INTEGRABILITY TEST FOR DISCRETE EQUATIONS VIA GENERALIZED SYMMETRIES.

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Abstract. In this article we present some integrability conditions for partial difference equations obtained using the formal symmetries approach. We apply them to find integrable partial difference equations contained in a class of equations obtained by the multiple scale analysis of the general multilinear dispersive difference equation defined on the square.

1. Introduction

DL met Marcos Moshinsky for the first time on his arrival in Mexico in February 1973. He went there with a two years fellowship of the CONACYT (Consejo Nacional de Ciencia y Tecnología), the Mexican Research Council. DL visit was part of his duties as military service in Italy and was following a visit to Mexico in the summer of 1972 by Francesco Calogero, with whom he graduated at the University of Rome *La Sapienza* in the spring of 1972 with a thesis on "Computation of bound state energy for nuclei and nuclear matter with One-Boson-Exchange Potentials" [5].

The bureaucratic process, preliminary to this visit, was long and tiring and it was successful only for the strong concern of DL uncle, Enzo Levi, professor of Hydrulics at UNAM [9] and for the help of Marcos Moshinsky who signed a contract for him which, as DL uncle wrote, was ready to substain with his own personal funds if no other source could be found.

During DL stay in Mexico he was able, with the help of Marcos, to appreciate the pleasures of research and to start his career in Mathematical Physics. With Marcos Moshinsky he published, during the two years stay at the Instituto de Fisica of UNAM, 4 articles [6, 14, 7, 8] but he also collaborated with some of the other physicists of the Institute [4, 11, 12]. This visit was fundamental to shape DL future life, both from the personal point of view and from the point of view of his career.

After 1975 DL met Marcos Moshinsky frequently both abroad and in Mexico. When DL was in Mexico, Marcos Moshinsky always invited him to his home to partecipate to his family dinners. As a memento of Marcos, let us add a picture of his with Pavel Winternitz, Petra Seligman and Decio Levi, taken in September 1999 in front of the Guest House of the CIC-AC where he stayed overnight after a seminar given at the Instituto de Fisica of UNAM, Cuernavaca branch.

Symmetries have played always an important role in physics and in the research of Marcos Moshinsky. This presentation shows how crucial can be the notion of symmetry in uncovering integrable structures in nonlinear partial difference equations.

The discovery of new integrable Partial Difference Equations ($P\Delta E$) is always a very challenging problem as, by proper continuous limits, we can obtain integrable Differential Difference Equations ($D\Delta E$) and Partial Differential Equations (PDE).

A very successful way to uncover integrable PDE has been the formal symmetry approach due to Shabat and his school in Ufa [22]. These results have been later extended to the case of $D\Delta E$ by Yamilov [31], a former student of Shabat.

Here we present some results on the application of the formal symmetry technique to $P\Delta E$.

The basic theory for obtaining symmetries of differential equations has been introduced by Sophus Lie at the end of the nineteen century and can be found together with its extension to generalized symmetries introduced by Emma Noether, for example in the book

FIGURE 1. Pavel Winternitz, Marcos Moshinsky, Petra Seligman and Decio Levi in front of the Guest House of CIC-AC in Cuernavaca in September 1999.

by Olver [26]. The extension of the classical theory to $P\Delta E$ can be found in the work of Levi and Winternitz [19, 21, 29].

Here in the following we outline the results of the geometric classification of $P\Delta E$ given by Adler, Bobenko and Suris [2] with some critical comments at the end. Then in Section 2 we derive the lowest integrability conditions starting from the request that the $P\Delta E$ admit generalized symmetries of sufficiently high order. In Section 3 we show how the test can be applied. Then we apply it to a simple class of $P\Delta E$ obtained by the multiple scale analysis of a generic multilinear dispersive equation defined on the square.

1.1. **Classification of linear affine discrete equations.** Adler, Bobenko and Suris [2] considered the following class of autonomous $P\Delta E$:

$$u_{i+1,j+1} = F(u_{i+1,j}, u_{i,j}, u_{i,j+1}), \tag{1}$$

where i, j are arbitrary integers. Eq. (1) is a discrete analogue of the hyperbolic equations

$$u_{xy} = F(u_x, u, u_y). \tag{2}$$

which are very important in many fields of physics. Up to now the general equation (2) has not been classified. Only the two particular cases:

$$u_{xy} = F(u); \qquad u_x = F(u,v)\ ,\ v_y = G(u,v),$$

which are essentially easier, have been classified by the formal symmetry approach [32, 33].

The ABS integrable lattice equations are defined as those autonomous affine linear (i.e. polynomial of degree one in each argument, i.e. multilinear) partial difference equations of the form

$$\mathcal{E}(u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}; \alpha, \beta) = 0, \tag{3}$$

where α and β are two constant parameters , whose *integrability* is based on the *consistency around a cube* (or 3D-consistency). Here and in the following, as the equations are autonomous, and thus translational invariant, we skip the indices i and j and write the equations around the origin.

The main idea of the consistency method is the following:

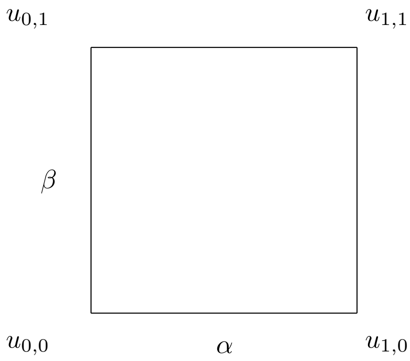

FIGURE 2. A square lattice

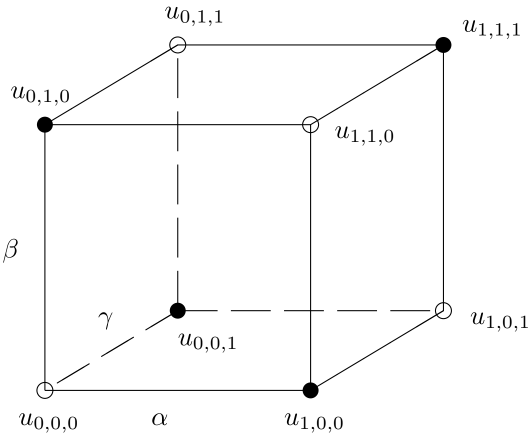

FIGURE 3. Three-dimensional consistency

(1) One starts from a square lattice and defines the three variables $u_{i,j}$ on the vertices (see Figure 2). By solving $\mathcal{E} = 0$ one obtains a rational expression for the fourth one.
(2) One adjoins a third direction, say k, and imagines the map giving $u_{1,1,1}$ as being the composition of maps on the various planes. There exist **three different ways** to obtain $u_{1,1,1}$ and the consistency constraint is that they all lead to the **same result**.
(3) Two further constraints have been introduced by Adler, Bobenko and Suris:
   - *D_4-symmetry*:

$$\begin{aligned}\mathcal{E}(u_{0,0},u_{1,0},u_{0,1},u_{1,1};\alpha,\beta) &= \pm\mathcal{E}(u_{0,0},u_{0,1},u_{1,0},u_{1,1};\beta,\alpha)\\ &= \pm\mathcal{E}(u_{1,0},u_{0,0},u_{1,1},u_{0,1};\alpha,\beta).\end{aligned}$$

   - *Tetrahedron property*: $u_{1,1,1}$ is independent of $u_{0,0,0}$.
(4) The equations are classified according to the following equivalence group:
   - A Möbius transformation.
   - Simultaneous **point** change of all variables.

As a result of this procedure all equations possess a symmetric (in the exchange of the first to the second index) Lax pair, Bäcklund transformations etc. . Thus the compatible equations are, for all purposes, completely integrable equations.

The ABS list read:

$$\text{(H1)} \qquad (u_{0,0}-u_{1,1})\,(u_{1,0}-u_{0,1})\,-\,\alpha\,+\,\beta\,=\,0, \tag{4}$$

The potential discrete KdV equation [15, 23]

$$\begin{aligned}
&\text{(H2)} \qquad (u_{0,0}-u_{1,1})(u_{1,0}-u_{0,1})+(\beta-\alpha)(u_{0,0}+u_{1,0}+u_{0,1}+u_{1,1})-\\
&\qquad\qquad -\alpha^2+\beta^2=0,\\
&\text{(H3)} \qquad \alpha(u_{0,0}u_{1,0}+u_{0,1}u_{1,1})-\beta(u_{0,0}u_{0,1}+u_{1,0}u_{1,1})+\delta(\alpha^2-\beta^2)=0,
\end{aligned} \tag{5}$$

$$\begin{aligned}
&\text{(Q1)} \qquad \alpha(u_{0,0}-u_{0,1})(u_{1,0}-u_{1,1})-\beta(u_{0,0}-u_{1,0})(u_{0,1}-u_{1,1})+\\
&\qquad\qquad +\delta^2\alpha\beta(\alpha-\beta)=0, \quad \text{The Schwarzian discrete KdV equation [16, 25]}\\
&\text{(Q2)} \qquad \alpha(u_{0,0}-u_{0,1})(u_{1,0}-u_{1,1})-\beta(u_{0,0}-u_{1,0})(u_{0,1}-u_{1,1})+\\
&\qquad\qquad +\alpha\beta(\alpha-\beta)(u_{0,0}+u_{1,0}+u_{0,1}+u_{1,1})-\alpha\beta(\alpha-\beta)(\alpha^2-\alpha\beta+\beta^2)=0,\\
&\text{(Q3)} \qquad (\beta^2-\alpha^2)(u_{0,0}u_{1,1}+u_{1,0}u_{0,1})+\beta(\alpha^2-1)(u_{0,0}u_{1,0}+u_{0,1}u_{1,1})-\\
&\qquad\qquad -\alpha(\beta^2-1)(u_{0,0}u_{0,1}+u_{1,0}u_{1,1})-\frac{\delta^2(\alpha^2-\beta^2)(\alpha^2-1)(\beta^2-1)}{4\alpha\beta}=0,\\
&\text{(Q4)} \qquad a_0u_{0,0}u_{1,0}u_{0,1}u_{1,1}+\\
&\qquad\qquad +a_1(u_{0,0}u_{1,0}u_{0,1}+u_{1,0}u_{0,1}u_{1,1}+u_{0,1}u_{1,1}u_{0,0}+u_{1,1}u_{0,0}u_{1,0})+\\
&\qquad\qquad +a_2(u_{0,0}u_{1,1}+u_{1,0}u_{0,1})+\bar{a}_2(u_{0,0}u_{1,0}+u_{0,1}u_{1,1})+\\
&\qquad\qquad +\tilde{a}_2(u_{0,0}u_{0,1}+u_{1,0}u_{1,1})+a_3(u_{0,0}+u_{1,0}+u_{0,1}+u_{1,1})+a_4=0,
\end{aligned}$$

where the seven parameters a_i's in (Q4) are related by 3 equations.

By a proper limiting procedure all equations of the ABS list are contained in eq. (Q4) [24]. The symmetries for the discrete equations of the ABS list have been constructed [28, 27] and are given by $D\Delta E$, subcases of Yamilov's discretization of the Krichever–Novikov equation (YdKN) [17, 31]:

$$\frac{du_0}{d\epsilon}=\frac{R(u_1,u_0,u_{-1})}{u_1-u_{-1}}, \quad R(u_1,u_0,u_{-1})=A_0u_1u_{-1}+B_0(u_1+u_{-1})+C_0,$$

where

$$\begin{aligned}
A_0&=c_1u_0^2+2c_2u_0+c_3,\\
B_0&=c_2u_0^2+c_4u_0+c_5,\\
C_0&=c_3u_0^2+2c_5u_0+c_6.
\end{aligned}$$

It is immediate to see that by defining $v_i=u_{i,j}$ and $\tilde{v}_i=u_{i,j+1}$, the equations of the ABS list are nothing else but Bäcklund transformations for particular subcases of the YdKN [13, 17]. The ABS equations do not exhaust all the possible Bäcklund transformations for the YdKN equation as the whole parameter space is not covered [17, 30]. Moreover, in the list of integrable $D\Delta E$ of Volterra type [31], there are equations different from the YdKN which may also have Bäcklund transformations of the form (1). So we have space for new integrable $P\Delta E$ which we will search by using the formal symmetry approach. An extension of the 3D consistency approach has been proposed by the same authors [3] allowing different equations in the different faces of the cube. However in this way ABS were able to provide only examples of new integrable $P\Delta E$ but not to present a complete classification scheme.

2. Construction of Integrability Conditions

We consider the class of autonomous $P\Delta E$

$$u_{1,1}=f_{0,0}=F(u_{1,0},u_{0,0},u_{0,1}) \quad (\partial_{u_{1,0}}F,\,\partial_{u_{0,0}}F,\,\partial_{u_{0,1}}F)\neq 0. \tag{6}$$

Introducing the two shifts operators, T_1 and T_2 such that $T_1u_{i,j}=u_{i+1,j}$, $T_2u_{i,j}=u_{i,j+1}$, it follows that the functions $u_{i,j}$ are related among themselves by eq. (6) and its shifted values

$$u_{i+1,j+1}=T_1^iT_2^jf_{0,0}=f_{i,j}=F(u_{i+1,j},u_{i,j},u_{i,j+1}).$$

So, the functions $u_{i,j}$ are not all independent. However we can introduce a set of independent functions $u_{i,j}$ in term of which all the others are expressed. A possible choice is given by $(u_{i,0},\, u_{0,j})$, for any arbitrary i, j integers.

A generalized symmetry, written in evolutionary form, is given by

$$\frac{d}{dt}u_{0,0} = g_{0,0} = G(u_{n,0}, u_{n-1,0}, \ldots, u_{n',0}, u_{0,k}, u_{0,k-1}, \ldots, u_{0,k'}), \quad n \geq n',\, k \geq k'. \tag{7}$$

where t is the group parameter. By shifting, we can write it in any point of the plane

$$\frac{d}{dt}u_{i,j} = T_1^i T_2^j g_{0,0} = g_{i,j} = G(u_{i+n,j}, \ldots, u_{i+n',j}, u_{i,j+k}, \ldots, u_{i,j+k'}).$$

In term of the functions $g_{i,j}$ we can write down the symmetry invariant condition

$$\left[g_{1,1} - \frac{df_{0,0}}{dt} \right]\Big|_{u_{1,1}=f_{0,0}} = 0. \tag{8}$$

i.e. $g_{1,1} = (g_{1,0}\partial_{u_{1,0}} + g_{0,0}\partial_{u_{0,0}} + g_{0,1}\partial_{u_{0,1}}) f_{0,0}$. This equation involves the independent variables $(u_{i,0},\, u_{0,j})$ appearing in $g_{0,0}$ shifted to points laying on lines neighboring the axis, i.e. $(u_{i,1},\, u_{1,j})$. For those function we can state the following Proposition [20], necessary to prove the subsequent Theorems:

Proposition 1. *The functions $u_{i,1}, u_{1,j}$ have the following structure:*

$$\begin{array}{lll}
i > 0: & u_{i,1} = u_{i,1}(u_{i,0}, u_{i-1,0}, \ldots, u_{1,0}, u_{0,0}, u_{0,1}), & \partial_{u_{i,0}} u_{i,1} = T_1^{i-1} f_{u_{1,0}}; \\
i < 0: & u_{i,1} = u_{i,1}(u_{i,0}, u_{i+1,0}, \ldots, u_{-1,0}, u_{0,0}, u_{0,1}), & \partial_{u_{i,0}} u_{i,1} = -T_1^i \frac{f_{u_{0,0}}}{f_{u_{0,1}}}; \\
j > 0: & u_{1,j} = u_{1,j}(u_{1,0}, u_{0,0}, u_{0,1}, \ldots, u_{0,j-1}, u_{0,j}), & \partial_{u_{0,j}} u_{1,j} = T_2^{j-1} f_{u_{0,1}}; \\
j < 0: & u_{1,j} = u_{1,j}(u_{1,0}, u_{0,0}, u_{0,-1}, \ldots, u_{0,j+1}, u_{0,j}), & \partial_{u_{0,j}} u_{1,j} = -T_2^j \frac{f_{u_{0,0}}}{f_{u_{1,0}}}.
\end{array} \tag{9}$$

In eq. (9) and in the following, $f_{u_{i,j}} = \frac{\partial f_{0,0}}{\partial u_{i,j}}$ and $g_{u_{i,j}} = \frac{\partial g_{0,0}}{\partial u_{i,j}}$. If a generalized symmetry of characteristic function $g_{0,0}$ depends on at least one variable of the form $u_{i,0}$, then $(g_{u_{n,0}}, g_{u_{n',0}}) \neq 0$, and the numbers n, n' are called the **orders of the symmetry**. The same can be said about the variables $u_{0,j}$ and the corresponding numbers k, k' if $(g_{u_{0,k}}, g_{u_{0,k'}}) \neq 0$.

Now we can state the following Theorem, whose proof can be found in [20]:

Theorem 1. *If the $P\Delta E$ $u_{1,1} = F$ possesses a generalized symmetry then the following relations must take place:*

$$n > 0, \qquad (T_1^n - 1) \log f_{u_{1,0}} = (1 - T_2) T_1 \log g_{u_{n,0}}, \tag{10}$$

$$n' < 0, \qquad (T_1^{n'} - 1) \log \frac{f_{u_{0,0}}}{f_{u_{0,1}}} = (1 - T_2) \log g_{u_{n',0}}, \tag{11}$$

$$k > 0, \qquad (T_2^k - 1) \log f_{u_{0,1}} = (1 - T_1) T_2 \log g_{u_{0,k}}, \tag{12}$$

$$k' < 0, \qquad (T_2^{k'} - 1) \log \frac{f_{u_{0,0}}}{f_{u_{1,0}}} = (1 - T_1) \log g_{u_{0,k'}}. \tag{13}$$

As

$$\begin{array}{ll}
T_l^m - 1 = (T_l - 1)(1 + T_l + \cdots + T_l^{m-1}), & m > 0, \\
T_l^m - 1 = (1 - T_l)(T_l^{-1} + T_l^{-2} + \cdots + T_l^m), & m < 0, \quad l = 1, 2,
\end{array}$$

it follows from Theorem 1 that we can write the equations (10, 11, 12, 13) as standard conservation laws. Thus, the assumption that a generalized symmetry exist implies the existence of some conservation laws.

If we assume that a second generalized symmetry exists, i.e. we can find a nontrival function $\tilde{G}$ such that

$$u_{0,0,\tilde{t}} = \tilde{g}_{0,0} = \tilde{G}(u_{\tilde{n},0}, u_{\tilde{n}-1,0}, \ldots, u_{\tilde{n}',0}, u_{0,\tilde{k}}, u_{0,\tilde{k}-1}, \ldots, u_{0,\tilde{k}'}), \tag{14}$$

where $\tilde{n}, \tilde{n}', \tilde{k}, \tilde{k}'$ are its orders, then we can state the following Theorem:

Theorem 2. *Let the $P\Delta E$ $u_{1,1}=F$ possess two generalized symmetries of orders (n,n',k,k') and $(\tilde n,\tilde n',\tilde k,\tilde k')$, $u_{00,t}=g_{00}$ and $u_{00,\tilde t}=\tilde g_{00}$, and let their orders satisfy one of the following conditions:*

$$\begin{array}{ll} \textit{Case 1}: \ n>0,\ \tilde n=n+1 & \textit{Case 2}: \ n'<0,\ \tilde n'=n'-1 \\ \textit{Case 3}: \ k>0,\ \tilde k=k+1 & \textit{Case 4}: \ k'<0,\ \tilde k'=k'-1 \end{array}$$

Then in correspondence with each of the previous cases the $P\Delta E$ $u_{1,1}=F$ admits a conservation law

$$(T_1-1)p_{0,0}^{(m)}=(T_2-1)q_{0,0}^{(m)}, \qquad m=1,2,3,4, \tag{15}$$

where

$$p_{0,0}^{(1)}=\log f_{u_{1,0}}, \qquad p_{0,0}^{(2)}=\log\frac{f_{u_{0,0}}}{f_{u_{0,1}}}, \qquad q_{0,0}^{(3)}=\log f_{u_{0,1}}, \qquad q_{0,0}^{(4)}=\log\frac{f_{u_{0,0}}}{f_{u_{1,0}}}. \tag{16}$$

So the assumption that the $P\Delta E$ $u_{1,1}=F$ have two generalized symmetries implies that we must have four necessary conditions of integrability, i.e. there must exist some functions of finite range $q_{0,0}^{(1)}, q_{0,0}^{(2)}, p_{0,0}^{(3)}, p_{0,0}^{(4)}$ satisfying the conservation laws (15) with $p_{0,0}^{(1)}, p_{0,0}^{(2)}, q_{0,0}^{(3)}, q_{0,0}^{(4)}$ defined by eq. (16). $q_{0,0}^{(1)}$ and $q_{0,0}^{(2)}$ may depend only on the variables $u_{i,0}$, and $p_{0,0}^{(3)}$ and $p_{0,0}^{(4)}$ on $u_{0,j}$.

Summarizing the results up to now obtained we can say that a nonlinear partial difference equation will be considered to be integrable if it has a generalized symmetry of finite order, i.e. depending on a finite number of fields. This provide some conditions which imply the existence of functions $p_{0,0}^{(m)}$ or $q_{0,0}^{(m)}$ of finite range whose existence is proved by *solving a total difference.*

For a $D\Delta E$, when all shifted variables are independent the proof that a total difference has a solution depending on a finite number of fields, i.e. is a finite range function, is carried out by applying the discrete analogue of the variational derivative, i.e. a function q_n is (up to a constant) a total difference of a function of finite range iff

$$\frac{\delta q_n}{\delta u_n}=\sum_j T^{-j}\frac{\partial q_n}{\partial u_{n+j}}=0, \tag{17}$$

see, e.g. [31]. For $P\Delta E$ this is no more valid as the shifted variables are not independent as they are related by the nonlinear $P\Delta E$, in our case $u_{1,1}=F(u_{1,0},u_{0,0},u_{0,1})$. This turns out to be the main problem for the application of the formal symmetry approach to $P\Delta E$.

To get a definite result we limit our considerations to five points generalized symmetries, i.e. when :

$$\dot u_{0,0}=g_{0,0}=G(u_{1,0},u_{-1,0},u_{0,0},u_{0,1},u_{0,-1}), \quad g_{u_{1,0}}g_{u_{-1,0}}g_{u_{0,1}}g_{u_{0,-1}}\neq 0. \tag{18}$$

The existence of a 5 points generalized symmetry will be taken by us as an *integrability criterion*. This may be a severe restriction as there might be integrable equations with symmetries depending on more lattice points. However just in this case we are able to get sufficiently easily a definite result and, as will be shown in the next Section, we can even solve a classification problem. In this case we can state the following Theorem, which specifies the results obtained so far to the case of five point symmetries:

Theorem 3. *If the $P\Delta E$ $u_{1,1}=F$ possesses a 5 points generalized symmetry, then the functions*

$$\begin{array}{ll} q_{0,0}^{(m)}=Q^{(m)}(u_{2,0},u_{1,0},u_{0,0}), & m=1,2, \\ p_{0,0}^{(m)}=P^{(m)}(u_{0,2},u_{0,1},u_{0,0}), & m=3,4, \end{array} \tag{19}$$

must satisfy the conditions (15, 16).

Then, using the relations (10–13) with $n=k=1$ and $n'=k'=-1$, we get the following relations between the solutions of the total difference conditions and the generalized symmetry G:

$$\begin{array}{ll} q_{0,0}^{(1)}=-T_1\log G_{,u_{1,0}}, & q_{0,0}^{(2)}=T_1\log G_{,u_{-1,0}}, \\ p_{0,0}^{(3)}=-T_2\log G_{,u_{0,1}}, & p_{0,0}^{(4)}=T_2\log G_{,u_{0,-1}}. \end{array} \tag{20}$$

So, to prove the integrability, which for us means find a generalized 5 point symmetry, for a nonlinear $P\Delta E$ $u_{11} = F$, we have to check the integrability conditions (15, 16). If they are satisfied, i.e. there exist some finite range functions $q_{0,0}^{(m)}$ and $p_{0,0}^{(m)}$, we can construct the partial derivatives of G. The compatibility of these partial derivatives of G, given by eqs. (20), provides the *additional integrability condition*

$$G_{,u_{1,0},u_{-1,0}} = G_{,u_{-1,0},u_{1,0}}, \qquad G_{,u_{0,1},u_{0,-1}} = G_{,u_{0,-1},u_{0,1}}. \tag{21}$$

If these additional integrability conditions are satisfied, we find $g_{0,0}$ up to an arbitrary unknown function of the form $\nu(u_{0,0})$, which may correspond to a Lie point symmetry. This function can be specified, using the determining equations (8).

The 5 point generalized symmetry $g_{0,0}$, so obtained, will be of the form:

$$g_{0,0} = \Phi(u_{1,0}, u_{0,0}, u_{-1,0}) + \Psi(u_{0,1}, u_{0,0}, u_{0,-1}) + \nu(u_{0,0}). \tag{22}$$

3. Application of the test: an example

To check the integrability conditions (15, 16) we need to find the finite range functions $q_{0,0}^{(m)}$ $(m = 1, 2)$ and $p_{0,0}^{(m)}$ $(m = 3, 4)$. This is not an easy task even if they are linear first order difference equations. A solution always exists but nothing ensure us a priory that the solution is a finite range function. So let us present a scheme for solving explicitly the integrability conditions we found for the equations on the square i.e. for finding the functions $q_{0,0}^{(1)}$, $q_{0,0}^{(2)}$, $p_{0,0}^{(3)}$ and $p_{0,0}^{(4)}$.

As an example of this procedure let us consider the solution of eq. (15) for $m = 1$, where

$$p_{0,0}^{(1)} = \log(f_{u_{1,0}}),\; q_{0,0}^{(1)} = Q^{(1)}(u_{2,0}, u_{1,0}, u_{0,0}),\; T_2 q_{0,0}^{(1)} = Q^{(1)}(u_{2,1}, u_{1,1}, u_{0,1}). \tag{23}$$

In eq. (23) we have the dependent variables $u_{2,1}$ and $u_{1,1}$ where $u_{2,1} = F(u_{2,0}, u_{1,0}, u_{1,1})$ while $u_{1,1} = F(u_{1,0}, u_{0,0}, u_{0,1})$. So eq. (15) for $m = 1$ will contain the unknow function F which characterize the class of equations we are considering twice, one time to calculate $u_{1,1}$ in terms of independent variables and then to calculate $u_{2,1}$ in term of $u_{1,1}$ and of the independent variables. This double dependence makes the calculations extremely difficult. To overcome this difficulty we take into account that we are considering autonomous equations which are shift invariant. So we can substitute eq. (15) for $m = 1$ with the following equivalent independent equations

$$p_{0,0}^{(m)} - p_{-1,0}^{(m)} = Q^{(m)}(u_{1,1}, u_{0,1}, u_{-1,1}) - Q^{(m)}(u_{1,0}, u_{0,0}, u_{-1,0}) \tag{24}$$

$$p_{0,-1}^{(m)} - p_{-1,-1}^{(m)} = Q^{(m)}(u_{1,0}, u_{0,0}, u_{-1,0}) - Q^{(m)}(u_{1,-1}, u_{0,-1}, u_{-1,-1}) \tag{25}$$

where, to simplify the notation, we introduce in the following the functions

$$u_{1,1} = f^{(1,1)}(u_{1,0}, u_{0,0}, u_{0,1}), \qquad u_{-1,1} = f^{(-1,1)}(u_{-1,0}, u_{0,0}, u_{0,1}),$$
$$u_{1,-1} = f^{(1,-1)}(u_{1,0}, u_{0,0}, u_{0,-1}), \qquad u_{-1,-1} = f^{(-1,-1)}(u_{-1,0}, u_{0,0}, u_{0,-1}),$$

to indicate $f_{0,0}$ and its analogues. Moreover, we introduce the following two differential operators

$$\mathcal{A} = \partial_{u_{0,0}} - \frac{f^{(1,1)}_{u_{0,0}}}{f^{(1,1)}_{u_{1,0}}}\partial_{u_{1,0}} - \frac{f^{(-1,1)}_{u_{0,0}}}{f^{(-1,1)}_{u_{-1,0}}}\partial_{u_{-1,0}}, \tag{26}$$

$$\mathcal{B} = \partial_{u_{0,0}} - \frac{f^{(1,-1)}_{u_{0,0}}}{f^{(1,-1)}_{u_{1,0}}}\partial_{u_{1,0}} - \frac{f^{(-1,-1)}_{u_{0,0}}}{f^{(-1,-1)}_{u_{-1,0}}}\partial_{u_{-1,0}}.$$

in such a way that the functional equations (24, 25) reduce to *differential monomials* [1]:

$$\mathcal{A}Q^{(m)}(u_{1,1}, u_{0,1}, u_{-1,1}) = 0, \qquad \mathcal{B}Q^{(m)}(u_{1,-1}, u_{0,-1}, u_{-1,-1}) = 0, \tag{27}$$

$$\mathcal{A}Q^{(m)}(u_{1,0}, u_{0,0}, u_{-1,0}) = r^{(m,1)}, \qquad \mathcal{B}Q^{(m)}(u_{1,0}, u_{0,0}, u_{-1,0}) = r^{(m,2)}. \tag{28}$$

Eqs. (27) are, by construction, identically satisfied while eqs. (28) provide a set of equations for the derivatives of $Q^{(m)}(u_{1,0}, u_{0,0}, u_{-1,0})$ with respect to its three arguments. By commuting the two operators (26) we can obtain a third equation for the derivatives of $Q^{(m)}(u_{1,0}, u_{0,0}, u_{-1,0})$ with respect to its three arguments:

$$[\mathcal{A}, \mathcal{B}]\, Q^{(m)}(u_{1,0}, u_{0,0}, u_{-1,0}) = r^{(m,3)}. \tag{29}$$

Eqs. (28, 29), if independent, define uniquely the derivatives of the function $Q^{(m)}(u_{1,0}, u_{0,0}, u_{-1,0})$ and, if their consistency is satisfied, from them we get the functions themselves.

In a similar manner from $(T_1 - 1)p^{(m)}_{0,0} = (T_2 - 1)q^{(m)}_{0,0}$ with $m = 3, 4$ we get the function $p^{(m)}_{0,0} = P^{(m)}(u_{0,2}, u_{0,1}, u_{0,0})$ and consequently the symmetry (22).

This procedure works if the function F is known, i.e. if we check a given equation for its integrability. It also works if F is known up to some unknown arbitrary constants to be specified. In such case we solve a classification problem with unknown constants. However, the problem is much more difficult if F depends on unknown arbitrary functions of one, two or three variables. In such a case the coefficients of the operators (26) and functions $r^{(m,k)}$ will depend on unknown functions, and $r^{(m,k)}$ may even depend on the composition of unknown functions. In this case a more complicated procedure might be necessary.

3.1. **A concrete example.** Let us consider the following $P\Delta E$ [10]

$$\begin{aligned} 2(u_{0,0} + u_{1,1}) &+ u_{1,0} + u_{0,1} + \gamma[4u_{0,0}u_{1,1} + 2u_{1,0}u_{0,1} + 3(u_{0,0} + u_{1,1})(u_{1,0} + u_{0,1})] + \\ &+ (\xi_2 + \xi_4)u_{0,0}u_{1,1}(u_{1,0} + u_{0,1}) + (\xi_2 - \xi_4)u_{1,0}u_{0,1}(u_{0,0} + u_{1,1}) + \\ &+ \zeta u_{0,0}u_{1,1}u_{1,0}u_{0,1} = 0. \end{aligned} \tag{30}$$

Eq. (30) is a dispersive multi–linear partial difference equation which passes the A_3 multiple scales integrability test [18]. Applying the Möbious transformation $u_{i,j} = 1/(\hat{u}_{i,j} - \gamma)$ we can rewrite it in a simplified form as

$$(u_{0,0}u_{1,1} + \alpha)(u_{1,0} + u_{0,1}) + (2u_{1,0}u_{0,1} + \beta)(u_{0,0} + u_{1,1}) + \delta = 0, \tag{31}$$

where α, β and δ are well defined functions of γ, ξ_2, ξ_4 and ζ. We now apply to eq. (31) the procedure outlined at the beginning of this section. Eq. (31) depends on three free parameters and we look for conditions on the three parameters, if any, such that the equation admits generalized symmetries. We get that the conditions are satisfied only in two cases:

(1) $\alpha = 2\beta \neq 0$, $\delta = 0$ and, as $\beta \neq 0$ we can always set $\beta = 1$. This choice of the parameters α and β corresponds to $\xi_2 = 3\xi_4 + 3\gamma^2$ and $\zeta = 12\gamma\xi_4$ in eq. (30). The corresponding integrable $P\Delta E$ reads:

$$(u_{0,0}u_{1,1} + 2)(u_{1,0} + u_{0,1}) + (2u_{1,0}u_{0,1} + 1)(u_{0,0} + u_{1,1}) = 0. \tag{32}$$

In correspondence with the eq. (32) we get the generalized symmetry

$$\begin{aligned} u_{0,0;t} &= \frac{(u_{0,0}^2 - 2)(2u_{0,0}^2 - 1)}{u_{0,0}} \left\{ A \left[\frac{1}{u_{1,0}u_{0,0} + 1} - \frac{1}{u_{-1,0}u_{0,0} + 1} \right] + \right. \\ &+ \left. B \left[\frac{1}{u_{0,1}u_{0,0} + 1} - \frac{1}{u_{0,-1}u_{0,0} + 1} \right] \right\}. \end{aligned} \tag{33}$$

(2) $\beta = 2\alpha \neq 0$, $\delta = 0$ and, as $\alpha \neq 0$ we can always set $\alpha = 1$. This choice of the parameters α and β corresponds to $\xi_2 = 6\gamma^2 - 3\xi_4$ and $\zeta = 12\gamma(\gamma^2 - \xi_4)$ in eq. (30). The corresponding integrable $P\Delta E$ reads:

$$(1 + u_{0,0}u_{1,1})(u_{1,0} + u_{0,1}) + 2(1 + u_{0,1}u_{1,0})(u_{0,0} + u_{1,1}) = 0. \tag{34}$$

In correspondence with the eq. (34) we get the generalized symmetry

$$u_{0,0;t} = A(u_{0,0}^2 - 1)\frac{u_{1,0} - u_{-1,0}}{u_{-1,0}u_{1,0} - 1} + B(u_{0,0}^2 - 1)\frac{u_{0,1} - u_{0,-1}}{u_{0,-1}u_{0,1} - 1} \tag{35}$$

Here A, B are constant coefficients, and in both cases, $(A = 0, B \neq 0)$ and $(A \neq 0, B = 0)$, the nonlinear $D\Delta E$ (33, 35) are, up to a point transformation, equations belonging to the classification of Volterra type equations done by Yamilov [31]. This shows that the eqs. (32, 34) do not belong to the ABS classification. Moreover this calculation shows that the

A_3 integrability in the multiple scale integrability test is not sufficient to select integrable $P\Delta E$ on the square having five points generalized symmetries.

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References

[1] J. Aczel, *Lectures on Functional Equations and Their Applications*, (New York: Academic, 1966); J. Aczel (ed) *Functional Equations: History, Applications and Theory*, (Amsterdam: Reidel,1984).
[2] V.E. Adler, A.I. Bobenko and Yu.B. Suris, *Classification of integrable equations on quad-graphs. The consistency approach*, Comm. Math. Phys. **233** (2003) 513–543.
[3] V.E. Adler, A.I. Bobenko and Yu.B. Suris, *Discrete nonlinear hyperbolic equations. Classification of integrable cases*, Funk. Anal. Prilozh. **43** (2009) 3–23 (in Russian); English transl. in Funct. Anal. Appl. **43** (2009) 3–17.
[4] T.A. Brody, R. Cid, J. L. Jimenez, D. Levi, J. R. Martinez, P. Pereyra, R. Rechtman and M. Rosales, *La Mechanica Cuantica y sus interpretaciones* , Rev. Mex. Fis. **25** (1976) E31–E62.
[5] F. Calogero, D. Levi, *One-boson-exchange potentials and nuclear structure*, Physical Review **C8** (1973) 83–92.
[6] E. Chacon, D. Levi and M. Moshinsky, *Relations between the hyper-spherical harmonic and the harmonic oscillator methods for the three body problem*, Rev. Mex. Fis. **22** (1973) 291–319.
[7] E. Chacon, D. Levi and M. Moshinsky, *Equivalence of a class of Wigner coefficients of SU(1,1) with those of SU(2)*, J. Math. Phys. **16** (1975) 1876–1879.
[8] E. Chacon, D. Levi and M. Moshinsky, *Lie algebras in the Schrödinger picture and radial matrix elements*, J. Math. Phys. **17** (1976) 1919–1929.
[9] *Inteligencia y Pasión: Fundadores del Instituto de Ingeniería*, ed. R. Díaz de Cossío Carbajal, (UNAM, Mexico D.F. , 2006).
[10] R. Hernandez Heredero, D. Levi and C. Scimiterna, private communication.
[11] M. de Llano and D. Levi, *Closed form of the second virial coefficient for Sutherland potential*, J. Chem. Phys. **63** (1975) 4561–4562.
[12] M. de Llano and D. Levi, *Generalized Van der Waals theory*, Rev. Mex. Fis. **25** (1976) 61–67.
[13] D. Levi, *Nonlinear differential-difference equations as Bäcklund transformations*, J. Phys. A: Math. Gen. **14** (1981), 1083–1098.
[14] D. Levi and M. Moshinsky, *Relations between hyper-spherical and harmonic-oscillator many-body matrix elements*, Nuovo Cimento **A20** (1974) 107–114.
[15] D. Levi and M. Petrera, *Continuous symmetries of the lattice potential KdV equation*, J. Phys. A:Math. Theor. **40** (2007) 4141–4159.
[16] D. Levi, M. Petrera and C. Scimiterna, *The lattice Schwarzian KdV equation and its symmetries*, J. Phys. A: Math. Theor. **40** (2007) 12753–12761.
[17] D. Levi, M. Petrera, C. Scimiterna and R. Yamilov, *On Miura Transformations and Volterra-Type Equations Associated with the Adler-Bobenko-Suris Equations*, SIGMA **4** (2008) 077, 14 pages.
[18] D. Levi and C. Scimiterna, *Integrability test of discrete nonlinear Schrd̈inger equations via multiscale reduction*, Appl. Anal. **89** (2010) 507–527.
[19] D. Levi and P. Winternitz, *Continuous symmetries of difference equations*, J. Phys. A: Math. Gen. **39** (2006) R1–R63.
[20] D Levi and R I Yamilov, *The generalized symmetry method for discrete equations*, J. Phys. A: Math. Theor. **42** (2009) 454012.
[21] D. Levi and R. I. Yamilov, Generalized Lie symmetries for difference equations, in *Symmetries and Integrability of Difference Equations*, London Mathematical Society Lecture Note Series No. 381, eds. D. Levi, P. Olver, Z. Thomova, P. Winternitz, (CUP, Cambridge, 2011).
[22] A.V. Mikhailov, A.B. Shabat and R.I. Yamilov, *The symmetry approach to the classification of nonlinear equations. Complete lists of integrable systems*, Uspekhi Mat. Nauk **42** (1987) 3–53 (in Russian); English transl. in Russian Math. Surveys **42** (1987) 1–63.
A.V. Mikhailov, A.B. Shabat, V.V. Sokolov, *The symmetry approach to classification of integrable equations*, In: *What is integrability?*, ed. V.E. Zakharov, (Berlin, Springer-Verlag, 1991) 115–184.
[23] F.W. Nijhoff and H.W. Capel, *The discrete Korteweg-de Vries equation*, Acta Appl. Math. **39** (1995) 133–158.
[24] F. Nijhoff, J. Atkinson and J. Hietarinta, *Soliton solutions for ABS lattice equations: I. Cauchy matrix approach*, J. Phys. A: Math Theor. **42** (2009) 404005.
[25] F.W. Nijhoff, G.R.W. Quispel and H.W. Capel, *Direct linearization of nonlinear difference-difference equations*, Phys. Lett. A **97** (1983) 125–128.

[26] P. Olver, *Applications of Lie Groups to Differential Equations*, (Springer-Verlag, New York, 1993).
[27] O.G. Rasin and P.E. Hydon, *Symmetries of integrable difference equations on the quad-graph*, Stud. Appl. Math. **119** (2007) 253–269.
[28] A. Tongas, D. Tsoubelis and P. Xenitidis, *Affine linear and D_4 symmetric lattice equations: symmetry analysis and reductions*, Jour. Phys. A **40** (2007) 13353–13384.
[29] P. Winternitz, Lie point symmetries of difference equations and symmetry preserving discretization of differential equations, in *Symmetries and Integrability of Difference Equations*, London Mathematical Society Lecture Note Series 381,eds. D. Levi, P. Olver, Z. Thomova, P. Winternitz, (CUP, Cambridge, 2011).
[30] P. Xenitidis, *Integrability and symmetries of difference equations: the Adler-Bobenko-Suris case*, Proceedings of the 4th Workshop "Group Analysis of Differential Equations and Integrable Systems", (2009) 226–242, arXiv: 0902.3954v1 [nlin.SI].
[31] R.I. Yamilov, *Classification of discrete evolution equations*, Uspekhi Mat. Nauk **38** (1983) 155–156 (in Russian).
R. Yamilov, *Symmetries as integrability criteria for differential difference equations*, J. Phys. A: Math. Gen. **39** (2006) R541–R623.
[32] A.V. Zhiber and A.B. Shabat, *The KleinGordon equations with nontrivial group* Dokl. Akad. Nauk. SSSR **247** (1979) 11037 (in Russian).
[33] A.V. Zhiber and A.B. Shabat, *Systems of equations ux = p(u, v), vy = q(u, v) having symmetries* Dokl. Akad. Nauk. SSSR **277** (1984) 29–33 (in Russian); Engl transl. Sov. Math. Dokl. **36** (1984) 23–6.

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