On natural Poisson bivectors on the sphere

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Abstract

We discuss the concept of natural Poisson bivectors, which allows us to consider the overwhelming majority of known integrable systems on the sphere in framework of bi-Hamiltonian geometry.

1 Introduction

The Hamilton-Jacobi theory seems to be one of the most powerful methods of investigation the dynamics of mechanical (holonomic and nonholonomic) and control systems. Besides its fundamental aspects such as its relation to the action integral and generating functions of symplectic maps, the theory is known to be very useful in integrating the Hamilton equations using the variables separation technique. The milestones of this technique include the works of Stäckel, Levi-Civita, Eisenhart, Woodhouse, Kalnins, Miller, Benenti and others. The majority of results was obtained for a very special class of integrable systems, important from the physical point of view, namely for the systems with quadratic in momenta integrals of motion. The Kowalevski, Chaplygin and Goryachev's results on separation of variables for the systems with higher order integrals of motion missed out of this scheme.

Bi-Hamiltonian structures can be seen as a dual formulation of integrability and separability, in the sense that they substitute a hierarchy of compatible Poisson structures to the hierarchy of functions in involution, which may be treated either as integrals of motion or as variables of separation for some dynamical system. The Eisenhart-Benenti theory was embedded into the bi-Hamiltonian set-up using the lifting of the conformal Killing tensor that lies at the heart of Benenti's construction [7, 12, 15]. The concept of natural Poisson bivectors allows us to generalize this construction and to study systems with quadratic and higher order integrals of motion in framework of a single theory [31]. Second new ingredient of this theory is the non-Stäckel separated relations [29, 30].

The aim of this note is to bring together all the known examples of natural Poisson bivectors on the sphere, because a good example is the best sermon. Corresponding integrable natural systems on two-dimensional unit sphere $\mathbb{S}^2$ are related to rigid body dynamics. In order to describe these systems we will use the angular momentum vector $J = (J_1, J_2, J_3)$ and the Poisson vector $x = (x_1, x_2, x_3)$ in a moving frame of coordinates attached to the principal axes of inertia [4]. The Poisson brackets between these variables

$$\{J_i\,, J_j\} = \varepsilon_{ijk} J_k\,, \qquad \{J_i\,, x_j\} = \varepsilon_{ijk} x_k\,, \qquad \{x_i\,, x_j\} = 0\,, \tag{1.1}$$

may be associated to the Lie-Poisson algebra of the three-dimensional Euclidean algebra $e(3)$ with two Casimir elements

$$C_1 = |x|^2 \equiv \sum_{k=1}^{3} x_k^2, \qquad C_2 = \langle x, J\rangle \equiv \sum_{k=1}^{3} x_k J_k. \tag{1.2}$$

As usual all the results are presented up to the linear canonical transformations, which consist of rotations

$$x \to \alpha\, U\, x\,, \qquad J \to U\, J\,,$$

where α is an arbitrary parameter and U is an orthogonal constant matrix, and shifts

$$x \to x\,, \qquad J \to J + S\,x\,,$$

where S is an arbitrary 3×3 skew-symmetric constant matrix [4, 16].

If the square integral of motion $C_2 = (x, J)$ is equal to zero, rigid body dynamics may be restricted on the unit sphere $\mathbb{S}^2$ and we can use standard spherical coordinate system on it's cotangent bundle $T^*\mathbb{S}^2$

$$\begin{array}{lll} x_1 = \sin\phi\sin\theta, & x_2 = \cos\phi\sin\theta, & x_3 = \cos\theta\,, \\ J_1 = \dfrac{\sin\phi\cos\theta}{\sin\theta}\,p_\phi - \cos\phi\,p_\theta\,, & J_2 = \dfrac{\cos\phi\cos\theta}{\sin\theta}\,p_\phi + \sin\phi\,p_\theta\,, & J_3 = -p_\phi\,. \end{array} \tag{1.3}$$

We use these variables in order to determine and classify the natural Poisson bivectors on $T^*\mathbb{S}^2$ up to the point canonical transformations.

As far as the organization of this paper is concerned, in Section 2 we briefly introduce the notions of bihamiltonian geometry relevant for subsequent sections. In particular, we discuss the concept of natural Poisson bivectors on cotangent bundles to Riemannian manifolds, which allows us to generalize classical Eisenhart-Benenti theory. In Section 3 we present the main theorem of the bi-Hamiltonian classification of bi-integrable systems on the sphere. Section 4 is devoted to the Stäckel separable systems coming from auxiliary bi-Hamiltonian systems In Section 5 we discuss construction of non-Stäckel systems, whose separability, to the best of our knowledge, has not been considered in literature yet.

2 Some issues in the geometry of bi-Hamiltonian manifolds

We start this section recalling some well known facts in the theory of bi-Hamiltonian manifolds [12].

Definition 1 *A bi-Hamiltonian manifold M is a smooth manifold M endowed with a pair of compatible Poisson bi-vectors P and P' such that*

$$[P, P'] = 0, \qquad [P', P'] = 0, \tag{2.1}$$

where $[.,.]$ is the Schouten bracket.

This means that every linear combination of P and P' is still a Poisson bivector.

If P is invertible Poisson bivector on M, one can introduce the so-called Nijenhuis operator (or hereditary, or recursion)

$$N = P'P^{-1}\,, \tag{2.2}$$

together with its transpose $N^* = P^{-1}P'$. By definition, N (resp., N^*) is an endomorphism of the tangent bundle to M (resp., of the cotangent bundle). As a remarkable consequence of the compatibility between P and P', the Nijenhuis torsion of N identically vanishes. So, from the classical Frölicher-Nijenhuis theory, we know that its eigenvalues are integrable distributions.

Throughout this paper, we will assume that recursion operator N has, at every point , the maximal number $n = \frac{1}{2}\,\mathrm{dim}M$ of different functionally independent eigenvalues $u_1, \ldots, u_n$, so that the eigenspace relative to any eigenvalue is two-dimensional. In this case M is said to be a regular bi-Hamiltonian manifold.

2.1 Bi-Hamiltonian structures on cotangent bundles

According to [32] a torsionless (1,1) tensor field L on a smooth manifold Q gives rise to a (second) Poisson structure on the cotangent space $M = T^*Q$, compatible with the canonical one.

Let θ be the Liouville 1-form on T^*Q and $\omega = d\theta$ the standard symplectic 2-form on T^*Q, whose associated Poisson bivector will be denoted with P. If we choose some local coordinates $(q_1, \ldots, q_n)$

on Q and the corresponding symplectic coordinates $(q_1,\ldots,q_n,p_1,\ldots,p_n)$ on T^*Q then we get the following local expressions

$$\theta = p_1dq_1+\ldots p_ndq_n\,,\qquad \text{and}\qquad P=\left(\begin{array}{cc} 0 & \mathrm{I}\\ -\mathrm{I} & 0\end{array}\right). \tag{2.3}$$

Using a torsionless tensor field L one can deform θ to a 1-form θ' and P to bivector P':

$$\theta'=\sum_{i,j=1}^n L_{ij}p_idq_j,\qquad \text{and}\qquad P'=\left(\begin{array}{cc} 0 & L_{ij}\\ & \\ -L_{ij} & \displaystyle\sum_{k=1}^n\left(\frac{\partial L_{ki}}{\partial q_j}-\frac{\partial L_{kj}}{\partial q_i}\right)p_k\end{array}\right). \tag{2.4}$$

The vanishing of L torsion entails that P' (2.4) is a Poisson bivector compatible with P.

Let us consider natural integrable by Liouville system on Q for which Hamilton function

$$H_1=T+V=\sum_{i,j=1}^n \mathrm{g}_{ij}\,p_i\,p_j+V(q_1,\ldots,q_n) \tag{2.5}$$

is the sum of the geodesic Hamiltonian T and potential energy V. If the corresponding Hamiltonian-Jacobi equation is separable in orthogonal coordinate systems on Q, then in framework of the Eisenhart-Benenti theory the second Poisson bivector P' (2.4) is defined by a conformal Killing tensor L of gradient type with pointwise simple eigenvalues associated with the metric g, see [1, 2, 3, 7, 15].

According to [31], elementary generalization of the Eisenhart-Benenti theory consists of introduction a concept of natural Poisson bivectors.

Definition 2 *The natural Poisson bivector P' on T^*Q is a sum of the geodesic Poisson bivector P'_T and the potential Poisson bivector defining by tensor field $\Lambda(q_1,\ldots,q_n)$ on Q*

$$P'=P'_T+\left(\begin{array}{cc} 0 & \Lambda_{ij}\\ & \\ -\Lambda_{ji} & \displaystyle\sum_{k=1}^n\left(\frac{\partial \Lambda_{ki}}{\partial q_j}-\frac{\partial \Lambda_{kj}}{\partial q_i}\right)p_k\end{array}\right). \tag{2.6}$$

By definition P'_T is the Poisson bivector compatible with canonical ones, so that

$$[P,P'_T]=[P'_T,P'_T]=0\,, \tag{2.7}$$

and the Nijenhuis torsion of Λ vanishes as a consequence of the compatibility P, P'_T and P'.

In this note we suppose that geodesic Poisson bivector P'_T has the following special form

$$P'_T=\left(\begin{array}{cc} \displaystyle\sum_{k=1}^n \mathrm{x}_{jk}(q)\frac{\partial \Pi_{jk}}{\partial p_i}-\mathrm{y}_{ik}(q)\frac{\partial \Pi_{ik}}{\partial p_j} & \Pi_{ij}\\ & \\ -\Pi_{ji} & \displaystyle\sum_{k=1}^n\left(\frac{\partial \Pi_{ki}}{\partial q_j}-\frac{\partial \Pi_{kj}}{\partial q_i}\right)\mathrm{z}_k(p)\end{array}\right). \tag{2.8}$$

In fact in this case P' is unambiguously determined by two $n\times n$ matrices Π and Λ on $2n$-dimensional space T^*Q because x,y and z are easily calculated by Π.

Firstly we have to solve equations (2.7) with respect to functions $\Pi_{ij}(p,q)$, $\mathrm{x}_{jk}(q)$, $\mathrm{y}_{ik}(q)$ and $\mathrm{z}_k(p)$. Then we can try to add to derived geodesic bivector P'_T some compatible potential part defined by $\Lambda(q)$.

Definition 3 *Matrices Π and Λ are compatible to each other if the natural Poisson bivector P' (2.6,2.8) satisfies the equations (2.1).*

Using this scheme we can easily classify all the Poisson bivectors (2.6,2.8) on the low-dimensional manifolds Q in consecutive order solving equations (2.7) and (2.1).

2.2 Bi-integrable systems on cotangent bundles

Let us consider a family of bi-integrable systems for which there are functionally independent integrals of motion $H_1,\ldots,H_n$ in the bi-involution

$$\{H_i,H_j\}=\{H_i,H_j\}'=0\,,\qquad i,j=1,\ldots,n, \tag{2.9}$$

with respect to a pair of compatible Poisson brackets $\{.,.\}$ and $\{.,.\}'$ defined by P and P'. There are three distinct constructions of bi-integrable systems starting with a pair of compatible Poisson bivectors on T^*Q, see [31] .

Firstly it is easy to see that on the cotangent bundle T^*Q canonical Poisson bivector P (2.3) is invertible and symplectic form ω is its inverse. It allows us to construct recursion operator N (2.2) and, as usual, functions

$$\mathcal{H}_k=\frac{1}{2k}\,\mathrm{tr}\,N^k \tag{2.10}$$

form a bi-Hamiltonian hierarchy on T^*Q, i.e. the Lenard relations hold

$$P'd\mathcal{H}_k=Pd\mathcal{H}_{k+1}\,,\qquad \text{for all}\qquad k\geq 1\,.$$

This follows from $N^*d\mathcal{H}_k=d\mathcal{H}_{k+1}$ and it is well-known to imply the involutivity of the $\mathcal{H}_1,\ldots,\mathcal{H}_n$ with respect to both Poisson brackets [12] .

Second special but more fundamental construction of integrable systems was originally formulated by Jacobi when he invented elliptic coordinates and successfully applied them to solve several important mechanical problems: *"The main difficulty in integrating a given differential equation lies in introducing convenient variables, which there is no rule for finding. Therefore, we must travel the reverse path and after finding some notable substitution, look for problems to which it can be successfully applied".*

In framework of the Jacobi method we consider $\mathcal{H}_i$ as constants of motion for an *auxiliary* bi-Hamiltonian system on T^*Q and treat functionally independent eigenvalues u_j of N

$$B(\lambda)=\Bigl(\det(N-\lambda\mathrm{I})\Bigr)^{1/2}=(\lambda-u_1)(\lambda-u_2)\cdots(\lambda-u_n)\,, \tag{2.11}$$

as "convenient variables" for a huge family of *separable* bi-integrable systems on T^*Q associated with various separated relations

$$\Phi_i(u_i,p_{u_i},H_1,\ldots,H_n)=0\;,\quad i=1,\ldots,n\;,\qquad \text{with }\det\left[\frac{\partial\phi_i}{\partial H_j}\right]\neq 0\,. \tag{2.12}$$

Here $u=(u_1,\ldots,u_n)$ and $p_u=(p_{u_1},\ldots,p_{u_n})$ are canonical variables of separation

$$\{u_i,p_{u_j}\}=\delta_{ij}\qquad\text{and}\qquad \{u_i,p_{u_j}\}'=\delta_{ij}\,u_i\,. \tag{2.13}$$

The Poisson brackets (2.13) entail that solutions $H_1,\ldots,H_n$ of the separated relations (2.12) are functionally independent integrals of motion in the bi-involution (2.9), see [25, 27]. Of course, this construction will be justified only if we are capable to obtain separable Hamilton functions $H=H_i$, which have natural form in initial (p,q) variables (2.5).

The third construction of integrals of motion in the bi-involution on irregular bi-Hamiltonian manifolds is discussed in [18, 31].

Remark 1 According to [12], in a neighborhood of a point of a regular bi-Hamiltonian manifold T^*Q variables $p_{u_1},\ldots,p_{u_n}$ may be always found by quadratures and the auxiliary bi-Hamiltonian system with integrals $\mathcal{H}_j$ (2.10) is trivial. In fact the bi-Hamiltonian systems (2.10) associated with distinct recursion operators N and $\widehat{N}$ are related by a trivial canonical transformation

$$u_j=\hat{u}_j\,,\qquad p_{u_j}=\hat{p}_{u_j} \tag{2.14}$$

because

$$\mathcal{H}_k=\sum_{j=1}^n u_j^k\,,\qquad\text{and}\qquad \widehat{\mathcal{H}}_k=\sum_{j=1}^n \hat{u}_j^k\,.$$

So, all the bi-Hamiltonian systems on T^*Q are equivalent to each other at least locally. On the other hand in the Jacobi method distinct separation relations (2.12) give rise to different separable systems with respect to transformation (2.14).

3 Special natural Poisson bivectors on the sphere

Geometrically invariant equations (2.1) have infinitely many solutions P' even if we add to them equations (2.9) with some fixed integrals of motion H_k [27]. In fact in order to get a search algorithm of effectively computable solutions we have to narrow the search space by using some non-invariant additional assumptions.

Firstly we stint ourselves by a family of natural Poisson bivectors (2.6) with geodesic part (2.8). Then, it is easy to see that the geodesic Hamiltonian

$$T=\sum_{i,j=1}^{n} \mathrm{g}_{ij}(q)\, p_i\, p_j$$

on the cotangent bundle T^*Q is the second order homogeneous polynomial in momenta, so we assume that entries of Π are the similar homogeneous polynomials

$$\Pi_{ij}=\sum_{k,m=1}^{n} c_{ij}^{km}(q)p_k p_m\,. \tag{3.1}$$

On two-dimensional unit sphere $Q=\mathbb{S}^2$ we use spherical coordinates (1.3) such as

$$q=(q_1,q_2)=(\phi,\theta) \qquad \text{and} \qquad p=(p_1,p_2)=(p_\phi,p_\theta)\,.$$

Nevertheless, even at $n=2$ equations (2.7) have too much distinct solutions in the form (2.8,3.1).

So, at the third step we introduce a family of partial solutions for which all the entries of P'_T (2.8) are independent on variable ϕ, i.e. at

$$c_{ij}^{km}(q)=c_{ij}^{km}(\theta), \qquad \mathrm{x}_{jk}(\phi,\theta)=\mathrm{x}_{jk}(\theta)\,, \qquad \mathrm{y}_{ik}(\phi,\theta)=\mathrm{y}_{ik}(\theta)\,. \tag{3.2}$$

It looks like reasonable assumption because the geodesic Hamiltonian T

$$\begin{aligned} T &= a_1J_1^2+a_2J_2^2+a_3J_3^2=\Bigl(a_3-\cot^2\theta(a_1\sin^2\phi+a_2\cos^2\phi)\Bigr)p_\phi^2\,,\\ &- \sin 2\phi\cot\theta(a_1-a_2)p_\theta\, p_\phi+(a_1\cos^2\phi+a_2\sin^2\phi)p_\theta^2 \end{aligned} \tag{3.3}$$

is independent on variable ϕ at $a_1=a_2$. If a_k are constants it means that two diagonal elements of inertia tensor of the body $a_1^{-1}=a_2^{-1}$ are equal to each other and we discuss *symmetric* rigid body [4].

Proposition 1 *If assumptions (3.1-3.2) hold we have three families of solutions P'_T (2.8) at*

$$\begin{aligned} &\textit{Case } 1. \qquad \Pi_{ij}=0;\\ &\textit{Case } 2. \qquad \mathrm{z}_1=0\,, \qquad \mathrm{z}_2=0\,;\\ &\textit{Case } 3. \qquad \mathrm{z}_1=\frac{p_\phi}{3}\,, \quad \mathrm{z}_2=\frac{p_\theta}{3}\,. \end{aligned} \tag{3.4}$$

The proof consist of direct solution of the equations $[P,P'_T]=0$.

Of course, non-invariant assumptions (3.1,3.2) depend on a choice of coordinate system and we miss a lot of another solutions of (2.1), which may be interesting in applications.

Remark 2 On the n-dimensional sphere there are the same three families of solutions, i.e. factor $1/3$ in (3.4) is independent on dimension of the sphere.

At the first case $P'_T=0$ and we can immediately look for compatible potential part $\Lambda(\phi,\theta)$ and the variables of separation $u_{1,2}$ (2.11), which are related with initial variables by the point canonical transformations

$$u_i=f_i(\phi,\theta), \qquad p_{u_i}=g_i(\phi,\theta)\,p_\phi+h_i(\phi,\theta)\,p_\theta\,. \tag{3.5}$$

As a consequence, the geodesic Hamiltonian is a second order homogeneous polynomial in physical and separated momenta and the theory of projectively equivalent metrics in classical differential geometry study essentially the same object [3].

In second case Π depends on six functions g, h and one parameter $\gamma = 0, 1$:

$$\Pi = \begin{pmatrix} \gamma p_\phi^2 & g_1(\theta)p_\phi^2 + g_2(\theta)p_\phi\, p_\theta + g_3(\theta)p_\theta^2 \\ \\ 0 & h_1(\theta)p_\phi^2 + h_2(\theta)p_\phi\, p_\theta + h_3(\theta)p_\theta^2 \end{pmatrix} \tag{3.6}$$

up to the point transformations $p_k \to \alpha_k p_1 + \beta_k p_2$. These functions g, h, together with functions x, y from (2.8), are solutions of the six non-linear differential equations (2.7).

In third case Π depends on nine functions f, g, h and one parameter $\gamma = 0, 1$:

$$\Pi = \begin{pmatrix} f_1(\theta)p_\phi^2 + f_2(\theta)p_\phi\, p_\theta + f_3(\theta)p_\theta^2 & g_1(\theta)p_\phi^2 + g_2(\theta)p_\phi\, p_\theta + g_3(\theta)p_\theta^2 \\ \\ \frac12 f_2(\theta)p_\phi^2 + 2f_3(\theta)p_\phi\, p_\theta + \gamma\Bigl(f_3(\theta) + h_3(\theta)\Bigr)^{3/2} p_\theta^2 & h_1(\theta)p_\phi^2 + h_2(\theta)p_\phi\, p_\theta + h_3(\theta)p_\theta^2 \end{pmatrix}. \tag{3.7}$$

Functions f, g, h, together with functions x, y from (2.8), are solutions of the 19 non-linear differential equations (2.7).

In both cases we can get a complete classification of the corresponding bi-Hamiltonian systems (2.10) . On the other hand, classification of separable integrable systems demands additional assumptions on the form of the separated relations. So, in next Sections we present matrices Π and Λ only for the following well-known separable systems on the sphere

- Case 1 - Lagrange top, Neumann system and systems separable in the elliptic coordinates;
- Case 2 - Goryachev system, Matveev-Dullin system, Kowalevsky top, Chaplygin system;
- Case 3 - Goryachev-Chaplygin top, Sokolov system, Kowalevsky-Goryachev-Chaplygin gyrostat;

which may be natively embedded into the proposed scheme as separable bi-integrable systems.

Remark 3 In fact we can not only include the so-called Gaffet system in the proposed scheme.

3.1 Case 2 - example of a bi-Hamiltonian system

Let us consider bi-Hamiltonian systems with the geodesic Hamiltonian (3.3) at $a_1 = a_2$. In this case we have to put $h_2(\theta) = 0$ in Π (3.6), and then equations (2.7) have some distinct solutions; among them we pick out solution defined by the following matrix

$$\Pi = \begin{pmatrix} \gamma\, p_\phi^2 & \gamma\left(1 - \dfrac{h_3'(\theta)\, F}{\alpha\sqrt{h_3(\theta}} + F^2\right) p_\phi\, p_\theta \\ \\ 0 & \gamma\left(1 + F^2\right) p_\phi^2 + h_3(\theta)\, p_\theta^2 \end{pmatrix}, \qquad F = \tan\left(\alpha \int \frac{d\theta}{\sqrt{h_3(\theta)}} + \beta\right)$$

If $a_1 = a_2 = 1$, then $h_3(\theta) = 1$ and we have

$$\Pi = \begin{pmatrix} p_\phi^2 & (1 + \tan^2\alpha\theta)\, p_\phi\, p_\theta \\ \\ 0 & (1 + \tan^2\alpha\theta)\, p_\phi^2 + p_\theta^2 \end{pmatrix}, \qquad \mathrm{y}_{12}(\theta) = \frac{2\alpha \mathrm{x}_{22}(\theta) - \cos\alpha\theta \sin\alpha\theta}{\alpha}. \tag{3.8}$$

The corresponding geodesic Hamiltonian (2.10) is equal to

$$\mathcal{T} = \frac12 \,\mathrm{tr}\, N = (2 + \tan^2\alpha\theta)p_\phi^2 + p_\theta^2\,.$$

At $\alpha = 1$ matrix Π (3.8) is consistent with the following potential matrix

$$\Lambda = \begin{pmatrix} f(\phi) & g(\phi,\theta) \\ \dfrac{f'(\phi)\sin\theta}{2\cos\theta} - g(\phi,\theta) & \dfrac{2\cos 2\phi(2\cos^2\theta + 1)g(\phi,\theta)}{\sin 2\phi \sin 2\theta} + \dfrac{f(\phi)}{\cos^2\theta} + a\tan^2\theta \end{pmatrix}, \qquad (3.9)$$

where

$$\begin{aligned} f(\phi) &= a\cot^2\phi + \frac{b}{\sin^2\phi} + \frac{c}{\sin^2\phi\cos^2\phi} + \frac{2d\cos^2\phi(2\cos^2\phi - 3)}{\sin^2\phi}, \\ g(\phi,\theta) &= \frac{2d\sin^3\theta\sin 2\phi}{\cos\theta}. \end{aligned}$$

So, auxiliary bi-Hamiltonian system associated with Π (3.8) and Λ (3.9) has the following Hamilton function (2.10)

$$\begin{aligned} \mathcal{H}_1 &= \mathcal{T} + \frac{a\big((x_1^2+x_2^2) - x_3^2(x_1^2-x_2^2)\big)}{x_1^2x_3^2} + \frac{(1+x_3^2)(x_1^2+x_2^2)\big(bx_2^2 + c(x_1^2+x_2^2)\big)}{x_1^2x_2^2x_3^2} \\ &- \frac{2d(x_1^2+x_2^2+2x_1^2x_3^2)\big((x_1^2+x_2^2) - x_3^2(x_1^2-x_2^2)\big)}{(x_1^2+x_2^2)x_1^2x_3^2}. \end{aligned}$$

Second integral of motion $\mathcal{H}_2$ is a fourth order polynomial in momenta. Description of the variables of separation and the corresponding separable bi-integrable system is an open problem

Similar bi-Hamiltonian systems with potential matrices Π (3.7) will be considered in the next Section.

4 The Stäckel systems on the sphere

Functionally independent integrals of motion $(H_1, \ldots, H_n)$ are Stäckel separable if the corresponding separation relations are given by the *affine* equations in H_j, that is,

$$\Phi_i(u_i, p_{u_i}, H_1, \ldots, H_n) = \sum_{j=1}^{n} S_{ij}(p_{u_i}, u_i)H_j - U_i(p_{u_i}, u_i) = 0, \qquad i = 1, \ldots, n, \qquad (4.1)$$

with S being an invertible matrix. Here functions S_{ij} and U_i depend only on one pair (p_{u_i}, u_i) of canonical variables of separation [21].

Definition 4 *If the separated relations (4.1) are coincided to each other, i.e.*

$$\Phi_i(u_i, p_{u_i}, H_1, \ldots, H_n) = \Phi(u_i, p_{u_i}, H_1, \ldots, H_n), \qquad i = 1, \ldots, n,$$

we have the so-called uniform Stäckel system.

Usually this single separated relation appears as a characteristic equation (spectral curve) of the Lax matrix [22].

In Stäckel case if we know a pair of compatible Poisson bivectors P and P', we can determine variables of separation (u, p_u) (2.11) and try to *calculate* integrals of motion $H_1, \ldots, H_2$ as solutions of the following equations [12]

$$P'dH_i = P\sum_{j=1}^{n} F_{ij}\, dH_j, \qquad \text{where} \qquad F = S^{-1}\,\mathrm{diag}\,(q_1, \ldots, q_n)\, S, \qquad i = 1, \ldots, n. \qquad (4.2)$$

Here S is some standard Stäckel matrix.

Remark 4 In fact at $n = 2$ we use only two Stäckel matrices

$$S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \text{and} \qquad S = \begin{pmatrix} 1 & 1 \\ q_1 & q_2 \end{pmatrix},$$

up to canonical transformations of variables of separation and time [22].

4.1 Case 1 - Lagrange top

If the spherical coordinates ϕ,θ (1.3) are variables of separation, one gets the simplest natural Poisson bivector P' (2.8) at

$$\Pi=0\,,\qquad\text{and}\qquad \Lambda=\left(\begin{array}{cc}\phi&0\\0&\theta\end{array}\right). \tag{4.3}$$

The auxiliary bi-Hamiltonian system is trivial

$$\mathcal{H}_1=\phi+\theta\,,\qquad \mathcal{H}_2=\frac12\,(\phi^2+\theta^2).$$

On the other hand, substituting variables of separation $u_1=\phi$ and $u_2=\theta$ into the non-uniform Stäckel relations

$$\Phi_1=\left(a+\frac{\cos^2\theta}{\sin^2\theta}\right)H_2-H_1+p_\theta^2+b\cos\theta=0\,,\qquad \Phi_2=p_\phi^2-H_2=0\,,$$

one gets integrals of motion for the Lagrange top in rotating frame

$$H_1=J_1^2+J_2^2+aJ_3^2+bx_3\,,\qquad H_2=J_3^2\,,\qquad a,b\in\mathbb{R}\,,$$

Remark 5 According to [28] bi-vector P' (2.8) associated with Λ (4.3) admits extension from cotangent bundle $T^*\mathbb{S}^2$ to the symplectic leaves of the Lie algebra $e^*(3)$ at $(x,J)\neq 0$.

4.2 Case 1 - Neumann system

Let us put $P'_T=0$ in (2.6) and consider some particular solution P' of the equations (2.1) defined by the following non-symmetric matrix

$$\Lambda=\left(\begin{array}{cc}a_1\cos^2\phi+a_2\sin^2\phi & \dfrac{(a_1-a_2)\sin2\phi}{2}\,\dfrac{\cos\theta}{\sin\theta}\\[2ex] \dfrac{(a_1-a_2)\sin2\phi}{2}\cos\theta\sin\theta & a_3\sin^2\theta+(a_1\sin^2\phi+a_2\cos^2\phi)\cos^2\theta\end{array}\right) \tag{4.4}$$

with three arbitrary parameters $a_k\in\mathbb{R}$. As above, the auxiliary bi-Hamiltonian system has trivial integrals of motion $\mathcal{H}_k$ (2.10), which are functions only on the configurational space $\mathbb{S}^2$.

On the other hand, coordinates of separation u_j (2.11) are the standard elliptic coordinates on the sphere

$$\frac{x_1^2}{\lambda-a_1}+\frac{x_2^2}{\lambda-a_2}+\frac{x_3^2}{\lambda-a_3}=\frac{(\lambda-u_1)(\lambda-u_2)}{(\lambda-a_1)(\lambda-a_2)(\lambda-a_3)}\,. \tag{4.5}$$

By substituting these variables in the Stäckel relations

$$u_iH_1-H_2-4(a_1-u_i)(a_2-u_i)(a_3-u_i)\,p_{u_i}^2+U_i(u_i)=0,\qquad i=1,2\,,$$

one gets bi-integrable systems with quadratic in momenta integrals of motion

$$H_1=J_1^2+J_2^2+J_3^2+V(x)\,,\qquad H_2=a_1J_1^2+a_2J_2^2+a_3J_3^2+W(x)\,,$$

which are in the bi-involution (2.9) with respect to both Poisson brackets. Here $V(x)$ and $W(x)$ are easy calculated from the Stäckel potentials $U_{1,2}$. For instance, if

$$U(u)=u(u-a_1-a_2-a_3)\,,$$

then one gets the Neumann system with the following integrals of motion

$$\begin{aligned}H_1&=J_1^2+J_2^2+J_3^2+a_1x_1+a_2x_2+a_3x_3\,,\\ H_2&=a_1J_1^2+a_2J_2^2+a_3J_3^2-a_2a_3x_1-a_1a_3x_2-a_1a_2x_3\,.\end{aligned} \tag{4.6}$$

Remark 6 Bi-vector P' (2.8) associated with Λ (4.4) also satisfies equations (2.1) at $(x,J)\neq 0$, but in this case we lose bi-involutivity (2.9) of integrals of motion $H_{1,2}$ (4.6) for the Clebsch system on the whole phase space $e^*(3)$. Of course, the corresponding elliptic coordinates on $e^*(3)$ remain variables of separation, but we can not get interesting natural Hamiltonians using these variables [26].

4.3 Case 2 - Goryachev and Dullin-Matveev systems

At $\gamma=0$ in (3.6) we have particular solution of the equations (2.7) defined by matrix

$$\Pi=\left(\begin{array}{cc} 0 & -\dfrac{\mathrm{i}}{2}\left(\dfrac{\partial}{\partial\theta}+\dfrac{2h(\theta)}{g(\theta)}\right)F \\ & \\ 0 & F \end{array}\right),\qquad F=\Bigl(g(\theta)p_\theta-\mathrm{i}h(\theta)p_\phi\Bigr)^2,\quad \mathrm{i}=\sqrt{-1}\,, \tag{4.7}$$

depending on arbitrary functions $g(\theta)$ and $h(\theta)$ and by functions

$$\mathrm{x}_{22}=-\frac{g(\theta)}{2h(\theta)}\,,\qquad \mathrm{y}_{12}=0,\qquad \mathrm{z}_k=0\,.$$

This matrix Π is consistent with the diagonal potential matrix

$$\Lambda=\alpha\,\exp\left(\mathrm{i}\phi-\int\frac{h(\theta)}{g(\theta)}\,d\theta\right)\left(\begin{array}{cc}1&0\\ & \\ 0&1\end{array}\right). \tag{4.8}$$

The corresponding bi-Hamiltonian systems (2.10) are non-physical and, therefore, we immediately proceed to consideration of the coordinates of separation $v_{1,2}=\sqrt{u_{1,2}}$ following to [33]. If we introduce polynomial

$$\mathcal{B}(\lambda)=(\lambda-v_1)(\lambda-v_2)=\lambda^2-\mathrm{i}\sqrt{F}\lambda+\Lambda_{1,1}\,.$$

instead of characteristic polynomial $B(\lambda)=(\lambda-v_1^2)(\lambda-v_2^2)$ (2.11) of recursion operator N, then it is easy to prove that

$$\{\mathcal{B}(\lambda),\mathcal{A}(\mu)\}=\frac{\lambda}{\mu-\lambda}\left(\frac{\mathcal{B}(\lambda)}{\lambda}-\frac{\mathcal{B}(\mu)}{\mu}\right),\qquad \{\mathcal{A}(\lambda),\mathcal{A}(\mu)\}=0\,,$$

where

$$\mathcal{A}(\lambda)=\int\frac{\mathrm{i}d\theta}{g(\theta)}-\frac{\mathrm{i}p_\phi}{\lambda}\,.$$

It entails that

$$p_{v_j}=\mathcal{A}(\lambda=v_j)\,,\qquad j=1,2,$$

are canonically conjugated to v_j momenta and that the corresponding Poisson brackets read as

$$\{v_i,p_{v_j}\}=\delta_{ij}\,,\qquad \{v_i,p_{v_j}\}'=\delta_{ij}v_i^2\,.$$

Now we have to substitute this family of variables of separation into separated relations and try to get natural Hamiltonians. For instance, let us take

$$g(\theta)=\sin\theta\,f(\theta)\,,\qquad h(\theta)=\cos\theta\,f(\theta)\,,$$

substitute

$$\lambda=v_j\qquad \mu=\frac{2\mathrm{i}}{3}\,v_j\,p_j,\qquad j=1,2,$$

into the single Stäckel equation

$$\Phi(\lambda,\mu)=\mu H_1+H_2-\mu^3-\lambda^3-b\lambda+\frac{\alpha^2}{\lambda}=0, \tag{4.9}$$

and solve a pair of the resulting equations with respect to $H_{1,2}$. If $a_1=a_2$ in the geodesic Hamiltonian (3.3) then

$$f(\theta)=\frac{\cos^{1/3}(\theta)}{\sin^2\theta}\,,$$

and we obtain integrals of motion for the Gorychev system on the sphere [14]

$$\begin{aligned} H_1 &= J_1^2+J_2^2+\frac{4}{3}J_3^2+\frac{2\mathrm{i}\alpha x_1}{x_3^{2/3}}-\frac{b}{x_3^{2/3}}\,, \\ H_2 &= \frac{2J_3}{3}\left(J_1^2+J_2^2+\frac{8}{9}J_3^2-\frac{b}{x_3^{2/3}}\right)-2\mathrm{i}\alpha x_3^{1/3}J_1+\frac{4\mathrm{i}\alpha}{3x_3^{2/3}}x_1J_3\,. \end{aligned} \tag{4.10}$$

In the similar manner if we take

$$g(\theta)=2(1-\cos\theta)\,f(\theta)\,,\qquad h(\theta)=\sin\theta\,f(\theta)\,,$$

and separation relations

$$\Phi(\lambda,\mu)=\mu H_1+H_2-\mu(\mu^2-\lambda^2)-b\lambda+\frac{\alpha^2}{\lambda}=0, \tag{4.11}$$

then we get Hamilton function for the Dullin-Matveev system [11]

$$H_1=J_1^2+J_2^2+\left(\frac{7}{12}+\frac{x_3}{2(x_3+|x|)}\right)J_3^2+\frac{2\mathrm{i}\alpha x_1}{(x_3+|x|)^{5/6}}-\frac{b}{(x_3+|x|)^{1/3}}\,,$$

at

$$f(\theta)=\frac{(\cos\theta+1)^{5/3}}{2\sin^3\theta}\,.$$

Other examples of natural bi-integrable systems associated with bivectors (4.7-4.8) may be found in [33]. If $\widetilde{P}'$ is the linear in momenta Poisson bivector from [33], then our natural Poisson bivector is equal to $P'=\widetilde{P}'P^{-1}\widetilde{P}'$.

Remark 7 We want to underline that the Goryachev and Dullin-Matveev systems are related with the non-hyperelliptic curve defined by separated relations (4.9) and (4.11)

4.4 Case 3 - spherical top and Chaplygin system

At $\gamma=0$ in (3.6) we have a particular solution of the equations (2.7) defined by matrix

$$\Pi=\begin{pmatrix} p_\phi^2 & \dfrac{\alpha-\sin^2\theta}{\cos^2\theta\sin^2\theta}\,p_\phi\,p_\theta \\ 0 & \dfrac{\alpha}{\sin^2\theta}\,p_\phi^2+\dfrac{\alpha-\sin^2\theta}{\cos^2\theta}\,p_\theta^2 \end{pmatrix},\qquad \alpha\in\mathbb{R}\,, \tag{4.12}$$

and functions

$$\mathrm{y}_{12}=\sin\theta\cos\theta+\frac{2\alpha\cos^2\theta}{\sin^2\theta-\alpha}\,\mathrm{x}_{22}\,.$$

In this case coordinates of separation $u_{1,2}$ (2.11) are equal to

$$u_1=p_\phi^2\,,\qquad u_2=\frac{\alpha p_\phi^2}{\sin^2\theta}-\frac{(\sin^2\theta-\alpha)p_\theta^2}{\cos^2\theta}\,,$$

so that conjugated momenta read as

$$p_{u_1}=\frac{\arctan\left(\dfrac{p_\theta\tan\theta}{p_\phi}\right)-\phi}{2p_\phi}\,,\qquad p_{u_2}=\frac{\sin\theta\cos\theta\arctan\left(\dfrac{\sin^2\theta\,p_\theta}{\sqrt{\alpha\cos^2\theta p_\phi^2-\sin^2\theta(\sin^2\theta-\alpha)p_\theta^2}}\right)}{2\sqrt{\alpha\cos^2\theta p_\phi^2-\sin^2\theta(\sin^2\theta-\alpha)p_\theta^2}}\,.$$

By substituting these variables of separation into the non-Stäckel relations

$$\Phi_1=\sqrt{u_1}-H_2=0\,,\qquad \Phi_2=\alpha H_1-u_2\Big(1-(\alpha-1)\tan^2(2p_{u_2}\sqrt{u_2})\Big)+\alpha f(\theta)=0\,,$$

where

$$\theta=\arccos\sqrt{\frac{u_2-\alpha H_2^2}{u_2}+\frac{\alpha(H_2^2-u_2)(1-\cos 4p_{u_2}\sqrt{u_2})}{2u_2}}$$

one gets generalized Lagrange top with integrals of motion

$$H_1=J_1^2+J_2^2+J_3^2+f(x_3)\,,\qquad H_2=J_3\,. \tag{4.13}$$

Other separated relations

$$\begin{aligned}\Phi_1(u_1,p_{u_1}) &= \frac{2}{\sqrt{u_1}\,\sin\left(4p_{u_1}\sqrt{u_1}\right)}\,H_2-H_1+u_1=0\,,\\ \Phi_2(u_1,p_{u_1}) &= \alpha H_1-u_2\Big(1-(\alpha-1)\tan^2(2p_{u_2}\sqrt{u_2})\Big)=0\,.\end{aligned} \tag{4.14}$$

give rise to integrals of motion for the spherical top

$$H_1=T=J_1^2+J_2^2+J_3^2\,,\qquad H_2=J_1J_2J_3\,. \tag{4.15}$$

There are only two potential matrices compatible with Π (4.12)

$$\Lambda^{(1)}=\begin{pmatrix} f(\phi) & 0\\[2mm] \dfrac{f'(\phi)(\sin^2\theta-\alpha)}{2\sin\theta\cos\theta} & \dfrac{\alpha f(\phi)}{\sin^2\theta}\end{pmatrix}$$

and

$$\Lambda^{(2)}=\begin{pmatrix} a\sin 2\phi+b\cos 2\phi & -\dfrac{\cos\theta}{\alpha\sin\theta}(\alpha-\sin^2\theta)(a\cos 2\phi-b\sin 2\phi)\\[2mm] -\dfrac{\sin\theta}{\alpha\cos\theta}(\alpha-\sin^2\theta)(a\cos 2\phi-b\sin 2\phi) & -\dfrac{(\alpha-2\sin^2\theta)(a\sin 2\phi+b\cos 2\phi)}{\alpha}\end{pmatrix}.$$

In the first case the auxiliary bi-Hamiltonian system with the Hamilton function (2.10)

$$\mathcal{H}_1^{(1)}=\left(1+\frac{\alpha-1}{x_3^2}\right)\left(J_1^2+J_2^2\right)+2J_3^2+f\left(\frac{x_1}{x_2}\right)\left(1+\frac{\alpha}{x_1^2+x_2^2}\right)\,, \tag{4.16}$$

is a deformation of the geodesic Hamiltonian for the Kowalevski top at $\alpha=1$ and $f=0$. By substituting the corresponding coordinates of separation (2.11)

$$\hat{u}_1=u_1+f(\phi)\,,\quad \hat{p}_{u_1}=p_{u_1}-\frac{1}{2}\int^{\phi}\frac{dx}{p_\phi^2+f(\phi)-f(x)}\,,\qquad \hat{u}_2=u_2+\frac{\alpha f(\phi)}{\sin^2\theta}\,,\quad \hat{p}_{u_2}=p_{u_2}\,,$$

into $\Phi_1 = \hat{u}_1 - \widehat{H}_2 = 0$ and the second separated relation Φ_2 in (4.14), one gets a generalization of the spherical top defined by the following integrals of motion

$$\widehat{H}_1 = J_1^2 + J_2^2 + J_3^2 + \frac{f\left(\dfrac{x_1}{x_2}\right)}{x_1^2 + x_2^2}, \qquad \widehat{H}_2 = J_3^2 + \frac{f\left(\dfrac{x_1}{x_2}\right)}{x_1^2 + x_2^2 + x_3^2}.$$

In the second case matrices Π (4.12) and $\Lambda^{(2)}$ give rise to the auxiliary bi-Hamiltonian system with the Hamilton function

$$\mathcal{H}_1^{(2)} = \left(1 + \frac{\alpha - 1}{x_3^2}\right)\left(J_1^2 + J_2^2\right) + 2J_3^2 + 4\alpha^{-1} a x_1 x_2 - 2\alpha^{-1} b (x_1^2 - x_2^2)\,. \tag{4.17}$$

It is a new deformation of the well-known Chaplygin system [5]. In the next Section we consider another natural Poisson bivector for this system and corresponding non-Stäckel separated relations.

Remark 8 According to [23], there is a non-canonical map, which relates integrals of motion (4.15) with integrals of motion for the Gaffet system [13]

$$H_1 = J_1^2 + J_2^2 + J_3^2 - \frac{1}{(x_1 x_2 x_3)^{2/3}}, \qquad H_2 = J_1 J_2 J_3 + \frac{x_2 x_3 J_1 + x_1 x_3 J_2 + x_1 x_2 J_3}{(x_1 x_2 x_3)^{2/3}}.$$

It is easy to prove that the natural Poisson bivector P' (2.6) with the second order homogeneous geodesic part Π (3.1) is missed for this system.

4.5 Case 3 - Goryachev-Chaplygin top and Sokolov system

At $\gamma = 0$ in (3.7) Equations (2.7) have a particular solution P'_T (2.8) defined by the following symmetric matrix

$$\Pi = \begin{pmatrix} p_\theta^2 + p_\phi^2(4 + 3\cot^2\alpha\theta) & 2p_\phi\, p_\theta \\ 2p_\phi\, p_\theta & p_\theta^2 - p_\phi^2 \cot^2\alpha\theta \end{pmatrix}, \qquad \alpha \in \mathbb{R}, \tag{4.18}$$

and by the functions

$$\mathrm{x}_{22} = \mathrm{y}_{12} = -\frac{\cos\alpha\theta\, \sin\alpha\theta}{\alpha}, \qquad \mathrm{z}_k = \frac{p_k}{3}.$$

There is only one potential matrix compatible with Π (4.18)

$$\Lambda = \frac{a}{\cos^2\alpha\theta}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The corresponding auxiliary bi-Hamiltonian system is defined by the Hamilton function (2.10)

$$\frac{1}{2}\mathcal{H}_1 = (2 + \cot^2\alpha\theta) p_\phi^2 + p_\theta^2 + \frac{a}{\cos^2\alpha\theta}.$$

If $\alpha = 1$, we have a deformation of the geodesic Hamiltonian for the Kowalevski top [17]

$$\frac{1}{2}\,\mathcal{H}_1 = J_1^2 + J_2^2 + 2J_3^2 + \frac{a}{x_3^2}. \tag{4.19}$$

This auxiliary bi-Hamiltonian system gives rise to the variables of separation $u_{1,2}$ (2.11)

$$u_{1,2} = \left(p_\phi \pm \sqrt{\frac{p_\phi^2}{\sin^2\theta} + p_\theta^2 + \frac{a}{\cos^2\theta}}\right)^2 = \left(J_3 \pm \sqrt{J_1^2 + J_2^2 + J_3^2 + \frac{a}{x_3^2}}\right)^2, \qquad \alpha = 1.$$

At $a = 0$ these coordinates were found in [6]. By substituting the the generalized Chaplygin variables

$$v_{1,2} = \sqrt{u_{1,2}}\,, \qquad p_{v_{1,2}} = -\frac{1}{2\mathrm{i}}\ln\Big(v_{1,2}(\mathrm{i}x_1 - x_2) - (\mathrm{i}J_1 - J_2)x_3\Big) + \frac{\ln(v_{1,2}^2 - a)}{4\mathrm{i}}, \tag{4.20}$$

into the uniform Stäckel relations

$$\Phi_{1,2}(v,p_v)=H_1v+H_2+b\sqrt{v^2-a}\sin 2p_v-v^3-cv^2=0\,,\qquad v=v_{1,2},\quad p_v=p_{v_{1,2}}\,,$$

one gets integrals of motion for the generalized Goryachev-Chaplygin gyrostat [8, 6]

$$\begin{aligned}H_1&=J_1^2+J_2^2+4J_3^2+2cJ_3+bx_1+\frac{a}{x_3^2}\\ H_2&=(2J_3+c)\left(J_1^2+J_2^2+\frac{a}{x_3^2}\right)-bx_3J_1\,.\end{aligned}$$

By substituting the same variables (4.20) into the following non-uniform Stäckel relations

$$\Phi_{1,2}(v_{1,2},p_{v_{1,2}})=\widehat{H}_1\pm\widehat{H}_2+b\,\sqrt{v_{1,2}^2-a}\,\sin 2p_{v_{1,2}}-v_{1,2}^2-cv_{1,2}=0\,,\qquad(4.21)$$

we obtain the generalized Sokolov system [20] defined by integrals of motion

$$\begin{aligned}\widehat{H}_1&=J_1^2+J_2^2+2J_3^2+cJ_3+b(J_3x_1-x_3J_1)+\frac{a}{x_3^2}\,,\\ \widehat{H}_2&=\Bigl(2J_3+c+bx_1\Bigr)\sqrt{J_1^2+J_2^2+J_3^2+\frac{a}{x_3^2}}\,,\end{aligned}$$

up to the canonical transformation discussed in [16].

Remark 9 It is easy to see that $\widehat{H}_2$ is non-polynomial function and if we substitute polynomial integral of motion $\widehat{H}_2^2$ in (4.21) one gets the non-Stäckel separated relations.

4.6 Case 3 - Kowalevski-Goryachev-Chaplygin gyrostat

The geodesic matrix Π (4.18) for the Goryachev-Chaplygin top may be deformed

$$\widehat{\Pi}=\Pi+\beta\begin{pmatrix}0&\dfrac{\cos\alpha\theta}{\sin^3\alpha\theta}\,p_\phi^2\\[2ex] 0&0\end{pmatrix},\qquad(4.22)$$

if

$$\mathrm{y}_{11}(\theta)=\mathrm{x}_{21}(\theta)-\frac{\beta}{2\alpha}\,,\qquad y_{12}(\theta)=-\frac{\cos^2\alpha\theta}{\sin^2\alpha\theta}\,\mathrm{x}_{22}(\theta)-\frac{\cos\alpha\theta}{\alpha\sin\alpha\theta}\,,\qquad \mathrm{z}_k=\frac{p_k}{3}.$$

In generic case matrix $\widehat{\Pi}$ (4.22) is compatible with the potential matrix

$$\widehat{\Lambda}^{(1)}=\frac{a\,\mathrm{e}^{-\frac{4\alpha\phi}{\beta}}}{\sin^2\alpha\theta}\begin{pmatrix}\cos^2\alpha\theta-4&-\frac{\beta\cos\alpha\theta}{\sin\alpha\theta}\\[1ex] \frac{4\cos\alpha\theta\,\sin\alpha\theta}{\beta}&\cos^2\alpha\theta\end{pmatrix}+\frac{b\sin^2\alpha\theta}{\cos^2\alpha\theta}\begin{pmatrix}1&-\frac{\beta}{\sin\alpha\theta\,\cos\alpha\theta}\\[1ex] 0&1\end{pmatrix}\qquad(4.23)$$

The corresponding auxiliary bi-Hamiltonian system is defined by the Hamiltonian

$$\frac12\mathcal{H}_1=(2+\cot^2\alpha\theta)p_\phi^2+p_\theta^2+\frac{a(\cos^2\alpha\theta-2)\mathrm{e}^{-\frac{4\alpha\phi}{\beta}}}{\sin^2\alpha\theta}+\frac{b\sin^2\alpha\theta}{\cos^2\alpha\theta}\,.$$

So, at $\alpha=1$ we have another deformation of the geodesic Hamiltonian for the Kowalevski top [17]

$$\frac12\mathcal{H}_1=J_1^2+J_2^2+2J_3^2-\frac{a(x_1^2+x_2^2+1)}{x_1^2+x_2^2}\,\mathrm{e}^{-\frac{4\arctan(x_1/x_2)}{\beta}}+\frac{b(x_1^2+x_2^2)}{x_3^2}\,.$$

At $\beta=\pm 2\,\mathrm{i}$ there is one more particular potential matrix compatible with $\widehat{\Pi}$ (4.22)

$$\widehat{\Lambda}^{(2)}=\gamma\,\mathrm{e}^{\pm\mathrm{i}\alpha\phi}\begin{pmatrix}\pm\,\mathrm{i}\,\sin\alpha\theta & \cos\alpha\theta\\ 0 & 0\end{pmatrix}. \tag{4.24}$$

In this particular case we can substitute the coordinates of separation $u_{1,2}$ (2.11) and the corresponding momenta $p_{u_{1,2}}$ into the uniform Stäckel relation

$$\Phi(u,p_u)=u^6+H_1u^4+H_2u^2+a+\sqrt{b(u)}\sin 2p_u=0\,, \tag{4.25}$$

and obtain integrals of motion for the Kowalevski-Goryachev-Chaplygin gyrostat with the following Hamilton function

$$H_1=J_1^2+J_2^2+2J_3^2+2c_1J_3+c_2x_1+c_3(x_1^2-x_2^2)+\frac{c_4}{x_3^2}\,, \tag{4.26}$$

see [5, 10, 17, 34]. Here $b(u)$ (4.25) is a special polynomial of eight order in u with coefficients depending on a and c_k, see details in [24].

Remark 10 In this case in order to get the conjugated momenta $p_{u_{1,2}}$ and the separated relation we used the Lax matrices and the reflection equation algebra, that drastically simplified all the calculations.

In generic case at $\beta\neq\pm 2\,\mathrm{i}$ description of the variables of separation and the corresponding bi-integrable system is an open problem.

5 The non-Stäckel systems on the sphere

There are many non-Stäckel bi-integrable systems for which separated relations (2.12) have more complicated forms. For instance, all the bi-Hamiltonian systems (2.10) are non-Stäckel systems for which separated relations are the n-th order polynomials in integrals of motion $\mathcal{H}_k$.

Namely, if $n=2$, the separated relations for bi-Hamiltonian systems (2.10) are the quadratic polynomials in $\mathcal{H}_1$

$$\Phi_j(u_j,p_{u_j})=u_j^2+(\mathcal{H}_1-u_j)^2-\mathcal{H}_2=0\,,\qquad j=1,2.$$

On the other hand, instead of two possibly distinct Stäckel relations $\Phi_{1,2}=0$ we can always consider two copies of single non-Stäckel equation

$$\widehat{\Phi}(u,p_u)=\Phi_1(u,p_u)\cdot\Phi_2(u,p_u)\,,\qquad u=u_{1,2},\quad p_u=p_{u_{1,2}}\,,$$

and its deformations

$$\widetilde{\Phi}_{1,2}(u,p_u)=\Phi_1(u,p_u)\cdot\Phi_2(u,p_u)+W_{1,2}(u,p_u)\,. \tag{5.1}$$

As usual, this construction will be justified only if we are capable to obtain natural Hamilton functions.

5.1 Case 2 - Kowalevski top and Chaplygin system

Let us consider a geodesic matrix P'_T (2.8) determined by the matrix Π (3.6)

$$\Pi=\frac{1}{\sin^\alpha\theta\,\cos^2\theta}\begin{pmatrix}0 & \dfrac{2\,p_\phi\,p_\theta}{\alpha}\\ 0 & \cos^2\theta\,p_\phi^2+\sin^2\theta\,p_\theta^2\end{pmatrix},\qquad \alpha\in\mathbb{R}\,, \tag{5.2}$$

at $\gamma=0$ and by functions

$$\mathrm{y}_{12}(\theta)=\cos\theta\Bigl(\sin\theta+\alpha\mathrm{x}_{22}(\theta)\cos\theta\Bigr)\,,\qquad \mathrm{z}_{1,2}=0\,.$$

There is only one potential matrix consistent with Π (5.2)

$$\Lambda=\begin{pmatrix} a\cos\alpha\phi-b\sin\alpha\phi & \Big(a\sin\alpha\phi-b\cos\alpha\phi\Big)\cot\theta\\ \\ \Big(a\sin\alpha\phi-b\cos\alpha\phi\Big)\tan\theta & -a\cos\alpha\phi+b\sin\alpha\phi\end{pmatrix}\,,\qquad a,b\in\mathbb{R}\,. \tag{5.3}$$

The corresponding coordinates of separation $u_{1,2}$ (2.11) are the roots of the polynomial

$$\begin{aligned} B(\lambda) &= \lambda^2-\frac{p_\theta^2\sin^2\theta+p_\phi^2\cos^2\theta}{\sin^\alpha\theta\cos^2\theta}\,\lambda-\frac{(a\cos\alpha\phi-b\sin\alpha\phi)(p_\theta^2\sin^2\theta+p_\phi^2\cos^2\theta)}{\sin^\alpha\theta\cos^2\theta}\\ &- \frac{2\sin\theta(a\sin\alpha\phi+b\cos\alpha\phi)p_\phi\,p_\theta}{\sin^\alpha\theta\cos^2\theta}-a^2-b^2\,. \end{aligned}\tag{5.4}$$

Following to [29, 30] we can introduce auxiliary polynomial

$$A(\lambda)=\frac{\sin\theta p_\theta}{\alpha\cos\theta}\,\lambda+\frac{a\sin\alpha\phi+b\cos\alpha\phi}{\alpha}\,p_\phi-\frac{\sin\theta(a\cos\alpha\phi-b\sin\alpha\phi)}{\alpha\cos\theta}\,p_\theta\,,$$

such as

$$\{B(\lambda),A(\mu)\}=\frac{1}{\lambda-\mu}\Big((\mu^2-a^2-b^2)B(\lambda)-(\lambda^2-a^2-b^2)B(\mu)\Big)\,,\qquad \{A(\lambda),A(\mu)\}=0\,.$$

It entails that

$$p_{u_j}=-\frac{1}{u_j^2-a^2-b^2}\,A(\lambda=u_j)\,,\qquad j=1,2,$$

are the canonically conjugated momenta satisfying to the Poisson brackets (2.13). At $\alpha=2$ these variables have been considered by Chaplygin [5].

By substituting these variables of separation into a pair of the Stäckel relations

$$\Phi_1=(u_1^2-a^2-b^2)p_{u_1}^2+H_1-H_2=0\,,\qquad \Phi_2=(u_2^2-a^2-b^2)p_{u_2}^2+H_1+H_2=0\,,$$

one gets separable bi-integrable system with the Hamilton function

$$2\alpha^2\,H_1=p_\phi^2-\tan^2\theta\,p_\theta^2+2(a\cos\alpha\phi+b\cos\alpha\phi)\,\cos^\alpha\theta\,,\qquad \alpha\in\mathbb{R}\,. \tag{5.5}$$

According to [29, 30] if we want to obtain more interesting Hamiltonians, we have to consider non-Stäckel separated relations (5.1). Namely, at $\alpha=1$ using non-Stäckel separated relations

$$\Phi(u,p_u)=\Big((u^2-a^2-b^2)p_u^2+H_1-H_2\Big)\Big((u^2-a^2-b^2)p_u+H_1+H_2\Big)+cu^2+du=0 \tag{5.6}$$

one gets Hamilton function of the generalized Kowalevski top [17]

$$H^{kow}=2H_1=\left(1-\frac{c+1}{x_3^2}\right)(J_1^2+J_2^2)+2J_3^2+2ax_2+2bx_1-\frac{d}{\sqrt{x_1^2+x_2^2}}\,. \tag{5.7}$$

At $\alpha=2$ we can use another non-Stäckel separated relations

$$\Phi(u,p_u)=\Big((u^2-a^2-b^2)p_u^2+cu+H_1-H_2\Big)\Big((u^2-a^2-b^2)p_u^2+cu+H_1+H_2)+du=0 \tag{5.8}$$

in order to get Hamiltonian of the generalized Chaplygin system [5, 10]

$$H^{ch}=8H_1=\left(1-\frac{4c+1}{x_3^2}\right)(J_1^2+J_2^2)+2J_3^2-2a(x_1^2-x_2^2)-2bx_1x_2-\frac{2d}{1+4c-x_3^2}\,. \tag{5.9}$$

At $c=-\alpha^{-2}$ we have geodesic Hamiltonian $T=J_1^2+J_2^2+2J_3^2$ with the constant inertia tensor.

Remark 11 By substituting these variables of separation into another separation relations we can obtain various generalizations of bi-integrable Hamiltonians (5.5,5.7,5.9).

5.2 Additive deformations of the natural Poisson bivectors

Let us consider trivial canonical transformation

$$p_\theta \to p_\theta + f(\theta)\,, \tag{5.10}$$

which preserved canonical Poisson bivector P (2.3). This mapping shifts the natural Poisson bivector P' (2.6) associated with matrices Π (5.2) and Λ (5.3) by the rule

$$\widehat{P}' = P' + g(\theta)\begin{pmatrix} 0 & 0 & 0 & \left(\dfrac{\alpha\cos^2\theta - 1}{\alpha\sin^2\theta} + \dfrac{\cot\theta}{\alpha}\ln' g\right)p_\phi \\ * & 0 & 0 & p_\theta + \dfrac{\cos^2\theta\,\sin^{\alpha-2}\theta}{4}\,g(\theta) \\ * & * & 0 & \dfrac{\sin^{\alpha-2}\theta(a\sin\alpha\phi + b\cos\alpha\phi)}{2}\Bigl(\sin\theta\cos\theta\ln' g - 1\Bigr) \\ * & * & * & 0 \end{pmatrix},$$

where

$$g(\theta) = -\frac{2f(\theta)\sin^{2-\alpha}\theta}{\cos^2\theta} \qquad \text{and} \qquad \ln' g = \frac{1}{g(\theta)}\frac{dg(\theta)}{d\theta}\,.$$

The Poisson bivector $\widehat{P}'$ gives rise to the "shifted" variables of separation

$$\hat{u} = u|_{p_\theta \to p_\theta + f(\theta)}\,, \qquad \hat{p_u} = p_u|_{p_\theta \to p_\theta + f(\theta)}\,. \tag{5.11}$$

If we substitute these variables of separation into the old separated relations (5.6) and (5.8) one gets non-natural Hamiltonians, which are related to the old Hamiltonians (5.7) and (5.9) by canonical transformation (5.10).

In order to get new natural Hamiltonians we have to appropriately modify the separated relations. For instance, let us take

$$f(\theta) = \frac{\sqrt{\beta}\,\tan^{\alpha-1}\theta}{\cos^\alpha\theta}\,.$$

At $\alpha = 1$ by substituting variables of separation (5.11) into the new separated relations

$$\widehat{\Phi} = \Phi - \beta H_1 + \beta^2 + \sqrt{\beta}(\hat{u}^2 - a^2 - b^2)\hat{p}_u, \qquad \hat{u} = \hat{u}_{1,2}\,, \quad \hat{p}_u = \hat{p}_{u_{1,2}}\,,$$

where Φ is given by (5.6), one gets generalization of the Hamilton function (5.7)

$$\widehat{H}^{kow} = \left(1 - \frac{c+1}{x_3^2}\right)(J_1^2 + J_2^2) + 2J_3^2 + 2ax_2 + 2bx_1 - \frac{d}{\sqrt{x_1^2 + x_2^2}} - \frac{\beta}{x_3}\,,$$

At $\alpha = 2$ the "shifted" separated relations

$$\widehat{\Phi} = \Phi + \sqrt{\beta}(\hat{u}^2 - a^2 - b^2)\hat{p}_u\,, \qquad \hat{u} = \hat{u}_{1,2}\,, \quad \hat{p}_u = \hat{p}_{u_{1,2}}\,,$$

where Φ is given by (5.8), yield similar generalization of the Hamiltonian (5.9)

$$\widehat{H}^{ch} = \left(1 - \frac{4c+1}{x_3^2}\right)(J_1^2 + J_2^2) + 2J_3^2 - 2a(x_1^2 - x_2^2) - 2bx_1x_2 - \frac{2d}{1 + 4c - x_3^2} + \beta\left(\frac{1}{x_3^4} - \frac{1}{x_3^6}\right).$$

These Hamiltonians at $c = -\alpha^{-2}$ and another Hamiltonians associated with various functions $f(\theta)$ may be found in [34].

Construction of the non-Stäckel separated relations, which gives rise to natural Hamiltonians associated with more complicated functions $f(\theta)$, will be discussed in the forthcoming publication.

Remark 12 Other natural Poisson bivectors studied in the previous Sections may be shifted on the similar linear in momenta terms. As above, it allows us to get various generalizations of the considered bi-integrable systems.

6 Conclusion

We proved that almost all known integrable systems on the two-dimensional unit sphere $\mathbb{S}$ may be studied in the framework of a single theory of natural Poisson bivectors. This collection of examples may be helpful for investigations of the invariant geometric properties of geodesic Π and potential Λ matrices, which allows us to obviate a necessity of the direct solutions of the equations (2.1) and (2.7). Moreover, it can possibly be a suitable step towards the construction of Poisson bivectors on more generic symplectic and Poisson manifolds .

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