), Fokas & Fuchssteiner (1981), but equation (1) had not been really noticed until the publication Camassa & Holm (1993) where the authors proposed it as a model for shallow water waves. Moreover, Camassa and Holm have shown that in the case $\omega = 0$ equation (1) possesses a peculiar solution with a cusp

$$U = v \exp(-|y - vt|) \tag{2}$$

which they called a "peakon". Moreover, they also found exact multi-peakon solutions. The paper of Camassa and Holm has triggered an enormous avalanche of publications with the developments of mathematical theory for this new class of integrable systems and speculations about possible applications of the Camassa-Holm equation to the theory of shallow water waves (including the problem of tsunami, Lakshmanan (2007)) and other long wave asymptotic theories.

There were a number of publications where authors raised some criticism to the original derivation and proposed new versions of the derivations starting from the reduction of the Green-Naghdi model, reduction of the generalised Serre model or a direct multiscaling asymptotic expansion of the Euler equation (see for example Johnson (2002), Dullin *et al.* (2003), Constantine & Lannes (2009), Dias & Milewski (2010) Camassa *et al.* (1994)). The main concern of these derivations was to achieve a fine adjustment of the coefficients in the first few terms of the asymptotic expansion in order to make the equation corresponding to the truncated expansion integrable. The aim of this paper is to point out that the Camassa-Holm equation does not correspond to any dominant balance in the asymptotic expansion and thus it does not represent a long wave asymptotic for water waves. We have analysed the above derivations and found them inconsistent with the basic principles of asymptotic theory. Indeed, one cannot keep the first principal asymptotic contribution (corresponding to the Korteweg–de Vries theory) together with the next asymptotic correction, then truncate the expansion and balance these two contributions by a re-scaling. Such a re-scaling contains the small asymptotic parameter and thus violates the original assumption about the ratio of the water depth and the characteristic wave length. We have computed the next (neglected in the above derivations) term of the expansion and have shown that after the proposed re-scaling it is of the same order as any term accounted in the Camassa-Holm equation. Moreover, the parameter ω in (1) appears in denominators of the neglected terms and thus cannot be set to zero (for the existence of peakon solutions) in any long-wave asymptotic theory. Here we should mention that the fact that the peakon solutions are irrelevant for shallow water waves was well understood earlier and published in Johnson (2002), Dullin *et al.* (2003). Using the exact soliton solution of the Camassa-Holm equation we have shown that the neglected terms are of the same order as terms accounted in equation (1).

2 Long wave expansion and the Camassa-Holm Equation

In this section, following Whitham (1974), we give a sketch of long wave asymptotic expansion beyond the Korteweg–de Vries (KdV) theory . We shall illustrate the derivation of the Camassa-Holm equation following Dullin *et al.* (2003), Dullin *et al.* (2004) and point out where the inconsistency occurs. Also we will show that the terms neglected in the theory of the Camassa-Holm equation are of the same order as any term accounted in (1). We claim that the inconsistency cannot be removed if one uses a reduction of the Green-Naghdi model (which itself is an approximation), or by choosing the value of the velocity potential inside of the flow (Johnson (2002)) or by any other method. We shall neglect the surface tension, since it does not affect our argument but simply makes expressions look more complicated. Also we shall neglect viscosity and compressibility of water.

Two dimensional motion of vorticity free fluid is described by the potential $\Phi(x', z', t')$ of the velocity field $\mathbf{u}(x', z', t') = \nabla\Phi$. In the bulk of the fluid the potential satisfies the Laplace equation

$$\Phi_{x'x'} + \Phi_{z'z'} = 0$$

with the boundary condition $\Phi_{z'} = 0$ at the bottom $z' = 0$. At the free surface of the fluid $z' = h_0 + H(x', t')$ there are kinematic and dynamic boundary conditions

$$\Phi_{z'} = H_{x'}\Phi_{x'} + H_{t'}$$
$$\Phi_{t'} + \frac{1}{2}\left(\Phi^2_{x'} + \Phi^2_{z'}\right) + gH = 0,$$

where g is the acceleration due to gravity and h_0 is the undisturbed depth of water.

Long wave asymptotic expansion assumes a small parameter $\epsilon = h_0^2/L^2$ where L is a typical wavelength of the wave. Another dimensionless small parameter of the theory is $\mu = a_0/h_0$ where a_0 is a typical amplitude of the wave. We shall assume that $\epsilon \leq \mu \ll 1$. Actually one can set $\mu = \epsilon$ and develop the theory with one parameter, but following Johnson (2002), Dullin *et al.* (2003), Constantine & Lannes (2009) we shall keep both parameters for better control over the terms.

Introducing dimensionless variables

$$x' = Lx,\ z' = h_0 z,\ t' = \frac{L}{\sqrt{gh_0}}t,\ H = \mu h_0 \eta,\ \Phi = \mu l\sqrt{gh_0}\phi,$$

we re-write the above system of equations in the form:

$$\epsilon\phi_{xx} + \phi_{zz} = 0 \tag{3}$$

$$[\phi_z]_{z=0} = 0 \tag{4}$$

$$\left[\frac{1}{\epsilon}\phi_z - \mu\eta_x\phi_x - \eta_t\right]_{z=1+\mu\eta(x,t)} = 0 \tag{5}$$

$$\left[\phi_t + \frac{1}{2}\left(\mu\phi_x^2 + \frac{\mu}{\epsilon}\phi_z^2\right) + \eta\right]_{z=1+\mu\eta(x,t)} = 0 \tag{6}$$

Starting from here we shall develop asymptotic expansion as a series in $\epsilon^n\mu^m,\ n,m \in \mathbb{Z}_+$. We shall illustrate the derivation the Camassa-Holm equation with corrections in three steps. We begin with the derivation of the Boussinesq expansion up to the order $\epsilon^n\mu^m,\ n+m=3$. Then, following Whitham (1974), we make a reduction to the KdV theory describing unidirectional wave propagation. Finally we transform the equation obtained to the form (1), keeping terms of order $\epsilon^n\mu^m,\ n+m=3$ to demonstrate that the Camassa-Holm equation does not represent a long wave asymptotic for surface waves.

2.1 The Boussinesq expansion

The Boussinesq expansion aims to eliminate the dependence on the vertical coordinate z and reduce (3) - (6) to a system of equations on the elevation $\eta = \eta(x,t)$ and the horizontal component of the velocity field at the bottom $w = \phi(x,0,t)_x$. In this Section we shall follow the construction presented in detail in Whitham (1974) (Chapter 13.11), but will keep more terms in the expansion.

It follows from the Laplace equation (3) and the boundary condition at the bottom (4) that

$$\phi(x,z,t) = \sum_{n=0}^{\infty}\epsilon^n(-1)^n\frac{z^{2n}}{(2n)!}\frac{\partial^{2n}F}{\partial x^{2n}}, \tag{7}$$

where $F(x,t)$ is the value of the potential ϕ at the bottom[1], and thus $w = F_x$.

In order to reduce the system (3) - (6) to two equations for functions $\eta(x,t)$ and $w(x,t)$ we substitute $\phi(x,z,t)$ (7), in (5) and (6), then we differentiate in x the equations obtained from (6) and replace F_x by w in the both equations. Keeping terms of order $\epsilon^n\mu^m,\ n+m \le 3$ we get

$$\begin{aligned} 0 &= \eta_t + w_x + \mu\left(\eta w\right)_x - \frac{\epsilon}{6}w_{xxx} \\ &-\frac{\mu\,\epsilon}{2}\left(\eta w_{xx}\right)_x + \frac{\epsilon^2}{120}w_{xxxxx} \\ &-\frac{\mu^2\epsilon}{2}\left(w_{xx}\eta^2\right)_x + \frac{\mu\,\epsilon^2}{24}\left(w_{xxxx}\eta\right)_x - \frac{\epsilon^3}{5040}\,w_{xxxxxxx} \end{aligned} \tag{8}$$

and

$$\begin{aligned} 0 &= w_t + \eta_x + \mu\,ww_x - \frac{\epsilon}{2}w_{txx} \\ &-\frac{\mu\,\epsilon}{2}\left(2w_{tx}\eta + ww_{xx} - w_x^2\right)_x + \frac{\epsilon^2}{24}w_{txxxx} \\ &-\frac{\mu^2\epsilon}{2}\left(2ww_{xx}\eta - 2{w_x}^2\eta + w_{tx}\eta^2\right)_x \\ &+\frac{\mu\,\epsilon^2}{24}\left(4\,w_{txxx}\eta + 3\,w_{xx}^2 - 4\,w_xw_{xxx} + w_{xxxx}w\right)_x - \frac{\epsilon^3}{720}\,w_{txxxxxx} \end{aligned} \tag{9}$$

[1] In Dullin *et al.* (2003) the authors use a different geometry, namely the bottom is set at $z=-1$. Their solution does not satisfy the boundary condition at $z=-1$ (presumably due to a misprint). To rectify the misprint one has to replace z by $z+1$ in the right hand side of (2.7),(2.8) in Dullin *et al.* (2003).

respectively. First two lines in (8),(9) coincide with equation (2.9) in Dullin *et al.* (2003)[2]. For the purpose of our paper we are keeping the next order in the expansion. There are no obstructions to find higher order terms if required.

2.2 Reduction To Unidirectional Waves. The KdV theory with higher asymptotic corrections

System (8),(9) describes waves propagating in both direction. There are many ways to reduce it to unidirectional wave propagation. In order to be consistent with Dullin *et al.* (2003) we employ the method proposed in Whitham (1974). Namely, we shall assume

$$w = \eta + \sum_{k=1}^{\infty}\sum_{n=0}^{k} \mu^n \epsilon^{k-n} f_{kn}[\eta]$$

and request that equations (8),(9) coincide upon this assumption, that would enable us to determine the coefficients $f_{kn}[\eta]$. Keeping terms with $k \le 3$ we get

$$\begin{aligned} w = \quad & \eta - \frac{\mu}{4}\eta^2 + \frac{\epsilon}{3}\eta_{xx} \\ & + \frac{\mu^2}{8}\eta^3 + \frac{\epsilon\,\mu}{16}\left(3\,\eta_x^2 + 8\,\eta\eta_{xx}\right) + \frac{\epsilon^2}{10}\eta_{xxxx} \\ & - \frac{5\,\mu^3}{64}\eta^4 + \frac{\mu^2\epsilon}{32}\left(4\,\eta^2\eta_{xx} + 3\,\eta\eta_x^2 + 6D_x^{-1}\left(\eta_x^3\right)\right) \\ & + \frac{\mu\,\epsilon^2}{1440}\left(504\,\eta\eta_{xxxx} + 1091\,\eta_x\eta_{xxx} + 652\,\eta_{xx}^2\right) + \frac{61\,\epsilon^3}{1890}\eta_{xxxxxx}\,. \end{aligned} \tag{10}$$

Here D_x^{-1} denotes integration. Assuming $\eta_x^3 \to 0$ rapidly enough as $x \to -\infty$ one can set $D_x^{-1}\left(\eta_x^3\right) = \int_{-\infty}^{x} \eta_x^3\,dx$. The first line of the expansion (10) one can find in Whitham (1974), terms f_{2n} were derived in Marchant & Smyth (1990) and Johnson (2002), here we extend the expansion to the terms f_{3n} of order $\mu^n\epsilon^m,\ n+m=3$.

Substitution of (10) in either (8) or (9) leads to equation

$$\begin{aligned} 0 = \eta_t + \eta_x + \frac{3\mu}{2}\eta_x\eta + \frac{\epsilon}{6}\eta_{xxx} & \\ - \frac{3\mu^2}{8}\eta^2\eta_x + \epsilon\,\mu\left(\frac{5}{12}\eta\,\eta_{xxx} + \frac{23}{24}\eta_x\eta_{xx}\right) + \frac{19\epsilon^2}{360}\eta_{xxxxx} & \\ + \frac{3\mu^3}{16}\eta^3\eta_x + \epsilon\,\mu^2\left(\frac{23}{16}\eta\,\eta_x\eta_{xx} + \frac{5}{16}\eta^2\eta_{xxx} + \frac{19}{32}{\eta_x}^3\right) & \\ + \epsilon^2\mu\left(\frac{1079}{1440}\eta_{xxxx}\eta_x + \frac{317}{288}\eta_{xx}\eta_{xxx} + \frac{19}{80}\eta_{xxxxx}\eta\right) + \frac{55\epsilon^3}{3024}\eta_{xxxxxxx}. & \end{aligned} \tag{11}$$

The first line of this expansion is the standard Korteweg–de Vries equation, corrections in the second line is the well known result (see Marchant & Smyth (1990), Johnson (2003)). For the purpose of our paper we proceed to the terms of order $\mu^n\epsilon^m,\ n+m=3$.

[2]In Dullin *et al.* (2003) in the first equation (2.9) there is a misprint in the sign at the term proportional to δ^4 (in our paper it is the term proportional to ϵ^2 in (8)).

2.3 Asymptotic near-identity transformation

Following Dullin *et al.* (2003) we shall apply the Galilean and asymptotically invertible near-identity transformations to equation (11). The purpose of these transformations is to bring the first two lines of equation (11) in the form, which can be re-scaled to equation (1).

First we apply the Galilean transformation

$$X = x - \delta t, \qquad T = t, \tag{12}$$

where δ is a constant which will be determined later. Then we perform the Kodama transformation

$$\eta(X,T) = u + \mu\left(\alpha_1 u^2 + \alpha_2 u_X D_X^{-1}(u)\right) + \epsilon\beta u_{XX} \tag{13}$$

to a new dependent variable $u = u(X,T)$. And finally we apply the Helmholtz operator $\mathcal{H} = 1 - \epsilon\gamma\partial_X^2$ to the equation obtained.

In the Galilean transformation and the Helmholtz operator we set $\delta = 9/19$ and $\gamma = 19/60$ in order to vanish the coefficients at the terms u_{XXX} and u_{XXXXX} respectively. The choice $\alpha_1 = 7/20,\ \alpha_2 = -1/5,\ \beta = 1/30$ excludes the term $u^2 u_X$ and guarantees (see details in Dullin *et al.* (2003)) that the first line of the resulting equation

$$\begin{aligned}
0 = u_T + \frac{10}{19}u_X + \frac{3\,\mu}{2}\,uu_X - \frac{19\,\epsilon}{60}u_{TXX} - \frac{\mu\epsilon}{120}\left(38\,u_{XX}u_X + 19\,u_{XXX}u\right)\\
+\frac{223\,\epsilon^3}{151200}u_{XXXXXXX} - \frac{3\,\mu^3}{100}u_{XX}D_X^{-1}(u)\left(u^2 - 2\,u_X D_X^{-1}(u)\right)\\
+\frac{\mu^2\epsilon}{2400}\left(976\,uu_Xu_{XX} - 48uu_{XXXX}D_X^{-1}(u) + 48\,{u_{XX}}^2 D_X^{-1}(u) + 680u_{XXX}u^2\right.\\
\left.+2765\,{u_X}^3\right) + \frac{\mu\epsilon^2}{3600}\left(903\,u_{XXXX}u_X + 316u_{XXXXX}u + 305u_{XX}u_{XXX}\right)
\end{aligned} \tag{14}$$

can be re-scaled to (1).

Until this stage the asymptotic theory is consistent and equation (14) is asymptotically equivalent to the KdV expansion. The first line of (14) was derived in Dullin *et al.* (2003), our contribution is in the retaining of the next corrections in the asymptotic expansion (the last three lines in (14)).

3 Derivation of the Camassa-Holm equation and its inconsistency with the asymptotic expansion

Analysing the publications with derivations of the Camassa-Holm equation as a long wave asymptotic expansion we notice that they have a similar pattern. Starting from the Euler equation or a certain well established model of water waves (the Green-Naghdi model, the generalised Serre model, etc) the authors arrive to the equation similar (up to an inessential re-scaling with constant coefficients) to the first line of (14). Then they truncate the expansion at this level and re-scale it to the form (1). We shall do the same re-scaling, but accounting the next correction.

The most general re-scaling of variables that transforms the first line of equation (14) into (1) is

$$u = A\,U, \qquad y = \frac{2}{19}\,\frac{\sqrt{285}}{\sqrt{\epsilon}}X, \qquad \tau = \frac{1}{19}\,\frac{\mu A\sqrt{285}}{\sqrt{\epsilon}}\,T, \tag{15}$$

where A is an arbitrary constant and $\omega = 10/(19\,A\,\mu)$. The re-scaling (15) balances the terms in the the equation by eliminating the small parameters ϵ and μ (except the the term with ω). It is

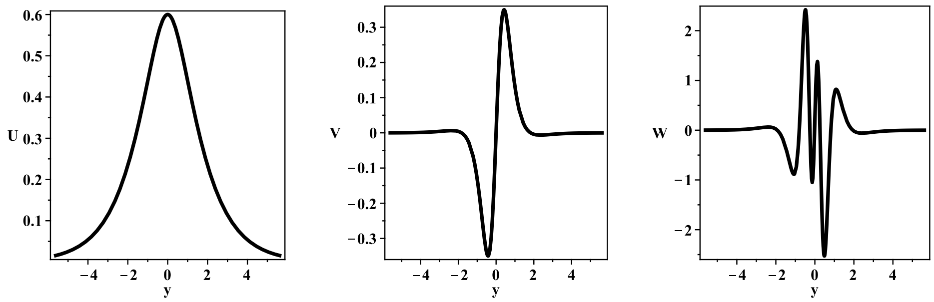

Figure 1: (From left to right) The soliton shape ($c = 1$, $\omega = 0.2$), the value of UU_{yyy} term evaluated on this soliton solution, the value of the correction (16).

easy to see that after this re-scaling the small parameters disappear from the correction (the last three lines in (14)), which takes the form

$$\frac{12}{361\,\omega^2}U_{yy}U_y\left(D_y^{-1}(U)\right)^2-\frac{6}{361\,\omega^2}U^2U_{yy}D_y^{-1}(U)+\frac{2440}{361\,\omega}U_yUU_{yy}$$
$$+\frac{2765}{722\,\omega}(U_y)^3+\frac{24}{361\,\omega}(U_{yy})^2D_y^{-1}(U)-\frac{24}{361\,\omega}UU_{yyyy}D_y^{-1}(U)+\frac{340}{361\,\omega}U^2U_{yyy}$$
$$+\frac{1806}{361}U_yU_{yyyy}+\frac{610}{361}U_{yy}U_{yyy}+\frac{632}{361}UU_{yyyyy}+\frac{446}{2527}\omega\,U_{yyyyyyy}\,. \quad (16)$$

One can also demonstrate that the small parameters disappear from all higher corrections.

One could hope that for the exact soliton or peakon solutions the correction term (16) and all higher corrections vanish. It is obviously not the case for peakons, moreover the constant ω, which has to be set zero (for the existence of peakons) is in the denominator. It does not happen with solitons either. To show that we used the exact soliton solution of equation (1) taken from Johnson (2003):

$$U(y,\tau)=\frac{(c-2\,\omega)}{1+2\omega c^{-1}\sinh^2(\theta)}$$

where c is an arbitrary constant satisfying condition $c > 2\omega$ and θ is a function of $y - c\tau$ implicitly given by equation

$$y-c\tau=\frac{2\theta}{\sqrt{1-2\omega c^{-1}}}+\ln\left(\frac{\cosh\left(\theta-\operatorname{arctanh}\sqrt{1-2\omega c^{-1}}\right)}{\cosh\left(\theta+\operatorname{arctanh}\sqrt{1-2\omega c^{-1}}\right)}\right).$$

Taking $c = 1,\ \omega = 0.2$ we compared a contribution from one of the terms of the Camassa-Holm equation with the value of the correction (16). It is shown on Fig.1 that the correction (16) is much bigger then a contribution from the term of the equation (1) evaluated on the soliton solution with this choice of parameters and this fact does not depend on the choice of the term.

The re-scaling (15) is a basic error, which leads to the inconsistency with the long wave asymptotic theory. Indeed, suppose we have started from equation (1) and have found its solution of a size or characteristic wavelength $\lambda \sim 1$. Re-scaling this solution to the variable X we find from (15) that the size in this variable is $\Lambda = \frac{19\sqrt{\epsilon}}{2\sqrt{285}}\lambda \approx 0.56\sqrt{\epsilon}\lambda$. Since the Galilean transformation (12) does not change the scale, the wave has the same size in the variable x. Coming back to the physical dimensional variable x' we realise that the size of the wave is $\lambda' = L\Lambda \approx 0.56\lambda h_0$ which is in contradiction with the long wave assumption $h_0^2/(\lambda')^2 \approx 3.2/\lambda^2 \ll 1$.

One can consider solutions of (1) of an extremely large characteristic length $\lambda \gg 1$, but with the same accuracy such solutions can be described by the Korteweg–de Vries equation with the first correction (11) (whose integration is not much different from the KdV itself ,Kodama (1985), Hiraoka & Kodama (2009)). For $\lambda \gg 1$ it follows from Dullin *et al.* (2004) that equations (1) and (11) are asymptotically equivalent, but the theory of the latter is much simpler and well developed.

4 Conclusion

The Camassa-Holm model for shallow water waves does not represent the long-wave asymptotic. Any solution of this model, which is not asymptotically close to a solution of the Korteweg–de Vries equation is an artefact of the model and irrelevant to the theory of shallow water waves. In the literature there are many papers with implicit criticism of various aspects of the derivation of the Camassa-Holm model and its validity as well as attempts to rectify them, but they only contribute further to the confusion. Serious concerns about the asymptotic sense of the Camassa-Holm equation as a water wave theory has been raised in Johnson (2002). In our paper we put an end to desperate attempts to justify the Camassa-Holm model using the long wave asymptotic theory. Based on our consideration it is not difficult to conclude, that neither the Camassa-Holm model nor the Degasperis-Procesi equation (Degasperis & Procesi (1999)) can represent a long wave asymptotic in problems of Hydrodynamic, Physics of Condensed Matter, Plasmas, etc, and that the peakon solutions are irrelevant for the long wave asymptotic theory.

Having said so, we do not want to undermine the mathematical value of equation (1). It is a fine example of an integrable multi-Hamiltonian system with interesting associated spectral theory, unusual properties, etc. It may have other applications due to the discovery by Misiolek (1998) (see also Khesin & Misiolek (2003)) that equation (1) is the Euler equation for the geodesic flow on the Virasoro group with respect to the right-invariant Sobolev H^1 -metric. As a mathematical object, the class of integrable equations discovered by Fuchssteiner and Fokas is a valuable contribution to the theory of differential equations.

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