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Integrable Origins of Higher Order Painlevé Equations

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ABSTRACT

Higher order Painlevé equations invariant under extended affine Weyl groups $A_n^{(1)}$ are obtained through self-similarity limit of a class of pseudo-differential Lax hierarchies with symmetry inherited from the underlying generalized Volterra lattice structure.

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1 Introduction

This paper investigates a self-similarity limit of a special class of pseudo-differential Lax hierarchies of the constrained KP hierarchy with symmetry structure defined by Bäcklund transformations induced by a discrete structure of Volterra type lattice. The underlying integrable hierarchy is realized in terms of $2M$ Lax coefficients e_i, c_i , $i = 1, \dots, M$ forming M “Darboux-Poisson” canonical pairs with respect to the second Gelfand-Dickey bracket of the underlying constrained Kadomtsev-Petviashvili (KP) hierarchy.

It is shown that in the self-similarity limit the second t_2 -flow equations of that hierarchy transform to the higher order Painlevé equations :

$$f'_i = f_i (f_{i+1} - f_{i+2} + f_{i+3} - f_{i+4} + \dots - f_{i-1}) + \alpha_i, \quad i = 0, 1, \dots, 2M \quad (1.1)$$

under a change of variables from e_i, c_i , $i = 1, \dots, M$ to f_i , $i = 1, \dots, 2M$, which is described in subsection 4.1. Equation (1.1) introduced $f_0 = -\sum_{i=1}^{2M} f_i - 2x$ and constants α_i satisfying $\sum_{i=0}^{2M} \alpha_i = -2$. The system satisfies periodicity conditions $f_{2M+i} = f_{i-1}$, $\alpha_{2M+i} = \alpha_{i-1}$, $i = 0, 1, 2, \dots, 2M$. These equations are invariant under Bäcklund transformations forming the extended affine Weyl group $A_{2M}^{(1)}$. The extended affine Weyl group $A_n^{(1)}$ is generated by $n + 1$ transformations s_0, s_1, \dots, s_n in addition to cyclic permutation π which together satisfy relations

$$\begin{aligned} s_i s_j s_i &= s_j s_i s_j \quad (j = i \pm 1), & s_i s_j &= s_j s_i \quad (j \neq i \pm 1) \\ \pi s_i &= s_{i+1} \pi, & \pi^{n+1} &= 1, & s_i^2 &= 1. \end{aligned}$$

The symmetric Painlevé equations with their extended affine Weyl symmetry group $A_n^{(1)}$ first appeared in Adler’s paper [1] in the setting of periodic dressing chains and later were discussed in great details by Noumi and Yamada [8, 9] (see also [10]).

Imposing second-class constraints on the second KP Poisson bracket structure via Dirac scheme reduces a number of $2M$ Lax coefficients to $2M - 1$ (or in general $2M - k$) coefficients and via self-similarity reduction reproduces Painlevé equations with the extended affine Weyl symmetry $A_{2M-1}^{(1)}$. Here we just present results of the special case of the extended affine Weyl symmetry $A_3^{(1)}$ and the corresponding Painlevé V equation.

In section 2 we present the underlying integrable Lax hierarchy focusing on the second flow equations and Bäcklund transformation keeping the Lax equations invariant. In Section 3 the self-similarity limit is taken and Hamiltonians governing the t_2 flow equations in this limit are derived. Next, in Section 4 the Hamiltonians found in Section 3 are shown to reproduce Hamiltonian structure of the higher Painlevé equations invariant under the extended affine Weyl symmetry $A_{2M}^{(1)}$ when expressed in terms of canonical variables. This is illustrated for special cases of $M = 1, 2, 3$ for which the generators of the extended affine Weyl symmetry group are derived from the Bäcklund transformation of Section 2. The Dirac reduction scheme when applied on the original integrable KP hierarchy allows reduction of the model with $A_{2M}^{(1)}$ symmetry down to a model characterized by $A_{2M-1}^{(1)}$ symmetry. This is illustrated in Section 5 for $M = 2$ when the reduced model is nothing but the Painlevé V system. Concluding comments are given in Section 6.

2 The Integrable Hierarchy, its Second Flow and Generalized Volterra Symmetry Structure

2.1 A “half-integer” Lattice

It is well-known that symmetry of many continuum KP-type hierarchies are governed by discrete lattice-like structures. A standard example is provided by the AKNS hierarchy and the Toda lattice structure of its Bäcklund transformations leading to Hirota type equations for the Toda chain of tau-functions [2]. There also exists the so-called two-boson formulation of the AKNS hierarchy which is invariant under symmetry transformations on a “half-integer” lattice which generalizes Toda lattice [2]. We now present a general “half-integer” lattice (or the generalized Volterra lattice) following closely reference [3]. The foundation of this formalism rests on two spectral equations:

$$\lambda^{1/2} \tilde{\Psi}_{n+\frac{1}{2}} = \Psi_{n+1} + \mathcal{A}_{n+1}^{(0)} \Psi_n + \sum_{p=1}^M \mathcal{A}_{n-p+1}^{(p)} \Psi_{n-p} \quad (2.1)$$

$$\lambda^{1/2} \Psi_n = \tilde{\Psi}_{n+\frac{1}{2}} + \mathcal{B}_n^{(0)} \tilde{\Psi}_{n-\frac{1}{2}} \quad (2.2)$$

and “time” evolution equations:

$$\tilde{\Psi}_{n+\frac{1}{2}} = (\partial - \mathcal{B}_n^{(0)} - \mathcal{A}_n^{(0)}) \tilde{\Psi}_{n-\frac{1}{2}} \quad ; \quad \Psi_{n+1} = (\partial - \mathcal{B}_n^{(0)} - \mathcal{A}_{n+1}^{(0)}) \Psi_n \quad (2.3)$$

which both involve objects labeled by integers and half-integers. After removing the term $\sum_{p=1}^M \mathcal{A}_{n-p+1}^{(p)} \Psi_{n-p}$ from equation (2.1) the above system yields the Volterra chain equations. For that reason we will refer to equations (2.1)–(2.3) as a generalized Volterra system. As shown in [3], upon eliminating the half-integer modes, the generalized Volterra system (2.1)–(2.3) reduces to the Toda lattice equations. From (2.1)–(2.3) we find:

$$\begin{aligned} \lambda^{1/2} \tilde{\Psi}_{n+\frac{1}{2}} = & \left(\partial - \mathcal{B}_n^{(0)} + \sum_{p=1}^M \mathcal{A}_{n-p+1}^{(p)} (\partial - \mathcal{B}_{n-p}^{(0)} - \mathcal{A}_{n-p+1}^{(0)})^{-1} \dots \right. \\ & \left. \dots (\partial - \mathcal{B}_{n-1}^{(0)} - \mathcal{A}_n^{(0)})^{-1} \right) \Psi_n \end{aligned} \quad (2.4)$$

and

$$\lambda^{1/2} \Psi_n = (\partial - \mathcal{A}_n^{(0)}) \tilde{\Psi}_{n-\frac{1}{2}} \quad (2.5)$$

Eliminating half-integer modes from the last two relations yields a spectral equation of a form

$$\lambda \Psi_n = L_n^{(M+1)} \Psi_n \quad (2.6)$$

with Lax operator $L_n^{(M+1)}$ given by recurrence relation:

$$L_n^{(M+1)} = e^{\int \mathcal{B}_{n-1}^{(0)}} \left(\partial - \mathcal{A}_n^{(0)} + \mathcal{B}_{n-1}^{(0)} \right) L_n^{(M)} (\partial - \mathcal{A}_n^{(0)})^{-1} e^{-\int \mathcal{B}_{n-1}^{(0)}} \quad (2.7)$$

where

$$L_n^{(M)} = \partial + \sum_{p=1}^M \mathcal{A}_{n-p}^{(p)} (\partial + \mathcal{B}_{n-1}^{(0)} - \mathcal{B}_{n-p-1}^{(0)} - \mathcal{A}_{n-p}^{(0)})^{-1} \cdots (\partial + \mathcal{B}_{n-1}^{(0)} - \mathcal{B}_{n-2}^{(0)} - \mathcal{A}_{n-1}^{(0)})^{-1} \quad (2.8)$$

Using equation (2.5) it is easy to shift the spectral equation (2.6) to the half-integer lattice:

$$\lambda \tilde{\Psi}_{n-\frac{1}{2}} = \tilde{L}_n^{(M+1)} \tilde{\Psi}_{n-\frac{1}{2}}, \quad \tilde{L}_n^{(M+1)} = (\partial - \mathcal{A}_n^{(0)})^{-1} L_n^{(M+1)} (\partial - \mathcal{A}_n^{(0)}) \quad (2.9)$$

The similarity transformation responsible for transformation from integer to half-integer lattice will be shown below to play a central role as a Bäcklund transformation of the higher order Painlevé equations.

2.2 Basic facts about $2M$ -bose constrained KP hierarchy

The recurrence relation (2.7) is realized by the $2M$ -bose constrained KP hierarchy with Lax operators L_M , $M = 1, 2, \dots$:

$$L_M = (\partial - e_M) \prod_{k=M-1}^1 \left(\partial - e_k - \sum_{l=k+1}^M c_l \right) \left(\partial - \sum_{l=1}^M c_l \right) \times \prod_{k=1}^M \left(\partial - e_k - \sum_{l=k}^M c_l \right)^{-1} \quad (2.10)$$

given here in terms of the ‘‘Darboux-Poisson’’ canonical pairs $(c_k, e_k)_{k=1}^M$. Recall that the KP hierarchy is endowed with bi-Hamiltonian Poisson bracket structures resulting from the two compatible Hamiltonian structures on the algebra of pseudo-differential operators. Remarkably, for the above Lax hierarchy the second bracket of hierarchy is realized in terms of $(c_k, e_k)_{k=1}^M$ as a Heisenberg Poisson bracket algebra:

$$\{e_i(x), c_j(y)\}_2 = -\delta_{ij} \delta_x(x-y), \quad i, j = 1, 2, \dots, M \quad (2.11)$$

The Lax operator (2.10) realizes the recursive relation (2.7) rewritten in this context as follows:

$$L_M = e^{\int c_M} (\partial + c_M - e_M) L_{M-1} (\partial - e_M)^{-1} e^{-\int c_M}, \quad L_0 \equiv \partial \quad (2.12)$$

for $M = 1, 2, \dots$. The corresponding second flow equations can be obtained from the second bracket structure as follows

$$\frac{\partial f}{\partial t_2} = \{f, H_2\}_2, \quad (2.13)$$

where the Hamiltonian H_2 is an integral of the coefficient $u_1(M)$ appearing in front of ∂^{-2} in the Lax operator (2.12) when cast in a conventional KP form:

$$L_M = \partial + u_0(M) \partial^{-1} + u_1(M) \partial^{-2} + \dots$$

As a consequence of equation (2.12) we obtain the recursive relations for the coefficients :

$$\begin{aligned} u_0(M) &= u_0(M-1) + (e'_M + e_M c_M) \\ u_1(M) &= u_1(M-1) + u'_0(M-1) + 2u_0(M-1)c_M + (e'_M + e_M c_M)(e_M + c_M) . \end{aligned}$$

with solutions

$$\begin{aligned} u_0(M) &= \sum_{i=1}^M (e'_i + e_i c_i) \\ u_1(M) &= \sum_{i=1}^{M-1} (M-i) (e'_i + e_i c_i)' + 2 \sum_{i=1}^{M-1} u_0(i) c_{i+1} + \sum_{i=1}^M (e'_i + e_i c_i) (e_i + c_i) \end{aligned}$$

The Darboux-Bäcklund transformation of the Lax operator L_M defined in equation (2.10) takes a form

$$L_M \rightarrow (\partial - e_M)^{-1} L_M (\partial - e_M)$$

and since $e_M \sim \mathcal{A}_n^{(0)}$ we see from equation (2.5) that it represents transformation on the Volterra lattice from integer modes to half-integer modes. In terms of coefficients this results for coefficients with highest indices in :

$$\begin{aligned} g(e_M) &= e_{M-1} + c_M, \quad g(c_M) = -e_{M-1} + e_M - \frac{c'_M}{c_M} \\ g(e_{M-1}) &= e_{M-2} + e_{M-1} - e_M + c_M + c_{M-1} + \frac{c'_M}{c_M} \\ g(c_{M-1}) &= -e_{M-2} + e_{M-1} - \frac{\left(-e_{M-1} + e_M - c_M - c_{M-1} - \frac{c'_M}{c_M}\right)'}{\left(-e_{M-1} + e_M - c_M - c_{M-1} - \frac{c'_M}{c_M}\right)} \end{aligned} \quad (2.14)$$

in addition to

$$g\left(e_k + \sum_{l=k+1}^M c_l\right) = e_{k-1} + \sum_{l=k}^M c_l, \quad 2 \leq k \leq M, \quad g\left(e_1 + \sum_{l=2}^M c_l\right) = \sum_{l=1}^M c_l. \quad (2.15)$$

For a special example of the so-called two Bose system with $M = 1$ (which below will be shown to correspond to the symmetric Painlevé IV equations) the Lax operator is:

$$L_1 = (\partial - e_1)(\partial - c_1)(\partial - e_1 - c_1)^{-1}$$

Such a Lax operator possesses a Darboux-Bäcklund symmetry:

$$L_1 \rightarrow (\partial - e_1)^{-1} L_1 (\partial - e_1) = (\partial - c_1)(\partial - e_1 + c_{1x}/c_1)(\partial - e_1 - c_1 + c_{1x}/c_1)^{-1}$$

which keeps its form unchanged and transforms e_1, c_1 as follows :

$$g(e_1) = c_1, \quad g(c_1) = e_1 - c_{1x}/c_1, \quad (2.16)$$

3 Hamiltonians and t_2 Flow Equations in the Self-similarity Limit of the $2M$ -bose constrained KP hierarchy

Second flow equation (2.13) results in the following expressions for the Lax coefficients:

$$\begin{aligned}\frac{\partial c_j}{\partial t_2} &= \frac{d}{dx} \left(c'_j - c_j^2 - 2e_j c_j + 2 \sum_{i=j+1}^M c'_i - 2 \sum_{i=j}^{M-1} c_i c_{i+1} \right), \quad j = 1, \dots, M \\ \frac{\partial e_j}{\partial t_2} &= \frac{d}{dx} \left(-e'_j - e_j^2 - 2e_j c_j - 2u_0(j-1) - 2e_j \sum_{i=j+1}^M c_i \right), \quad j = 1, \dots, M\end{aligned}\tag{3.1}$$

Effectively, the action of the self-similarity reduction replaces $\partial f / \partial t_2$ with $-(xf)_x / 2$. Integrating all equations obtained through taking the self-similarity limit we find

$$\begin{aligned}e'_j + 2 \sum_{i=1}^{j-1} e'_i &= 2xe_j - e_j^2 - 2e_j \left(\sum_{i=j}^{M-1} c_{i+1} \right) - 2 \sum_{i=1}^j e_i c_i + \bar{\kappa}_j \\ c'_j + 2 \sum_{i=j+1}^M c'_i &= -2xc_j + c_j^2 + 2c_j \sum_{i=j}^{M-1} c_{i+1} + 2c_j e_j + \kappa_j\end{aligned}\tag{3.2}$$

for $j = 1, \dots, M$ and with integration constants $\kappa_j, \bar{\kappa}_j$. The above equations are Hamiltonian in a sense that :

$$e'_j + 2 \sum_{i=1}^{j-1} e'_i = \frac{\partial \mathcal{H}_M}{\partial c_j}, \quad c'_j + 2 \sum_{i=j+1}^M c'_i = -\frac{\partial \mathcal{H}_M}{\partial e_j}\tag{3.3}$$

with

$$\mathcal{H}_M = - \sum_{j=1}^M e_j c_j (e_j + c_j - 2x) - 2 \sum_{1 \leq j < i \leq M} e_j c_j c_i + \sum_{j=1}^M \bar{\kappa}_j c_j - \sum_{j=1}^M \kappa_j e_j\tag{3.4}$$

Note, that the Hamiltonian \mathcal{H}_M defined in (3.4) satisfy

$$\frac{d\mathcal{H}_M}{dx} = \sum_{j=1}^M \left(\frac{\partial \mathcal{H}_M}{\partial e_j} e'_j + \frac{\partial \mathcal{H}_M}{\partial c_j} c'_j \right) + \frac{\partial \mathcal{H}_M}{\partial x} = \frac{\partial \mathcal{H}_M}{\partial x} = 2 \sum_{j=1}^M e_j c_j\tag{3.5}$$

as follows from the fact that the first two terms on the right hand side cancel.

It follows that equations (3.2) can be rewritten as

$$e'_j = 2xe_j + 4x \sum_{k=1}^{j-1} (-1)^{j-k} e_k - e_j \left(e_j + 2 \sum_{k=j}^M c_k \right) + \sum_{k=1}^{j-1} (-1)^{j-k+1} e_k \left(2e_k + 2c_k + 4 \sum_{l=k+1}^M c_l \right) + \bar{k}_j \quad (3.6)$$

$$c'_j = -2xc_j + 4x \sum_{k=j+1}^M (-1)^{j-k+1} c_k + c_j \left(c_j + 2e_j + 2 \sum_{k=j+1}^M c_k \right) + \sum_{k=j+1}^M (-1)^{j-k} c_k \left(2c_k + 4e_k + 4 \sum_{l=k+1}^M c_l \right) + k_j \quad (3.7)$$

for $j = 1, \dots, M$ with appropriately redefined constants k_j, \bar{k}_j :

$$\kappa_j = k_j + 2 \sum_{i=j+1}^M k_i, \quad \bar{\kappa}_j = \bar{k}_j + 2 \sum_{i=1}^{j-1} \bar{k}_i, \quad j = 1, \dots, M$$

Equations (3.6)-(3.7) can be expressed as actions of two vector fields on \mathcal{H}_M :

$$e'_j = E_j(\mathcal{H}_M), \quad c'_j = C_j(\mathcal{H}_M) \quad (3.8)$$

where vector fields E_j, C_j are

$$E_j = \sum_{i=1}^j (-1)^{j-i} \left(\frac{\partial}{\partial c_i} - \frac{\partial}{\partial c_{i-1}} \right), \quad C_j = - \sum_{i=j}^M (-1)^{i-j} \left(\frac{\partial}{\partial e_i} - \frac{\partial}{\partial e_{i+1}} \right) \quad (3.9)$$

We now can define the Poisson brackets through

$$\{e_j, F\} = E_j(F), \quad \{c_j, F\} = C_j(F) \quad (3.10)$$

which results in a Poisson bracket structure :

$$\{e_j, c_i\} = \delta_{j,i} + 2E_{j,i} \quad (3.11)$$

where $E_{j,i}$ is an element of strictly lower-triangular matrix and equal to:

$$E_{j,i} = \begin{cases} (-1)^{j-i} & j > i, \\ 0 & j \leq i. \end{cases}$$

The equations of motion (3.6)-(3.7) are reproduced through

$$e'_j = \{e_j, \mathcal{H}_M\}, \quad c'_j = \{c_j, \mathcal{H}_M\}. \quad (3.12)$$

4 Connection to Higher Order Painlevé Equations

4.1 General Construction

Let $q_i, p_i, i = 1, \dots, M$ be canonical coordinates satisfying the canonical brackets

$$\{q_i, p_j\} = -\delta_{ij}, \quad \{q_i, q_j\} = 0 = \{p_i, p_j\}, \quad i = 1, \dots, M.$$

Relations

$$q_i = f_{2i}, \quad p_i = \sum_{k=1}^i f_{2k-1}, \quad i = 1, \dots, M \quad (4.1)$$

define new variables $f_k, k = 1, \dots, 2M$ and map the canonical brackets into the following Poisson brackets :

$$\{f_i, f_{i+1}\} = 1, \quad \{f_i, f_{i-1}\} = -1, \quad i = 1, \dots, 2M.$$

We now propose a conversion table mapping $e_i, c_i, i = 1, \dots, M$ into a special set of canonical coordinates as well as Painlevé variables $f_k, k = 1, \dots, 2M$ that will satisfy the higher order Painlevé equations (1.1).

First, we list the result for $e_i, i = 1, \dots, M$:

$$\begin{aligned} e_M &= q_M + p_M + 2x + \frac{k_M}{p_M - p_{M-1}} = \sum_{i=1}^M f_{2i-1} + f_{2M} + 2x + \frac{k_M}{f_{2M-1}} \\ e_{M-1} &= -p_{M-1} = -\sum_{i=1}^{M-1} f_{2i-1} \\ e_{M-2} &= -q_1 - \dots - q_{M-2} = -f_2 - \dots - f_{2M-4} \\ e_{M-3} &= -p_{M-2} + p_1 = -f_3 - f_5 - \dots - f_{2M-5} \\ &\dots = \dots \\ e_2 &= -f_{M-2} - f_M = \begin{cases} -q_{M/2} - q_{M/2-1} & M \text{ even} \\ -p_{(M+1)/2} + p_{(M-3)/2} & M \text{ odd} \end{cases} \\ e_1 &= -f_{M-1} = \begin{cases} -p_{M/2} + p_{M/2-1} & M \text{ even} \\ -q_{(M-1)/2} & M \text{ odd} \end{cases} \end{aligned} \quad (4.2)$$

and next for c_i $i = 1, \dots, M$:

$$\begin{aligned}
c_M &= -p_M + p_{M-1} = -f_{2M-1}, \\
c_{M-1} &= p_M - p_{M-1} - q_M - q_{M-1} = f_{2M-1} - f_{2M-2} - f_{2M} \\
c_{M-2} &= p_{M-2} + q_1 + q_2 + \dots + q_{M-3} + q_{M-2} + 2q_{M-1} + 2q_M + 2x \\
c_{M-3} &= -p_{M-2} - p_1 - q_1 - q_2 - \dots - q_{M-4} - q_{M-3} - 2q_{M-2} \\
&\quad - 2q_{M-1} - 2q_M - 2x, \\
c_{M-4} &= p_1 + p_{M-3} + q_2 + \dots + q_{M-4} + q_{M-3} + \\
&\quad + 2q_{M-2} + 2q_{M-1} + 2q_M + 2x \\
&\quad \dots = \dots \\
c_{M-2k} &= p_{k-1} + p_{M-k-1} + q_k + \dots + q_{M-k-1} + 2q_{M-k} + 2q_{M-k+1} + \dots \\
&\quad \dots + 2q_{M-1} + 2q_M + 2x, \quad k = 1, 2, 3, \dots \\
c_{M-(2k-1)} &= -(p_{k-1} + p_{M-k} + q_{k-1} + \dots + q_{M-k-1} + 2q_{M-k} + \dots \\
&\quad + 2q_{M-1} + 2q_M + 2x), \quad k = 2, 3, \dots
\end{aligned} \tag{4.3}$$

Thus from eq. (4.3) it follows that

$$c_1 = \begin{cases} -(p_{M/2-1} + p_{M/2} + q_{M/2-1} + 2q_{M/2} + \dots + 2q_M + 2x) & M \text{ even} \\ p_{(M-3)/2} + p_{(M-1)/2} + q_{(M-1)/2} & \\ + 2q_{(M+1)/2} + \dots + 2q_M + 2x & M \text{ odd} \end{cases} \tag{4.4}$$

The Hamiltonian \mathcal{H}_M defined in (3.4) reads in terms of q_i, p_i , $i = 1, \dots, M$ defined in equations (4.2)-(4.3) as follows

$$\begin{aligned}
\mathcal{H}_M &= \sum_{j=1}^M p_j q_j (p_j + q_j + 2x) + 2 \sum_{1 \leq j < i \leq M} p_j q_j q_i \\
&\quad - \sum_{j=1}^M \alpha_{2j} p_j + \sum_{j=1}^M q_j \left(\sum_{k=1}^j \alpha_{2k-1} \right)
\end{aligned} \tag{4.5}$$

in agreement with reference [11]. The corresponding Hamilton equations:

$$\begin{aligned}
p'_i &= \frac{\partial \mathcal{H}_M}{\partial q_i} = p_i \left(p_i + 2 \sum_{j=i}^M q_j + 2x \right) + 2 \sum_{j=1}^{i-1} p_j q_j + \sum_{j=1}^i \alpha_{2j-1} \\
q'_i &= -\frac{\partial \mathcal{H}_M}{\partial p_i} = -q_i \left(2p_i + q_i + 2 \sum_{j>i}^M q_j + 2x \right) + \alpha_{2i}
\end{aligned}$$

are equivalent to the higher Painlevé equations as given in equation (1.1) with identification of variables provided by relation (4.1).

In the following subsections of this section we will illustrate the above general result for $M = 1, 2, 3$.

4.2 The case of $M = 1$ and Painlevé IV Equations

For $M = 1$ the equations (3.6)-(3.7) take the form of the Levi system :

$$\begin{aligned} e_1' &= 2xe_1 - (e_1 + 2c_1)e_1 + \bar{k}_1 \\ c_1' &= -2xc_1 + (c_1 + 2e_1)c_1 + k_1, \end{aligned} \quad (4.6)$$

which is kept invariant under transformations (2.16) when accompanied by transformations $g(k_1) = 2 - \bar{k}_1$, $g(\bar{k}_1) = -k_1$ of integration constants. Note that eqs. (4.6) are Hamiltonian in the following sense

$$e_1' = \frac{\partial \mathcal{H}_1}{\partial c_1} = \{e_1, \mathcal{H}_1\}, \quad c_1' = -\frac{\partial \mathcal{H}_1}{\partial e_1} = \{c_1, \mathcal{H}_1\},$$

where as follows from the previous subsection :

$$\mathcal{H}_1 = 2xe_1c_1 - e_1^2c_1 - e_1c_1^2 + \bar{k}_1c_1 - k_1e_1, \quad \{e_1, c_1\} = 1. \quad (4.7)$$

Also we derive from

$$\frac{\partial \mathcal{H}_1}{\partial x} = 2e_1c_1$$

that

$$\mathcal{H}_{1xx} = 2e_1'c_1 + 2e_1c_1' = 2e_1^2c_1 - 2c_1^2e_1 + 2\bar{k}_1c_1 + 2k_1e_1$$

and

$$2(\mathcal{H}_1 - x\mathcal{H}_{1x}) = 2(-e_1^2c_1 - c_1^2e_1 + \bar{k}_1c_1 - k_1e_1)$$

Accordingly,

$$\mathcal{H}_{1xx} + 2(\mathcal{H}_1 - x\mathcal{H}_{1x}) = 4c_1(e_1c_1 - \bar{k}_1) = 2c_1(\mathcal{H}_{1x} - 2\bar{k}_1)$$

and

$$\mathcal{H}_{1xx} - 2(\mathcal{H}_1 - x\mathcal{H}_{1x}) = 4e_1(e_1c_1 + k_1) = 2e_1(\mathcal{H}_{1x} + 2k_1).$$

Thus, \mathcal{H}_1 satisfies the Jimbo-Miwa equation [7] of Painlevé IV system:

$$(\mathcal{H}_{1xx} + 2(\mathcal{H}_1 - x\mathcal{H}_{1x}))(\mathcal{H}_{1xx} - 2(\mathcal{H}_1 - x\mathcal{H}_{1x})) = 2\mathcal{H}_{1x}(\mathcal{H}_{1x} - 2\bar{k}_1)(\mathcal{H}_{1x} + 2k_1)$$

Connection of $M = 1$ example (4.6) to $A_2^{(1)}$ symmetric Painlevé IV set of equations

$$\begin{aligned} f_{0x} &= f_0(f_1 - f_2) + \alpha_0 \\ f_{1x} &= f_1(f_2 - f_0) + \alpha_1 \\ f_{2x} &= f_2(f_0 - f_1) + \alpha_2 \end{aligned}$$

with $\alpha_0 + \alpha_1 + \alpha_2 = -2$ can be made explicit by setting

$$f_i = -c_1, \quad f_{i+1} = -e_1 + \frac{c_1x}{c_1}, \quad f_{i+2} = e_1 + c_1 - \frac{c_1x}{c_1} - 2x, \quad \alpha_i = k_1, \quad \alpha_{i+2} = -k_1 - \bar{k}_1$$

for $i = 0, 1, 2$ and with the Darboux-Bäcklund transformation g defined in (2.16) and accordingly mapping f_i to f_{i+1} , $\alpha_i \rightarrow -\alpha_{i+1}$ and $\alpha_{i+2} \rightarrow \alpha_i + \alpha_{i+1}$. This is consistent with realization of g as $g = \pi s_i$ for $i = 0, 1, 2$, where generators s_i of the affine Weyl group $A_2^{(1)}$ act as $s_i(\alpha_{i+2}) = \alpha_i + \alpha_{i+2}$. The above solutions together with the idea of introducing permutation symmetry of the extended affine Weyl group $A_2^{(1)}$ by associating f_i 's to any of the solutions of the Levi system was discussed in [4].

4.3 The Four-Bose system and $A_4^{(1)}$ Painlevé Equations

We now consider a four-boson case with $M = 2$ and $(c_k, e_k)_{k=1}^2$ subject to equations

$$\begin{aligned} e'_1 &= 2xe_1 - (e_1 + 2c_1 + 2c_2)e_1 + \bar{k}_1 \\ e'_2 &= 2xe_2 - 4xe_1 - (e_2 + 2c_2)e_2 + (2c_1 + 2e_1 + 4c_2)e_1 + \bar{k}_2 \\ c'_1 &= -2xc_1 + 4xc_2 + (c_1 + 2e_1)c_1 + (2c_1 - 2c_2 - 4e_2)c_2 + k_1 \\ c'_2 &= -2xc_2 + (c_2 + 2e_2)c_2 + k_2 \end{aligned} \quad (4.8)$$

as follows from equations (3.6)-(3.6). The corresponding Hamiltonian is

$$\begin{aligned} \mathcal{H}_2 &= -e_1c_1(e_1 + c_1 + 2c_2) - e_2c_2(e_2 + c_2) + 2x \sum_{i=1}^2 e_i c_i \\ &\quad - k_2e_2 + \bar{k}_1c_1 - (k_1 + 2k_2)e_1 + (\bar{k}_2 + 2\bar{k}_1)c_2. \end{aligned} \quad (4.9)$$

We find that in the case of $M = 2$ the vector fields $E_j, C_j, i = 1, 2$ are

$$E_1 = \frac{\partial}{\partial c_1}, \quad E_2 = \frac{\partial}{\partial c_2} - 2\frac{\partial}{\partial c_1}, \quad C_1 = 2\frac{\partial}{\partial e_2} - \frac{\partial}{\partial e_1}, \quad C_2 = -\frac{\partial}{\partial e_2}$$

and according to (3.10) they lead to the Poisson brackets :

$$\{e_1, c_1\} = 1, \quad \{e_1, c_2\} = 0, \quad \{e_2, c_1\} = -2, \quad \{e_2, c_2\} = 1 \quad (4.10)$$

consistent with equations of motion (4.8) through the Poisson brackets :

$$e'_j = \{e_j, \mathcal{H}_2\}, \quad c'_j = \{c_j, \mathcal{H}_2\}, \quad j = 1, 2. \quad (4.11)$$

The symmetry transformations (2.14)-(2.15) read here :

$$\begin{aligned} g(e_2) &= e_1 + c_2, & g(e_1) &= e_1 - e_2 + c_2 + c_1 + \frac{c'_2}{c_2} \\ g(c_2) &= -e_1 + e_2 - \frac{c'_2}{c_2}, & g(c_1) &= e_1 - \frac{\left(-e_1 + e_2 - c_2 - c_1 - \frac{c'_2}{c_2}\right)'}{\left(-e_1 + e_2 - c_2 - c_1 - \frac{c'_2}{c_2}\right)}, \end{aligned} \quad (4.12)$$

which keep equations (4.8) invariant for :

$$\begin{aligned} g(k_1) &= -2 + \bar{k}_1 + 2\bar{k}_2, & g(\bar{k}_1) &= 2 - \bar{k}_1 - \bar{k}_2 - k_1 - 3k_2, \\ g(k_2) &= 2 - \bar{k}_1 - \bar{k}_2, & g(\bar{k}_2) &= -4 + 3\bar{k}_1 + 2\bar{k}_2 + 2k_1 + 5k_2. \end{aligned} \quad (4.13)$$

In order to see the meaning of this transformation from the group theoretic point of view we cast equations (4.8) into the symmetric $A_4^{(1)}$ Painlevé equations:

$$f'_i = f_i(f_{i+1} - f_{i+2} + f_{i+3} - f_{i+4}) + \alpha_i, \quad i = 0, \dots, 4 \quad (4.14)$$

with conditions $f_i = f_{i+5}$ and $\sum_{i=0}^4 \alpha_i = -2$. We propose the following identification

$$\begin{aligned}
f_1 &= -e_1 \\
f_2 &= g(f_1) = - \left(e_1 - e_2 + c_2 + c_1 + \frac{c'_2}{c_2} \right) = -e_1 - e_2 - c_1 - 2c_2 + 2x - \frac{k_2}{c_2} \\
f_3 &= -c_2 \\
f_4 &= g(f_3) = - \left(-e_1 + e_2 - \frac{c'_2}{c_2} \right) = e_1 + e_2 + c_2 - 2x + \frac{k_2}{c_2} \\
f_0 &= -f_1 - f_2 - f_3 - f_4 - 2x = e_1 + c_1 + 2c_2 - 2x
\end{aligned}$$

and

$$\begin{aligned}
\alpha_1 &= -\bar{k}_1, & \alpha_2 &= 2 - \bar{k}_1 - \bar{k}_2 - k_1 - 3k_2 \\
\alpha_3 &= k_2, & \alpha_4 &= -2 + \bar{k}_1 + \bar{k}_2 \\
\alpha_0 &= -2 + \sum_{i=1}^4 \alpha_i = -2 + \bar{k}_1 + k_1 + 2k_2.
\end{aligned}$$

Alternatively we can write relations between $e_i, c_i, i = 1, 2$ and $f_i, i = 0, 1, \dots, 4$ as

$$\begin{aligned}
e_1 &= -f_1, & e_2 &= -f_1 - f_4 + f'_3/f_3 = -f_0 - f_2 + \frac{\alpha_3}{f_3} \\
c_1 &= f_3 - f_2 - f_4, & c_2 &= -f_3
\end{aligned} \tag{4.15}$$

in agreement with equations (4.2) and (4.3). Accordingly one can rewrite the g -transformation from (4.12) as

$$\begin{aligned}
g(f_1) &= f_2, & g(f_3) &= f_4 \\
g(f_2) &= -f_3 + f_2 + f_4 - \frac{f'_4}{f_4} + \frac{f'_2}{f_2} \\
&= f_3 - \frac{\alpha_4}{f_4} + \frac{\alpha_2}{f_2} \\
g(f_4) &= f_1 + f_3 - f_2 + \frac{f'_4}{f_4} = f_0 + \frac{\alpha_4}{f_4}.
\end{aligned} \tag{4.16}$$

Comparing with definitions of transformations $s_i, i = 1, 2, 3, 4$ (see f.i. [10]) we see that expression for transformation g from (4.12)-(4.13) agrees with

$$g = \pi s_1 + \pi s_3 - \pi$$

as applied on both f 's and α 's.

More generally, associating $-e_1$ and $-c_2$ to f_i and f_{i+2} , respectively, with i taking all the values $i = 1, 2, 3, 4$ we obtain identifications:

$$g = \pi s_i + \pi s_{i+2} - \pi \equiv g_i$$

for each realization, where s_i are generators of the affine Weyl $A_4^{(1)}$. This relates g with $s_i + s_{i+2}$ and by varying i over all its values makes possible to recover all the affine Weyl $A_4^{(1)}$ generators s_i from one Darboux-Bäcklund transformation g . For instance $s_1 = (-I + \pi(g_1 + g_2 + g_3 - g_0 - g_4))/2$.

4.4 $M = 3$ Bose System and the symmetric $A_6^{(1)}$ Painlevé Equations

For $M = 3$, equations (3.6)-(3.7) become

$$\begin{aligned}
e'_1 &= 2xe_1 - (e_1 + 2c_1 + 2c_2 + 2c_3)e_1 + \bar{k}_1 \\
e'_2 &= 2xe_2 - 4xe_1 - (e_2 + 2c_2 + 2c_3)e_2 - (-2c_1 - 2e_1 - 4c_2 - 4c_3)e_1 + \bar{k}_2 \\
e'_3 &= 2xe_3 - 4xe_2 + 4xe_1 - (e_3 + 2c_3)e_3 - (-2c_2 - 2e_2 - 4c_3)e_2 - (2c_1 + 2e_1 + 4c_2 + 4c_3)e_1 + \bar{k}_3 \\
c'_1 &= -2xc_1 + 4xc_2 - 4xc_3 + (c_1 + 2e_1)c_1 + (2c_1 - 2c_2 - 4e_2)c_2 + (2c_1 - 4c_2 + 2c_3 + 4e_3)c_3 + k_1 \\
c'_2 &= -2xc_2 + 4xc_3 + (c_2 + 2e_2)c_2 + (2c_2 - 2c_3 - 4e_3)c_3 + k_2 \\
c'_3 &= -2xc_3 + (c_3 + 2e_3)c_3 + k_3.
\end{aligned} \tag{4.17}$$

These equations are Hamiltonian as in eqs.(3.3) with

$$\begin{aligned}
\mathcal{H}_3 &= 2x e_1 c_1 + 2x e_2 c_2 + 2x e_3 c_3 - e_1 c_1 (e_1 + c_1 + 2c_2 + 2c_3) \\
&\quad - e_2 c_2 (e_2 + c_2 + 2c_3) - e_3 c_3 (e_3 + c_3) + \bar{k}_1 c_1 + (\bar{k}_2 + 2\bar{k}_1) c_2 \\
&\quad + (\bar{k}_3 + 2\bar{k}_1 + 2\bar{k}_2) c_3 - (k_1 + 2k_3 + 2k_2) e_1 - (k_2 + 2k_3) e_2 - k_3 e_3.
\end{aligned} \tag{4.18}$$

The symmetric $A_6^{(1)}$ Painlevé equations

$$f'_i = f_i(f_{i+1} - f_{i+2} + f_{i+3} - f_{i+4} + f_{i+5} - f_{i+6}) + \alpha_i, \quad i = 0, 1, 2, \dots, 6, \tag{4.19}$$

with condition $f_i = f_{i+7}$ and with

$$f_0 = -\sum_{i=1}^6 f_i - 2x$$

are satisfied by

$$\begin{aligned}
f_0 &= e_1 + e_2 + c_2 + 2c_3 - 2x \\
f_1 &= e_1 + c_1 + 2c_2 + 2c_3 - 2x \\
f_2 &= -e_1 \\
f_3 &= -e_2 - e_1 - c_1 - 2c_2 - 2c_3 + 2x \\
f_4 &= -e_2 + e_3 - c_2 - c_3 - \frac{c'_3}{c_3} \\
&= -e_2 - e_3 - c_2 - 2c_3 + 2x - \frac{k_3}{c_3} \\
f_5 &= -c_3 \\
f_6 &= -\left(e_3 - e_2 - \frac{c'_3}{c_3}\right) \\
&= e_2 + e_3 + c_3 - 2x + \frac{k_3}{c_3}
\end{aligned} \tag{4.20}$$

Furthermore

$$\begin{aligned}
\alpha_0 &= -2 + k_2 + 2k_3 + \bar{k}_2 + \bar{k}_1 = -2 + \kappa_2 + \bar{\kappa}_2 - \bar{\kappa}_1 \\
\alpha_1 &= -2 + k_1 + 2k_2 + 2k_3 + \bar{k}_1 = -2 + \kappa_1 + \bar{\kappa}_1 \\
\alpha_2 &= -\bar{k}_1 = -\bar{\kappa}_1 \\
\alpha_3 &= 2 - k_1 - 2k_2 - 2k_3 - \bar{k}_2 - \bar{k}_1 = 2 - \kappa_1 - \bar{\kappa}_2 + \bar{\kappa}_1 \\
\alpha_4 &= 2 - \bar{k}_2 - \bar{k}_3 - k_2 - 3k_3 = 2 - \bar{\kappa}_3 + \bar{\kappa}_2 - \kappa_2 - \kappa_3 \\
\alpha_5 &= k_3 = \kappa_3 \\
\alpha_6 &= -2 + \bar{k}_2 + \bar{k}_3 = -2 + \bar{\kappa}_3 - \bar{\kappa}_2
\end{aligned}$$

in terms of objects e_i, c_i, k_i, \bar{k}_i , $i = 1, 2, 3$ from $M = 3$ equations (4.17).

The logic of deriving the above association is as follows. We initially set $f_5 = f_{2M-1} = -c_M = -c_3$ and $f_6 = f_{2M} = g(f_5)$. This is suggested by equation of motion for c_M which has the unique $2xc_M$ term on the right hand side making it a good candidate for f_i , since Painlevé equations have this structure after elimination of f_0 . But equations of motion for f_5 and f_6 after elimination of f_0 only involve sums $f_1 + f_3$ and $f_2 + f_4$, respectively. Therefore these quantities are derived from these equations to be $f_1 + f_3 = -e_2 = -e_{M-1}$ and $f_2 + f_4 = g(f_1 + f_3) = -g(e_2)$. Next, we turn into equation for f_1 in (4.19), which we rewrite as

$$f'_1 = f_1(-2e_1 - 2c_3 - 2c_2 + 2x + f_1) + \alpha_1$$

after we substituted $f_5 = -c_3$, $f_1 + f_3 = -e_2$ and $f_0 = -(f_2 + f_4) - e_3 - k_3/c_3$ and the values for f_6 and $f_2 + f_4$ determined previously by g transformations. We find that the solution to these equations is given by

$$\begin{aligned}
f_1 &= e_1 + c_1 + 2c_2 + 2c_3 - 2x \\
\alpha_1 &= 2 + k_1 + 2k_2 + 2\bar{k}_3 + \bar{k}_1
\end{aligned}$$

From that result we derive f_3 as $-e_2 - f_1$ and from eq. for f_3 we derived f_4 and $f_2 = -e_1$.

Applying similarity transformation (2.2)

$$L_3 \rightarrow (\partial - e_3)^{-1} L_3 (\partial - e_3)$$

on the Lax operator obtained from (2.10) by setting $M = 3$.

$$\begin{aligned}
g(e_3) &= e_2 + c_3, & g(c_3) &= -e_2 + e_3 - \frac{c'_3}{c_3} \\
g(e_2) &= e_1 + e_2 - e_3 + c_3 + c_2 + \frac{c'_3}{c_3} \\
g(c_2) &= -e_1 + e_2 - \frac{\left(-e_2 + e_3 - c_3 - c_2 - \frac{c'_3}{c_3}\right)'}{\left(-e_2 + e_3 - c_3 - c_2 - \frac{c'_3}{c_3}\right)}
\end{aligned} \tag{4.21}$$

in addition to

$$g(e_1 + c_2 + c_3) = \sum_{l=1}^3 c_l.$$

These Bäcklund transformations amount to the following transformations when applied on f_i :

$$\begin{aligned} g(f_2) &= f_3 - \frac{\alpha_4}{f_4}, & g(f_1 + f_3) &= f_2 + f_4 \\ g(f_0) &= f_1, & g(f_4) &= f_5 + \frac{\alpha_4}{f_4} - \frac{\alpha_6}{f_6} \\ g(f_5) &= f_6, & g(f_6) &= f_0 + \frac{\alpha_6}{f_6}, \end{aligned}$$

which agrees with the action of

$$g = \pi s_5 + \pi s_3 - \pi \quad (4.22)$$

as applied on f_i and α_i .

Now the Bäcklund transformation of c_1 is equal to

$$g(c_1) = e_1 - \frac{(g(c_2) + g(c_3) - c_1 - c_2 - c_3)'}{g(c_2) + g(c_3) - c_1 - c_2 - c_3} \quad (4.23)$$

in terms of $g(c_2)$ and $g(c_3)$ from eqs. (4.21).

In terms of f_i and α_i 's it takes a form

$$\begin{aligned} g(c_1) &= -f_2 - \frac{\left(f_3 - \frac{\alpha_4}{f_4}\right)'}{f_3 - \frac{\alpha_4}{f_4}} \\ &= -f_1 - f_4 - f_6 + f_0 + f_5 + \frac{\alpha_4}{f_4} - \frac{\alpha_3 + \alpha_4}{f_3 - \frac{\alpha_4}{f_4}} \end{aligned}$$

This gives rise to the following transformation of f_1 :

$$g(f_1) = f_2 - \frac{\alpha_3 + \alpha_4}{f_3 - \frac{\alpha_4}{f_4}}$$

Thus representation of the Bäcklund transformation g in terms of generators of the extended affine Weyl group $A_6^{(1)}$ from (4.22) needs to be augmented as follows:

$$g = \pi s_5 + \pi s_3 - \pi + s_4(\pi s_2 - \pi) = \pi s_5 + \pi s_3 s_2 - \pi$$

as applied on f_i and α_i .

By generalizing eq. (4.20) by substituting $i = 5$ with arbitrary i between 0 and M one obtains

$$\begin{aligned} g_i &= \pi s_i + \pi s_{i-2} s_{i-3} - \pi, \quad i = 0, 1, \dots, 6 \\ f_i &= -c_M \end{aligned}$$

providing a scheme to reproduce all generators s_0, s_1, \dots, s_6 of the affine Weyl group of $A_6^{(1)}$ by one Bäcklund transformation g .

5 Reduction of $M = 2$ case. Painlevé V

We will follow reference [3] and perform a Dirac reduction of $M = 2$ case (see subsection 4.3), by redefining variables as follows :

$$(e_1, c_1, e_2, c_2) \rightarrow (\tilde{e}_1 = e_1, \tilde{c}_1 = c_1, \tilde{e}_2 = (e_2 - c_2)/2, \tilde{c}_2 = (c_2 - e_2)/2),$$

which is equivalent to setting a second-class constraint

$$c = c_2 = -e_2$$

with the Dirac bracket:

$$\{c, c\} = \frac{1}{2}\delta_x(x - y).$$

The self-similarity reduction applied on the resulting t_2 evolution equations (3.1) yields

$$-2xc_1 = c'_1 + 2c' - c_1^2 - 2e_1c_1 - 2c_1c + k_1 \quad (5.1)$$

$$-2xe_1 = -e'_1 - e_1^2 - 2e_1c_1 - 2e_1c + \bar{k}_1 \quad (5.2)$$

$$-2xc = e'_1 + e_1c_1 + k. \quad (5.3)$$

Eliminating c and c_1 from equations (5.1)- (5.3) yields the following expression for $y = e_1/2x$:

$$y_{zz} = -\frac{1}{z}y_z + \left(\frac{1}{2y} + \frac{1}{2(y-1)}\right)y_z^2 - \frac{\alpha y}{z^2(y-1)} - \frac{\beta(y-1)}{z^2y} - \frac{\gamma}{z}y(y-1) - \delta y(y-1)(2y-1) \quad (5.4)$$

with constants

$$\alpha = \frac{1}{8}(k+1)(k+\bar{k}_1+1) + \frac{\bar{k}_1^2}{32} = \frac{1}{8}\left(k+1 + \frac{\bar{k}_1}{2}\right)^2 \quad (5.5)$$

$$\beta = -\frac{\bar{k}_1^2}{32} = -\frac{1}{8}\left(\frac{\bar{k}_1}{2}\right)^2, \quad \gamma = \frac{k+k_1+1}{2\sigma}, \quad \delta = -\frac{1}{2\sigma^2}$$

after a change of coordinate $x \rightarrow z$ such that $z = \sigma x^2$.

The above equation takes on a conventional form of the Painlevé V equation for $w = y/(y-1)$ and $\delta = -1/2$.

To study the Darboux-Bäcklund transformation of the Painlevé V system we perform the similarity transformation

$$L \rightarrow (\partial + c)^{-1} L (\partial + c)$$

on the Lax operator for the reduced 4-boson system [3]. This induces the following transformations for variables of the reduced subspace:

$$g(e_1) = e_1 + c_1 + 2c$$

$$g(c_1) = e_1 - \frac{(c_1 + e_1 + 2c)'}{c_1 + e_1 + 2c}$$

$$g(c) = -e_1 - c$$

It follows that g transforms the constants k, k_1, \bar{k}_1 as

$$\begin{aligned} g(k) &= -k - \bar{k}_1 \\ g(\bar{k}_1) &= k_1 + \bar{k}_1 + 2k \\ g(k_1) &= \bar{k}_1 - 2 \end{aligned} \tag{5.6}$$

Next, applying this transformation to solution $w = y/(y - 1)$ of a conventional Painlevé V equation yields

$$\begin{aligned} g(w) &= 1 - \frac{2z\sigma w}{F} \\ F &= +zw_z - \frac{1}{2}w^2 \left(k + 1 + \frac{\bar{k}_1}{2} \right) + w \left(\frac{1}{2}(k + 1) + z\sigma \right) + \frac{1}{4}\bar{k}_1 \end{aligned} \tag{5.7}$$

In terms of quantities

$$c_g = \frac{1}{2} \left(k + 1 + \frac{\bar{k}_1}{2} \right), \quad a_g = \frac{1}{4}\bar{k}_1$$

with properties

$$c_g^2 = 2\alpha, \quad a_g^2 = -2\beta$$

the function F from relation (5.7) can be rewritten as

$$F = +zw_z - w^2 c_g + w(c_g - a_g + z\sigma) + a_g$$

in complete agreement with [5, 6].

6 Outlook

We have here derived the higher order Painlevé equations by taking self-similarity limit of the special class of integrable models and shown how the extended affine Weyl groups $A_n^{(1)}$ symmetries are induced by Bäcklund transformations generated by translations on the underlying “half-integer” Volterra lattice. The Hamiltonian of the integrable model reduced by the self-similarity procedure has been explicitly shown to transform under change of variables into the Hamiltonian for the higher Painlevé equations. In the forthcoming publication we plan to provide explicit proof for formulas governing such change of variables and include in the formalism the Painlevé equations with the extended affine Weyl groups $A_{2n-1}^{(1)}$ symmetries. We will also employ a link between on the one hand integrable hierarchies and on the other hand higher order Painlevé equations to derive the corresponding higher order Painlevé hierarchies.

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