T-systems and Y-systems in integrable systems

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Abstract

The T and Y-systems are ubiquitous structures in classical and quantum integrable systems. They are difference equations having a variety of aspects related to commuting transfer matrices in solvable lattice models, q-characters of Kirillov-Reshetikhin modules of quantum affine algebras, cluster algebras with coefficients, periodicity conjectures of Zamolodchikov and others, dilogarithm identities in conformal field theory, difference analogue of L-operators in KP hierarchy, Stokes phenomena in 1d Schrödinger problem, AdS/CFT correspondence, Toda field equations on discrete space-time, Laplace sequence in discrete geometry, Fermionic character formulas and combinatorial completeness of Bethe ansatz, Q-system and ideal gas with exclusion statistics, analytic and thermodynamic Bethe ansätze, quantum transfer matrix method and so forth. This review article is a collection of short reviews on these topics which can be read more or less independently.

Contents

1. Introduction

1.1. **T and Y-systems.** The T-system is a difference equation among commuting variables $T_m^{(a)}(u)$, most typically looking as ($m \in \mathbb{Z}_{\geq 0}$)

$$T_m^{(a)}(u-1)T_m^{(a)}(u+1) = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) + T_m^{(a-1)}(u)T_m^{(a+1)}(u).$$

Originally it was found as a functional relation in 2d solvable lattice models in statistical mechanics [1]. In this context, $T_m^{(a)}(u)$ is a commuting row transfer matrix in the sense of Baxter [2] labeled with (a, m) and having the spectral parameter u[1].

The Y-system is another difference equation, typically like ($m \in \mathbb{Z}_{\geq 1}$)

$$Y_m^{(a)}(u-1)Y_m^{(a)}(u+1) = \frac{(1+Y_m^{(a-1)}(u))(1+Y_m^{(a+1)}(u))}{(1+Y_{m-1}^{(a)}(u)^{-1})(1+Y_{m+1}^{(a)}(u)^{-1})}.$$

It was extracted as a universal functional relation in thermodynamic Bethe ansatz (TBA) for solvable lattice models as well as (1+1)d integrable quantum field theory models [3, 4, 5]. In this context, $Y_m^{(a)}(u)$ stands for the Boltzmann factor of an excitation mode in the sense of Yang-Yang [6] labeled with (a, m) and having the rapidity u.

As such, the both systems originate in Yang-Baxter quantum integrable systems but are apparently concerned with the objects that are not related too directly. The first curiosity is nevertheless that the formal substitution

$$Y_m^{(a)}(u) = \frac{T_m^{(a-1)}(u)T_m^{(a+1)}(u)}{T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u)}$$

provides a solution to the Y-system in terms of the T-system. Moreover, such a canonical pair of companion systems can be formulated uniformly for all the classical simple Lie algebras $\mathfrak{g}$ [1][2]. Now we can give a deferred explanation of the

[1]By T we meant Transfer matrices, but it can either be thought as Toda or Tau etc.

[2]Actually to be understood as Yangian $Y(\mathfrak{g})$ or untwisted quantum affine algebra $U_q(\hat{\mathfrak{g}})$. Twisted case is also known. See Remark 2.1.

superscript a; it runs over the vertices of the Dynkin diagram of $\mathfrak{g}$. The above formulas are just the examples from type A[3] .

In the relevant developments across the centuries, the T and Y-systems have turned out to be ubiquitous structures with a wealth of applications. For instance, they emerge in q-characters for Kirillov-Reshetikhin modules of quantum affine algebras, exchange relations in cluster algebras with coefficients, periodicity conjectures of Zamolodchikov and others, dilogarithm identities in conformal field theory (CFT) and their functional generalizations, dressed vacuum forms in analytic Bethe ansatz, Stokes phenomena in ordinary differential equations, anomalous scaling dimensions of $\mathcal{N} = 4$ super Yang-Mills operators, area of minimal surface in AdS, Laplace sequence of quadrilateral lattice in discrete geometry, tau functions in lattice Toda field equations, Fermionic formulas for branching coefficients and weight multiplicities for Lie algebra characters, combinatorial completeness of string hypothesis in Bethe ansatz, Q-system and grand partition function of ideal gas with exclusion statistics, quantum transfer matrix approach to finite temperature problems and so on.

This review article is a collection of brief expositions of these topics where the T and Y-systems have played key roles. It consists of sections of moderate length which are not mutually dependent too much. A more detailed account of the contents can be found in Section 1.2.

As an overview, T-systems are fundamental structures reflecting symmetries and algebraic aspects of the problems rather directly. They can also accommodate various gauge/normalization freedom of concrete models. On the other hand, Y-systems are more universal being more or less free from such degrees of freedom. They are suitable for practical applications with appropriate analyticity input. In fact, the connection between the T and Y-systems mentioned previously has opened a route to establish TBA type integral equations directly from transfer matrices without recourse to the TBA itself. In this sense, Y-systems are the format in which the symmetries encoded in the T-systems are most efficiently utilized as a practical implement.

In the light of ever growing perspectives, what sort of equations or structures are to be recognized as T or Y-systems is actually a matter of time-dependent option. For instance from an algebraic point of view (leaving analytic aspects), T-systems have been generalized broadly to the quantum affinization of quantum Kac-Moody algebras by Hernandez [7] (Section 4.6). Cluster algebra with coefficients by Fomin and Zelevinsky [8] offers a comprehensive scheme to generalize and control the T and Y-systems simultaneously by quivers (Section 5). Nonetheless, this paper is mostly devoted to the description of basic results concerning the aforementioned "classic" T and Y-systems associated with $\mathfrak{g}$. We therefore look forward to the next review to come, hopefully someday by some author, bringing a delightful renewal.

1.2. **Contents and brief guide.** Here are abstracts of the subsequent sections. They will be followed by another brief guide to the paper.

Section 2. The T and Y-systems for untwisted and twisted quantum affine algebras are presented. They have unrestricted and level ℓ restricted versions. Those for Yangian are formally the same with the unrestricted ones for the untwisted

[3]The T-system for type A formally coincides with what is known as the Hirota-Miwa equation in soliton theory, which was an unexpected link also to classical integrable systems.

quantum affine algebra $U_q(\hat{\mathfrak{g}})$, where $\mathfrak{g}$ denotes a finite dimensional simple Lie algebra throughout the paper. We also include the $U_q(sl(r|s))$ case. This section is meant to be the reference of these systems throughout the paper. The first property, T-system provides a solution to Y-system, is stated. Subsequent sections will mainly be concerned with the untwisted case $U_q(\hat{\mathfrak{g}})$[4].

Section 3. The T-system was originally discovered as functional relations among commuting transfer matrices for solvable lattice models in statistical mechanics. We give an elementary exposition of such contexts for the both vertex and restricted solid-on-solid (RSOS) models along with their fusion procedure. The two types of models are related to the unrestricted and restricted T-systems, respectively.

Section 4. We describe the background of the T-system in the representation theory of quantum affine algebras such as classification of irreducible finite dimensional representations, Kirillov-Reshetikhin modules and q-characters. The fundamental results are that q-characters of the Kirillov-Reshetikhin modules satisfy the T-system (Theorem 4.8) and the description of the Grothendieck ring $\mathrm{Rep}\, U_q(\hat{\mathfrak{g}})$ by the T-system (Theorem 4.9). A broad extention of the T-system to the quantum affinization of quantum Kac-Moody algebras is also mentioned. The results of this section are not necessary elsewhere except the basics of q-characters which will be mentioned in tableau sum formulas (Section 7), analytic Bethe ansatz (Section 8) and Q-system (Section 13).

Section 5. The cluster algebra with coefficients is built upon cluster variables and coefficient tuples obeying certain exchange relations controlled by a quiver. We demonstrate how such a setup encodes the T and Y-systems simultaneously in an essential way. It opens a fruitful link with the cluster category theory, which led to a final proof of the dilogarithm identities in conformal field theory and the periodicity conjecture on the both systems for arbitrary level and $\mathfrak{g}$.

Section 6. Jacobi-Trudi type determinant formulas are listed for T-systems for non exceptional $\mathfrak{g}$. The type C_r and D_r cases involve Pfaffians as well.

Section 7. Tableau sum formulas are presented for T-systems for non exceptional $\mathfrak{g}$ along the context of q-characters.

Section 8. We argue the relation between q-characters and eigenvalue formulas (dressed vacuum forms) of transfer matrices in solvable lattice models by analytic Bethe ansatz. Combined with the results in Section 7, it leads to solutions of T-systems in terms of the Baxter Q-functions. We mainly concern vertex models and include a brief argument on RSOS models.

Section 9. We introduce a difference analogue of L-operators in soliton theory to construct solutions to the T-systems for $\mathfrak{g} = A_r$ and C_r by Casoratian (difference analogue of Wronskians). The Baxter Q-functions are identified with a special class of Casoratians and generalized to a wider family of functions that admit Bäcklund transformations. Analogous difference L-operators are presented also for B_r, D_r and $sl(r|s)$.

Section 10. A restricted T-system for A_1 emerges in Stokes phenomena of 1d Schrödinger equation with a specific potential. Similar facts hold also for the T-system for A_r and a class of $(r+1)$th order ordinary differential equation (ODE). Wronskians for these equations evaluated at the origin play an analogous role to the Casoratians in Section 9 (Wronskian-Casoratian duality). We describe these

[4] Thus in most situations we will say simply T and Y-systems for $\mathfrak{g}$ instead of $U_q(\hat{\mathfrak{g}})$.

features that stay within an elementary algebraic part in the so-called ODE/IM (integrable models) correspondence.

Section 11. This section is most hep-th oriented. We briefly digest applications of some specific T and Y-systems in the two topics from the AdS/CFT correspondence. The first one is from the gauge theory about the anomalous scaling dimensions (planar AdS/CFT spectrum) of $\mathcal{N} = 4$ super Yang-Mills operators. The second one is the area of the minimal surface in AdS from the string theory, which is relevant to gluon planar scattering amplitudes. The analysis in the latter topic involves the Stokes phenomena related to a generalized sinh-Gordon equation, which may be viewed as a generalization of the ODE/IM correspondence mentioned in Section 10.

Section 12. Continuous limits of the T-system for $\mathfrak{g}$ yield the difference-differential or 2d differential equations known as the (lattice) Toda field equation. Their Hamiltonian structure is presented for general $\mathfrak{g}$. We also discuss an aspect from classical discrete geometry, where the Y-system for A_∞ arises as the Laplace sequence of quadrilateral lattice, the discrete geometry analogue of the conjugate net.

Section 13. T-system without spectral parameter is called Q-system[5]. We systematically construct certain power series solutions to the (generalized) Q-system by multi-variable Lagrange inversion. As a corollary of this and results from Section 4, the so-called Fermionic character formula for the Kirillov-Reshetikhin modules is fully established for all $\mathfrak{g}$. Physically, this problem is also connected to the grand partition function of ideal gas with exclusion statistics. These results are reviewed in conjunction with the intimately related subject known as combinatorial completeness of Bethe ansatz for $U_q(\hat{\mathfrak{g}})$ both at $q = 1$ and $q = 0$, where the case $q = 1$ goes back to Bethe [9], the God farther of the subject, himself.

Section 14. We explain how the Y-system for $\mathfrak{g}$ emerges from the TBA equation associated with $U_q(\hat{\mathfrak{g}})$ q being a root of unity derived in Section 15. Various relations among the TBA kernels are summarized. The constant Y-system is introduced and related to the Q-system. They are essential ingredients in the dilogarithm identity (Section 5.1) and the TBA analysis of RSOS models (Section 15). As a related issue, we briefly discuss the Q-system at root of unity including Conjecture 14.2.

Section 15. The $U_q(\hat{\mathfrak{g}})$ Bethe equation with q a root of unity is relevant to the critical RSOS models sketched in Section 3.3. We outline the TBA analysis to evaluate the high temperature entropy by the level restricted Q-system (Section 14.5–14.6) and central charges by the dilogarithm identity (Section 5.1). The TBA equation obtained here uniformly for general $\mathfrak{g}$ is the origin of our Y-system as shown in Section 14.1 and 14.3.

Section 16. The finite size or finite temperature problems in solvable lattice models are analyzed efficiently by the use of T and Y-systems without relying on TBA approach and string hypothesis. We illustrate various such methods along the simplest vertex and RSOS models based on $\mathfrak{g} = A_1$. We also include a simple application of the periodicity of the level 0 restricted T-system to the calculation of correlation lengths of vertex models in Section 16.1.

Let us close the introduction with yet another brief guide of the contents. As we already mentioned, Section 2 is the collection of the basic data; concrete forms of the T and Y-systems that will be considered in the paper and definitions/notations concerning the root system of $\mathfrak{g}$. With regard to the subsequent sections, it is

[5] This Q is unrelated with Baxter's Q-functions. See Section 13.8 for the origin of the name.

too demanding to assume the familiarity of the contents in earlier sections. So we have avoided such a style and tried to make each section into a more or less independently readable review on a specific topic around 10 pages. Most of them contain bibliographical notes at the end, which hopefully help the readers gain more perspectives into the subjects and activities around.

There are nevertheless several sections that are intimately related or partly dependent of course. Roughly, they may be grouped (non exclusively) under the following theme.

- Solvable lattice models and their analysis: Section 3, 8, 15, 16.
- Kirillov-Reshetikhin modules and their q-characters: Section 4, 7, 8, 13.
- Variety of solutions to T-system: Section 6, 7, 8, 9.
- Stokes phenomena: Section 10, 11.
- Q-system and constant Y-system: Section 13, 14.
- Y-system and TBA: Section 11, 14, 15.

2. T and Y-systems for quantum affine algebras and Yangians

We present the T-system and Y-system associated with untwisted and twisted quantum affine algebras. They have unrestricted and level restricted versions. Those for Yangian are formally the same with the unrestricted ones for the untwisted quantum affine algebras. We also include the case $U_q(sl(r|s))$. This section is devoted to the presentation of these systems with the basic data on root systems. Thus we will only state their first property, T-system provides a solution to Y-system, in Theorem 2.5. leaving the exposition of variety of aspects in subsequent sections.

2.1. **Untwisted case.** Let $\mathfrak{g}$ be a simple Lie algebra associated with a Dynkin diagram of finite type. We set $I = \{1, \ldots, r\}$ with $r = \operatorname{rank} \mathfrak{g}$ and enumerate the vertices of the Dynkin diagrams as Figure 1. We follow [10] *except for* E_6, for which we choose the one naturally corresponding to the enumeration of the twisted affine diagram $E_6^{(2)}$ in Section 2.4. With a slight abuse of notation, we will write for example $\mathfrak{g} = A_r$ to mean that $\mathfrak{g}$ is the one associated with the Dynkin diagram of type A_r. The cases A_r, D_r, E_6, E_7 and E_8 are referred to as simply laced.

We set numbers t and t_a $(a \in I)$ by

$$t = \begin{cases} 1 & \mathfrak{g} : \text{simply laced}, \\ 2 & \mathfrak{g} = B_r, C_r, F_4, \\ 3 & \mathfrak{g} = G_2, \end{cases} \qquad t_a = \begin{cases} 1 & \mathfrak{g} : \text{nonsimply laced}, \\ 1 & \mathfrak{g} : \text{nonsimply laced}, \ \alpha_a\text{: long root}, \\ t & \mathfrak{g} : \text{nonsimply laced}, \ \alpha_a\text{: short root}. \end{cases} \tag{2.1}$$

Let $\alpha_a, \omega_a \, (a \in I)$ be the simple roots and the fundamental weights of $\mathfrak{g}$. We fix a bilinear form $(\ |\ )$ on the dual space of the Cartan subalgebra normalized as

$$(\alpha_a|\alpha_a) = \frac{2}{t_a}, \quad (\alpha_a|\omega_b) = \frac{\delta_{ab}}{t_a}. \tag{2.2}$$

Let $C = (C_{ab})$, $C_{ab} = 2(\alpha_a|\alpha_b)/(\alpha_a|\alpha_a)$, be the Cartan matrix of $\mathfrak{g}$. We have $C_{ab} = t_a(\alpha_a|\alpha_b)$, $\alpha_a = \sum_{b=1}^r C_{ba}\omega_b$ and $(C^{-1})_{ab} = t_a(\omega_a|\omega_b)$. We denote by h and $h^\vee$ the Coxeter number and the dual Coxeter number of $\mathfrak{g}$, respectively. They are listed in the following with the dimension of $\mathfrak{g}$.

$\mathfrak{g}$	A_r	B_r	C_r	D_r	E_6	E_7	E_8	F_4	G_2
$\dim\mathfrak{g}$	$r(r+2)$	$r(2r+1)$	$r(2r+1)$	$r(2r-1)$	78	133	248	52	14
h	$r+1$	$2r$	$2r$	$2r-2$	12	18	30	12	6
$h^\vee$	$r+1$	$2r-1$	$r+1$	$2r-2$	12	18	30	9	4

(2.3)

The relation $\dim\mathfrak{g} = (1+h)\mathrm{rank}\,\mathfrak{g}$ holds as is well known.

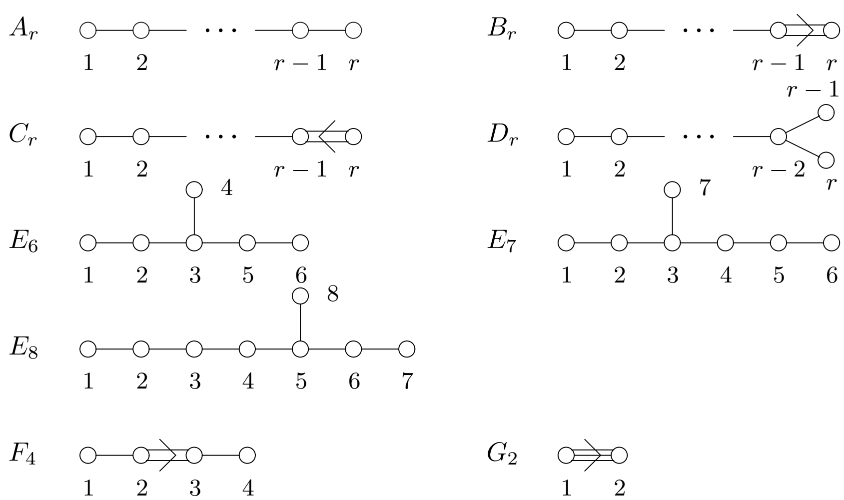


FIGURE 1. The Dynkin diagrams for $\mathfrak{g}$ and their enumerations.

The unrestricted T-system for $\mathfrak{g}$ is the following relations among the commuting variables $\{T^{(a)}_m(u) \mid a \in I, m \in \mathbb{Z}_{\geq 1}, u \in U\}$, where $T^{(0)}_m(u) = T^{(a)}_0(u) = 1$ if they occur in the RHS.

For simply laced $\mathfrak{g}$,

$$T^{(a)}_m(u-1)T^{(a)}_m(u+1) = T^{(a)}_{m-1}(u)T^{(a)}_{m+1}(u) + \prod_{b\in I: C_{ab}=-1} T^{(b)}_m(u). \tag{2.4}$$

For example in type A_r, it has the form

$$T^{(a)}_m(u-1)T^{(a)}_m(u+1) = T^{(a)}_{m-1}(u)T^{(a)}_{m+1}(u) + T^{(a-1)}_m(u)T^{(a+1)}_m(u), \tag{2.5}$$

for $1 \leq a \leq r$ with $T^{(r+1)}_m(u) = 1$. In particular, for A_1 it reads

$$T_m(u-1)T_m(u+1) = T_{m-1}(u)T_{m+1}(u) + 1 \tag{2.6}$$

with the simplified notation $T_m(u) = T^{(1)}_m(u)$.

For $\mathfrak{g} = B_r$,

$$
\begin{aligned}
T_m^{(a)}(u-1)T_m^{(a)}(u+1) &= T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) \\
&\quad + T_m^{(a-1)}(u)T_m^{(a+1)}(u) \quad (1 \le a \le r-2), \\
T_m^{(r-1)}(u-1)T_m^{(r-1)}(u+1) &= T_{m-1}^{(r-1)}(u)T_{m+1}^{(r-1)}(u) + T_m^{(r-2)}(u)T_{2m}^{(r)}(u), \\
T_{2m}^{(r)}\left(u-\tfrac{1}{2}\right)T_{2m}^{(r)}\left(u+\tfrac{1}{2}\right) &= T_{2m-1}^{(r)}(u)T_{2m+1}^{(r)}(u) \\
&\quad + T_m^{(r-1)}\left(u-\tfrac{1}{2}\right)T_m^{(r-1)}\left(u+\tfrac{1}{2}\right), \\
T_{2m+1}^{(r)}\left(u-\tfrac{1}{2}\right)T_{2m+1}^{(r)}\left(u+\tfrac{1}{2}\right) &= T_{2m}^{(r)}(u)T_{2m+2}^{(r)}(u) + T_m^{(r-1)}(u)T_{m+1}^{(r-1)}(u).
\end{aligned}
\tag{2.7}
$$

For $\mathfrak{g} = C_r$,

$$
\begin{aligned}
T_m^{(a)}\left(u-\tfrac{1}{2}\right)T_m^{(a)}\left(u+\tfrac{1}{2}\right) &= T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) \\
&\quad + T_m^{(a-1)}(u)T_m^{(a+1)}(u) \quad (1 \le a \le r-2), \\
T_{2m}^{(r-1)}\left(u-\tfrac{1}{2}\right)T_{2m}^{(r-1)}\left(u+\tfrac{1}{2}\right) &= T_{2m-1}^{(r-1)}(u)T_{2m+1}^{(r-1)}(u) \\
&\quad + T_{2m}^{(r-2)}(u)T_m^{(r)}\left(u-\tfrac{1}{2}\right)T_m^{(r)}\left(u+\tfrac{1}{2}\right), \\
T_{2m+1}^{(r-1)}\left(u-\tfrac{1}{2}\right)T_{2m+1}^{(r-1)}\left(u+\tfrac{1}{2}\right) &= T_{2m}^{(r-1)}(u)T_{2m+2}^{(r-1)}(u) \\
&\quad + T_{2m+1}^{(r-2)}(u)T_m^{(r)}(u)T_{m+1}^{(r)}(u), \\
T_m^{(r)}(u-1)T_m^{(r)}(u+1) &= T_{m-1}^{(r)}(u)T_{m+1}^{(r)}(u) + T_{2m}^{(r-1)}(u).
\end{aligned}
\tag{2.8}
$$

For $\mathfrak{g} = F_4$,

$$
\begin{aligned}
T_m^{(1)}(u-1)T_m^{(1)}(u+1) &= T_{m-1}^{(1)}(u)T_{m+1}^{(1)}(u) + T_m^{(2)}(u), \\
T_m^{(2)}(u-1)T_m^{(2)}(u+1) &= T_{m-1}^{(2)}(u)T_{m+1}^{(2)}(u) + T_m^{(1)}(u)T_{2m}^{(3)}(u), \\
T_{2m}^{(3)}\left(u-\tfrac{1}{2}\right)T_{2m}^{(3)}\left(u+\tfrac{1}{2}\right) &= T_{2m-1}^{(3)}(u)T_{2m+1}^{(3)}(u) \\
&\quad + T_m^{(2)}\left(u-\tfrac{1}{2}\right)T_m^{(2)}\left(u+\tfrac{1}{2}\right)T_{2m}^{(4)}(u), \\
T_{2m+1}^{(3)}\left(u-\tfrac{1}{2}\right)T_{2m+1}^{(3)}\left(u+\tfrac{1}{2}\right) &= T_{2m}^{(3)}(u)T_{2m+2}^{(3)}(u) + T_m^{(2)}(u)T_{m+1}^{(2)}(u)T_{2m+1}^{(4)}(u), \\
T_m^{(4)}\left(u-\tfrac{1}{2}\right)T_m^{(4)}\left(u+\tfrac{1}{2}\right) &= T_{m-1}^{(4)}(u)T_{m+1}^{(4)}(u) + T_m^{(3)}(u).
\end{aligned}
\tag{2.9}
$$

For $\mathfrak{g} = G_2$,

$$
\begin{aligned}
T_m^{(1)}(u-1)T_m^{(1)}(u+1) &= T_{m-1}^{(1)}(u)T_{m+1}^{(1)}(u) + T_{3m}^{(2)}(u), \\
T_{3m}^{(2)}\left(u-\tfrac{1}{3}\right)T_{3m}^{(2)}\left(u+\tfrac{1}{3}\right) &= T_{3m-1}^{(2)}(u)T_{3m+1}^{(2)}(u) \\
&\quad + T_m^{(1)}\left(u-\tfrac{2}{3}\right)T_m^{(1)}(u)T_m^{(1)}\left(u+\tfrac{2}{3}\right), \\
T_{3m+1}^{(2)}\left(u-\tfrac{1}{3}\right)T_{3m+1}^{(2)}\left(u+\tfrac{1}{3}\right) &= T_{3m}^{(2)}(u)T_{3m+2}^{(2)}(u) \\
&\quad + T_m^{(1)}\left(u-\tfrac{1}{3}\right)T_m^{(1)}\left(u+\tfrac{1}{3}\right)T_{m+1}^{(1)}(u), \\
T_{3m+2}^{(2)}\left(u-\tfrac{1}{3}\right)T_{3m+2}^{(2)}\left(u+\tfrac{1}{3}\right) &= T_{3m+1}^{(2)}(u)T_{3m+3}^{(2)}(u) \\
&\quad + T_m^{(1)}(u)T_{m+1}^{(1)}\left(u-\tfrac{1}{3}\right)T_{m+1}^{(1)}\left(u+\tfrac{1}{3}\right).
\end{aligned}
\tag{2.10}
$$

We note that these relations are *not* bilinear in general under the boundary condition stated before (2.4). The second terms on the RHS can be of order 0,1,2 and 3 in $T^{(a)}_m(u)$.

The variable $u \in U$ is called the *spectral parameter*. The set U can be either the complex plane $\mathbb{C}$, or the cylinder $\mathbb{C}_\xi := \mathbb{C}/(2\pi\sqrt{-1}/\xi)\mathbb{Z}$ such that $2\pi\sqrt{-1}/\xi \notin \mathbb{Q}$. The choice will not matter seriously, but reflects the underlying algebra.

Remark 2.1. In Section 4 we will see that the T-system for $\mathfrak{g}$ is actually associated with the untwisted quantum affine algebra $U_q(\hat{\mathfrak{g}})$ with $q = e^{\hbar}$ when $U = \mathbb{C}_{t\hbar}$. The choice $U = \mathbb{C}$ corresponds to the Yangian $Y(\mathfrak{g})$ in a similar sense. In this paper we will mostly concern $U_q(\hat{\mathfrak{g}})$ case. Thus we have simply chosen to say T-system for $\mathfrak{g}$ rather than T-system for $U_q(\hat{\mathfrak{g}})$. The latter terminology is more balanced when the twisted case is considered in Section 2.4. Note that the choice $U = \mathbb{C}_\xi$ effectively imposes an additional periodicity $T^{(a)}_m(u) = T^{(a)}_m(u + \frac{2\pi\sqrt{-1}}{\xi})$. By the assumption $2\pi\sqrt{-1}/\xi \notin \mathbb{Q}$, this does not interfere with the T-system. Similar remarks apply to the Y-system in the sequel.

The unrestricted Y-system for $\mathfrak{g}$ is the following relations among commuting variables $\{Y^{(a)}_m(u) \mid a \in I, m \in \mathbb{Z}_{\geq 1}, u \in U\}$, where $Y^{(0)}_m(u) = Y^{(a)}_0(u)^{-1} = 0$ if they occur in the RHS.

For simply laced $\mathfrak{g}$,

$$Y^{(a)}_m(u-1)Y^{(a)}_m(u+1) = \frac{\prod_{b\in I: C_{ab}=-1}(1+Y^{(b)}_m(u))}{(1+Y^{(a)}_{m-1}(u)^{-1})(1+Y^{(a)}_{m+1}(u)^{-1})}. \tag{2.11}$$

For $\mathfrak{g} = B_r$,

$$Y^{(a)}_m(u-1)Y^{(a)}_m(u+1) = \frac{(1+Y^{(a-1)}_m(u))(1+Y^{(a+1)}_m(u))}{(1+Y^{(a)}_{m-1}(u)^{-1})(1+Y^{(a)}_{m+1}(u)^{-1})} \quad (1 \leq a \leq r-2), \tag{2.12}$$

$$Y^{(r-1)}_m(u-1)Y^{(r-1)}_m(u+1) = \frac{\begin{array}{c}(1+Y^{(r-2)}_m(u))(1+Y^{(r)}_{2m-1}(u))(1+Y^{(r)}_{2m+1}(u))\\ \times(1+Y^{(r)}_{2m}\left(u-\frac{1}{2}\right))(1+Y^{(r)}_{2m}\left(u+\frac{1}{2}\right))\end{array}}{(1+Y^{(r-1)}_{m-1}(u)^{-1})(1+Y^{(r-1)}_{m+1}(u)^{-1})},$$

$$Y^{(r)}_{2m}\left(u-\tfrac{1}{2}\right)Y^{(r)}_{2m}\left(u+\tfrac{1}{2}\right) = \frac{1+Y^{(r-1)}_m(u)}{(1+Y^{(r)}_{2m-1}(u)^{-1})(1+Y^{(r)}_{2m+1}(u)^{-1})},$$

$$Y^{(r)}_{2m+1}\left(u-\tfrac{1}{2}\right)Y^{(r)}_{2m+1}\left(u+\tfrac{1}{2}\right) = \frac{1}{(1+Y^{(r)}_{2m}(u)^{-1})(1+Y^{(r)}_{2m+2}(u)^{-1})}.$$

For $\mathfrak{g} = C_r$,

$$Y^{(a)}_m\left(u-\tfrac{1}{2}\right)Y^{(a)}_m\left(u+\tfrac{1}{2}\right) = \frac{(1+Y^{(a-1)}_m(u))(1+Y^{(a+1)}_m(u))}{(1+Y^{(a)}_{m-1}(u)^{-1})(1+Y^{(a)}_{m+1}(u)^{-1})} \quad (1 \leq a \leq r-2), \tag{2.13}$$

$$Y^{(r-1)}_{2m}\left(u-\tfrac{1}{2}\right)Y^{(r-1)}_{2m}\left(u+\tfrac{1}{2}\right) = \frac{(1+Y^{(r-2)}_{2m}(u))(1+Y^{(r)}_m(u))}{(1+Y^{(r-1)}_{2m-1}(u)^{-1})(1+Y^{(r-1)}_{2m+1}(u)^{-1})},$$

$$Y^{(r-1)}_{2m+1}\left(u-\tfrac{1}{2}\right)Y^{(r-1)}_{2m+1}\left(u+\tfrac{1}{2}\right)=\frac{1+Y^{(r-2)}_{2m+1}(u)}{(1+Y^{(r-1)}_{2m}(u)^{-1})(1+Y^{(r-1)}_{2m+2}(u)^{-1})},$$

$$Y^{(r)}_m(u-1)Y^{(r)}_m(u+1)=\frac{\begin{array}{l}(1+Y^{(r-1)}_{2m+1}(u))(1+Y^{(r-1)}_{2m-1}(u))\\ \times(1+Y^{(r-1)}_{2m}\left(u-\tfrac{1}{2}\right))(1+Y^{(r-1)}_{2m}\left(u+\tfrac{1}{2}\right))\end{array}}{(1+Y^{(r)}_{m-1}(u)^{-1})(1+Y^{(r)}_{m+1}(u)^{-1})}.$$

For $\mathfrak{g}=F_4$,

$$Y^{(1)}_m(u-1)Y^{(1)}_m(u+1)=\frac{1+Y^{(2)}_m(u)}{(1+Y^{(1)}_{m-1}(u)^{-1})(1+Y^{(1)}_{m+1}(u)^{-1})},\tag{2.14}$$

$$Y^{(2)}_m(u-1)Y^{(2)}_m(u+1)=\frac{\begin{array}{l}(1+Y^{(1)}_m(u))(1+Y^{(3)}_{2m-1}(u))(1+Y^{(3)}_{2m+1}(u))\\ \times(1+Y^{(3)}_{2m}\left(u-\tfrac{1}{2}\right))(1+Y^{(3)}_{2m}\left(u+\tfrac{1}{2}\right))\end{array}}{(1+Y^{(2)}_{m-1}(u)^{-1})(1+Y^{(2)}_{m+1}(u)^{-1})},$$

$$Y^{(3)}_{2m}\left(u-\tfrac{1}{2}\right)Y^{(3)}_{2m}\left(u+\tfrac{1}{2}\right)=\frac{(1+Y^{(2)}_m(u))(1+Y^{(4)}_{2m}(u))}{(1+Y^{(3)}_{2m-1}(u)^{-1})(1+Y^{(3)}_{2m+1}(u)^{-1})},$$

$$Y^{(3)}_{2m+1}\left(u-\tfrac{1}{2}\right)Y^{(3)}_{2m+1}\left(u+\tfrac{1}{2}\right)=\frac{1+Y^{(4)}_{2m+1}(u)}{(1+Y^{(3)}_{2m}(u)^{-1})(1+Y^{(3)}_{2m+2}(u)^{-1})},$$

$$Y^{(4)}_m\left(u-\tfrac{1}{2}\right)Y^{(4)}_m\left(u+\tfrac{1}{2}\right)=\frac{1+Y^{(3)}_m(u)}{(1+Y^{(4)}_{m-1}(u)^{-1})(1+Y^{(4)}_{m+1}(u)^{-1})}.$$

For $\mathfrak{g}=G_2$,

$$Y^{(1)}_m(u-1)Y^{(1)}_m(u+1)=\frac{\begin{array}{l}(1+Y^{(2)}_{3m-2}(u))(1+Y^{(2)}_{3m+2}(u))\\ \times(1+Y^{(2)}_{3m-1}\left(u-\tfrac{1}{3}\right))(1+Y^{(2)}_{3m-1}\left(u+\tfrac{1}{3}\right))\\ \times(1+Y^{(2)}_{3m+1}\left(u-\tfrac{1}{3}\right))(1+Y^{(2)}_{3m+1}\left(u+\tfrac{1}{3}\right))\\ \times(1+Y^{(2)}_{3m}\left(u-\tfrac{2}{3}\right))(1+Y^{(2)}_{3m}\left(u+\tfrac{2}{3}\right))\\ \times(1+Y^{(2)}_{3m}(u))\end{array}}{(1+Y^{(1)}_{m-1}(u)^{-1})(1+Y^{(1)}_{m+1}(u)^{-1})},\tag{2.15}$$

$$Y^{(2)}_{3m}\left(u-\tfrac{1}{3}\right)Y^{(2)}_{3m}\left(u+\tfrac{1}{3}\right)=\frac{1+Y^{(1)}_m(u)}{(1+Y^{(2)}_{3m-1}(u)^{-1})(1+Y^{(2)}_{3m+1}(u)^{-1})},$$

$$Y^{(2)}_{3m+1}\left(u-\tfrac{1}{3}\right)Y^{(2)}_{3m+1}\left(u+\tfrac{1}{3}\right)=\frac{1}{(1+Y^{(2)}_{3m}(u)^{-1})(1+Y^{(2)}_{3m+2}(u)^{-1})},$$

$$Y^{(2)}_{3m+2}\left(u-\tfrac{1}{3}\right)Y^{(2)}_{3m+2}\left(u+\tfrac{1}{3}\right)=\frac{1}{(1+Y^{(2)}_{3m+1}(u)^{-1})(1+Y^{(2)}_{3m+3}(u)^{-1})}.$$

We stress that the T and Y-systems for nonsimply laced $\mathfrak{g}$ are *not* just a folding of simply laced cases.

We also remark that T and Y-systems for B_2 and C_2 are equivalent and transformed to each other by $T^{(1)}_m(u)\leftrightarrow T^{(2)}_m(u)$ and $Y^{(1)}_m(u)\leftrightarrow Y^{(2)}_m(u)$ reflecting the fact $B_2\simeq C_2$.

2.2. **Restriction.** We fix an integer $\ell \geq 2$ called *level*. Let t_a be the number in (2.1). The level ℓ restricted T-system for $\mathfrak{g}$ (with the unit boundary condition) is the relations (2.4)–(2.10) naturally restricted to $\{T^{(a)}_m(u) \mid a \in I, 1 \leq m \leq t_a\ell - 1, u \in U\}$ by imposing $T^{(a)}_{t_a\ell}(u) = 1$ (the unit boundary condition).

The level ℓ restricted Y-system for $\mathfrak{g}$ is the relations (2.11)–(2.15) naturally restricted to $\{Y^{(a)}_m(u) \mid a \in I, 1 \leq m \leq t_a\ell - 1, u \in U\}$ by imposing $Y^{(a)}_{t_a\ell}(u)^{-1} = 0$.

Note that for $\mathfrak{g}$ nonsimply laced, the above restriction makes sense also at $\ell = 1$. The resulting T and Y-systems become equivalent to the level t restricted T and Y-systems for A_n with $n = \sharp\{a \in I \mid t_a = t\}$ under the rescaling of the spectral parameter $u \to u/t$. One can also consider the level 0 case formally. See around (16.2).

Example 2.2. We write down the level 2 restricted T and Y-systems for A_2:

$$T^{(1)}_1(u-1)T^{(1)}_1(u+1) = 1 + T^{(2)}_1(u), \quad T^{(2)}_1(u-1)T^{(2)}_1(u+1) = 1 + T^{(1)}_1(u),$$
$$Y^{(1)}_1(u-1)Y^{(1)}_1(u+1) = 1 + Y^{(2)}_1(u), \quad Y^{(2)}_1(u-1)Y^{(2)}_1(u+1) = 1 + Y^{(1)}_1(u).$$

Thus they are identical.

Example 2.3. We write down the level 2 restricted T-system for C_2:

$$\begin{aligned}
&T^{(1)}_1(u-\tfrac{1}{2})T^{(1)}_1(u+\tfrac{1}{2}) = T^{(1)}_2(u) + T^{(2)}_1(u),\\
&T^{(1)}_2(u-\tfrac{1}{2})T^{(1)}_2(u+\tfrac{1}{2}) = T^{(1)}_1(u)T^{(1)}_3(u) + T^{(2)}_1(u-\tfrac{1}{2})T^{(2)}_1(u+\tfrac{1}{2}),\\
&T^{(1)}_3(u-\tfrac{1}{2})T^{(1)}_3(u+\tfrac{1}{2}) = T^{(1)}_2(u) + T^{(2)}_1(u),\\
&T^{(2)}_1(u-1)T^{(2)}_1(u+1) = 1 + T^{(1)}_2(u).
\end{aligned}$$

Example 2.4. Level ℓ restricted T-system for A_{r-1} has the form

$$T^{(a)}_m(u-1)T^{(a)}_m(u+1) = T^{(a)}_{m-1}(u)T^{(a)}_{m+1}(u) + T^{(a-1)}_m(u)T^{(a+1)}_m(u)$$

for $1 \leq a \leq r-1$ and $1 \leq m \leq \ell - 1$. It is invariant under the simultaneous transformation $T^{(a)}_m(u) \mapsto T^{(m)}_a(\pm u + \text{const})$ and $r \leftrightarrow \ell$. The similar property holds also for the level ℓ restricted Y-system for A_{r-1}. This symmetry is called the *level-rank duality*.

2.3. **Relation between T and Y-systems.** The unrestricted T-system for $\mathfrak{g}$ has the form

$$T^{(a)}_m(u-\tfrac{1}{t_a})T^{(a)}_m(u+\tfrac{1}{t_a}) = T^{(a)}_{m-1}(u)T^{(a)}_{m+1}(u) + \prod_{(b,k,v)} T^{(b)}_k(v)^{N(a,m,u|b,k,v)}, \quad (2.16)$$

where the last term is a finite product. Then, it is easy to see that the unrestricted Y-system for the same $\mathfrak{g}$ takes the form

$$Y^{(a)}_m(u-\tfrac{1}{t_a})Y^{(a)}_m(u+\tfrac{1}{t_a}) = \frac{\prod_{(b,k,v)}(1+Y^{(b)}_k(v))^{N(b,k,v|a,m,u)}}{(1+Y^{(a)}_{m-1}(u)^{-1})(1+Y^{(a)}_{m+1}(u)^{-1})}. \quad (2.17)$$

The same relation holds also between the level ℓ restricted T and Y-systems.

Let us write (2.16) simply as

$$T^{(a)}_m(u-\tfrac{1}{t_a})T^{(a)}_m(u+\tfrac{1}{t_a}) = T^{(a)}_{m-1}(u)T^{(a)}_{m+1}(u) + M^{(a)}_m(u). \quad (2.18)$$

Theorem 2.5 ([1])**.** *Suppose $T^{(a)}_m(u)$ satisfies the unrestricted T-system for $\mathfrak{g}$. Then*

$$Y^{(a)}_m(u) = \frac{M^{(a)}_m(u)}{T^{(a)}_{m-1}(u)T^{(a)}_{m+1}(u)} \tag{2.19}$$

is a solution of the unrestricted Y-system for $\mathfrak{g}$. The same claim holds between the level ℓ restricted T and Y-systems.

Sketch of proof. This can be directly verified by substituting the resulting relations

$$1 + Y^{(a)}_m(u) = \frac{T^{(a)}_m\big(u - \frac{1}{t_a}\big)T^{(a)}_m\big(u + \frac{1}{t_a}\big)}{T^{(a)}_{m-1}(u)T^{(a)}_{m+1}(u)}, \tag{2.20}$$

$$1 + Y^{(a)}_m(u)^{-1} = \frac{T^{(a)}_m\big(u - \frac{1}{t_a}\big)T^{(a)}_m\big(u + \frac{1}{t_a}\big)}{M^{(a)}_m(u)} \tag{2.21}$$

into the Y-system. Here we demonstrate the calculation for simply laced $\mathfrak{g}$.

$$\begin{aligned}
&Y^{(a)}_m(u-1)Y^{(a)}_m(u+1)\\
&= \frac{\prod_{b:C_{ab}=-1} T^{(b)}_m(u-1)T^{(b)}_m(u+1)}{T^{(a)}_{m-1}(u-1)T^{(a)}_{m+1}(u-1)T^{(a)}_{m-1}(u+1)T^{(a)}_{m+1}(u+1)}\\
&= \frac{\prod_{b:C_{ab}=-1}(T^{(b)}_{m-1}(u)T^{(b)}_{m+1}(u) + \prod_{c:C_{bc}=-1} T^{(c)}_m(u))}{T^{(a)}_{m-2}(u)T^{(a)}_m(u) + \prod_{b:C_{ab}=-1} T^{(b)}_{m-1}(u)}\\
&\quad \times \frac{1}{T^{(a)}_m(u)T^{(a)}_{m+2}(u) + \prod_{b:C_{ab}=-1} T^{(b)}_{m+1}(u)}\\
&= \frac{\prod_{b:C_{ab}=-1}(1 + Y^{(b)}_m(u))}{(1 + Y^{(a)}_{m-1}(u)^{-1})(1 + Y^{(a)}_{m+1}(u)^{-1})}.
\end{aligned}$$

This calculation is valid also at $m = 1$ by formally setting $T^{(a)}_{-1}(u) = 0$. For level ℓ restricted case, it is valid similarly by formally setting $T^{(a)}_{\ell+1}(u) = 0$.

Theorem 2.5 has a natural account from the viewpoint of cluster algebra with coefficients. See Remark 5.5.

Example 2.6. We write down the relation (2.19) for the level 2 restricted T-system for C_2. From Example 2.3, they read

$$Y^{(1)}_1(u) = \frac{T^{(2)}_1(u)}{T^{(1)}_2(u)}, \qquad Y^{(1)}_2(u) = \frac{T^{(2)}_1(u-\frac{1}{2})T^{(2)}_1(u+\frac{1}{2})}{T^{(1)}_1(u)T^{(1)}_3(u)},$$
$$Y^{(1)}_3(u) = \frac{T^{(2)}_1(u)}{T^{(1)}_2(u)}, \qquad Y^{(2)}_1(u) = T^{(1)}_2(u).$$

Thus the specific construction (2.19) automatically imposes the condition $Y^{(1)}_1(u) = Y^{(1)}_3(u)$. On the other hand, the level restricted Y-system alone does *not* restrict itself to such a situation in general.

Remark 2.7. Consider a slight modification of the general T-system relation (2.18) into

$$T^{(a)}_m(u - \tfrac{1}{t_a})T^{(a)}_m(u + \tfrac{1}{t_a}) = T^{(a)}_{m-1}(u)T^{(a)}_{m+1}(u) + g^{(a)}_m(u)M^{(a)}_m(u), \tag{2.22}$$

where $g_m^{(a)}(u)$ is any function satisfying

$$g_m^{(a)}(u - \tfrac{1}{t_a})g_m^{(a)}(u + \tfrac{1}{t_a}) = g_{m-1}^{(a)}(u)g_{m+1}^{(a)}(u). \tag{2.23}$$

Then it is easily checked that the substitution

$$Y_m^{(a)}(u) = \frac{g_m^{(a)}(u)M_m^{(a)}(u)}{T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u)} \tag{2.24}$$

is still a solution of the same Y-system.

2.4. **Twisted case.** Let us proceed to the T and Y-systems associated with the twisted quantum affine algebras following [11, 12]. In this subsection and the next, X_N exclusively denotes a Dynkin diagram of type A_N $(N \geq 2)$, D_N $(N \geq 4)$, or E_6. We keep the enumeration of the nodes of X_N by the set $I = \{1, \dots, N\}$ as in Figure 1. For a pair $(X_N, \kappa) = (A_N, 2)$, $(D_N, 2)$, $(E_6, 2)$, or $(D_4, 3)$, we define the diagram automorphism $\sigma : I \to I$ of X_N of order κ as follows: $\sigma(a) = a$ *except for the following cases in our enumeration*:

$$\begin{aligned}
&\sigma(a) = N + 1 - a \quad (a \in I) && (X_N, \kappa) = (A_N, 2),\\
&\sigma(N-1) = N,\ \sigma(N) = N - 1 && (X_N, \kappa) = (D_N, 2),\\
&\sigma(1) = 6,\ \sigma(2) = 5,\ \sigma(5) = 2,\ \sigma(6) = 1 && (X_N, \kappa) = (E_6, 2),\\
&\sigma(1) = 3,\ \sigma(3) = 4,\ \sigma(4) = 1 && (X_N, \kappa) = (D_4, 3).
\end{aligned} \tag{2.25}$$

Let I/σ be the set of the σ-orbits of nodes of X_N. We choose, at our discretion, a complete set of representatives $I_\sigma \subset I$ of I/σ as

$$I_\sigma = \begin{cases} \{1, 2, \dots, r\} & (X_N, \kappa) = (A_{2r-1}, 2), (A_{2r}, 2), (D_{r+1}, 2),\\ \{1, 2, 3, 4\} & (X_N, \kappa) = (E_6, 2),\\ \{1, 2\} & (X_N, \kappa) = (D_4, 3). \end{cases} \tag{2.26}$$

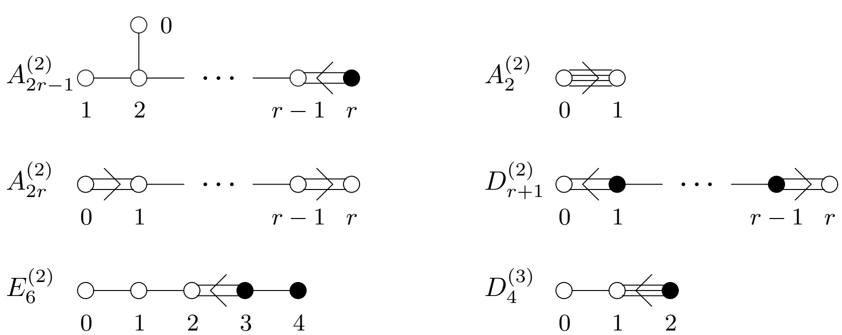
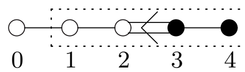
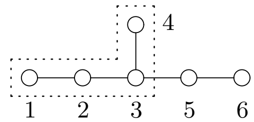


FIGURE 2. The Dynkin diagrams $X_N^{(\kappa)}$ of twisted affine type and their enumerations by $I_\sigma \cup \{0\}$. For a filled node a, $\sigma(a) = a$ (i.e., $\kappa_a = \kappa$) holds.

Let $X_N^{(\kappa)} = A_{2r-1}^{(2)}$ $(r \geq 2), A_{2r}^{(2)}$ $(r \geq 1), D_{r+1}^{(2)}$ $(r \geq 3), E_6^{(2)}$, or $D_4^{(3)}$ be a Dynkin diagram of twisted affine type [10]. We enumerate the nodes of $X_N^{(\kappa)}$ with $I_\sigma \cup \{0\}$ as in Figure 2, where I_σ is the one for (X_N, κ). By this, we have established the identification of the non-0th nodes of the diagram $X_N^{(\kappa)}$ with the nodes of the diagram X_N belonging to the set I_σ. For example, for $E_6^{(2)}$, the correspondence is as follows:

$E_6^{(2)}$ 

E_6 

The filled nodes 3,4 in $E_6^{(2)}$ correspond to the fixed nodes by σ in E_6. We use this identification throughout. (The 0th node of $X_N^{(\kappa)}$ is irrelevant in our setting here.)

We define κ_a $(a \in I_\sigma)$ as

$$\kappa_a = \begin{cases} 1 & \sigma(a) \neq a, \\ \kappa & \sigma(a) = a. \end{cases} \tag{2.27}$$

Note that $X_N^{(2)} = A_{2r}^{(2)}$ is the unique case in which $\kappa_a = 1$ for any $a \in I_\sigma$. By $U_q(X_N^{(\kappa)})$ we mean the quantized universal enveloping algebra [13] of the twisted affine Lie algebra of type $X_N^{(\kappa)}$ [10].

Let us proceed to the unrestricted T-systems. Choose $\hbar \in \mathbb{C} \setminus 2\pi\sqrt{-1}\mathbb{Q}$ arbitrarily. The unrestricted T-system for $U_q(X_N^{(\kappa)})$ is the following relations for commuting variables $\{T_m^{(a)}(u) \mid a \in I_\sigma, m \in \mathbb{Z}_{\geq 1}, u \in \mathbb{C}_{\kappa_a \hbar}\}$, where $\Omega = 2\pi\sqrt{-1}/\kappa\hbar$, and $T_m^{(0)}(u) = T_0^{(a)}(u) = 1$ if they occur in the RHS in the relations:

For $X_N^{(\kappa)} = A_{2r-1}^{(2)}$,

$$\begin{aligned} T_m^{(a)}(u-1)T_m^{(a)}(u+1) &= T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) \\ &\quad + T_m^{(a-1)}(u)T_m^{(a+1)}(u) \quad (1 \leq a \leq r-1), \\ T_m^{(r)}(u-1)T_m^{(r)}(u+1) &= T_{m-1}^{(r)}(u)T_{m+1}^{(r)}(u) + T_m^{(r-1)}(u)T_m^{(r-1)}(u+\Omega). \end{aligned} \tag{2.28}$$

For $X_N^{(\kappa)} = A_{2r}^{(2)}$,

$$\begin{aligned} T_m^{(a)}(u-1)T_m^{(a)}(u+1) &= T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) \\ &\quad + T_m^{(a-1)}(u)T_m^{(a+1)}(u) \quad (1 \leq a \leq r-1), \\ T_m^{(r)}(u-1)T_m^{(r)}(u+1) &= T_{m-1}^{(r)}(u)T_{m+1}^{(r)}(u) + T_m^{(r-1)}(u)T_m^{(r)}(u+\Omega). \end{aligned} \tag{2.29}$$

For $X_N^{(\kappa)} = D_{r+1}^{(2)}$,

$$\begin{aligned} T_m^{(a)}(u-1)T_m^{(a)}(u+1) &= T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) \\ &\quad + T_m^{(a-1)}(u)T_m^{(a+1)}(u) \quad (1 \leq a \leq r-2), \\ T_m^{(r-1)}(u-1)T_m^{(r-1)}(u+1) &= T_{m-1}^{(r-1)}(u)T_{m+1}^{(r-1)}(u) \\ &\quad + T_m^{(r-2)}(u)T_m^{(r)}(u)T_m^{(r)}(u+\Omega), \\ T_m^{(r)}(u-1)T_m^{(r)}(u+1) &= T_{m-1}^{(r)}(u)T_{m+1}^{(r)}(u) + T_m^{(r-1)}(u). \end{aligned} \tag{2.30}$$

For $X_N^{(\kappa)} = E_6^{(2)}$,

$$\begin{aligned}
T_m^{(1)}(u-1)T_m^{(1)}(u+1) &= T_{m-1}^{(1)}(u)T_{m+1}^{(1)}(u) + T_m^{(2)}(u), \\
T_m^{(2)}(u-1)T_m^{(2)}(u+1) &= T_{m-1}^{(2)}(u)T_{m+1}^{(2)}(u) + T_m^{(1)}(u)T_m^{(3)}(u), \\
T_m^{(3)}(u-1)T_m^{(3)}(u+1) &= T_{m-1}^{(3)}(u)T_{m+1}^{(3)}(u) + T_m^{(2)}(u)T_m^{(2)}(u+\Omega)T_m^{(4)}(u), \\
T_m^{(4)}(u-1)T_m^{(4)}(u+1) &= T_{m-1}^{(4)}(u)T_{m+1}^{(4)}(u) + T_m^{(3)}(u).
\end{aligned} \tag{2.31}$$

For $X_N^{(\kappa)} = D_4^{(3)}$,

$$\begin{aligned}
T_m^{(1)}(u-1)T_m^{(1)}(u+1) &= T_{m-1}^{(1)}(u)T_{m+1}^{(1)}(u) + T_m^{(2)}(u), \\
T_m^{(2)}(u-1)T_m^{(2)}(u+1) &= T_{m-1}^{(2)}(u)T_{m+1}^{(2)}(u) \\
&\qquad + T_m^{(1)}(u)T_m^{(1)}(u-\Omega)T_m^{(1)}(u+\Omega).
\end{aligned} \tag{2.32}$$

The domain $\mathbb{C}_{\kappa_a\hbar}$ of the parameter u effectively imposes the following periodicity:

$$T_m^{(a)}(u) = \begin{cases} T_m^{(a)}(u+\kappa\Omega) & \sigma(a) \neq a, \\ T_m^{(a)}(u+\Omega) & \sigma(a) = a. \end{cases} \tag{2.33}$$

Remark 2.8. The T-system for $U_q(X_N^{(\kappa)})$ is obtainable from the T-system for $\mathfrak{g} = X_N$ by a folding in the following sense. Denoting the variable in the latter by $\tilde{T}_m^{(a)}(u)$ with $a \in I$, one imposes the condition $\tilde{T}_m^{(\sigma^k(a))}(u) = \tilde{T}_m^{(a)}(u+k\Omega)$ and identifies $\tilde{T}_m^{(a)}(u)$ with $a \in I_\sigma \subset I$ as the variable $T_m^{(a)}(u)$ in the former. The same remark applies also to the Y-system given in the sequel.

The unrestricted Y-system for $U_q(X_N^{(\kappa)})$ is the following relations for the commuting variables $\{Y_m^{(a)}(u) \mid a \in I_\sigma, m \in \mathbb{Z}_{\geq 1}, u \in \mathbb{C}_{\kappa_a\hbar}\}$, where $\Omega = 2\pi\sqrt{-1}/\kappa\hbar$, and $Y_m^{(0)}(u) = Y_0^{(a)}(u)^{-1} = 0$ if they occur in the RHS in the relations:

For $X_N^{(\kappa)} = A_{2r-1}^{(2)}$,

$$\begin{aligned}
Y_m^{(a)}(u-1)Y_m^{(a)}(u+1) &= \frac{(1+Y_m^{(a-1)}(u))(1+Y_m^{(a+1)}(u))}{(1+Y_{m-1}^{(a)}(u)^{-1})(1+Y_{m+1}^{(a)}(u)^{-1})} \\
&\qquad\qquad (1 \leq a \leq r-1), \\
Y_m^{(r)}(u-1)Y_m^{(r)}(u+1) &= \frac{(1+Y_m^{(r-1)}(u))(1+Y_m^{(r-1)}(u+\Omega))}{(1+Y_{m-1}^{(r)}(u)^{-1})(1+Y_{m+1}^{(r)}(u)^{-1})}.
\end{aligned} \tag{2.34}$$

For $X_N^{(\kappa)} = A_{2r}^{(2)}$,

$$\begin{aligned}
Y_m^{(a)}(u-1)Y_m^{(a)}(u+1) &= \frac{(1+Y_m^{(a-1)}(u))(1+Y_m^{(a+1)}(u))}{(1+Y_{m-1}^{(a)}(u)^{-1})(1+Y_{m+1}^{(a)}(u)^{-1})} \\
&\qquad\qquad (1 \leq a \leq r-1), \\
Y_m^{(r)}(u-1)Y_m^{(r)}(u+1) &= \frac{(1+Y_m^{(r-1)}(u))(1+Y_m^{(r)}(u+\Omega))}{(1+Y_{m-1}^{(r)}(u)^{-1})(1+Y_{m+1}^{(r)}(u)^{-1})}.
\end{aligned} \tag{2.35}$$

For $X_N^{(\kappa)} = D_{r+1}^{(2)}$,

$$Y_m^{(a)}(u-1)Y_m^{(a)}(u+1) = \frac{(1+Y_m^{(a-1)}(u))(1+Y_m^{(a+1)}(u))}{(1+Y_{m-1}^{(a)}(u)^{-1})(1+Y_{m+1}^{(a)}(u)^{-1})} \tag{2.36}$$

$$(1 \le a \le r-2),$$

$$Y_m^{(r-1)}(u-1)Y_m^{(r-1)}(u+1) = \frac{(1+Y_m^{(r-2)}(u))(1+Y_m^{(r)}(u))(1+Y_m^{(r)}(u+\Omega))}{(1+Y_{m-1}^{(r-1)}(u)^{-1})(1+Y_{m+1}^{(r-1)}(u)^{-1})},$$

$$Y_m^{(r)}(u-1)Y_m^{(r)}(u+1) = \frac{1+Y_m^{(r-1)}(u)}{(1+Y_{m-1}^{(r)}(u)^{-1})(1+Y_{m+1}^{(r)}(u)^{-1})}.$$

For $X_N^{(\kappa)} = E_6^{(2)}$,

$$Y_m^{(1)}(u-1)Y_m^{(1)}(u+1) = \frac{1+Y_m^{(2)}(u)}{(1+Y_{m-1}^{(1)}(u)^{-1})(1+Y_{m+1}^{(1)}(u)^{-1})}, \tag{2.37}$$

$$Y_m^{(2)}(u-1)Y_m^{(2)}(u+1) = \frac{(1+Y_m^{(1)}(u))(1+Y_m^{(3)}(u))}{(1+Y_{m-1}^{(2)}(u)^{-1})(1+Y_{m+1}^{(2)}(u)^{-1})},$$

$$Y_m^{(3)}(u-1)Y_m^{(3)}(u+1) = \frac{(1+Y_m^{(2)}(u))(1+Y_m^{(2)}(u+\Omega))(1+Y_m^{(4)}(u))}{(1+Y_{m-1}^{(3)}(u)^{-1})(1+Y_{m+1}^{(3)}(u)^{-1})},$$

$$Y_m^{(4)}(u-1)Y_m^{(4)}(u+1) = \frac{1+Y_m^{(3)}(u)}{(1+Y_{m-1}^{(4)}(u)^{-1})(1+Y_{m+1}^{(4)}(u)^{-1})}.$$

For $X_N^{(\kappa)} = D_4^{(3)}$,

$$Y_m^{(1)}(u-1)Y_m^{(1)}(u+1) = \frac{1+Y_m^{(2)}(u)}{(1+Y_{m-1}^{(1)}(u)^{-1})(1+Y_{m+1}^{(1)}(u)^{-1})}, \tag{2.38}$$

$$Y_m^{(2)}(u-1)Y_m^{(2)}(u+1) = \frac{(1+Y_m^{(1)}(u))(1+Y_m^{(1)}(u-\Omega))(1+Y_m^{(1)}(u+\Omega))}{(1+Y_{m-1}^{(2)}(u)^{-1})(1+Y_{m+1}^{(2)}(u)^{-1})}.$$

2.5. **Restriction and relations between T and Y-systems.** Fix an integer $\ell \ge 2$ called *level*. The level ℓ restricted T-system for $U_q(X_N^{(\kappa)})$ (with the unit boundary condition) is the relations (2.28)–(2.32) naturally restricted to $\{T_m^{(a)}(u) \mid a \in I_\sigma, 1 \le m \le \ell-1, u \in \mathbb{C}_{\kappa_a \hbar}\}$ by imposing $T_\ell^{(a)}(u) = 1$ (the unit boundary condition).

The level ℓ restricted Y-system for $U_q(X_N^{(\kappa)})$ is the relations (2.34)–(2.38) naturally restricted to $\{Y_m^{(a)}(u) \mid a \in I_\sigma, 1 \le m \le \ell-1, u \in \mathbb{C}_{\kappa_a \hbar}\}$ by imposing $Y_\ell^{(a)}(u)^{-1} = 1$.

The properties stated in Theorem 2.5 and Remark 2.7 also hold between the T and Y-systems of for $U_q(X_N^{(\kappa)})$. On the other hand, the correspondence like (2.16) and (2.17) in the untwisted case is not valid.

2.6. $\boldsymbol{U_q(sl(r|s))}$ **case.** Among a variety of Lie super algebras, we present the T-system and the Y-system related to $U_q(sl(r|s))$ as a typical example. For brevity we employ the following notation within this subsection.

$$H_{r,s} = (\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}) \setminus (\mathbb{Z}_{\ge r} \times \mathbb{Z}_{\ge s}), \quad \overline{H}_{r,s} = (\mathbb{Z}_{\ge 0} \times \mathbb{Z}_{\ge 0}) \setminus (\mathbb{Z}_{>r} \times \mathbb{Z}_{>s}). \tag{2.39}$$

These sets are often called fat hook. The T-system for $U_q(sl(r|s))$ is the following relations among the commuting variables $\{T^{(a)}_m(u) \mid (a,m) \in \overline{H}_{r,s}, u \in U\}$.

$$T^{(a)}_m(u-1)T^{(a)}_m(u+1) = T^{(a-1)}_m(u)T^{(a+1)}_m(u) + T^{(a)}_{m-1}(u)T^{(a)}_{m+1}(u), \tag{2.40}$$
$$T^{(r)}_{s+1}(u) = T^{(r+1)}_s(u). \tag{2.41}$$

The relation (2.40) is imposed for all $(a,m) \in \overline{H}_{r,s} \setminus \{(0,0)\}$, where if any $T^{(b)}_k(u)$ with $(b,k) \notin \overline{H}_{r,s}$ is contained in the RHS, it should be understood as 0.

$$T^{(b)}_k(u) = 0 \quad \text{if } (b,k) \notin \overline{H}_{r,s}. \tag{2.42}$$

This leads to the simple recursion relations for the sequences corresponding to the boundary $\overline{H}_{r,s} \setminus H_{r,s}$.

$$\begin{aligned} T^{(a)}_m(u-1)T^{(a)}_m(u+1) &= T^{(a)}_{m+1}(u)T^{(a)}_{m-1}(u) \qquad (a,m) \in (r, \mathbb{Z}_{>s}) \cup (0, \mathbb{Z}_{>0}), \\ T^{(a)}_m(u-1)T^{(a)}_m(u+1) &= T^{(a-1)}_m(u)T^{(a+1)}_m(u) \quad (a,m) \in (\mathbb{Z}_{>r}, s) \cup (\mathbb{Z}_{>0}, 0). \end{aligned} \tag{2.43}$$

The extra relation (2.41) leads by induction to

$$T^{(r)}_{s+a}(u) = T^{(r+a)}_s(u) \quad a \geq 0. \tag{2.44}$$

In the applications, the variables appearing in (2.43) and (2.44) are chosen appropriately reflecting the normalization of the system. The relation (2.40) is the same as type A case. The essential difference from it lies in (2.42) and (2.44).

Let us proceed to the Y-system. We assume $r \geq s \geq 2$ first. The Y-system for $U_q(sl(r|s))$ is the following relations among the commuting variables $\{\Upsilon^{(a)}_1(u), \Upsilon^{(a)}_2(u) \mid a \in \mathbb{Z}_{\geq 1}, u \in U\} \cup \{Y^{(a)}_m(u) \mid (a,m) \in H_{r,s}, u \in U\}$.

$$Y^{(a)}_m(u-1)Y^{(a)}_m(u+1) = \frac{(1+Y^{(a)}_{m+1}(u))(1+Y^{(a)}_{m-1}(u))}{(1+Y^{(a-1)}_m(u)^{-1})(1+Y^{(a+1)}_m(u)^{-1})} \quad (a,m) \in H_{r,s}, \tag{2.45}$$

$$\Upsilon^{(1)}_1(u-1)\Upsilon^{(1)}_1(u+1) = \Upsilon^{(2)}_2(u)(1+Y^{(1)}_{s-1}(u)), \tag{2.46}$$

$$\Upsilon^{(a)}_1(u-1)\Upsilon^{(a)}_1(u+1) = \Upsilon^{(a+1)}_1(u)\Upsilon^{(a-1)}_1(u)\frac{1+Y^{(a)}_{s-1}(u)}{1+Y^{(a-1)}_s(u)} \quad a \geq 2, \tag{2.47}$$

$$\Upsilon^{(a)}_2(u-1)\Upsilon^{(a)}_2(u+1) = \Upsilon^{(a+1)}_2(u)\Upsilon^{(a-1)}_2(u)(1+Y^{(a)}_{s-1}(u)) \quad a \geq 2, \tag{2.48}$$

$$\Upsilon^{(1)}_1(u) = \Upsilon^{(1)}_2(u), \quad \Upsilon^{(r)}_1(u) = Y^{(r)}_s(u). \tag{2.49}$$

In the RHS of these relations, any factor $(1+Y^{(b)}_k(u)^{\pm 1})$ with $(b,k) \notin H_{r,s}$ is to be understood as 1. When $r > s = 1$, the equations (2.46) and (2.48) are absent. The Y-system for $s \geq r \geq 2$ is given by (2.45)–(2.49) by interchanging r and s.

There is a simple relation between the T-system and Y-system analogous to Theorem 2.5. Suppose that $T^{(a)}_m(u)$ is a solution to the T-system. Then the combinations

$$Y^{(a)}_m(u) = \frac{T^{(a)}_{m+1}(u)T^{(a)}_{m-1}(u)}{T^{(a+1)}_m(u)T^{(a-1)}_m(u)} \qquad (a,m) \in H_{r,s}, \tag{2.50}$$

$$\Upsilon^{(a)}_1(u) = \frac{T^{(a)}_{s-1}(u)}{T^{(a-1)}_s(u)}, \qquad \Upsilon^{(a)}_2(u) = \frac{T^{(a)}_{s-1}(u)}{T^{(0)}_{s+a-1}(u)} \tag{2.51}$$

satisfy the Y-system. In particular, (2.49) holds due to (2.41). When $s \geq r \geq 2$, the parallel fact holds by interchanging r and s and the role of indices a and m in $T^{(a)}_m(u)$ and $Y^{(a)}_m(u)$ everywhere. In view of the symmetry of the sets (2.39), we do not introduce the level restriction.

Remark 2.9. The above set of relations seem different from those given in [14] for $gl(2|2)$, where a special relation $T^{(r)}_{s-2} \propto T^{(r-2)}_s$ valid only for this case is utilized. Thanks to this, $\Upsilon^{(a)}_1$ $(a \neq r)$ and $\Upsilon^{(a)}_2$ are not necessarily needed. The two sets of Y-systems nevertheless lead to an identical set of thermodynamic Bethe ansatz equations[6]. The Y-system (2.45)–(2.49) is consistent with the thermodynamic Bethe ansatz equations in [15] under the identification $N, K \leftrightarrow r, s$ and

$$Y^{(a)}_{s-m} = \mathrm{e}^{-\zeta^{(a)}_m/T}\,(1 \leq a, 1 \leq m \leq s-1), \qquad Y^{(a)}_s = \mathrm{e}^{-\epsilon_a/T}\,(1 \leq a \leq r),$$
$$Y^{(a)}_{s+j} = \mathrm{e}^{\kappa^{(j)}_a/T}\,(1 \leq j, 1 \leq a \leq r-1).$$

2.7. **Bibliographical notes.** The Hirota relation (2.5) for transfer matrices in A_r case first appeared in [1], where the T-system for $\mathfrak{g}$ was introduced as functional relations among the commuting transfer matrices $\{T^{(a)}_m(u)\}$. The models relevant to the unrestricted and restricted versions are the vertex and the restricted solid-on-solid (RSOS) type models, respectively. In such a setting, T-system acquires some scalar coefficients depending on the normalization of $T^{(a)}_m(u)$ as in Remark 2.7. The unit boundary condition is also modified accordingly. Actually in [1], the restricted T-system was introduced by imposing a slightly weaker condition $T^{(a)}_{t_a\ell+1}(u) = 0$. The T-system for the twisted case was introduced in [11] in a similar context. Our presentation here follows [16, 12]. The T-system unifies the many functional relations studied earlier individually. See Sections 3-4 for more detail.

The level ℓ restricted Y-system for $\mathfrak{g}$ was introduced by [3] for simply laced $\mathfrak{g}$ with $\ell = 2$ as a universal property of the thermodynamic Bethe ansatz (TBA) equation in the context of integrable perturbations of conformal field theories. Then, it was extended to the general case by [4] based on the TBA equation related to RSOS models for $U_q(\hat{\mathfrak{g}})$ [17]. This procedure is detailed in Section 14. The Y-system for simply laced $\mathfrak{g}$ was also given by [5] independently. For more literatures in the similar context, see Section 14.7. The transformation (2.19) between the T and Y-systems first appeared in [18] for the simplest case $\mathfrak{g} = A_1$, and extended in [1] to general $\mathfrak{g}$. T-systems related to Lie super algebras and super symmetric models have been studied in various contexts. See for example [14, 19, 20, 21, 22, 23, 24] and reference therein.

[6]There are typos in [14] for $gl(2|2)$, around (5.4) and (5.5).

3. T-system among commuting transfer matrices

The aim of this section is to introduce the basic examples of solvable lattice models, both vertex and restricted solid-on-solid (RSOS) type, and demonstrate how the T-system is obtained for their transfer matrices in connection to the fusion procedure. Although these issues are nowadays well recognized to be intimately related to the representation theory of quantum groups, we defer such a description to Section 4 avoiding too many definitions from the onset. Our presentation here is based on explicit calculations in trigonometric parameterization along the simplest example from $\mathfrak{g} = A_1$ The exception is the last subsection 3.7, where we will formally argue the general features of those models associated with general $\mathfrak{g}$ quoting known facts on Kirillov-Reshetikhin modules and Q-system from Sections 4, 13.6 and 14.6.

3.1. **Vertex models and fusion.** We recall the 6 vertex model and its fusion without much recourse to the representation theory[7]. Consider the two dimensional square lattice, where each edge is assigned with a local variable belonging to $\{1, 2\}$. Around each vertex, we allow the following 6 configurations with the respective Boltzmann weights.

$$\begin{array}{cccccc}
1 & 2 & 1 & 2 & 2 & 1 \\
1 \,+\, 1 & 2 \,+\, 2 & 2 \,+\, 2 & 1 \,+\, 1 & 1 \,+\, 2 & 2 \,+\, 1 \\
1 & 2 & 1 & 2 & 1 & 2 \\
1-q^2z & 1-q^2z & q(1-z) & q(1-z) & z(1-q^2) & 1-q^2.
\end{array} \tag{3.1}$$

The other 10 configurations are assigned with 0 Boltzmann weight. Let $V = \mathbb{C}v_1 \oplus \mathbb{C}v_2$. Then (3.1) is arranged in the quantum R matrix $R(z) \in \mathrm{End}(V \otimes V)$ as

$$\begin{aligned}
&R(z) = a(z)\sum_i E_{ii}\otimes E_{ii} + b(z)\sum_{i\neq j} E_{ii}\otimes E_{jj} + c(z)\Bigl(z\sum_{i<j} + \sum_{i>j}\Bigr)E_{ji}\otimes E_{ij},\\
&a(z) = 1-q^2z, \quad b(z) = q(1-z), \quad c(z) = 1-q^2.
\end{aligned} \tag{3.2}$$

Here the indices run over $\{1, 2\}$ and E_{ij} is the matrix unit acting as $E_{ij}v_k = \delta_{jk}v_i$. The R matrix $R(z)$ is associated with the quantum affine algebra $U_q = U_q(A^{(1)}_1)$ [13]. In fact, $\check{R}(z) := PR(z)$ commutes with $\Delta(U_q)$, where P denotes the transposition of the components[8]. A more account will be given in Section 4.3. Schematically (3.2) is expressed as

$$R(z) = \sum_{ijkl}\Bigl(j \,\underset{k}{\overset{l}{+}}{}_{z}\, i\Bigr)E_{ij}\otimes E_{kl}, \qquad \check{R}(z) = \sum_{ijkl}\Bigl(j \,\underset{i}{\overset{l}{+}}{}_{z}\, k\Bigr)E_{ij}\otimes E_{kl}, \tag{3.3}$$

where the z dependence is exhibited. The Yang-Baxter equation

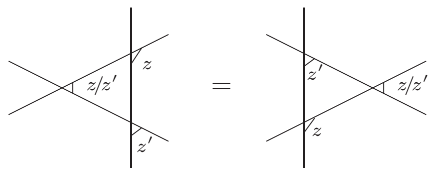
$$R_{23}(z')R_{13}(z)R_{12}(z/z') = R_{12}(z/z')R_{13}(z)R_{23}(z')$$

holds [2], where the indices signify the components in the tensor product as $\overset{1}{V} \otimes \overset{2}{V} \otimes \overset{3}{V}$ on which the both sides act. It is depicted as

---

[7] Some terminology will be refined after (3.16).

[8] The asymmetry between the last two in (3.1) is due to our choice of the coproduct (4.9). It fits the crystal base theory making the limit $q \to 0$ of (3.7) well defined, although this fact will not be used in this paper.



(3.4)

Starting from the 6 vertex model [25, 26], one can construct higher spin solvable vertex models by the fusion procedure [27]. Let V_m be the irreducible U_q module spanned by the m fold $q-$symmetric tensors. Concretely, $V_1 = V$ and V_m with $m \geq 2$ is realized as the quotient $V^{\otimes m}/A$, where $A = \sum_j V^{\otimes j} \otimes \mathrm{Im}\, \check{R}(q^{-2}) \otimes V^{\otimes m-2-j}$. It is easy to see $\mathrm{Im}\, \check{R}(q^{-2}) = \mathrm{Ker}\, \check{R}(q^2) = \mathbb{C}(v_1 \otimes v_2 - qv_2 \otimes v_1)$. We take the base vector of V_m as $v_2^{\otimes x_2} \otimes v_1^{\otimes x_1} \mod A$, where $x_i \in \mathbb{Z}_{\geq 0}$ and $x_1 + x_2 = m$. The base will also be denoted by $x = (x_1, x_2)$ for brevity. Obviously $\dim V_m = m + 1$.

The Yang-Baxter equation (3.4) with $z' = zq^2$ tells that $\mathrm{Im}\, \check{R}(q^{-2}) \subset \overset{1}{V} \otimes \overset{2}{V}$ is preserved under the action of $R_{13}(zq^2)R_{23}(z)$. Therefore its action on $(\overset{1}{V} \otimes \overset{2}{V}) \otimes \overset{3}{V}$ can be restricted to $V_2 \otimes V_1 = \big((V \otimes V)/\mathrm{Im}\, \check{R}(q^{-2})\big) \otimes V$. Similarly, by using (3.4) repeatedly, it is shown that the composition

$$\frac{R_{1,m+1}(zq^{m-1})R_{2,m+1}(zq^{m-3}) \cdots R_{m,m+1}(zq^{-m+1})}{a(zq^{m-3})a(zq^{m-5}) \cdots a(zq^{-m+1})} \tag{3.5}$$

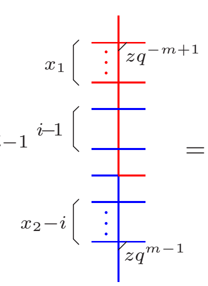
can be restricted to $V_m \otimes V_1$. The resulting operator, the fusion R matrix $R^{(m,1)}(z) \in \mathrm{End}(V_m \otimes V_1)$, is given by

$$R^{(m,1)}(z)(x \otimes v_j) = \sum_{k=1,2} \Big(x \overset{j}{\underset{k}{\boldsymbol{+}}}_{z}\, y \Big)\, y \otimes v_k, \tag{3.6}$$

$$x \overset{j}{\underset{k}{\boldsymbol{+}}}_{z}\, y = \begin{cases} q^{m-x_k} - q^{x_k+1}z & j = k, \\ (1 - q^{2x_1})z & (j,k) = (2,1), \\ 1 - q^{2x_2} & (j,k) = (1,2), \end{cases} \tag{3.7}$$

where $y = (y_1, y_2)$ is specified by the weight conservation (so called "ice rule") as $y_i = x_i + \delta_{ij} - \delta_{ik}$. By the definition $R^{(1,1)}(z) = R(z)$ and (3.7) reduces to (3.1) for $m = 1$. In the case $(j,k) = (1,2)$ for example, the matrix element $1 - q^{2x_2}$ is obtained from the following calculation (D =denominator in (3.5)):

$$\frac{1}{D} \sum_{i=1}^{x_2} q^{i-1}$$



$$= \sum_{i=1}^{x_2} \frac{q^{i-1}(1-q^2)a(zq^{m-1}) \prod_{n=x_1+1}^{x_1+i-1} b(zq^{-m-1+2n})}{\prod_{n=x_1+1}^{x_1+i} a(zq^{-m-1+2n})}. \tag{3.8}$$

The red and blue edges are assigned with the local states 1 and 2, respectively. The incoming state (left column) represents $v_2^{\otimes x_2} \otimes v_1^{\otimes x_1}$. The factor q^{i-1} accounts for the effect of rearranging the outgoing state into the base form by using the relation

$v_1 \otimes v_2 \equiv q v_2 \otimes v_1 \mod A$ as

$$v_2^{\otimes x_2 - i} \otimes v_1 \otimes v_2^{\otimes i-1} \otimes v_1^{\otimes x_1} \equiv q^{i-1} v_2^{\otimes y_2} \otimes v_1^{\otimes y_1} \in V_m,$$

where $y = (y_1, y_2) = (x_1 + 1, x_2 - 1)$ for $(j, k) = (1, 2)$.

One can fuse $R^{(m,1)}(z)$ further along the other component of the tensor product in a completely parallel fashion. The composition

$$R^{(m,1)}_{0,n}(zq^{n-1}) \cdots R^{(m,1)}_{0,2}(zq^{-n+3}) R^{(m,1)}_{0,1}(zq^{-n+1}) \in \mathrm{End}(V_m \otimes V_1^{\otimes n}) \tag{3.9}$$

can be restricted to $V_m \otimes V_n$. The result yields the quantum R matrix $R^{(m,n)}(z) \in \mathrm{End}(V_m \otimes V_n)$. The R matrices so obtained again satisfy the Yang-Baxter equation in $\mathrm{End}(V_l \otimes V_m \otimes V_n)$:

$$R^{(m,n)}_{23}(z') R^{(l,n)}_{13}(z) R^{(l,m)}_{12}(z/z') = R^{(l,m)}_{12}(z/z') R^{(l,n)}_{13}(z) R^{(m,n)}_{23}(z'). \tag{3.10}$$

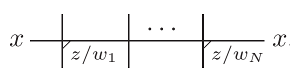
It is depicted as (3.4) with the three lines to be interpreted as representing V_l, V_m and V_n.

The quantum R matrix $R^{(m,n)}(z)$ gives rise to a fusion vertex model on a planar square lattice by the same rule as the diagrams (3.3) and (3.6). The local variables on the horizontal and vertical edges are taken from V_m and V_n, respectively.

3.2. **Transfer matrices.** Here we use the additive spectral parameter u as well as the multiplicative one z. They are related as $z = q^u$. We introduce the row to row transfer matrix

$$T_m(u) = \mathrm{Tr}_{V_m}\left(R^{(m,s_N)}_{0,N}(z/w_N) \cdots R^{(m,s_1)}_{0,1}(z/w_1)\right)$$

$$= \sum_{x \in V_m}$$



(3.11)

The horizontal line is associated with V_m which is called the auxiliary space. The trace over it corresponds to the periodic boundary condition. There are N vertical lines corresponding to $V_{s_1} \otimes \cdots \otimes V_{s_N}$ which is called the quantum space. The $T_m(u)$ is a linear operator acting on the quantum space. The data s_i, w_i represent the inhomogeneity in the spins and coupling constants.

The first consequence of the Yang-Baxter equation (3.10) is the commutativity of the transfer matrices acting on the common quantum space (common s_i and w_i in the present context)

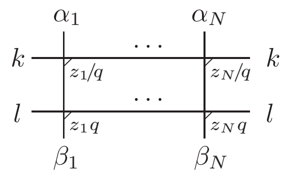
$$[T_m(u), T_n(v)] = 0. \tag{3.12}$$

Let us take $s_i = 1$ for all i for simplicity and demonstrate the functional relation

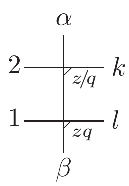
$$\begin{aligned} &T_1(u+1)T_1(u-1) = T_0(u)T_2(u) + g_1(u)\mathrm{id}, \\ &T_0(u) = \prod_{i=1}^{N} a(z_i/q), \quad g_1(u) = \prod_{i=1}^{N} a(z_i q) b(z_i/q), \end{aligned} \tag{3.13}$$

where $z_i = z/w_i$. This corresponds to the T-system for A_1 (2.6) with $m = 1$ modified by a model dependent factors $T_0(u)$ and $g_1(u)$. Consider the diagram for $T_1(u+1)T_1(u-1)$ corresponding to the matrix element for the transition $v_{\alpha_1} \otimes$

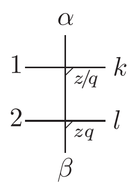
$\cdots \otimes v_{\alpha_N} \mapsto v_{\beta_1} \otimes \cdots \otimes v_{\beta_N}$:

$$\sum_{k,l=1,2}$$  (3.14)

Given α_i, β_i, the sum over k, l is regarded as the trace of an operator acting on the auxiliary space $V_1 \otimes V_1$ horizontally. The space $V_1 \otimes V_1$ possesses the invariant subspace $\operatorname{Im} \check{R}(q^{-2}) = \mathbb{C}(v_1 \otimes v_2 - q v_2 \otimes v_1)$ which propagates to the right owing to the Yang-Baxter equation (3.4). In fact, the following identity can be checked directly.



$- \quad q\times$



$$= \delta_{\alpha\beta} a(zq) b(z/q) \times \begin{cases} 1 & (k,l) = (2,1), \\ -q & (k,l) = (1,2), \\ 0 & \text{otherwise}. \end{cases} \tag{3.15}$$

Thus $\operatorname{Im} \check{R}(q^{-2})$ contributes to $\operatorname{Tr}_{V_1 \otimes V_1}$ (3.14) as $\prod_{i=1}^{N} \delta_{\alpha_i, \beta_i} a(z_i q) b(z_i/q)$, giving the second term in the RHS of (3.13). The other contribution to the trace is from $(V_1 \otimes V_1)/\operatorname{Im} \check{R}(q^{-2}) = V_2$. This is equal to $T_0(u) T_2(u)$ by the definition, where the factor $T_0(u)$ is due to the denominator in (3.5) with $m = 2$. In this way one observes that the exact sequence

$$0 \to \operatorname{Im} \check{R}(q^{-2}) \to V_1 \otimes V_1 \to V_2 \to 0 \tag{3.16}$$

plays a key role in deriving (3.13).

In Section 4.2, we will introduce the Kirillov-Reshetikhin module $W_m^{(a)}(u)$ for general quantum affine algebra $U_q(\hat{\mathfrak{g}})$. The case $\mathfrak{g} = A_1$ relevant here, denoted by $W_m(u) = W_m^{(1)}(u)$, will be described explicitly in Section 4.3. In such a formalism, one endows each line in the diagrams like (3.14)–(3.15) with a spectral parameter $z = q^u$ which corresponds to a Kirillov-Reshetikhin module $W_m(u)$. The R matrix $R^{(m,n)}(z) \in \operatorname{End}(V_m \otimes V_n)$ is actually to be understood as $R^{(m,n)}(z_1/z_2) \in \operatorname{End}(W_m(u_1) \otimes W_n(u_2))$ with $z_i = q^{u_i}$. Up to an overall scalar, it is characterized by the intertwining property $\Delta(g) P R^{(m,n)}(z_1/z_2) = P R^{(m,n)}(z_1/z_2) \Delta(g)$ where g is any element from $U_q(A_1^{(1)})$ and Δ is the coproduct (4.9) [13]. Accordingly, we say that the transfer matrix $T_m(u)$ (3.11) has the auxiliary space $W_m(u)$ and acts on the quantum space $W_{s_1}(v_1) \otimes \cdots \otimes W_{s_N}(v_N)$ with $w_i = q^{v_i}$.

The exact sequence (3.16) will also be refined into the one among tensor product of Kirillov-Reshetikhin modules. See (4.16). The T-system relation $T_m(u+1) T_m(u-1) = T_{m+1}(u) T_{m-1}(u) + g_m(u)\mathrm{id}$ for general m follows from Theorem 4.2 with $n = j = m$. An additional feature here is that one actually needs to consider the central extension of $U_q(A_1^{(1)})$ to properly cope with the factor $g_m(u)$. We refer to [1, section 2.2] for this point. See also [28].

To summarize, the Kirillov-Reshetikhin module of the quantum affine algebra and their exact sequence form the representation theoretical background for the R matrix, fusion procedure and the T-system among commuting family of transfer matrices.

3.3. **Restricted solid-on-solid (RSOS) models and fusion.** Besides vertex models, there are another class of solvable lattice models called Interaction Round Face (IRF or simply face) models [2]. The relation of the two classes of models has been studied from various viewpoints [29, 30, 31, 32, 33]. Here we recall the 8 vertex solid-on-solid (8VSOS) model [34]. It is the fundamental example associated with $U_q(A_1^{(1)})$ at q a root of unity and serves as the prototype of restricted solid-on-solid (RSOS) models. It generalizes to $U_q(\hat{\mathfrak{g}})$ for any $\mathfrak{g}$ in principle. We illustrate the fusion procedure [35] and the derivation of the simplest case of the T-system for the commuting transfer matrices [36, 37]. The contents are parallel with the 6 vertex model discussed in the previous subsection. For simplicity we concentrate on the critical case[9].

Consider the two dimensional square lattice, where each site is assigned with a local state belonging to $\mathbb{Z}$. On the two local states a, b on neighboring sites, the condition $|a-b| = 1$ is imposed. With the allowed configuration round a face, the following Boltzmann weights are assigned [34].

$$
\begin{aligned}
&W\left(\begin{matrix} a & a\mp1 \\ a\pm1 & a \end{matrix}\middle| u\right) = \frac{[2+u]_{q^{1/2}}}{[2]_{q^{1/2}}}, \quad W\left(\begin{matrix} a\pm1 & a \\ a & a\pm1 \end{matrix}\middle| u\right) = \frac{[2\xi+2a\mp u]_{q^{1/2}}}{[2\xi+2a]_{q^{1/2}}}, \\
&W\left(\begin{matrix} a\pm1 & a \\ a & a\mp1 \end{matrix}\middle| u\right) = \frac{[2\xi+2a\pm 2]_{q^{1/2}}[u]_{q^{1/2}}}{[2\xi+2a]_{q^{1/2}}[2]_{q^{1/2}}},
\end{aligned}
\tag{3.17}
$$

where u is the spectral parameter, q and ξ are generic constants which will be specialized when considering the restriction in Section 3.5. The function $[u]_{q^{1/2}}$ is given by replacing $q \to q^{1/2}$ in

$$
[u]_q = \frac{q^u - q^{-u}}{q - q^{-1}}. \tag{3.18}
$$

The Boltzmann weights (3.17) are depicted as

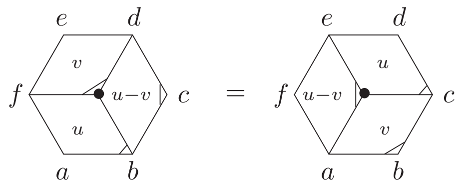
$$
\begin{array}{ccc} b & & c \\ & \boxed{u} & \\ a & & d \end{array} = W\left(\begin{matrix} b & c \\ a & d \end{matrix}\middle| u\right). \tag{3.19}
$$

It satisfies the (generalized) star-triangle relation [2] which plays the role of the Yang-Baxter equation in face models:

$$
\begin{aligned}
&\sum_g W\left(\begin{matrix} f & g \\ a & b \end{matrix}\middle| u\right) W\left(\begin{matrix} e & d \\ f & g \end{matrix}\middle| v\right) W\left(\begin{matrix} g & d \\ b & c \end{matrix}\middle| u-v\right) \\
&= \sum_g W\left(\begin{matrix} e & d \\ g & c \end{matrix}\middle| u\right) W\left(\begin{matrix} g & c \\ a & b \end{matrix}\middle| v\right) W\left(\begin{matrix} f & e \\ a & g \end{matrix}\middle| u-v\right).
\end{aligned}
\tag{3.20}
$$

[9]The RSOS models allow elliptic Boltzmann weights in general. The critical case means the trigonometric case of them. The fusion procedure and the T-system are equally valid in the elliptic case as well.

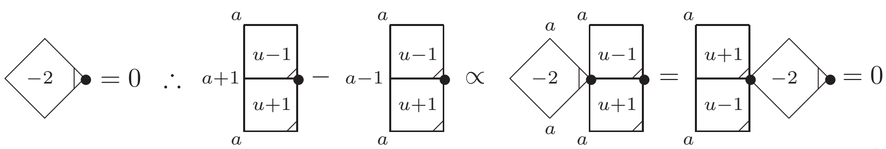
The sum over g consists of at most two terms in each side because of the neighboring condition, e.g., $|f-g| = |b-g| = |d-g| = 1$ for the LHS. We depict (3.20) as



(3.21)

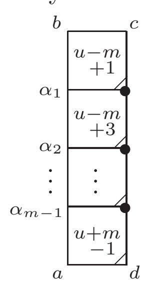
where • stands for the sum over the local state. The faces drawn together are to be understood as the product of the attached Boltzmann weights.

One can apply the fusion procedure to the 8VSOS model [35]. Note the properties



(3.22)

where the second equality from the right is due to the star-triangle relation. This implies that for $m = 2$, the quantity

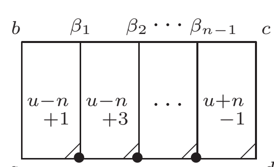


(3.23)

is independent of $\alpha_1, \ldots, \alpha_{m-1}$ as long as they are chosen so that $|\alpha_i - \alpha_{i+1}| = 1$ ($\alpha_0 = b$, $\alpha_m = a$). The independence for general m can be shown similarly. Thus (3.23) only depends on the local states a, b, c, d on the corners. We define the fused Boltzmann weight $W_{m,1}\left(\begin{matrix} b & c \\ a & d \end{matrix}\middle| u\right)$ to be (3.23) divided by $\prod_{j=1}^{m-1} \frac{[u+m+1-2j]_{q^{1/2}}}{[2]_{q^{1/2}}}$. By induction on m, the following formulas are easily established ($W_{1,1} = W$).

$$
\begin{aligned}
W_{m,1}\left(\begin{matrix} b & b\mp 1 \\ a & a\mp 1 \end{matrix}\middle| u\right) &= \frac{[2\xi + a + b \pm m]_{q^{1/2}}\,[1 \pm (a-b) + u]_{q^{1/2}}}{[2]_{q^{1/2}}[2\xi + 2a]_{q^{1/2}}}, \\
W_{m,1}\left(\begin{matrix} b & b\pm 1 \\ a & a\mp 1 \end{matrix}\middle| u\right) &= \frac{[m \pm (a-b)]_{q^{1/2}}\,[2\xi + a + b \pm 1 \pm u]_{q^{1/2}}}{[2]_{q^{1/2}}[2\xi + 2a]_{q^{1/2}}}.
\end{aligned}
\tag{3.24}
$$

One can fuse them further in the horizontal direction. A similar argument shows that the quantity



(3.25)

is independent of $\beta_1, \dots, \beta_{n-1}$ as long as $|\beta_i - \beta_{i+1}| = 1$ ($\beta_0 = b$, $\beta_n = c$). Here each rectangle stands for the weight $W_{m,1}$ (3.24) with the specified spectral parameters. Thus we define $W_{m,n}\left(\begin{matrix} b & c \\ a & d \end{matrix}\middle| u\right)$ to be (3.25). By construction, it is zero unless

$$b-a, c-d \in \{-m, -m+2, \dots, m\}, \quad c-b, d-a \in \{-n, -n+2, \dots, n\}. \tag{3.26}$$

The the star-triangle relation (3.20) is generalized to

$$\begin{aligned} &\sum_g W_{l,n}\left(\begin{matrix} f & g \\ a & b \end{matrix}\middle| u\right) W_{m,n}\left(\begin{matrix} e & d \\ f & g \end{matrix}\middle| v\right) W_{l,m}\left(\begin{matrix} g & d \\ b & c \end{matrix}\middle| u-v\right) \\ &= \sum_g W_{l,n}\left(\begin{matrix} e & d \\ g & c \end{matrix}\middle| u\right) W_{m,n}\left(\begin{matrix} g & c \\ a & b \end{matrix}\middle| v\right) W_{l,m}\left(\begin{matrix} f & e \\ a & g \end{matrix}\middle| u-v\right). \end{aligned} \tag{3.27}$$

3.4. **Relation to vertex models.** The trigonometric face models under consideration are related to the 6 vertex model and its fusion in Section 3.1 [30]. Let us explain it along the simplest cases (3.17) and (3.2). Let $a \in \mathbb{Z}_{\geq 3}$ and V_{a-1} be the spin $\frac{a-1}{2}$ representation of $U_q(A_1)$ in Section 3.1[10]. We use the coproduct (4.9) and the concrete form (4.10). In the irreducible decomposition $V_{a-1} \otimes V_1 = \bigoplus_{b=a\pm 1} V_{b-1}$, the highest weight vector $v_{a,b} \in V_{b-1}$ is given by $v_{a,a+1} = v_1^{a-1} \otimes v_1^1$ and $v_{a,a-1} = v_1^{a-1} \otimes v_2^1 - q^{a-1} v_2^{a-1} \otimes v_1^1$. Repeating this once more, one gets the highest weight vectors $v_{a,b,c}$ in the irreducible component V_{c-1} in the decomposition of $V_{a-1} \otimes V_1 \otimes V_1$ labeled with a, b, c such that $|a-b| = |b-c| = 1$. Explicitly, they read

$$\begin{aligned} v_{a,a+1,a+2} &= v_1^{a-1} \otimes v_1^1 \otimes v_1^1, \\ v_{a,a-1,a} &= [a-1]_q (v_1^{a-1} \otimes v_2^1 \otimes v_1^1 - q^{a-1} v_2^{a-1} \otimes v_1^1 \otimes v_1^1), \\ v_{a,a+1,a} &= [a]_q v_1^{a-1} \otimes v_1^1 \otimes v_2^1 - q^{a-1}[a-1]_q v_2^{a-1} \otimes v_1^1 \otimes v_1^1 - q^a v_1^{a-1} \otimes v_2^1 \otimes v_1^1, \\ v_{a,a-1,a-2} &= v_1^{a-1} \otimes v_2^1 \otimes v_2^1 - q^{a-1} v_2^{a-1} \otimes v_1^1 \otimes v_2^1 - q^{a-2} v_2^{a-1} \otimes v_2^1 \otimes v_1^1 \\ &\quad + q^{2a-4} v_3^{a-1} \otimes v_1^1 \otimes v_1^1. \end{aligned} \tag{3.28}$$

Now consider the operator $1 \otimes \check{R}(z)$ acting on $V_{a-1} \otimes V_1 \otimes V_1$. Since it commutes with $U_q(A_1)$, the image of the highest weight vectors are again highest. The face Boltzmann weights can be extracted from the matrix elements between those highest weight vectors as

$$\left(1 \otimes \check{R}(q^u)\right) v_{a,b,c} = -(q-q^{-1}) q^{1+\frac{u}{2}} \sum_d W\left(\begin{matrix} b & c \\ a & d \end{matrix}\middle| u\right) v_{a,d,c}. \tag{3.29}$$

Here $\xi = 0$ in the RHS and the sum is over b such that $|a-d| = |d-c| = 1$. A similar relation holds also between the fusion models.

Conversely, one can deduce the R matrix from the face Boltzmann weights as a limit where the site variables or effectively ξ tends to infinity. For instance, (3.7) is

[10] Actually, V_{a-1} can be the Verma module with the highest weight vector v_1^{a-1} such that $k_1 v_1^{a-1} = q^{a-1} v_1^{a-1}$ for generic a.

obtained from (3.24) as

$$
-(q-q^{-1})q^{\frac{m+1+u}{2}}\frac{(a,b)_m(b,c)_1}{(d,c)_m(a,d)_1}\lim_{q^\xi\to 0}W_{m,1}\left(\begin{matrix} b & c \\ a & d\end{matrix}\middle| u\right) = x \overset{j}{\underset{k}{+_{z}}} y, \tag{3.30}
$$

$$
(a,b)_m = q^{\frac{1}{8}(a-b)^2+\frac{1}{4}m(a+b)}, \quad z = q^u, \tag{3.31}
$$

$$
x=(x_1,x_2)=\left(\frac{m-a+b}{2},\frac{m+a-b}{2}\right), \quad j=\frac{3+b-c}{2},\ k=\frac{3+a-d}{2}. \tag{3.32}
$$

The factor in the LHS of (3.30) does not spoil the star-triangle relation.

3.5. **Restriction.** The (fusion) face models constructed thus far possess local states ranging over the infinite set $\mathbb{Z}$ and are called unrestricted. To obtain a model with finitely many local states, we make *restriction*. We introduce the integer $\ell \in \mathbb{Z}_{\geq 2}$ called *level*, and specialize the parameters as follows:

$$
\xi = 0, \quad q = \exp\left(\frac{\pi\sqrt{-1}}{\ell+2}\right), \quad [u]_{q^{1/2}} = \frac{\sin\frac{\pi u}{2(\ell+2)}}{\sin\frac{\pi}{2(\ell+2)}}. \tag{3.33}
$$

We further set $W_{m,n}\left(\begin{matrix} b & c \\ a & d\end{matrix}\middle| u\right) = 0$ unless the pairs $(a,b),(d,c)$ (resp. $(a,d),(b,c)$) are m-admissible (resp. n-admissible). We say that a pair (a,b) is m-admissible if

$$
b-a \in \{-m,-m+2,\dots,m\}, \tag{3.34}
$$

$$
a+b \in \{m+2,m+4,\dots,2\ell+2-m\}. \tag{3.35}
$$

Notice that the admissibility forces $a,b \in \{1,2,\dots,\ell+1\}$. The resulting $W_{m,n}\left(\begin{matrix} b & c \\ a & d\end{matrix}\middle| u\right)$ with $a,b,c,d \in \{1,2,\dots,\ell+1\}$ is called the restricted Boltzmann weight. One may wonder if $[0]_{q^{1/2}} = [2\ell+4]_{q^{1/2}} = 0$ may cause a divergence somewhere in the construction. However it has been proved [35] that the restricted Boltzmann weights are well-defined and satisfy the star-triangle relation (3.27) among themselves[11]. In this way one obtains the level ℓ RSOS model whose local states belong to $\{1,2,\dots,\ell+1\}$ and the fusion degree specified by m and n.

Let us comment on the admissibility condition among which the first one (3.34) already appeared in (3.26). When $\ell \to \infty$, the admissibility reduces to the Clebsch-Gordan rule:

$$
V_{a-1}\otimes V_m = \bigoplus_{b-1=|a-1-m|,\dots,a+m-3,a+m-1} V_{b-1}. \tag{3.36}
$$

The RHS contains precisely those b such that (a,b) is m-admissible at $\ell=\infty$. For ℓ finite, the necessity of $a+b \leq 2\ell+2-m$ can be seen for example in the first Boltzmann weight in (3.24). Under the specialization (3.33), it contains the factor $\sin\left(\frac{\pi(a+b+m)}{2\ell+4}\right)$ in the numerator. Thus the "next"' b for which $a+b = 2\ell+4-m$ "can not be reached". Such a truncation is also observed at the level of characters associated with (3.36). Denoting the q-dimension of V_{a-1} at root of unity by $\dim_q V_{a-1} = \sin\left(\frac{\pi a}{\ell+2}\right)/\sin\left(\frac{\pi}{\ell+2}\right)$, we have

$$
(\dim_q V_m)(\dim_q V_{a-1}) = \sum_{b:(a,b)\text{ is } m\text{-admissible}} \dim_q V_{b-1}. \tag{3.37}
$$

[11] Actually the statement holds for appropriately symmetrized $W_{m,n}$. See [35, section 2.2].

This truncated decomposition is also known as the fusion rule in the SU(2) level ℓ WZW conformal field theory [38].

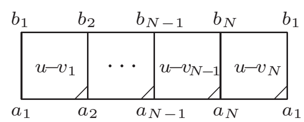
Finally we remark that given ℓ, one can not fuse too much. In fact, (3.34) and (3.35) fix the admissible pairs to $\{(a,a) \mid 1 \le a \le \ell+1\}$ at $m=0$ and to $\{(1,\ell+1),(\ell+1,1)\}$ at $m=\ell$. They lead to completely frozen models. Nontrivial situations correspond to the fusion degrees in the range $1 \le m \le \ell-1$. This is an origin of the truncation condition in the restricted T-system (Section 2.2) for $\mathfrak{g} = A_1$.

3.6. **Transfer matrices.** We consider the row to row transfer matrix $T_m(u)$ with periodic boundary condition whose elements $T_m(u)^{b_1,\ldots,b_N}_{a_1,\ldots,a_N}$ are given by

$$W_{m,s_1}\left(\begin{matrix} b_1 & b_2 \\ a_1 & a_2 \end{matrix}\middle| u-v_1\right) \cdots W_{m,s_{N-1}}\left(\begin{matrix} b_{N-1} & b_N \\ a_{N-1} & a_N \end{matrix}\middle| u-v_{N-1}\right) W_{m,s_N}\left(\begin{matrix} b_N & b_1 \\ a_N & a_1 \end{matrix}\middle| u-v_N\right).$$

No sum is involved. It is depicted as

$$T_m(u)^{b_1,\ldots,b_N}_{a_1,\ldots,a_N} = \tag{3.38}$$



Here $(a_i, a_{i+1}), (b_i, b_{i+1})$ are s_i-admissible $(a_{N+1} = a_1, b_{N+1} = b_1)$ and (a_i, b_i) is m-admissible for all i. The inhomogeneity s_i, v_i in fusion degrees and coupling constants are fixed and suppressed in the notation. The $T_m(u)$ is zero unless the parity condition $\sum_{i=1}^N s_i \equiv 0 \mod 2$ is satisfied. The star-triangle relation (3.27) implies the commutativity [2]

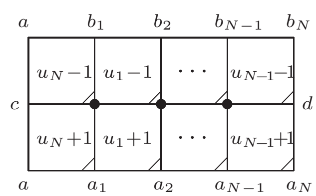
$$[T_m(u), T_n(v)] = 0. \tag{3.39}$$

Let us take $s_1 = 1$ for all i for simplicity and demonstrate the functional relation

$$\begin{aligned} &T_1(u+1)T_1(u-1) = T_0(u)T_2(u) + g_1(u)\mathrm{id}, \\ &T_0(u) = \prod_{i=1}^N \frac{[u_i+1]_{q^{1/2}}}{[2]_{q^{1/2}}}, \quad g_1(u) = \prod_{i=1}^N \frac{[u_i+3]_{q^{1/2}}[u_i-1]_{q^{1/2}}}{[2]^2_{q^{1/2}}}, \end{aligned} \tag{3.40}$$

where $u_i = u - v_i$. Set

$$\mathcal{L}_{c,d} = \tag{3.41}$$



where each face stands for $W = W_{1,1}$. To the difference $\mathcal{L}_{a+1,d} - \mathcal{L}_{a-1,d}$, one can apply the same trick as (3.22). In particular, the repeated use of the star-triangle relation and the property $W\left(\begin{matrix} b & c \\ a & d \end{matrix}\middle| -2\right) \propto \delta_{ac}$ tells that it vanishes unless $a_i = b_i$ for all i. Then the induction on N leads to the identity

$$\mathcal{L}_{a+1,d} - \mathcal{L}_{a-1,d} = \frac{[2a]_{q^{1/2}}}{[2a_N]_{q^{1/2}}} \prod_{i=1}^N \left(\delta_{a_i,b_i} \frac{[u_i+3]_{q^{1/2}}[u_i-1]_{q^{1/2}}}{[2]^2_{q^{1/2}}}\right) \times \begin{cases} 1 & d = a_N+1, \\ -1 & d = a_N - 1. \end{cases} \tag{3.42}$$

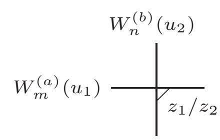
Now we are ready to evaluate the matrix elements of $T_1(u+1)T_1(u-1)$. When $a_N = b_N = a$, we have

$$
\begin{aligned}
(T_1(u+1)T_1(u-1))^{b_1,\ldots,b_N}_{a_1,\ldots,a_N} &= \mathcal{L}_{a-1,a-1} + \mathcal{L}_{a+1,a+1} \\
&= \mathcal{L}_{a-1,a-1} + \mathcal{L}_{a-1,a+1} + \mathcal{L}_{a+1,a+1} - \mathcal{L}_{a-1,a+1}
\end{aligned}
$$

The first two terms yield $T_0(u)T_2(u)^{b_1,\ldots,b_N}_{a_1,\ldots,a_N}$ by the definition (3.23). The other two terms are equal to $(g_1(u)\mathrm{id})^{b_1,\ldots,b_N}_{a_1,\ldots,a_N}$ due to (3.42) with $a = a_N$. When $a_N = b_N \pm 2$, one can more easily check (3.40) since $g_1(u)\mathrm{id}$ does not contribute.

3.7. **Vertex and RSOS models for general $\mathfrak{g}$.** We include a formal and partly conjectural description of solvable vertex and RSOS models and their T-system for general $\mathfrak{g}$. We will use the terminology introduced in later sections. (Therefore this technical section may better be skipped on the first reading.)

Let $W^{(a)}_m(u)$ be the Kirillov-Reshetikhin module (Section 4.2), where $a \in I$ (set of vertices on the Dynkin diagram of $\mathfrak{g}$) and $m \in \mathbb{Z}_{\geq 1}$. It is an irreducible finite dimensional representation of untwisted quantum affine algebra $U_q = U_q(\hat{\mathfrak{g}})$. Up to an overall scalar, there is the unique element, the R matrix, $R \in \mathrm{End}(W^{(a)}_m(u_1) \otimes W^{(b)}_n(u_2))$ characterized by the intertwining property $\Delta(U_q)PR = PR\Delta(U_q)$, where P is the transposition. It can in principle be constructed concretely by solving this linear equation, or by the fusion of the simpler cases $m = n = 1$ (cf. Theorem 4.3) or by taking the image of the universal R. Let us denote the resulting R matrix by $R^{(a,m;b,n)}(z_1/z_2)$, where $z_i = q^{tu_i}$, t is defined by (2.1) and the dependence through z_1/z_2 is due to the general theory.

$R^{(a,m;b,n)}(z_1/z_2) =$ 

(3.43)

As in (3.11), one introduces the row to row transfer matrix with the auxiliary space $W^{(a)}_m(u)$ by $(z = q^{tu})$

$$
T^{(a)}_m(u) = \mathrm{Tr}_{W^{(a)}_m(u)}\left(R^{(a,m;r_N,s_N)}_{0,N}(z/w_N)\cdots R^{(a,m;r_1,s_1)}_{0,1}(z/w_1)\right), \tag{3.44}
$$

which acts on the quantum space $W^{(r_1)}_{s_1}(v_1)\otimes\cdots\otimes W^{(r_N)}_{s_N}(v_N)$ with $w_i = q^{tv_i}$. They are all commutative, i.e., $[T^{(a)}_m(u), T^{(b)}_n(v)] = 0$ thanks to the Yang-Baxter relation. It is a corollary of the exact sequence underlying Theorem 4.8 and the argument on the central extension (cf. [1, section 2.2]) that $T^{(a)}_m(u)$ satisfies the unrestricted T-system for $\mathfrak{g}$ (2.22) with some scalars $T^{(a)}_0(u)$ and $g^{(a)}_m(u)$ appropriately chosen depending on the normalization of $T^{(a)}_m(u)$.

Let $\ell \in \mathbb{Z}_{\geq 2}$. From the R matrix one can in principle construct the face Boltzmann weights for level ℓ $U_q(\hat{\mathfrak{g}})$ RSOS model at $q = \exp\left(\frac{\pi\sqrt{-1}}{t(\ell+h^\vee)}\right)$[12]. Let us introduce

$$
P_+ = \mathbb{Z}_{\geq 0}\,\omega_1 \oplus \cdots \oplus \mathbb{Z}_{\geq 0}\,\omega_r, \quad P_\ell = \{\lambda \in P_+ \mid (\lambda|\text{maximal root}) \leq \ell\}, \tag{3.45}
$$

[12] Actually any primitive $2t(\ell+h^\vee)$ th root of unity. $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$ (2.3).

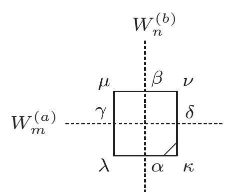
where ω_a is a fundamental weight of $\mathfrak{g}$ (Section 2.1). P_ℓ is the classical projection of the set of level ℓ dominant integral weights of the affine Lie algebra $\hat{\mathfrak{g}}$ at level ℓ [10]. For $\lambda \in P_+$, let V_λ be the irreducible $U_q(\mathfrak{g})$-module with highest weight λ. Let $\mathrm{res}\, W^{(a)}_m$ be the (not necessarily irreducible) $U_q(\mathfrak{g})$-module obtained by restricting the $U_q(\hat{\mathfrak{g}})$-module $W^{(a)}_m(u)$. It is independent of u. See around (4.23). When q is not a root of unity, one has the irreducible decomposition

$$V_\lambda \otimes \mathrm{res}\, W^{(a)}_m \otimes \mathrm{res}\, W^{(b)}_n = \bigoplus_{\mu \in P_+} \Omega(\lambda)_\mu \otimes V_\mu, \tag{3.46}$$

where $\Omega(\lambda)_\mu$ is the space of highest weight vectors of weight μ. Since $\check{R}(z) = PR^{(a,m;b,n)}(z)$ commutes with $U_q(\mathfrak{g})$, the space $\Omega(\lambda)_\mu$ is invariant under $\mathrm{id} \otimes \check{R}(z)$. Thus its matrix elements yield the Boltzmann weights of unrestricted SOS model as in (3.29). The star-triangle relation for them follows from this construction.

To make the restriction, we consider the case $q = \exp\left(\frac{\pi\sqrt{-1}}{t(\ell+h^\vee)}\right)$, where the decomposition (3.46) no longer holds [39, 40]. However, based on the observation for $\mathfrak{g} = A_1$ [30], we conjecture that if λ is taken from P_ℓ and $m < t_a\ell, n \le t_b\ell$, the quotient of the RHS of (3.46) by the type I modules [41, 42][13] reduces the sum $\mu \in P_+$ to $\mu \in P_\ell$, and $\mathrm{id} \otimes \check{R}(z)$ remains well defined on it. Then the RSOS Boltzmann weights are defined as the matrix elements of $\mathrm{id} \otimes \check{R}(z)$ on the quotient space, and satisfy the star-triangle relation.

The RSOS model so constructed has the fluctuating variables on edges as well as sites in general (cf. [43, Fig.1]).



$$\beta \in \Omega^{(b,n)}_{\mu\nu} \qquad \gamma \in \Omega^{(a,m)}_{\lambda\mu} \qquad \delta \in \Omega^{(a,m)}_{\kappa\nu} \qquad \alpha \in \Omega^{(b,n)}_{\lambda\kappa} \tag{3.47}$$

The site variables belong to P_ℓ. In fact for $\mathfrak{g} = A_1$, one may regard the set of site variables $\{1, 2, \ldots, \ell{+}1\}$ as $P_\ell = \{0, \omega_1, \ldots, \ell\omega_1\}$. To describe the edge variables, we consider the decomposition $V_\lambda \otimes \mathrm{res}\, W^{(a)}_m = \bigoplus_{\mu \in P_+} \overline{\Omega}^{(a,m)}_{\lambda\mu} \otimes V_\mu$ at generic q. When $q = \exp\left(\frac{\pi\sqrt{-1}}{t(\ell+h^\vee)}\right)$, we need to take the quotient of the RHS by the type I modules, and this induces the quotient $\Omega^{(a,m)}_{\lambda\mu}$ of $\overline{\Omega}^{(a,m)}_{\lambda\mu}$. The edge variable associated to $W^{(a)}_m$ belongs to the space $\Omega^{(a,m)}_{\lambda\mu}$. We set $\mathcal{A}^{(a,m)}_{\lambda\mu} = \dim \Omega^{(a,m)}_{\lambda\mu}$ and say that an (ordered) pair of site variables $(\lambda, \mu) \in P_\ell \times P_\ell$ is admissible under $W^{(a)}_m$ if $\mathcal{A}^{(a,m)}_{\lambda\mu} \ge 1$[14]. The matrix $\mathcal{A}^{(a,m)} = (\mathcal{A}^{(a,m)}_{\lambda\mu})_{\lambda,\mu \in P_\ell}$ is called the admissibility matrix of $W^{(a)}_m$.

---

[13]Indecomposable modules with $\dim_q = 0$. See (14.49).

[14] The type A_r is bit special in that $\mathcal{A}^{(a,m)}_{\lambda\mu} \in \{0, 1\}$ holds for any (a, m) and λ, μ, hence effectively no edge variable exists. However, the situation $\mathcal{A}^{(a,m)}_{\lambda\mu} \ge 2$ still happens for the fusion types more general than those specified by Kirillov-Reshetikhin modules [43].

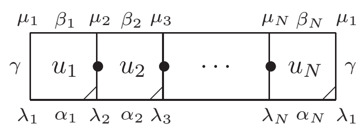
Let us formulate the row to row transfer matrix $T^{(a)}_m(u)$ that corresponds to the dual of the one for the vertex model (3.44). It acts on the space of paths

$$\mathcal{H}(N) = \bigoplus_{\lambda_i \in P_\ell} \Omega^{(r_1,s_1)}_{\lambda_1\lambda_2} \otimes \cdots \otimes \Omega^{(r_N,s_N)}_{\lambda_N\lambda_1}, \tag{3.48}$$

$$\dim \mathcal{H}(N) = \mathrm{Tr}\left(\mathcal{A}^{(r_1,s_1)} \cdots \mathcal{A}^{(r_N,s_N)}\right). \tag{3.49}$$

The matrix elements are depicted as follows ($u_i = u - v_i, \lambda_i = \lambda_{i+N}, \mu_i = \mu_{i+N}$):

$$T^{(a)}_m(u)^{\mu_1,\beta_1,\mu_2,\beta_2,\ldots,\mu_N,\beta_N}_{\lambda_1,\alpha_1,\lambda_2,\alpha_2,\ldots,\lambda_N,\alpha_N} = \sum_{\gamma\in\Omega^{(a,m)}_{\lambda_1\mu_1}}$$



(3.50)

Here the symbols α_i and β_i denote a basis of $\Omega^{(r_i,s_i)}_{\lambda_i\lambda_{i+1}}$ and $\Omega^{(r_i,s_i)}_{\mu_i\mu_{i+1}}$, respectively. The pairs $(\lambda_i, \lambda_{i+1})$ and (μ_i, μ_{i+1}) are both admissible under $W^{(r_i)}_{s_i}$, whereas (λ_i, μ_i) is so under $W^{(a)}_m$. The RHS stands for the product of the N Boltzmann weights attached to the elementary squares summed over the states on the vertical edges accommodating $\Omega^{(a,m)}_{\lambda_i\mu_i}$ for $i = 1, \ldots, N$. As for the weights, $\lambda_{i+1} - \lambda_i \equiv \mu_{i+1} - \mu_i \equiv s_i\omega_{r_i}$ mod the root lattice, therefore the $T^{(a)}_m(u)$ under consideration is vanishing unless

$$\sum_{i=1}^{N} s_i \left(C^{-1}\right)_{a\,r_i} \in \mathbb{Z} \quad \text{for all } a \in I, \tag{3.51}$$

where C is the Cartan matrix of $\mathfrak{g}$ (Section 2.1). Due to the star-triangle relation (including sums over edge variables), the commutativity $[T^{(a)}_m(u), T^{(b)}_n(v)] = 0$ holds. We conjecture that $T^{(a)}_m(u)$ satisfies the level ℓ restricted T-system for $\mathfrak{g}$ of the form (2.22) with some scalars $T^{(a)}_0(u)$ and $g^{(a)}_m(u)$ appropriately chosen depending on the normalization. In particular, this implies that the $|P_\ell|$ by $|P_\ell|$ matrices $\mathcal{A}^{(a,m)}$ with $a \in I, 0 \le m \le t_a\ell$ are commutative and satisfy the level ℓ restricted Q-system (cf. Section 14.5) with the boundary condition

$$\mathcal{A}^{(a,1)} = 1, \quad \mathcal{A}^{(a,t_a\ell+1)} = 0^{15}. \tag{3.52}$$

Let $\dim_q V_\lambda$ be the q-dimension of V_λ at $q = \exp\left(\frac{\pi\sqrt{-1}}{t(\ell+h^\vee)}\right)$ defined in (14.49). We set $Q^{(a)}_m = \dim_q \mathrm{res}\, W^{(a)}_m$, which supposedly satisfies the level ℓ restricted Q-system (14.5) (Conjecture 14.2). Now the generalization of (3.37) is given as

$$Q^{(a)}_m \dim_q V_\lambda = \sum_{\mu\in P_\ell} \mathcal{A}^{(a,m)}_{\lambda\mu} \dim_q V_\mu \quad (\lambda \in P_\ell). \tag{3.53}$$

Since $\dim_q V_\lambda > 0$ for any $\lambda \in P_\ell$, the Perron-Frobenius theorem tells that $Q^{(a)}_m$ is the largest eigenvalue of the admissibility matrix $\mathcal{A}^{(a,m)}$. Therefore in the homogeneous case where $(r_i, s_i) = (p, s)$ for all i, we find from (3.49) that

$$\lim_{N\to\infty} (\dim \mathcal{H}(N))^{1/N} = Q^{(p)}_s. \tag{3.54}$$

[15] This leads to $\prod_{b\in I}(\mathcal{A}^{(b,t_b\ell)})^{C_{ab}} = 1$ for any $a \in I$, which is a weaker constraint than $\mathcal{A}^{(a,t_a\ell)} = 1$ employed in the definition of the level ℓ restricted Q-system in Section 14.5).

This property will be re-derived in the TBA analysis in (15.20)

In general, the Boltzmann weights (3.47) are expressed in terms of the function $[u]_{q^{t/2}} \propto \sin\left(\frac{\pi u}{2(\ell+h^\vee)}\right)$. ($t$ is defined in (2.1).) This is indeed the case for A_1 as in (3.17) and in the other known examples. It is also consistent with the Bethe equation (8.25). Consequently, the transfer matrix with an appropriate normalization possesses the periodicity

$$T^{(a)}_m(u + 2(\ell + h^\vee)) = T^{(a)}_m(u). \tag{3.55}$$

We will see in Theorem 5.7 that the level ℓ restricted T-system in Section 2.2[16] alone compels this property.

3.8. **Bibliographical notes.** The integrability of the 6 vertex model (3.1) (first solved in [25, 26]) has been formulated in terms of the Yang-Baxter equation and commuting transfer matrices in [2]. Solutions of the Yang-Baxter equation that have been known by 1980 are surveyed in [44] from the perspective of the quantum inverse scattering method. Subsequent generalizations of trigonometric vertex models to type A [45, 46, 47] and many other $\mathfrak{g}$ [48, 49] have been assembled in the reprint volume [50]. The fusion of vertex models is formulated in [27]. See also [51]. The idea of utilizing the functional relations of transfer matrices goes back to Baxter [52, 2]. Some simplest examples of the T-system have been obtained for the XXZ chain [53], the $O(n)$-symmetry models [54] and vertex models associated with some other $\mathfrak{g}$ [55].

With regard to the RSOS models, the 8VSOS model is the fundamental example containing the Ising and (generalized) hard hexagon models as the level $\ell = 2, 3$ cases, respectively. The one point function [34] essentially gives rise to the character of the Virasoro minimal series, and this fact inspired intensive studies on the relations with conformal field theory and representation theory of quantum affine algebras. In the terminology in Section 3.7, the 8VSOS model corresponds to the level ℓ RSOS model for $\mathfrak{g} = A_1$ with fusion type $W^{(1)}_1$ (both on the horizontal and vertical edges).

Beyond the A_1 case, concrete constructions of RSOS models for untwisted affine Lie algebra $\hat{\mathfrak{g}}$ have been done for non exceptional series $\mathfrak{g} = A_r, B_r, C_r, D_r$ [56, 57] associated with $W^{(1)}_1$ ("vector representation") and $\mathfrak{g} = G_2$ [58] with $W^{(2)}_1$. The fusion of RSOS models have been worked out explicitly only for type A [35, 43]. One of the earliest examples of the T-system for RSOS models (except the Ising) is [36] for the generalized hard hexagon model. It was systematized to the general level restricted T-system for A_1 in [18]. See also [37] where the relation of the form "$T_m T_1 = T_{m-1} + T_{m+1}$" was given. In [59], the Jacobi-Trudi type functional relations (cf. Theorem 6.1 and 6.2) were given for the fusion RSOS models of type A_r. The T-system for A_r is extracted from them in [1], where the extention to all $\mathfrak{g}$ was proposed based on the connection to the Y-system and the Q-system. Finally, one can construct the quantum field theory analogue of the commuting transfer matrices that act on Virasoro Fock spaces and satisfy the T-system. See [60] for the original construction for $\mathfrak{g} = A_1$ and [61] for a recent application.

[16] In this case, the normalization is $T^{(a)}_0(u) = T^{(a)}_{t_a\ell}(u) = 1$.

4. T-system in quantum group theory

4.1. **Quantum affine algebra.** For simplicity we concentrate on the untwisted quantum affine algebra $U_q(\hat{\mathfrak{g}})$ until Section 4.5. We assume that $q \in \mathbb{C}^\times$ is not a root of unity and set $q = e^\hbar$, therefore the domain U of the spectral parameter u should be understood as $U = \mathbb{C}_{t\hbar}$. See Section 2.1. We set $\hat{I} = \{0\} \sqcup I$ and let $\hat{C} = (\hat{C}_{ij})_{i,j\in\hat{I}}$ be the Cartan matrix of the affine Kac-Moody algebra $\hat{\mathfrak{g}}$ [10]. For $i, j \in I$, one has $\hat{C}_{ij} = C_{ij}$ where the latter is an element of the Cartan matrix C of $\mathfrak{g}$. By definition, the (untwisted) quantum affine algebra $U_q(\hat{\mathfrak{g}})$ [62, 13] is the associative algebra over $\mathbb{C}$ with generators $x_i^\pm, k_i^{\pm 1}$, $(i \in \hat{I})$ and the relations:

$$\begin{aligned}
&k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i,\\
&k_i x_j^\pm k_i^{-1} = q_i^{\pm \hat{C}_{ij}} x_j^\pm, \quad [x_i^+, x_j^-] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},\\
&\sum_{\nu=0}^{1-\hat{C}_{ij}} (-1)^\nu \begin{bmatrix} 1-\hat{C}_{ij} \\ \nu \end{bmatrix}_{q_i} (x_i^+)^{1-\hat{C}_{ij}-\nu} x_j^\pm (x_i^\pm)^\nu = 0 \ \ (i \neq j).
\end{aligned} \tag{4.1}$$

Here $q_0 = q$ and $q_i = q^{t/t_i}$ for $i \in I$. For the notations t and t_i, see (2.1). Furthermore, for $0 \le n \le m$,

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[n]_q![m-n]_q!}, \quad [m]_q! = [1]_q[2]_q \cdots [m]_q. \tag{4.2}$$

See (3.18) for the definition of $[m]_q$. The algebra $U_q(\hat{\mathfrak{g}})$ is denoted by $U_q'(\hat{\mathfrak{g}})$ in some literature indicating that the analogue of the derivation operator in $\hat{\mathfrak{g}}$ has not been included. There are 2^{r+1} algebra automorphisms of $U_q(\hat{\mathfrak{g}})$ given on generators by

$$k_i \mapsto \sigma_i k_i, \ \ x_i^+ \mapsto \sigma_i x_i^+, \ \ x_i^- \mapsto x_i^- \tag{4.3}$$

for any set of signs $\sigma_0, \ldots, \sigma_r \in \{\pm 1\}$. Obviously, $U_q(\hat{\mathfrak{g}})$ contains $U_q(\mathfrak{g})$ as a subalgebra.

There is another realization of $U_q(\hat{\mathfrak{g}})$ called the Drinfeld new realization [63, 64]. Namely, $U_q(\hat{\mathfrak{g}})$ is isomorphic to the algebra with generators $x_{i,n}^\pm$ ($i \in I$, $n \in \mathbb{Z}$), $k_i^{\pm 1}$ ($i \in I$), $h_{i,n}$ ($i \in I$, $n \in \mathbb{Z}\backslash\{0\}$) and central elements $c^{\pm 1/2}$, with the following relations:

$$\begin{aligned}
&k_i k_j = k_j k_i, \quad k_i h_{j,n} = h_{j,n} k_i, \quad k_i x_{j,n}^\pm k_i^{-1} = q_i^{\pm C_{ij}} x_{j,n}^\pm,\\
&[h_{i,n}, x_{j,m}^\pm] = \pm \frac{1}{n} [nC_{ij}]_{q_i} c^{\mp |n|/2} x_{j,n+m}^\pm, \quad [h_{i,n}, h_{j,m}] = \delta_{n,-m} \frac{1}{n} [nC_{ij}]_{q_i} \frac{c^n - c^{-n}}{q_j - q_j^{-1}},\\
&x_{i,n+1}^\pm x_{j,m}^\pm - q_i^{\pm C_{ij}} x_{j,m}^\pm x_{i,n+1}^\pm = q_i^{\pm C_{ij}} x_{i,n}^\pm x_{j,m+1}^\pm - x_{j,m+1}^\pm x_{i,n}^\pm,\\
&[x_{i,n}^+, x_{j,m}^-] = \delta_{ij} \frac{c^{(n-m)/2} \phi_{i,n+m}^+ - c^{-(n-m)/2} \phi_{i,n+m}^-}{q_i - q_i^{-1}},\\
&\sum_{\pi \in \Sigma_s} \sum_{k=0}^{s} (-1)^k \begin{bmatrix} s \\ k \end{bmatrix}_{q_i} x_{i,n_{\pi(1)}}^\pm \ldots x_{i,n_{\pi(k)}}^\pm x_{j,m}^\pm x_{i,n_{\pi(k+1)}}^\pm \ldots x_{i,n_{\pi(s)}}^\pm = 0, \quad i \neq j
\end{aligned} \tag{4.4}$$

for all sequences of integers $n_1, \dots, n_s$, where $s = 1 - C_{ij}$, Σ_s is the symmetric group on s letters, and $\phi^{\pm}_{i,n}$'s are determined by the formal power series

$$\sum_{n=0}^{\infty} \phi^{\pm}_{i,\pm n}\zeta^{\pm n} = k_i^{\pm 1} \exp\left(\pm(q_i - q_i^{-1}) \sum_{m=1}^{\infty} h_{i,\pm m}\zeta^{\pm m}\right). \tag{4.5}$$

In the two realizations (4.1) and (4.4), the symbol $k_i^{\pm 1}$ $(i \in I)$ stands for the same generator under the isomorphism. $U_q(\hat{\mathfrak{g}})$ admits a Hopf algebra structure [62, 13].

4.2. **Finite dimensional representations.** A representation W of $U_q(\hat{\mathfrak{g}})$ is called *type 1* if the generators $k_0, k_1, \dots, k_r$ act semi simply on W with eigenvalues in $q^{\mathbb{Z}}$ and $c^{1/2}$ in (4.4) acts as 1 on W. A vector $v \in W$ is called a highest weight vector if

$$x^{+}_{i,n} \cdot v = 0, \qquad \phi^{\pm}_{i,n} \cdot v = \psi^{\pm}_{i,n} v, \qquad c^{1/2} v = v, \tag{4.6}$$

for some complex numbers $\psi^{\pm}_{i,n}$. A type 1 representation W is called a highest weight representation if $W = U_q(\hat{\mathfrak{g}}) \cdot v$ for some highest weight vector v.

Theorem 4.1 ([65, 66])**.** *(1) Every finite dimensional irreducible representation of $U_q(\hat{\mathfrak{g}})$ can be obtained from a type 1 representation by a twisting with an automorphism (4.3).*

(2) Every finite dimensional irreducible representation of $U_q(\hat{\mathfrak{g}})$ of type 1 is a highest weight representation.

(3) A type 1 highest weight representation with the highest weight vector v in (4.6) is finite dimensional if and only if there exist polynomials $\mathcal{P}_a(\zeta) \in \mathbb{C}[\zeta]$ $(a \in I)$ such that $\mathcal{P}_a(0) = 1$ and

$$\sum_{n \ge 0} \psi^{\pm}_{a,\pm n}\zeta^{\pm n} = q_a^{\deg \mathcal{P}_a} \frac{\mathcal{P}_a(\zeta q_a^{-1})}{\mathcal{P}_a(\zeta q_a)} \in \mathbb{C}[[\zeta^{\pm 1}]]. \tag{4.7}$$

The polynomials $\mathcal{P}_a(\zeta)$ are called Drinfeld polynomials after the analogous classification theorem by Drinfeld for Yangians [63].

The Kirillov-Reshetikhin module $W^{(a)}_m(u)$ $(a \in I, m \in \mathbb{Z}_{\ge 1}, u \in \mathbb{C}_{t\hbar})$ is the irreducible finite dimensional representation of $U_q(\hat{\mathfrak{g}})$ that corresponds to the Drinfeld polynomial

$$\mathcal{P}_b(\zeta) = \begin{cases} \prod_{s=1}^{m}(1 - \zeta q^{tu} q_a^{m+1-2s}) & \text{if } b = a, \\ 1 & \text{otherwise.} \end{cases} \tag{4.8}$$

This $W^{(a)}_m(u)$ is equal to $W^{(a)}_{m,q^{tu}q_i^{-m+1}}$ in [67, 68]. In particular, $W^{(1)}_1(u), \dots, W^{(r)}_1(u)$ are called fundamental representations.

4.3. **Example.** Consider the simplest example $U_q = U_q(A^{(1)}_1)$. In the realization (4.1), $\hat{I} = \{0, 1\}$ and the Cartan matrix is $\hat{C} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. The coproduct is given by

$$\Delta x^{+}_i = x^{+}_i \otimes 1 + k_i \otimes x^{+}_i, \quad \Delta x^{-}_i = x^{-}_i \otimes k_i^{-1} + 1 \otimes x^{-}_i, \quad \Delta k_i^{\pm 1} = k_i^{\pm 1} \otimes k_i^{\pm 1}. \tag{4.9}$$

For $m \in \mathbb{Z}_{\ge 0}$, let $W_m(u) = W^{(1)}_m(u)$ be the Kirillov-Reshetikhin module. Plainly, it is the $m + 1$ dimensional (i.e., spin $\frac{m}{2}$) irreducible representation $W_m(u) =$

$\mathbb{C}v_1^m \oplus \cdots \oplus \mathbb{C}v_{m+1}^m$ given by $(z = q^u)$

$$x_1^- v_j^m = [m+1-j] v_{j+1}^m, \quad x_1^+ v_j^m = [j-1] v_{j-1}^m, \quad k_1^{\pm 1} v_j^m = q^{\pm(m+2-2j)} v_j^m, \tag{4.10}$$

$$x_0^+ v_j^m = z[m+1-j] v_{j+1}^m, \quad x_0^- v_j^m = z^{-1}[j-1] v_{j-1}^m, \quad k_0^{\pm 1} v_j^m = q^{\mp(m+2-2j)} v_j^m, \tag{4.11}$$

where $[j] = [j]_q = \frac{q^j - q^{-j}}{q - q^{-1}}$ as in (3.18). In the Drinfeld new realization (4.4), the highest weight vector is identified with v_1^m and the eigenvalues in (4.6) read

$$\psi_{1,\pm n}^{\pm} = \begin{cases} q^{\pm m} & n = 0, \\ \pm(q^m - q^{-m})(zq^m)^{\pm n} & n \geq 1. \end{cases}$$

The relation (4.7) holds with the Drinfeld polynomial

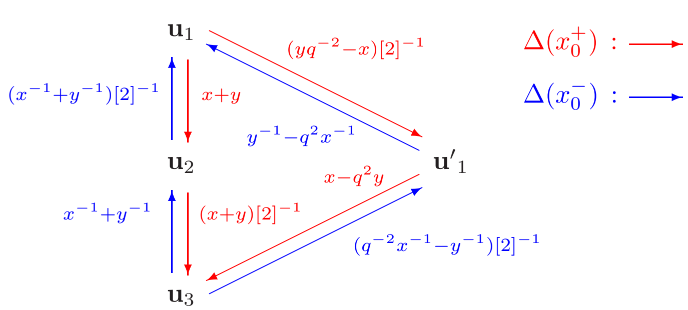
$$\mathcal{P}_1(\zeta) = (1 - \zeta q^{u-m+1})(1 - \zeta q^{u-m+3}) \cdots (1 - \zeta q^{u+m-1})$$

in agreement with (4.8).

The exact sequence (3.16) is refined along the definitions here. The vectors $v_i \in V_1$ and $x = (x_1, x_2) \in V_m$ in Section 3.1 are to be identified with v_i^1 and $v_{x_2+1}^m$ in (4.10)–(4.11), respectively. We introduce the base of $W_1(u) \otimes W_1(v)$ as

$$\begin{aligned} &\mathbf{u}_1 = v_1^1 \otimes v_1^1, \\ &\mathbf{u}_2 = \frac{1}{[2]} \Delta(x_1^-) \mathbf{u}_1 = \frac{v_1^1 \otimes v_2^1 + q^{-1} v_2^1 \otimes v_1^1}{[2]}, \qquad \mathbf{u'}_1 = v_1^1 \otimes v_2^1 - q v_2^1 \otimes v_1^1, \\ &\mathbf{u}_3 = \Delta(x_1^-) \mathbf{u}_2 = v_2^1 \otimes v_2^1. \end{aligned} \tag{4.12}$$

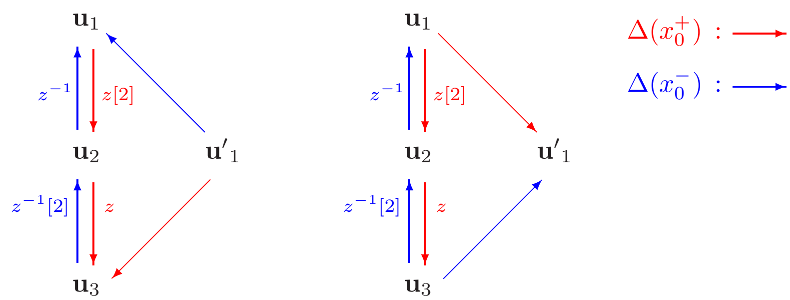
Under the action of $x_1^{\pm}, k_1^{\pm 1}$, the set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\{\mathbf{u'}_1\}$ behave as the triplet and the singlet representations as usual. On the other hand, with regard to $x_0^{\pm}$, they are mixed as follows. $(x = q^u, y = q^v)$



(4.13)

The diagram means $\Delta(x_0^+)\mathbf{u}_1 = (x+y)\mathbf{u}_2 + \frac{yq^{-2}-x}{[2]}\mathbf{u'}_1$ for instance. From (4.13), we find that $W_1(u) \otimes W_1(v)$ is irreducible if and only if $\frac{x}{y} \neq q^{\pm 2}$, namely $u - v \neq \pm 2$.

In the reducible cases, (4.13) looks as



(i) $z := qx = q^{-1}y$ (ii) $z := q^{-1}x = qy$.

(4.14)

In the both cases, $W_1(u) \otimes W_1(v)$ is indecomposable and the subspace $\mathbb{C}\mathbf{u}_1 \oplus \mathbb{C}\mathbf{u}_2 \oplus \mathbb{C}\mathbf{u}_3$ becomes isomorphic to $W_2(\frac{u+v}{2})$ corresponding to the multiplicative spectral parameter z. The difference is that $W_2(\frac{u+v}{2})$ is the irreducible submodule in case of (i) while it is the irreducible quotient for (ii). Denoting the trivial one dimensional module $\mathbb{C}\mathbf{u}'_1$ by W_0, we thus get the exact sequences of U_q-modules:

$$\text{(i)} \quad 0 \to W_2(u) \to W_1(u-1) \otimes W_1(u+1) \to W_0 \to 0, \tag{4.15}$$

$$\text{(ii)} \quad 0 \to W_0 \to W_1(u+1) \otimes W_1(u-1) \to W_2(u) \to 0. \tag{4.16}$$

The general case, which was first worked out in the context of Yangian, is summarized in

Theorem 4.2 ([69])**.** *$W_m(u) \otimes W_n(v)$ is reducible if and only if $|u - v| = m + n - 2j + 2$ for some $1 \le j \le \min(m, n)$. In these case, the following exact sequences are valid:*

$$\begin{aligned} 0 &\to W_{j-1}(u+m-j+1) \otimes W_{m+n-j+1}(v-m+j-1) \to W_m(u) \otimes W_n(v) \\ &\to W_{m-j}(u-j) \otimes W_{n-j}(v+j) \to 0 \end{aligned} \tag{4.17}$$

for $v - u = m + n - 2j + 2$.

$$\begin{aligned} 0 &\to W_{m-j}(u+j) \otimes W_{n-j}(v-j) \to W_m(u) \otimes W_n(v) \\ &\to W_{j-1}(u-m+j-1) \otimes W_{m+n-j+1}(v+m-j+1) \to 0 \end{aligned} \tag{4.18}$$

for $u - v = m + n - 2j + 2$.

4.4. ***q*-characters.** Let $\operatorname{Rep} U_q(\hat{\mathfrak{g}})$ be the Grothendieck ring of the category of the type 1 finite dimensional $U_q(\hat{\mathfrak{g}})$-modules. Such a module W allows the direct sum decomposition

$$W = \bigoplus_{\gamma = (\gamma^{\pm}_{a,\pm n})_{i \in I, n \ge 0}} W_\gamma, \quad W_\gamma = \{v \in W \mid \exists p \ge 0, \forall a \in I, n \ge 0, (\phi^{\pm}_{a,\pm n} - \gamma^{\pm}_{a,\pm n})^p v = 0\}.$$

It can be shown [70] that the generating function of the (generalized) eigenvalues is expressed as

$$\sum_{n > 0} \gamma^{\pm}_{a,\pm n} \zeta^{\pm n} = q_a^{\deg R_a^+ - \deg R_a^-} \frac{R_a^+(\zeta q_a^{-1}) R_a^-(\zeta q_a)}{R_a^+(\zeta q_a) R_a^-(\zeta q_a^{-1})} \in \mathbb{C}[[\zeta^{\pm 1}]] \tag{4.19}$$

in terms of some polynomials $R_a^{\pm}(\zeta)$ in ζ with constant term 1.

Let $\mathbb{Z}[Y_{a,z}^{\pm 1}]_{a\in I, z\in\mathbb{C}^\times}$ be the ring of integer coefficient Laurent polynomials in infinitely many algebraically independent variables $\{Y_{a,z} \mid a \in I, z \in \mathbb{C}^\times\}$. The Frenkel-Reshetikhin q-character χ_q is the injective ring morphism

$$\chi_q : \mathrm{Rep}\, U_q(\hat{\mathfrak{g}}) \to \mathbb{Z}[Y_{a,z}^{\pm 1}]_{a\in I, z\in\mathbb{C}^\times}, \qquad \chi_q(W) = \sum_\gamma \dim(W_\gamma) m_\gamma, \tag{4.20}$$

where the monomial m_γ is specified from $R_a^{\pm}(\zeta)$ (4.19) by

$$m_\gamma = \prod_{a\in I, z\in\mathbb{C}^\times} Y_{a,z}^{r^+_{a,z} - r^-_{a,z}}, \quad R_a^{\pm}(\zeta) = \prod_{z\in\mathbb{C}^\times} (1-\zeta z)^{r^{\pm}_{a,z}}. \tag{4.21}$$

Suppose that W is the irreducible representation with Drinfeld polynomials $\mathcal{P}_a(\zeta) = \prod_{s=1}^{m_a}(1-\zeta z_s^{(a)})$. Comparing (4.7) with (4.19) and (4.21), one finds that its q-character $\chi_q(W)$ contains the monomial $\prod_{a=1}^r \prod_{s=1}^{m_a} Y_{a,z_s^{(a)}}$ corresponding to the highest weight vector. Such a monomial is called a highest weight monomial. Thus in particular, the q-character of the Kirillov-Reshetikhin module $W_m^{(a)}(u)$ is a Laurent polynomial containing the highest weight monomial as

$$\chi_q(W_m^{(a)}(u)) = \prod_{s=1}^m Y_{a, zq_a^{m+1-2s}} + \cdots, \tag{4.22}$$

where we have set $z = q^{tu}$. The case $m = 1$ is called the fundamental q-character. For an analogous treatment of the Yangians, see [71].

Define $\mathrm{Ch}\, U_q(\hat{\mathfrak{g}})$ to be the image $\mathrm{Im}\,\chi_q$ and call it the q-character ring of $U_q(\hat{\mathfrak{g}})$. By the definition, $\mathrm{Ch}\, U_q(\hat{\mathfrak{g}})$ is an integral domain and a commutative ring isomorphic to $\mathrm{Rep}\, U_q(\hat{\mathfrak{g}})$. The following fact is well known.

Theorem 4.3 ([70], Corollary 2)**.** *The ring $\mathrm{Ch}\, U_q(\hat{\mathfrak{g}})$ is freely generated by the fundamental q-characters $\chi_q(W_1^{(a)}(u))$ $(a \in I, u \in U)$.*

Example 4.4. For $\mathfrak{g} = A_1$, the q-character of the Kirillov-Reshetikhin module $W_m^{(1)}(u)$ is given by ($z = q^u$, $Y_z = Y_{1,z}$)

$$\begin{aligned}
&\chi_q(W_1^{(1)}(u)) = Y_z + Y_{zq^2}^{-1},\\
&\chi_q(W_2^{(1)}(u)) = Y_{zq^{-1}}Y_{zq} + Y_{zq^{-1}}Y_{zq^3}^{-1} + Y_{zq}^{-1}Y_{zq^3}^{-1}, \quad \text{and in general,}\\
&\chi_q(W_m^{(1)}(u)) = \sum_{j=0}^{m} \prod_{i=1}^{m-j} Y_{zq^{-m-1+2i}} \prod_{k=1}^{j} Y_{zq^{m+3-2k}}^{-1}.
\end{aligned}$$

Example 4.5. We write down the fundamental q-characters $\chi_q(W_1^{(a)}(u))$ for $\mathfrak{g}$ with rank 2 ($z = q^{tu}$).

$$
\begin{aligned}
A_2: \chi_q(W_1^{(1)}(u)) &= Y_{1,z} + Y^{-1}_{1,zq^2}Y_{2,zq} + Y^{-1}_{2,zq^3},\\
\chi_q(W_1^{(2)}(u)) &= Y_{2,z} + Y_{1,zq}Y^{-1}_{2,zq^2} + Y^{-1}_{1,zq^3},\\
B_2: \chi_q(W_1^{(1)}(u)) &= Y_{1,z} + Y^{-1}_{1,zq^4}Y_{2,zq}Y_{2,zq^3} + Y_{2,zq}Y^{-1}_{2,zq^5} + Y_{1,zq^2}Y^{-1}_{2,zq^3}Y^{-1}_{2,zq^5} + Y^{-1}_{1,zq^6},\\
\chi_q(W_1^{(2)}(u)) &= Y_{2,z} + Y_{1,zq}Y^{-1}_{2,zq^2} + Y^{-1}_{1,zq^5}Y_{2,zq^4} + Y^{-1}_{2,zq^6},\\
C_2: \chi_q(W_1^{(a)}(u)) &= (\chi_q(W_1^{(3-a)}(u)) \text{ for } B_2)|_{Y_{1,z}\leftrightarrow Y_{2,z}} \quad (a = 1, 2),\\
G_2: \chi_q(W_1^{(1)}(u)) &= Y_{1,z} + Y_{2,zq}Y_{2,zq^3}Y_{2,zq^5}Y^{-1}_{1,zq^6} + Y_{2,zq}Y_{2,zq^3}Y^{-1}_{2,zq^7}\\
&+ Y_{1,zq^4}Y_{2,zq}Y^{-1}_{2,zq^5}Y^{-1}_{2,zq^7} + Y_{1,zq^2}Y_{1,zq^4}Y^{-1}_{2,zq^3}Y^{-1}_{2,zq^5}Y^{-1}_{2,zq^7}\\
&+ Y_{2,zq}Y_{2,zq^9}Y^{-1}_{1,zq^{10}} + Y_{1,zq^2}Y_{2,zq^9}Y^{-1}_{1,zq^{10}}Y^{-1}_{2,zq^3} + Y_{2,zq}Y^{-1}_{2,zq^{11}}\\
&+ Y_{1,zq^4}Y^{-1}_{1,zq^8} + Y_{2,zq^5}Y_{2,zq^7}Y_{2,zq^9}Y^{-1}_{1,zq^8}Y^{-1}_{1,zq^{10}} + Y_{1,zq^2}Y^{-1}_{2,zq^3}Y^{-1}_{2,zq^{11}}\\
&+ Y_{2,zq^5}Y_{2,zq^7}Y^{-1}_{1,zq^8}Y^{-1}_{2,zq^{11}} + Y_{2,zq^5}Y^{-1}_{2,zq^9}Y^{-1}_{2,zq^{11}}\\
&+ Y_{1,zq^6}Y^{-1}_{2,zq^7}Y^{-1}_{2,zq^9}Y^{-1}_{2,zq^{11}} + Y^{-1}_{1,zq^{12}},\\
\chi_q(W_1^{(2)}(u)) &= Y_{2,z} + Y_{1,zq}Y^{-1}_{2,zq^2} + Y^{-1}_{1,zq^7}Y_{2,zq^4}Y_{2,zq^6} + Y_{2,zq^4}Y^{-1}_{2,zq^8}\\
&+ Y_{1,zq^5}Y^{-1}_{2,zq^6}Y^{-1}_{2,zq^8} + Y^{-1}_{1,zq^{11}}Y_{2,zq^{10}} + Y^{-1}_{2,zq^{12}}.
\end{aligned}
$$

More examples will be given in Sections 7.1–7.4.

Any finite dimensional $U_q(\hat{\mathfrak{g}})$-module W defines a representation of the subalgebra $U_q(\mathfrak{g})$, which we denote by $\operatorname{res} W$. The (usual) character χ of the latter lives in $\mathbb{Z}[y_a^{\pm 1}]_{a\in I}$ with $y_a = e^{\omega_a}$ with ω_a being a fundamental weight. The q-character is a deformation of the character by z in that

$$\operatorname{res}\chi_q(W) = \chi(\operatorname{res} W), \tag{4.23}$$

where res in the LHS is to be understood as

$$
\begin{aligned}
\operatorname{res}\,:\, \mathbb{Z}[Y^{\pm 1}_{a,z}]_{a\in I, z\in\mathbb{C}^\times} &\to \mathbb{Z}[y_a^{\pm 1}]_{a\in I}\\
Y_{a,z} &\mapsto y_a.
\end{aligned} \tag{4.24}
$$

Note that $\operatorname{res} W$ is not necessarily an irreducible $U_q(\mathfrak{g})$-module even if W is so as a $U_q(\hat{\mathfrak{g}})$-module. Therefore the irreducible q-character $\chi_q(W_m^{(a)}(u))$ does not restrict to an irreducible character in general. In fact in Example 4.5, one observes

$$\operatorname{res}\chi_q(W_1^{(1)}(u)) = \begin{cases} \chi(V_{\omega_1}) + \chi(V_0) & \text{if } \mathfrak{g} = G_2 \text{ and } a = 1,\\ \chi(V_{\omega_a}) & \text{otherwise,}\end{cases} \tag{4.25}$$

where V_λ denotes the irreducible $U_q(\mathfrak{g})$-module with highest weight λ. The algebra $\mathfrak{g} = A_r$ is exceptional in that $\operatorname{res}\chi_q(W_m^{(a)}(u)) = \chi(V_{m\omega_a})$ holds for all a and m. See (7.7) and (13.63). A systematic treatment of such decompositions is related to the *Kirillov-Reshetikhin conjecture* which has been fully solved by now. See Section 13, especially Section 13.7.

For $a \in I$ and $z \in \mathbb{C}^\times$, set

$$A_{a,z} = Y_{a,zq_a^{-1}}Y_{a,zq_a}\prod_{b:C_{ba}=-1}Y^{-1}_{b,z}\prod_{b:C_{ba}=-2}Y^{-1}_{b,zq^{-1}}Y^{-1}_{b,zq}\prod_{b:C_{ba}=-3}Y^{-1}_{b,zq^{-2}}Y^{-1}_{b,z}Y^{-1}_{b,zq^2}. \tag{4.26}$$

By the definition, one has $\operatorname{res} A_{a,z} = \prod_{b\in I} y_b^{C_{ba}} = e^{\alpha_a}$ with α_a being a simple root.

Let S_a $(a \in I)$ be the screening operator [70]. Namely, S_a sends $\mathbb{Z}[Y^{\pm 1}_{a,z}]_{a\in I, z\in\mathbb{C}^\times}$ to the extended ring adjoined with the extra symbols $S_{a,z}$ with $a \in I, z \in \mathbb{C}^\times$. The action is given by

$$S_a \cdot Y_{b,z} = \delta_{ab} Y_{a,z} S_{a,z} \tag{4.27}$$

and the Leibniz rule $S_a\cdot(YZ) = (S_a\cdot Y)Z + Y(S_a\cdot Z)$. Thus for example $S_a\cdot Y^{-1}_{b,z} = -\delta_{ab}Y^{-1}_{a,z}S_{a,z}$. The symbol $S_{a,z}$ is assumed to obey the relation

$$S_{a,zq_a^2} = A_{a,zq_a} S_{a,z} \tag{4.28}$$

in the extended ring.

Theorem 4.6 ([70, 72])**.** *(1) The q-character of an irreducible finite dimensional $U_q(\hat{\mathfrak{g}})$-module W has the form $\chi_q(W) = m_+(1+\sum_p M_p)$, where m_+ is the highest weight monomial and each M_p is a monomial in $A^{-1}_{a,z}$, $a \in I, z\in\mathbb{C}^\times$, (i.e., it does not contain any positive power factors of $A_{a,z}$).*

(2) The image $\operatorname{Im}\chi_q(\simeq \operatorname{Ch} U_q(\hat{\mathfrak{g}}))$ of the q-character morphism (4.20) is equal to $\bigcap_{a=1}^r \operatorname{Ker} S_a$.

(1) is a natural analogue of its undeformed version $\operatorname{res}\chi_q(W) \in \operatorname{res}(m_+)(1+\sum_\alpha \mathbb{Z}_{\geq 0}\, e^{-\alpha})$, where $\operatorname{res}(m_+) = e^{\text{highest weight}}$ and the α-sum runs over $\mathbb{Z}_{\geq 0}\alpha_1 \oplus \cdots \oplus \mathbb{Z}_{\geq 0}\alpha_r \setminus \{0\}$.

The assertion (2) has a background in the characterization of the (deformed) W-algebra as the intersection of the kernel of screening operators [70].

Example 4.7. Let us illustrate Theorem 4.6 along $\mathfrak{g} = A_2$. The definition (4.26) reads

$$A_{1,z} = Y_{1,zq^{-1}}Y_{1,zq}Y^{-1}_{2,z}, \quad A_{2,z} = Y_{2,zq^{-1}}Y_{2,zq}Y^{-1}_{1,z}.$$

Take $\chi_q = \chi_q(W^{(1)}_1(u)) = Y_{1,z} + Y^{-1}_{1,zq^2}Y_{2,zq} + Y^{-1}_{2,zq^3}$ for A_2 in Example 4.5. The highest weight monomial is $Y_{1,z}$. χ_q is expressed as

$$\chi_q = Y_{1,q}(1 + A^{-1}_{1,zq} + A^{-1}_{1,zq}A^{-1}_{2,zq^2}) \tag{4.29}$$

in agreement with (1). With regard to (2), let us check that χ_q belongs to $\operatorname{Ker}S_1 \bigcap \operatorname{Ker}S_2$.

$$S_1\cdot\chi_q = Y_{1,z}S_{1,z} - Y^{-1}_{1,zq^2}Y_{2,zq}S_{1,zq^2} = Y_{1,z}S_{1,z} - Y^{-1}_{1,zq^2}Y_{2,zq}A_{1,zq}S_{1,z} = 0,$$

$$S_2\cdot\chi_q = Y^{-1}_{1,zq^2}Y_{2,zq}S_{2,zq} - Y^{-1}_{2,zq^3}S_{2,zq^3} = Y^{-1}_{1,zq^2}Y_{2,zq}S_{2,zq} - Y^{-1}_{2,zq^3}A_{2,zq^2}S_{2,zq} = 0.$$

4.5. **T-system and q-characters.** We continue to set $u \in U = \mathbb{C}_{t\hbar}$ in this subsection. The following is the fundamental result that relates the Kirillov-Reshetikhin modules and the T-system.

Theorem 4.8 ([67, 68])**.** *For any $\mathfrak{g}$, $T^{(a)}_m(u) = \chi_q(W^{(a)}_m(u))$ satisfies the unrestricted T-system for $\mathfrak{g}$.*

In fact, the exact sequence corresponding to the $\mathfrak{g}$-version of $j = n = m$ case of (4.18) has been obtained. It is an elementary exercise to check that the q-characters for $\mathfrak{g} = A_1$ in Example 4.4 satisfies the T-system (2.6).

Theorem 4.8 leads to a description of the ring $\operatorname{Rep} U_q(\hat{\mathfrak{g}}) \simeq \operatorname{Ch} U_q(\hat{\mathfrak{g}})$ by the q-characters of the Kirillov-Reshetikhin modules and the unrestricted T-system, which we shall now explain. Let $T = \{T^{(a)}_m(u) \mid a \in I, m \in \mathbb{Z}_{\geq 1}, u \in U\}$ denote the family of variables. Let $\mathcal{T}(\mathfrak{g})$ be the ring with generators $T^{(a)}_m(u)^{\pm 1}$ with the relations given by the T-system for $\mathfrak{g}$. Define $\mathcal{T}^\circ(\mathfrak{g})$ to be the subring of $\mathcal{T}(\mathfrak{g})$ generated by T.

Theorem 4.9 ([16])**.** *The ring $\mathcal{T}^\circ(\mathfrak{g})$ is isomorphic to $\operatorname{Rep} U_q(\hat{\mathfrak{g}})$ by the correspondence $T_m^{(a)}(u) \mapsto W_m^{(a)}(u)$.*

4.6. **T-system for quantum affinizations of quantum Kac-Moody algebras.** The T-systems has been generalized by Hernandez [7] to the quantum affinizations of a wide class of quantum Kac–Moody algebras studied in [63, 73, 74, 75, 76, 77]. Among many features distinct from the setting so far, most notable ones are that the category $\operatorname{Rep} U_q(\hat{\mathfrak{g}})$ and the tensor product $\otimes$ need to be replaced by $\operatorname{Mod}(U_q(\hat{\mathfrak{g}}))$ consisting of not necessarily finite dimensional modules and the fusion product $*_f$, respectively. Nevertheless, with an appropriate definition of the Kirillov-Reshetikhin modules and their q-characters, the latter satisfy the (generalized) T-system [7].

Here we only give the definition of the quantum affinization of quantum Kac-Moody algebras and write down the T-system, leaving many details to [7]. Instead, we include the explicit form of the corresponding Y-system [78] on which our presentation is mainly based.

We begin by resetting the definitions and notations such as $C, t, q_i, \mathfrak{g}, \hat{\mathfrak{g}}, U_q(\mathfrak{g})$ and $U_q(\hat{\mathfrak{g}})$ introduced so far[17]. Let $I = \{1, \dots, r\}$ and let $C = (C_{ij})_{i,j \in I}$ be a *generalized Cartan matrix* in [10]; namely, it satisfies $C_{ij} \in \mathbb{Z}$, $C_{ii} = 2$, $C_{ij} \leq 0$ for any $i \neq j$, and $C_{ij} = 0$ if and only if $C_{ji} = 0$. We assume that C is *symmetrizable*, i.e., there is a diagonal matrix $D = \operatorname{diag}(d_1, \dots, d_r)$ with $d_i \in \mathbb{Z}_{\geq 1}$ such that $B = DC$ is symmetric. We assume that there is no common divisor for $d_1, \dots, d_r$ except for 1.

Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of the Cartan matrix C [10]; namely, $\mathfrak{h}$ is a $(2r - \operatorname{rank} C)$ dimensional $\mathbb{Q}$-vector space, and $\Pi = \{\alpha_1, \dots, \alpha_r\} \subset \mathfrak{h}^*$, $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_r^\vee\} \subset \mathfrak{h}$ such that $\alpha_j(\alpha_i^\vee) = C_{ij}$. Let $q \in \mathbb{C}^\times$ be not a root of unity. We set $q_i = q^{d_i}$ $(i \in I)$ and use the symbols defined in (4.2). Let $U_q(\mathfrak{g})$ be the quantum Kac-Moody algebra [62, 13], which is a q-analogue of the Kac-Moody algebra $\mathfrak{g}$ associated with C [10].

The *quantum affinization* (without central elements) of the quantum Kac–Moody algebra $U_q(\mathfrak{g})$, denoted by $U_q(\hat{\mathfrak{g}})$, is the $\mathbb{C}$-algebra with generators $x_{i,n}^\pm$ $(i \in I,$

[17]This reset is only for the current subsection.

$n \in \mathbb{Z}$), k_h ($h \in \mathfrak{h}$), $h_{i,n}$ ($i \in I$, $n \in \mathbb{Z} \setminus \{0\}$) and the following relations:

$$k_h k_{h'} = k_{h+h'}, \qquad k_0 = 1, \qquad k_h \phi_i^{\pm}(z) = \phi_i^{\pm}(z) k_h,$$
$$k_h x_i^{\pm}(z) = q^{\pm \alpha_i(h)} x_i^{\pm}(z) k_h,$$
$$\phi_i^{+}(z) x_j^{\pm}(w) = \frac{q^{\pm B_{ij}} w - z}{w - q^{\pm B_{ij}} z} x_j^{\pm}(w) \phi_i^{+}(z),$$
$$\phi_i^{-}(z) x_j^{\pm}(w) = \frac{q^{\pm B_{ij}} w - z}{w - q^{\pm B_{ij}} z} x_j^{\pm}(w) \phi_i^{-}(z),$$
$$x_i^{+}(z) x_j^{-}(w) - x_j^{-}(w) x_i^{+}(z) = \frac{\delta_{ij}}{q_i - q_i^{-1}} \left(\delta\left(\frac{w}{z}\right) \phi_i^{+}(w) - \delta\left(\frac{z}{w}\right) \phi_i^{-}(z) \right),$$
$$(w - q^{\pm B_{ij}} z) x_i^{\pm}(z) x_j^{\pm}(w) = (q^{\pm B_{ij}} w - z) x_j^{\pm}(w) x_i^{\pm}(z),$$
$$\sum_{\pi \in \Sigma} \sum_{k=1}^{1-C_{ij}} (-1)^k \begin{bmatrix} 1 - C_{ij} \\ k \end{bmatrix}_{q_i} x_i^{\pm}(w_{\pi(1)}) \cdots x_i^{\pm}(w_{\pi(k)}) x_j^{\pm}(z)$$
$$\times\, x_i^{\pm}(w_{\pi(k+1)}) \cdots x_i^{\pm}(w_{\pi(1-C_{ij})}) = 0 \qquad (i \neq j). \tag{4.30}$$

In (4.30) Σ is the symmetric group for the set $\{1, \ldots, 1 - C_{ij}\}$. We have also used the following formal series:

$$x_i^{\pm}(z) = \sum_{n \in \mathbb{Z}} x_{i,n}^{\pm} z^n, \quad \phi_i^{\pm}(z) = k_{\pm d_i \alpha_i^{\vee}} \exp\left(\pm (q - q^{-1}) \sum_{n \geq 1} h_{i,\pm n} z^{\pm n} \right).$$

and the formal delta function $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$.

When C is of finite type, the above $U_q(\hat{\mathfrak{g}})$ is called an (*untwisted*) *quantum affine algebra* (*without central elements*) or *quantum loop algebra*; it is isomorphic to a subquotient of the previously introduced one (4.4) by the ideal generated by $c^{\pm 1/2} - 1$ [63, 64]. When C is of affine type, the quantum Kac-Moody algebra $U_q(\mathfrak{g})$ is the one in (4.1). Its quantum affinization $U_q(\hat{\mathfrak{g}})$ is called a *quantum toroidal algebra* (without central elements). In general, if C is not of finite type, $U_q(\hat{\mathfrak{g}})$ is no longer isomorphic to a subquotient of any quantum Kac–Moody algebra and has no Hopf algebra structure.

From now on we shall exclusively consider a symmetrizable generalized Cartan matrix C satisfying the following condition due to Hernandez [7]:

$$\text{If } C_{ij} < -1, \text{ then } d_i = -C_{ji} = 1, \tag{4.31}$$

where $D = \mathrm{diag}(d_1, \ldots, d_r)$ is the diagonal matrix symmetrizing C. We say that a generalized Cartan matrix C is *tamely laced* if it is symmetrizable and satisfies the condition (4.31). A generalized Cartan matrix C is *simply laced* if $C_{ij} = 0$ or -1 for any $i \neq j$. If C is simply laced, then it is symmetric, $d_a = 1$ for any $a \in I$, and it is tamely laced.

With a tamely laced generalized Cartan matrix C, we associate a *Dynkin diagram* in the standard way [10]: For any pair $i \neq j \in I$ with $C_{ij} < 0$, the vertices i and j are connected by $\max\{|C_{ij}|, |C_{ji}|\}$ lines, and the lines are equipped with an arrow from j to i if $C_{ij} < -1$. Note that the condition (4.31) means

(i) the vertices i and j are not connected if $d_i, d_j > 1$ and $d_i \neq d_j$,
(ii) the vertices i and j are connected by d_i lines with an arrow from i to j or not connected if $d_i > 1$ and $d_j = 1$,

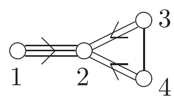
(iii) the vertices i and j are connected by a single line or not connected if $d_i = d_j$.

Example 4.10. (1) Any Cartan matrix of finite or affine type is tamely laced except for types $A_1^{(1)}$ and $A_{2\ell}^{(2)}$.

(2) The following generalized Cartan matrix C is tamely laced:

$$C = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -3 & 2 & -2 & -2 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}, \qquad D = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

The corresponding Dynkin diagram is



Define the integer t by

$$t = \mathrm{lcm}(d_1, \dots, d_r).$$

For $a, b \in I$, we write $a \sim b$ if $C_{ab} < 0$, i.e., a and b are adjacent in the corresponding Dynkin diagram. Let U be either $\frac{1}{t}\mathbb{Z}$, the complex plane $\mathbb{C}$, or the cylinder $\mathbb{C}_\xi := \mathbb{C}/(2\pi\sqrt{-1}/\xi)\mathbb{Z}$ for some $\xi \in \mathbb{C} \setminus 2\pi\sqrt{-1}\mathbb{Q}$, depending on the situation under consideration.

For a tamely laced generalized Cartan matrix C, the unrestricted T-system associated with C [7] is the following relations among the commuting variables $\{T_m^{(a)}(u) \mid a \in I, m \in \mathbb{Z}_{\geq 1}, u \in U\}$:

$$T_m^{(a)}\left(u - \frac{d_a}{t}\right) T_m^{(a)}\left(u + \frac{d_a}{t}\right) = T_{m-1}^{(a)}(u) T_{m+1}^{(a)}(u) + \prod_{b:b\sim a} T_{\frac{d_a}{d_b}m}^{(b)}(u) \qquad \text{if} \quad d_a > 1, \tag{4.32}$$

$$T_m^{(a)}\left(u - \frac{d_a}{t}\right) T_m^{(a)}\left(u + \frac{d_a}{t}\right) = T_{m-1}^{(a)}(u) T_{m+1}^{(a)}(u) + \prod_{b:b\sim a} S_m^{(b)}(u) \qquad \text{if} \quad d_a = 1, \tag{4.33}$$

where $T_0^{(a)}(u) = 1$ if they occur in the RHS in the relations. The symbol $S_m^{(b)}(u)$ is defined by

$$S_m^{(b)}(u) = \prod_{k=1}^{d_b} T^{(b)}_{1+E\left[\frac{m-k}{d_b}\right]}\left(u + \frac{1}{t}\left(2k - 1 - m + E\left[\frac{m-k}{d_b}\right] d_b\right)\right), \tag{4.34}$$

and $E[x]$ $(x \in \mathbb{Q})$ denotes the largest integer not exceeding x.

Explicitly, $S_m^{(b)}(u)$ is written as follows: For $0 \leq j < d_b$,

$$S_{d_b m+j}^{(b)}(u) = \left\{ \prod_{k=1}^{j} T_{m+1}^{(b)}\left(u + \frac{1}{t}(j + 1 - 2k)\right) \right\} \left\{ \prod_{k=1}^{d_b - j} T_m^{(b)}\left(u + \frac{1}{t}(d_b - j + 1 - 2k)\right) \right\}.$$

For example, for $d_b = 1$,

$$S_m^{(b)}(u) = T_m^{(b)}(u),$$

for $d_b = 2$,

$$S_{2m}^{(b)}(u) = T_m^{(b)}\left(u - \frac{1}{t}\right) T_m^{(b)}\left(u + \frac{1}{t}\right),$$

$$S_{2m+1}^{(b)}(u) = T_{m+1}^{(b)}(u) T_m^{(b)}(u),$$

for $d_b = 3$,

$$S_{3m}^{(b)}(u) = T_m^{(b)}\left(u - \frac{2}{t}\right) T_m^{(b)}(u) T_m^{(b)}\left(u + \frac{2}{t}\right),$$

$$S_{3m+1}^{(b)}(u) = T_{m+1}^{(b)}(u) T_m^{(b)}\left(u - \frac{1}{t}\right) T_m^{(b)}\left(u + \frac{1}{t}\right),$$

$$S_{3m+2}^{(b)}(u) = T_{m+1}^{(b)}\left(u - \frac{1}{t}\right) T_{m+1}^{(b)}\left(u + \frac{1}{t}\right) T_m^{(b)}(u),$$

and so on. The second terms in the RHS of (4.32) and (4.33) can be written in a unified way as follows [7]:

$$\prod_{b:b\sim a} \prod_{k=1}^{-C_{ab}} T^{(b)}_{-C_{ba}+E\left[\frac{d_a(m-k)}{d_b}\right]}\left(u + \frac{d_b}{t}\left(\frac{-2k+1}{C_{ab}} - C_{ba} + E\left[\frac{d_a(m-k)}{d_b}\right] - 1\right) - \frac{d_a m}{t}\right).$$

When C is of finite type $\mathfrak{g}$, the above T-system coincides with the one for $U_q(\hat{\mathfrak{g}})$ in Section 2.1. For C of affine type, it was also studied by [79] as a discrete Toda field equation.

Let us proceed to the Y-system. For a tamely laced generalized Cartan matrix C, the unrestricted Y-system associated with C is the following relations among the commuting variables $\{Y_m^{(a)}(u) \mid a \in I, m \in \mathbb{Z}_{\geq 1}, u \in U\}$, where $Y_0^{(a)}(u)^{-1} = 0$ if they occur in the RHS in the relations:

$$Y_m^{(a)}\left(u - \frac{d_a}{t}\right) Y_m^{(a)}\left(u + \frac{d_a}{t}\right) = \frac{\prod_{b:b\sim a} Z^{(b)}_{\frac{d_a}{d_b},m}(u)}{(1 + Y_{m-1}^{(a)}(u)^{-1})(1 + Y_{m+1}^{(a)}(u)^{-1})} \qquad \text{if} \quad d_a > 1, \tag{4.35}$$

$$Y_m^{(a)}\left(u - \frac{d_a}{t}\right) Y_m^{(a)}\left(u + \frac{d_a}{t}\right) = \frac{\prod_{b:b\sim a} \left(1 + Y^{(b)}_{\frac{m}{d_b}}(u)\right)}{(1 + Y_{m-1}^{(a)}(u)^{-1})(1 + Y_{m+1}^{(a)}(u)^{-1})} \qquad \text{if} \quad d_a = 1, \tag{4.36}$$

where for $p \in \mathbb{Z}_{\geq 1}$

$$Z_{p,m}^{(b)}(u) = \prod_{j=-p+1}^{p-1} \left\{ \prod_{k=1}^{p-|j|} \left(1 + Y_{pm+j}^{(b)}\left(u + \frac{1}{t}(p - |j| + 1 - 2k)\right)\right)\right\},$$

and $Y^{(b)}_{\frac{m}{d_b}}(u) = 0$ in (4.36) if $\frac{m}{d_b} \notin \mathbb{Z}_{\geq 1}$.

The Y-systems here are formally in the same form as (2.11)–(2.15) for the quantum affine algebras. However, p in $Z_{p,m}^{(b)}(u)$ here may be greater than 3. In the RHS of (4.35), $\frac{d_a}{d_b}$ is either 1 or d_a due to (4.31). The term $Z_{p,m}^{(b)}(u)$ is written more explicitly as follows: for $p = 1$,

$$Z_{1,m}^{(b)}(u) = 1 + Y_m^{(b)}(u),$$

for $p = 2$,

$$Z^{(b)}_{2,m}(u) = \big(1 + Y^{(b)}_{2m-1}(u)\big)\left(1 + Y^{(b)}_{2m}\left(u - \frac{1}{t}\right)\right)\left(1 + Y^{(b)}_{2m}\left(u + \frac{1}{t}\right)\right)\big(1 + Y^{(b)}_{2m+1}(u)\big),$$

for $p = 3$,

$$\begin{aligned} Z^{(b)}_{3,m}(u) = \big(1 + Y^{(b)}_{3m-2}(u)\big)&\left(1 + Y^{(b)}_{3m-1}\left(u - \frac{1}{t}\right)\right)\left(1 + Y^{(b)}_{3m-1}\left(u + \frac{1}{t}\right)\right) \\ &\times \left(1 + Y^{(b)}_{3m}\left(u - \frac{2}{t}\right)\right)\big(1 + Y^{(b)}_{3m}(u)\big)\left(1 + Y^{(b)}_{3m}\left(u + \frac{2}{t}\right)\right) \\ &\times \left(1 + Y^{(b)}_{3m+1}\left(u - \frac{1}{t}\right)\right)\left(1 + Y^{(b)}_{3m+1}\left(u + \frac{1}{t}\right)\right)\big(1 + Y^{(b)}_{3m+2}(u)\big), \end{aligned}$$

and so on. There are p^2 factors in $Z^{(b)}_{p,m}(u)$.

The T and Y-systems in this subsection satisfy formally the same relations as those explained in Section 2.3. Their restricted versions have also been formulated in [78].

4.7. **Bibliographical notes.** The origin of the Kirillov-Reshetikhin modules (they are named so in [80, Definition 1.1]) goes back to [81], where the spectral parameter dependence was not considered. The idea of treating them as one family of $Y(\mathfrak{g})$ or $U_q(\hat{\mathfrak{g}})$ modules with spectral parameter satisfying the T-system in the Grothendieck ring was initiated by [1], where the identification by Drinfeld polynomials was also given in the context of Yangian based on the result of [69]. Meanwhile, the representation theory of finite dimensional $U_q(\hat{\mathfrak{g}})$ modules was pushed forward by [82, 65], where the Kirillov-Reshetikhin modules were characterized and studied as *minimal affinizations* of $U_q(\hat{\mathfrak{g}})$ modules [83, 84, 85, 86, 87].

The relation between the Kirillov-Reshetikhin modules and T-systems became transparent after the introduction of q-character by [70]. The case of Yangian goes back to [71]. Theorem 4.8 is due to [67] for simply laced $\mathfrak{g}$ and [68] for general $\mathfrak{g}$. Under certain circumstances, there are algorithms to compute q-characters [72] or its further generalization called t-analogue of q-characters $\chi_{q,t}$ [88, 89] for ADE case. In particular, $\chi_{q,t}$ of all the fundamental representations has been produced [89], among which the E_8 case requires a supercomputer.

The T-systems for the quantum affinizations of quantum Kac–Moody algebras in Section 4.6 are due to [7]. The corresponding Y-system and formulation by cluster algebra are given in [78].

5. Formulation by cluster algebras

5.1. **Dilogarithm identities in conformal field theory.** Let $L(x)$ be the *Rogers dilogarithm function* [90, 91]

$$L(x) = -\frac{1}{2}\int_0^x \left\{\frac{\ln(1-y)}{y} + \frac{\ln y}{1-y}\right\} dy \quad (0 \le x \le 1). \tag{5.1}$$

It is well known that the following properties hold ($0 \le x, y \le 1$).

$$L(0) = 0, \quad L(1) = \frac{\pi^2}{6}, \tag{5.2}$$

$$L(x) + L(1-x) = \frac{\pi^2}{6}, \tag{5.3}$$

$$L(x) + L(y) + L(1-xy) + L\left(\frac{1-x}{1-xy}\right) + L\left(\frac{1-y}{1-xy}\right) = \frac{\pi^2}{2}. \tag{5.4}$$

In the series of works by Bazhanov, Kirillov, and Reshetikhin [53, 37, 81, 92, 59], they reached a remarkable conjecture on identities expressing the central charges of conformal field theories in terms of $L(x)$, and partly established it.

In what follows, $\mathfrak{g}$ denotes any one of the simple Lie algebras $A_r, B_r, \dots, G_2$ as in the previous sections. In Section 2.2, we defined the level ℓ restricted Y-system for $\mathfrak{g}$ for $\ell \in \mathbb{Z}_{\ge 1}$. Let us introduce the system of relations for the variable $\{Y^{(a)}_m \mid a \in I, 1 \le m \le t_a \ell - 1\}$ obtained from the level ℓ restricted Y-system by setting $Y^{(a)}_m(u) = Y^{(a)}_m$ dropping the dependence on the spectral parameter u. We call it the *level ℓ restricted constant Y-system*.

Theorem 5.1 ([93, 94])**.** *There exists a unique solution of the level ℓ restricted constant Y-system for $\mathfrak{g}$ satisfying $Y^{(a)}_m \in \mathbb{R}_{>0}$ for all $a \in I, 1 \le m \le t_a \ell - 1$.*

Theorem 5.1 was proved by [93] for simply laced case, and extended to nonsimply laced case by [94] using the same method. For more information on the constant Y-system, see Section 14.4 and 14.6.

The following theorem was originally conjectured by [81] and [59] for simply laced case, and conjectured by [92] and properly corrected by [17] for nonsimply laced case.

Theorem 5.2 (Dilogarithm identities [92, 95, 94, 96])**.** *Suppose that a family of positive real numbers $\{Y^{(a)}_m \mid a \in I, 1 \le m \le t_a \ell - 1\}$ satisfy the level ℓ constant Y-system for $\mathfrak{g}$. Then, we have the identities*

$$\frac{6}{\pi^2} \sum_{a \in I} \sum_{m=1}^{t_a \ell - 1} L\left(\frac{Y^{(a)}_m}{1 + Y^{(a)}_m}\right) = \frac{\ell \dim \mathfrak{g}}{\ell + h^\vee} - \operatorname{rank} \mathfrak{g}, \tag{5.5}$$

where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$ (2.3).

The rational number of the first term in the RHS of (5.5) is the central charge of the *Wess-Zumino-Witten conformal field theory* associated with $\mathfrak{g}$ with level ℓ [97, 98]. The rational number in the RHS of (5.5) itself is also the central charge of the *parafermion conformal field theory* associated with $\mathfrak{g}$ with level ℓ [99, 100]. The identity (5.5) is crucial to establish the connection between conformal field theories and various types of non conformal integrable models in various limits (cf. Section 15.3).

Example 5.3 ([53])**.** Consider the case $\mathfrak{g} = A_1$ and any ℓ, which is equivalent to the case $\mathfrak{g} = A_{\ell-1}$ and $\ell = 2$ by the *level-rank duality*. Then, one has the solution

$$Y^{(1)}_m = \frac{\sin^2 \frac{\pi}{\ell+2}}{\sin \frac{m\pi}{\ell+2} \sin \frac{(m+2)\pi}{\ell+2}}, \tag{5.6}$$

and the corresponding identity (5.5) reads

$$\frac{6}{\pi^2}\sum_{m=1}^{\ell-1} L\left(\frac{\sin^2\frac{\pi}{\ell+2}}{\sin^2\frac{(m+1)\pi}{\ell+2}}\right) = \frac{3\ell}{2+\ell} - 1. \tag{5.7}$$

This identity has been known and studied by various authors in various points of view. See [101, 102] and reference therein. In particular, the identity is derived [103, 104] from the following q-series expression [105] for the parafermion conformal character ("string function" in [10] multiplied with Dedekind's eta function):

$$\sum_{n_1,\dots,n_{\ell-1}=0}^{\infty} q^{\sum_{k,m=1}^{\ell-1} n_k n_m (\min(k,m)-\frac{km}{\ell})} \prod_{m=1}^{\ell-1}(q)_{n_m}^{-1}, \quad (q)_k := \prod_{j=1}^{k}(1-q^j), \tag{5.8}$$

where the sum is under the constraint $\sum_{m=1}^{\ell-1} mn_m \equiv 0 \bmod 2\ell$. In fact, a crude estimate by a saddle point method tells that as $q \to 1$, this series diverges as $\mathrm{const}\cdot(\overline{q})^{-c/24}$ where c is the LHS of (5.7) and $\overline{q} \to 0$ is the modular conjugate specified by $(\ln q)(\ln \overline{q}) = 4\pi^2$. Comparing this with the known asymptotics of the string function [10] yields (5.7). For general $\mathfrak{g}$, see around (14.43).

For $\mathfrak{g} = A_r$, Kirillov [92] gave the explicit expression of the solution (cf. Example 14.4), and proved the corresponding identity (5.5) by the analytic method, but an extension of the proof to the other cases seemed difficult.

In the 90's, people pursued a proof through lifting the dilogarithm identities to the Rogers-Ramanujan type identities as Example 5.3 (e.g., [106, 107, 108, 109, 110]). This created a new subject called the *Fermionic formula* of conformal characters and their variants, which turned out to be a rich subject itself, and it has been intensively studied to this day by its own right. See (ii) in Section 13.8. In spite of this successful development, the original problem of proving the dilogarithm identities (5.5) itself did not make much progress.

The scene changed after the introduction of a new class of commutative algebras called *cluster algebras* by Fomin-Zelevinsky [111] around 2000, which we explain in this section.

5.2. **Cluster algebras with coefficients.** Here we recall the definition of the cluster algebras with coefficients and some of their basic properties, following the convention in [8] with slight change of notations and terminology. See [8] for more detail and information.

Fix an arbitrary semifield $\mathbb{P}$, i.e., an abelian multiplicative group endowed with a binary operation of addition $\oplus$ which is commutative, associative, and distributive with respect to the multiplication [112]. Let $\mathbb{QP}$ denote the quotient field of the group ring $\mathbb{ZP}$ of $\mathbb{P}$. Let I be a finite set[18], and let $B = (B_{ij})_{i,j\in I}$ be a skew symmetrizable (integer) matrix; namely, there is a diagonal positive integer matrix D such that ${}^t(DB) = -DB$. Let $x = (x_i)_{i\in I}$ be an I-tuple of formal variables, and let $y = (y_i)_{i\in I}$ be an I-tuple of elements in $\mathbb{P}$. For the triplet (B, x, y), called the *initial seed*, the *cluster algebra $\mathcal{A}(B, x, y)$ with coefficients in* $\mathbb{P}$ is defined as follows.

Let (B', x', y') be a triplet consisting of skew symmetrizable matrix B', an I-tuple $x' = (x'_i)_{i\in I}$ with $x'_i \in \mathbb{QP}(x)$, and an I-tuple $y' = (y'_i)_{i\in I}$ with $y'_i \in \mathbb{P}$.

[18] This I does not necessarily correspond to the I in Section 2.1 for the index set of Dynkin diagrams.

For each $k \in I$, we define another triplet $(B'', x'', y'') = \mu_k(B', x', y')$, called the *mutation of* (B', x', y') *at* k, as follows.

(i) Mutations of matrix.

$$B''_{ij} = \begin{cases} -B'_{ij} & i = k \text{ or } j = k, \\ B'_{ij} + \frac{1}{2}(|B'_{ik}|B'_{kj} + B'_{ik}|B'_{kj}|) & \text{otherwise.} \end{cases} \tag{5.9}$$

(ii) Exchange relation of coefficient tuple.

$$y''_i = \begin{cases} {y'_k}^{-1} & i = k, \\ y'_i \dfrac{1}{(1 \oplus {y'_k}^{-1})^{B'_{ki}}} & i \neq k,\ B'_{ki} \geq 0, \\ y'_i (1 \oplus y'_k)^{-B'_{ki}} & i \neq k,\ B'_{ki} \leq 0. \end{cases} \tag{5.10}$$

(iii) Exchange relation of cluster.

$$x''_i = \begin{cases} \dfrac{y'_k \prod_{j: B'_{jk} > 0} {x'_j}^{B'_{jk}} + \prod_{j: B'_{jk} < 0} {x'_j}^{-B'_{jk}}}{(1 \oplus y'_k) x'_k} & i = k. \\ x'_i & i \neq k, \end{cases} \tag{5.11}$$

It is easy to see that μ_k is an involution, namely, $\mu_k(B'', x'', y'') = (B', x', y')$. Now, starting from the initial seed (B, x, y), iterate mutations and collect all the resulted triplets (B', x', y'). We call (B', x', y') the *seeds*, y' and y'_i a *coefficient tuple* and a *coefficient*, x' and x'_i, a *cluster* and a *cluster variable*, respectively. The *cluster algebra* $\mathcal{A}(B, x, y)$ *with coefficients in* $\mathbb{P}$ is the $\mathbb{ZP}$-subalgebra of the rational function field $\mathbb{QP}(x)$ generated by all the cluster variables.

It is standard to identify a *skew-symmetric* (integer) matrix $B = (B_{ij})_{i,j \in I}$ with a *quiver* Q *without loops or 2-cycles.* The set of the vertices of Q is given by I, and we put B_{ij} arrows from i to j if $B_{ij} > 0$. The mutation $Q'' = \mu_k(Q')$ of a quiver Q' is given by the following rule: For each pair of an incoming arrow $i \to k$ and an outgoing arrow $k \to j$ in Q', add a new arrow $i \to j$. Then, remove a maximal set of pairwise disjoint 2-cycles. Finally, reverse all arrows incident with k.

Let $\mathbb{P}_{\mathrm{univ}}(y)$ be the *universal semifield* of the I-tuple of generators $y = (y_i)_{i \in I}$, namely, the semifield consisting of the *subtraction-free* rational functions of formal variables y with usual multiplication and addition in the rational function field $\mathbb{Q}(y)$. We write $\oplus$ in $\mathbb{P}_{\mathrm{univ}}(y)$ as $+$ for simplicity.

From now on, unless otherwise mentioned, we set the semifield $\mathbb{P}$ *for* $\mathcal{A}(B, x, y)$ *to be* $\mathbb{P}_{\mathrm{univ}}(y)$*, where* y *is the coefficient tuple in the initial seed* (B, x, y)*.*

Let $\mathbb{P}_{\mathrm{trop}}(y)$ be the *tropical semifield* of $y = (y_i)_{i \in I}$, which is the abelian multiplicative group freely generated by y endowed with the addition $\oplus$

$$\prod_i y_i^{a_i} \oplus \prod_i y_i^{b_i} = \prod_i y_i^{\min(a_i, b_i)}. \tag{5.12}$$

There is a canonical surjective semifield homomorphism $\pi_{\mathbf{T}}$ (the *tropical evaluation*) from $\mathbb{P}_{\mathrm{univ}}(y)$ to $\mathbb{P}_{\mathrm{trop}}(y)$ defined by $\pi_{\mathbf{T}}(y) = y$. For any coefficient y'_i of $\mathcal{A}(B, x, y)$, let us write $[y'_i]_{\mathbf{T}} := \pi_{\mathbf{T}}(y'_i)$ for simplicity. We call $[y'_i]_{\mathbf{T}}$'s the *tropical coefficients* (the *principal coefficients* in [8]). They satisfy the exchange relation (5.10) by replacing y'_i with $[y'_i]_{\mathbf{T}}$ with $\oplus$ being the addition in (5.12). We also extend this homomorphism to the homomorphism of fields $\pi_{\mathbf{T}} : (\mathbb{QP}_{\mathrm{univ}}(y))(x) \to (\mathbb{QP}_{\mathrm{trop}}(y))(x)$.

To each seed (B', x', y') of $\mathcal{A}(B, x, y)$ we attach the *F-polynomials* $F'_i(y) \in \mathbb{Q}(y)$ $(i \in I)$ by the specialization of $[x'_i]_{\mathbf{T}}$ at $x_j = 1$ $(j \in I)$. It is, in fact, a polynomial in y with integer coefficients due to the Laurent phenomenon [8, Proposition 3.6]. For definiteness, let us take $I = \{1, \dots, n\}$. Then, x' and y' have the following factorized expressions [8, Proposition 3.13, Corollary 6.3] by the F-polynomials.

$$x'_i = \left(\prod_{j=1}^{n} x_j^{g'_{ji}} \right) \frac{F'_i(\hat{y}_1, \dots, \hat{y}_n)}{F'_i(y_1, \dots, y_n)}, \quad \hat{y}_i = y_i \prod_{j=1}^{n} x_j^{B_{ji}}, \tag{5.13}$$

$$y'_i = [y'_i]_{\mathbf{T}} \prod_{j=1}^{n} F'_j(y_1, \dots, y_n)^{B'_{ji}}. \tag{5.14}$$

The integer vector $\mathbf{g}'_i = (g'_{1i}, \dots, g'_{ni})$ $(i = 1, \dots, n)$ uniquely determined by (5.13) for each x'_i is called the *g-vector* for x'_i.

Let $\mathbf{i} = (i_1, \dots, i_r)$ be an I-sequence, namely, $i_1, \dots, i_r \in I$. We define the *composite mutation* $\mu_{\mathbf{i}}$ by $\mu_{\mathbf{i}} = \mu_{i_r} \cdots \mu_{i_2} \mu_{i_1}$, where the product means the composition.

Lemma 5.4. *Let $B = (B_{ij})_{i,j \in I}$ be a skew symmetrizable matrix and let $\mathbf{i} = (i_1, \dots, i_r)$ be an I-sequence. Suppose that $B_{i_a i_b} = 0$ for any $1 \le a, b \le r$. Then, the following facts hold.*

(a) For any permutation σ of $\{1, \dots, r\}$, we have

$$\mu_{\mathbf{i}}(B, x, y) = \mu_{(i_{\sigma(1)}, \dots, i_{\sigma(r)})}(B, x, y). \tag{5.15}$$

(b) Let $B' = \mu_{\mathbf{i}}(B)$. Then, $B'_{i_a i_b} = 0$ holds for any $1 \le a, b \le r$.

(c) Let $(B', x', y') = \mu_{\mathbf{i}}(B, x, y)$. Then, $(B, x, y) = \mu_{\mathbf{i}}(B', x', y')$.

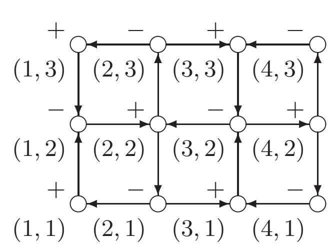
5.3. **T and Y-systems in cluster algebras.** All the T and Y-systems in Sections 2.1–2.5 are regarded as relations among a cluster among cluster variables and coefficients in certain cluster algebras $\mathcal{A}(B, x, y)$.

Let us mention two big advantages of cluster algebra formulation.

(a) The T and Y-systems are integrated in one algebra $\mathcal{A}(B, x, y)$, and commonly controlled by F-polynomials (together with tropical coefficients and g-vectors) through the formulas (5.13) and (5.14). This fact may be hardly realized just by treating the T and Y-systems only.

(b) The cluster algebra $\mathcal{A}(B, x, y)$ itself is further controlled by the *(generalized) cluster category* developed by [113, 114, 115, 116, 117, 118].

Here we concentrate on an example of level 4 restricted T and Y-systems for A_4 to present a basic idea. Let Q be the following quiver with index set $\mathcal{I} = \{1, 2, 3, 4\} \times \{1, 2, 3\}$. Note that we also attached the property $+/-$ to each vertex.



(5.16)

Below we identify Q with the corresponding skew symmetric matrix B as described in Section 5.2.

Let $\mathbf{i}_+$ (resp. $\mathbf{i}_-$) be a sequence of all the distinct elements of $\mathcal{I}$ with property $+$ (resp. $-$), where the order of the sequence is chosen arbitrarily thank to Lemma 5.4.

Then, the quiver Q has the following periodicity under the sequences of mutation $\mathbf{i}_+$ and $\mathbf{i}_-$:

$$Q \overset{\mu_{\mathbf{i}_+}}{\longleftrightarrow} Q^{\mathrm{op}} \overset{\mu_{\mathbf{i}_-}}{\longleftrightarrow} Q, \tag{5.17}$$

where Q^{op} is the *opposite quiver* of Q, namely, the quiver obtained from Q by inverting all the arrows.

Now we set $(Q(0), x(0), y(0)) := (Q, x, y)$ (the initial seed of $\mathcal{A}(Q, x, y)$), and consider the corresponding infinite sequence of mutations of *seeds*

$$\begin{aligned} \cdots \overset{\mu_{\mathbf{i}_+}}{\longleftrightarrow} (Q(-1), x(-1), y(-1)) \overset{\mu_{\mathbf{i}_-}}{\longleftrightarrow} (Q(0), x(0), y(0)) \overset{\mu_{\mathbf{i}_+}}{\longleftrightarrow} \\ (Q(1), x(1), y(1)) \overset{\mu_{\mathbf{i}_-}}{\longleftrightarrow} (Q(2), x(2), y(2)) \overset{\mu_{\mathbf{i}_+}}{\longleftrightarrow} \cdots, \end{aligned} \tag{5.18}$$

$$Q(u) = \begin{cases} Q & u \text{ is even} \\ Q^{\mathrm{op}} & u \text{ is odd}, \end{cases} \tag{5.19}$$

thereby introducing a family of clusters $x(u)$ $(u \in \mathbb{Z})$ and coefficients tuples $y(u)$ $(u \in \mathbb{Z})$.

For $((i,i'),u) \in \mathcal{I} \times \mathbb{Z}$, we write $((i,i'),u) : \mathbf{p}_+$ if $i + i' + u$ is even, or equivalently, if u is even and (i,i') has the property $+$, or u is odd and (i,i') has the property $-$. Plainly speaking, $((i,i'),u) : \mathbf{p}_+$ is a *forward mutation point* in (5.18).

For $((i,i'),u) \in \mathcal{I} \times \mathbb{Z}$, we set $((i,i'),u) : \tilde{\mathbf{p}}_+$ if $((i,i'),u+1) : \mathbf{p}_+$. Consequently, we have

$$((i,i'),u) : \tilde{\mathbf{p}}_+ \Longleftrightarrow ((i,i'),u \pm 1) : \mathbf{p}_+. \tag{5.20}$$

First, we explain how the Y-system appears in cluster algebra. The sequence of mutations (5.18) gives various relations among coefficients $y_{i,i'}(u)$ $(((i,i'),u) \in \mathcal{I} \times \mathbb{Z})$ by the exchange relation (5.10). Then, all these coefficients are products of the "generating" coefficients $y_{i,i'}(u)$ and $1 + y_{i,i'}(u)$ $(((i,i'),u) : \mathbf{p}_+)$. Furthermore, these generating coefficients obey some relations, which are the Y-system.

Let us write down the relations explicitly. Take $((i,i'),u) : \mathbf{p}_+$ and consider the mutation at $((i,i'),u)$, where $y_{i,i'}(u)$ is exchanged to $y_{i,i'}(u+1) = y_{i,i'}(u)^{-1}$, by (5.10). In the next step going from $Q(u+1)$ to $Q(u+2)$, the (forward) mutation points are those satisfying $((j,j'),u+1) : \mathbf{p}_+$. Therefore the above $y_{i,i'}(u+1)$ gets multiplied by factors $(1 + y_{j,j'}(u+1))$ if the quiver $Q(u+1)$ has an arrow from (i,i') to (j,j'), and $(1 + y_{j,j'}(u+1)^{-1})^{-1}$ if the quiver $Q(u+1)$ has an arrow from (j,j') to (i,i'). The result coincides with the coefficient $y_{i,i'}(u+2)$. In summary, we have the following relations: For $((i,i'),u) : \mathbf{p}_+$,

$$y_{i,i'}(u) y_{i,i'}(u+2) = \frac{(1 + y_{i-1,i'}(u+1))(1 + y_{i+1,i'}(u+1))}{(1 + y_{i,i'-1}(u+1)^{-1})(1 + y_{i,i'+1}(u+1)^{-1})}, \tag{5.21}$$

where $y_{0,i'}(u+1) = y_{5,i'}(u+1) = 0$ and $y_{i,0}(u+1)^{-1} = y_{i,4}(u+1)^{-1} = 0$ in the RHS. Or, equivalently, for $((i,i'),u) : \tilde{\mathbf{p}}_+$,

$$y_{i,i'}(u-1) y_{i,i'}(u+1) = \frac{(1 + y_{i-1,i'}(u))(1 + y_{i+1,i'}(u))}{(1 + y_{i,i'-1}(u)^{-1})(1 + y_{i,i'+1}(u)^{-1})}. \tag{5.22}$$

This certainly agrees with the level 4 restricted Y-system for A_4 under the identification of $y_{i,i'}(u)$ with $Y^{(i)}_{i'}(u)$.

Next, we explain how the T-system appears in cluster algebra. The sequence of mutations (5.18) gives various relations among cluster variables $x_{i,i'}(u)$ $(((i,i'),u) \in \mathcal{I} \times \mathbb{Z})$ by the exchange relation (5.11). All these coefficients are represented by the "generating" cluster variables $x_{i,i'}(u)$ $(((i,i'),u):\mathbf{p}_+)$. Furthermore, these generating cluster variables obey some relations, which are the T-system.

Let us write down the relations explicitly. Take $((i,i'),u):\mathbf{p}_+$ and consider the mutation at $((i,i'),u)$. Then, by (5.11) and the fact that $((i\pm 1,i'),u)$ and $((i,i'\pm 1),u)$ are *not* forward mutation points, we have

$$\begin{aligned} x_{i,i'}(u)x_{i,i'}(u+2) = &\frac{y_{i,i'}(u)}{1+y_{i,i'}(u)}x_{i-1,i'}(u+1)x_{i+1,i'}(u+1)\\ &+\frac{1}{1+y_{i,i'}(u)}x_{i,i'-1}(u+1)x_{i,i'+1}(u+1), \end{aligned} \tag{5.23}$$

where $x_{0,i'}(u+1)=x_{5,i'}(u+1)=x_{i,0}(u+1)=x_{i,4}(u+1)=1$ in the RHS. By introducing the "shifted cluster variables" $\tilde{x}_i(u):=x_i(u+1)$ for $((i,i'),u):\tilde{\mathbf{p}}_+$, these relations can be written in a more "balanced" form and become parallel to (5.22) as follows: For $((i,i'),u):\mathbf{p}_+$,

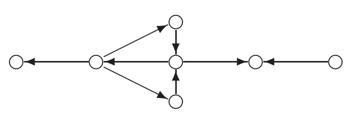
$$\begin{aligned} \tilde{x}_{i,i'}(u-1)\tilde{x}_{i,i'}(u+1) = &\frac{y_{i,i'}(u)}{1+y_{i,i'}(u)}\tilde{x}_{i-1,i'}(u)\tilde{x}_{i+1,i'}(u)\\ &+\frac{1}{1+y_{i,i'}(u)}\tilde{x}_{i,i'-1}(u)\tilde{x}_{i,i'+1}(u). \end{aligned} \tag{5.24}$$

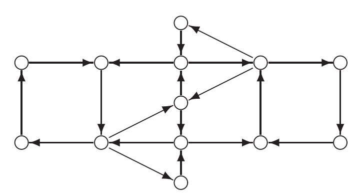
Let $\mathcal{A}(B,x)$ be the cluster algebra with trivial coefficients with initial seed (B,x). Namely, we set every coefficient to be 1 in the trivial semifield $\mathbf{1}=\{1\}$. Let $\pi_{\mathbf{1}}:\mathbb{P}_{\mathrm{univ}}(y)\to\mathbf{1}$ be the projection. Let $[x_i(u)]_{\mathbf{1}}$ be the image of $x_i(u)$ by the algebra homomorphism $\mathcal{A}(B,x,y)\to\mathcal{A}(B,x)$ induced from $\pi_{\mathbf{1}}$. By the specialization of (5.24), we have

$$[\tilde{x}_{i,i'}(u-1)]_{\mathbf{1}}[\tilde{x}_{i,i'}(u+1)]_{\mathbf{1}} = [\tilde{x}_{i-1,i'}(u)]_{\mathbf{1}}[\tilde{x}_{i+1,i'}(u)]_{\mathbf{1}} + [\tilde{x}_{i,i'-1}(u)]_{\mathbf{1}}[\tilde{x}_{i,i'+1}(u)]_{\mathbf{1}}. \tag{5.25}$$

This certainly agrees with the level 4 restricted T-system for A_4 under the identification of $[\tilde{x}_{i,i'}(u)]_{\mathbf{1}}$ with $T^{(i)}_{i'}(u)$.

For $\mathfrak{g}$ simply laced, the quiver relevant to the level ℓ restricted T and Y-systems is drawn similarly to (5.16) on the vertex set $\mathcal{I}=\{\text{nodes of the Dynkin diagram}\}\times\{1,2,\dots,\ell-1\}$. For $\mathfrak{g}$ nonsimply laced, it is slightly more involved [94, 96]. Here we only give examples for B_3 with level 2 (left) and level 3 (right).





Remark 5.5. Once we realize that the T and Y-systems are integrated in a single cluster algebra with coefficients as above, the relation between T and Y-systems in Theorem 2.5 becomes an immediate consequence of a more general relation between cluster variables and coefficients in [8, Prop. 3.9], where (2.19) is a special case of [8, eq. (3.7)] with the specialization of the base semifield $\mathbb{P}$ therein to the trivial semifield. See also [119, Prop. 5 .11] for the relation between more general T and Y-systems.

5.4. **Application to periodicity and dilogarithm identities.** As remarkable applications of the cluster algebra formulation, one can prove the periodicities of T and Y-systems and dilogarithm identities (5.5).

The following periodicity property was originally conjectured for type A_1 by [3], for simply laced case by Ravanini-Tateo-Valleriani [5], and for nonsimply laced case by Kuniba-Nakanishi-Suzuki [1].

Theorem 5.6 (Periodicity [120, 121, 122, 123, 124, 115, 125, 16, 94, 96])**.** *For any family of variables $\{Y_m^{(a)}(u) \mid a \in I, 1 \leq m \leq t_a\ell - 1, u \in \mathbb{Z}\}$ satisfying the level ℓ restricted Y-system for $\mathfrak{g}$, we have the periodicity*

$$Y_m^{(a)}(u + 2(h^\vee + \ell)) = Y_m^{(a)}(u). \tag{5.26}$$

To prove Theorem 5.6 in full generality, the use of the categorification of the cluster algebra by the cluster category by [117, 118] is essential.

Since the T-system is integrated in the same cluster algebra, one can simultaneously prove the periodicity of T-system as well, which was overlooked in the literature until recently [126, 16].

Theorem 5.7 (Periodicity [8, 127, 124, 115, 16, 94, 96])**.** *For any family of variables $\{T_m^{(a)}(u) \mid a \in I, 1 \leq m \leq t_a\ell - 1, u \in \mathbb{Z}\}$ satisfying the level ℓ restricted T-system for $\mathfrak{g}$, we have the periodicity*

$$T_m^{(a)}(u + 2(h^\vee + \ell)) = T_m^{(a)}(u). \tag{5.27}$$

Closely related to the periodicity of Y-systems, the following (significant) functional generalization of the dilogarithm identities (5.5) was originally conjectured for simply laced case by Gliozzi-Tateo [128].

Theorem 5.8 (Functional dilogarithm identities [120, 121, 129, 95, 94, 96])**.** *Suppose that a family of positive real numbers $\{Y_m^{(a)}(u) \mid a \in I, 1 \leq m \leq t_a\ell - 1, u \in \mathbb{Z}\}$ satisfy the level ℓ restricted Y-system for $\mathfrak{g}$. Then, we have the identities*

$$\frac{6}{\pi^2} \sum_{a \in I} \sum_{m=1}^{t_a\ell - 1} \sum_{u=0}^{2(h^\vee+\ell)-1} L\left(\frac{Y_m^{(a)}(u)}{1 + Y_m^{(a)}(u)} \right) = 2t(\ell h - h^\vee) \operatorname{rank} \mathfrak{g}, \tag{5.28}$$

where h is the Coxeter number of $\mathfrak{g}$ (2.3).

Example 5.9 ([128])**.** (i) In the simplest case, type A_1, the identity (5.28) is equivalent to (5.3).

(ii) In the next simplest case, type A_2, the identity (5.28) is equivalent to the 5-term relation (5.4).

Theorem 5.8 implies Theorem 5.2; namely, take a *constant solution* $Y_m^{(a)} = Y_m^{(a)}(u)$ of the Y-system with respect to the spectral parameter u. Then, one obtains (5.5) from (5.28).

See Section 5.5 for more precise account of contributions to Theorems 5.6, 5.7, and 5.8.

5.5. **Bibliographical notes.** The cluster algebraic formulation of Y-systems was given for the simply laced case with level 2 by [122], for the simply laced case with general level by [115], for the nonsimply laced case by [94, 96], and for the quantum affinizations of the tamely laced quantum Kac-Moody algebras by [78, 130]. The recognition of T-systems in the cluster algebras was made a little later than Y-systems in [131, 16, 132], though the simply laced case with level 2 clearly appeared in [133]. The formulation here is due to [78, 94, 96]. See [119] for a further generalization of T and Y-systems in view of cluster algebras.

Theorem 5.6 was proved for type A_r with level 2 by [120, 121], for the simply laced case with level 2 by [133], for type A_r with general level by [123] and [124], for the simply laced case with general level by [115, 125], and for all the cases with unified method by [94, 96].

Theorem 5.7 was proved for the simply laced case with level 2 by [133], for type A_r with general level by [127] and [124], for the simply laced case with general level by [115] and [16], for type C_r with general level by [16], and for all the cases with a unified method by [94, 96]. Actually in [16, 94, 96], refinements of Theorem 5.6 and 5.7 have been obtained concerning the property under the half shift $u \to u + h^\vee + \ell$.

Theorem 5.8 was proved for type A_r with level 2 by [120, 121], for the simply laced case with level 2 by [129], for the simply laced case with general level by [95] and for the nonsimply laced case by [94, 96]. See [119] for a further generalization of dilogarithm identities in view of cluster algebras.

There is a dilogarithm conjecture that generalize (5.5) involving $-24\times$(scaling dimensions) in addition to the central charge in the RHS. See [4] and [134, appendix D]. Some of them has been proved in [101, section 1.3, 1.4].

6. Jacobi-Trudi type formula

6.1. Introduction: Type A_r. In this section we exclusively consider unrestricted T-systems. By Theorem 4.3, we know that $T^{(a)}_m(u)$ is expressible as a polynomial in the fundamental ones $T^{(1)}_1(v), \dots, T^{(r)}_1(v)$ with various v. Such formulas can be derived directly. Consider for instance the unrestricted T-system for A_2:

$$\begin{aligned}
T^{(1)}_m(u-1)T^{(1)}_m(u+1) &= T^{(1)}_{m-1}(u)T^{(1)}_{m+1}(u) + T^{(2)}_m(u),\\
T^{(2)}_m(u-1)T^{(2)}_m(u+1) &= T^{(2)}_{m-1}(u)T^{(2)}_{m+1}(u) + T^{(1)}_m(u).
\end{aligned}$$

Setting $m = 1, 2$ and noting $T^{(1)}_0(u) = T^{(2)}_0(u) = 1$, one gets

$$\begin{aligned}
T^{(1)}_2(u) &= T^{(1)}_1(u-1)T^{(1)}_1(u+1) - T^{(2)}_1(u),\\
T^{(2)}_2(u) &= T^{(2)}_1(u-1)T^{(2)}_1(u+1) - T^{(1)}_1(u),\\
T^{(1)}_3(u) &= T^{(1)}_1(u-2)T^{(1)}_1(u)T^{(1)}_1(u+2) - T^{(1)}_1(u-2)T^{(2)}_1(u+1)\\
&\quad - T^{(1)}_1(u+2)T^{(2)}_1(u-1) + 1.
\end{aligned}$$

The formulas generated in this manner are systematized in a determinant form:

$$T^{(1)}_2(u) = \begin{vmatrix} T^{(1)}_1(u-1) & T^{(2)}_1(u) \\ 1 & T^{(1)}_1(u+1) \end{vmatrix}, \quad T^{(2)}_2(u) = \begin{vmatrix} T^{(2)}_1(u-1) & 1 \\ T^{(1)}_1(u) & T^{(2)}_1(u+1) \end{vmatrix},$$

$$T^{(1)}_3(u) = \begin{vmatrix} T^{(1)}_1(u-2) & T^{(2)}_1(u-1) & 1 \\ 1 & T^{(1)}_1(u) & T^{(2)}_1(u+1) \\ 0 & 1 & T^{(1)}_1(u+2) \end{vmatrix}.$$

Proceeding similarly, one gets

Theorem 6.1 ([59]). *For the unrestricted T-system for A_r, the following formula is valid:*

$$T_m^{(a)}(u) = \det(T_1^{(a-i+j)}(u+i+j-m-1))_{1\le i,j\le m} \tag{6.1}$$

where $T_1^{(a)}(u)=0$ unless $0\le a\le r+1$, and $T_1^{(0)}(u)=T_1^{(r+1)}(u)=1$.

The proof reduces to the Jacobi identity among the determinants

$$D[{}^{m+1}_{m+1}]D[{}^{1}_{1}] = D[{}^{1,m+1}_{1,m+1}]D + D[{}^{\ \ 1}_{m+1}]D[{}^{m+1}_{\ \ 1}], \tag{6.2}$$

where $D[{}^{i_1,i_2,\dots}_{j_1,j_2,\dots}]$ is the minor of D removing i_k's rows and j_k's columns.

Alternatively, one can also solve the T-system to express everything by $T_k^{(1)}(v)$ with various v and k. By the same method as before, one can easily systematize such formulas and establish

Theorem 6.2 ([59]). *For the unrestricted T-system for A_r (2.5) without assuming $T_m^{(r+1)}(u)=1$, the following formula is valid:*

$$T_m^{(a)}(u) = \det(T_{m-i+j}^{(1)}(u+i+j-a-1))_{1\le i,j\le a} \quad (1\le a\le r+1), \tag{6.3}$$

where $T_0^{(1)}(u)=1$ and $T_m^{(1)}(u)=0$ for $m<0$.

The formulas (6.1) and (6.3) are quantum analogue of the Jacobi-Trudi formula for Schur functions [135].

In the remainder of this section, we present the Jacobi-Trudi type formulas analogous to (6.1) for the T-systems for B_r, C_r and D_r. The result involves not only determinants but also Pfaffians for $T_m^{(r)}(u)$ in C_r and $T_m^{(r-1)}(u)$ and $T_m^{(r)}(u)$ in D_r.

6.2. Type B_r.

For any $k\in\mathbf{C}$, set

$$x_k^a = \begin{cases} T_1^{(a)}(u+k) & 1\le a\le r, \\ 1 & a=0. \end{cases} \tag{6.4}$$

We introduce the infinite dimensional matrices $\mathcal{T}=(\mathcal{T}_{ij})_{i,j\in\mathbb{Z}}$ and $\mathcal{E}=(\mathcal{E}_{ij})_{i,j\in\mathbb{Z}}$ as follows.

$$\mathcal{T}_{ij} = \begin{cases} x^{\frac{j-i}{2}+1}_{\frac{i+j}{2}-1} & \text{if } i\in 2\mathbb{Z}+1 \text{ and } \frac{i-j}{2}\in\{1,0,\dots,2-r\}, \\ -x^{\frac{i-j}{2}+2r-2}_{\frac{i+j}{2}-1} & \text{if } i\in 2\mathbb{Z}+1 \text{ and } \frac{i-j}{2}\in\{1-r,-r,\dots,2-2r\}, \\ -x^r_{r+i-\frac{5}{2}} & \text{if } i\in 2\mathbb{Z} \text{ and } j=i+2r-3, \\ 0 & \text{otherwise.} \end{cases} \tag{6.5}$$

$$\mathcal{E}_{ij} = \begin{cases} \pm 1 & \text{if } i=j-1\pm 1 \text{ and } i\in 2\mathbb{Z}, \\ x^r_{i-1} & \text{if } i=j-1 \text{ and } i\in 2\mathbb{Z}+1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.6}$$

For instance for B_3, they read

$$(\mathcal{T}_{ij})_{i,j\geq 1} = \begin{pmatrix} x_0^1 & 0 & x_1^2 & 0 & -x_2^2 & 0 & -x_3^1 & 0 & -1 & \\ 0 & 0 & 0 & 0 & -x_{5/2}^3 & 0 & 0 & 0 & 0 & \\ 1 & 0 & x_2^1 & 0 & x_3^2 & 0 & -x_4^2 & 0 & -x_5^1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & -x_{9/2}^3 & 0 & 0 & \\ 0 & 0 & 1 & 0 & x_4^1 & 0 & x_5^2 & 0 & -x_6^2 & \\ & & & & \vdots & & & & & \ddots \end{pmatrix}, \tag{6.7}$$

$$(\mathcal{E}_{ij})_{i,j\geq 1} = \begin{pmatrix} 0 & x_0^3 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & x_2^3 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & x_4^3 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \\ & & & \vdots & & & & \ddots \end{pmatrix}. \tag{6.8}$$

Let $\mathcal{T}|_{u\to u+s}$ be the overall shift of the lower index $x_k^a \to x_{k+s}^a$ in $\mathcal{T}$ in accordance with (6.4). As is evident from this example, the quantity x_k^a is contained in $\mathcal{T}|_{u\to u+s}$ at most once as its matrix element for any $1 \leq a \leq r$ and k. For example, the shift $s = 1$ is needed to accommodate x_1^1 as the (1,1) element of $\mathcal{T}|_{u\to u+s}$. In view of this, we employ the notation $\mathcal{T}_m(i,j,\pm x_k^a)$ to mean the m by m sub-matrix of $\mathcal{T}|_{u\to u+s}$, where s is chosen so that its (i,j) element becomes exactly $\pm x_k^a$. For example in (6.7),

$$\mathcal{T}_3(1,1,x_0^1) = \begin{pmatrix} x_0^1 & 0 & x_1^2 \\ 0 & 0 & 0 \\ 1 & 0 & x_2^1 \end{pmatrix}, \quad \mathcal{T}_3(1,1,x_1^1) = \begin{pmatrix} x_1^1 & 0 & x_2^2 \\ 0 & 0 & 0 \\ 1 & 0 & x_3^1 \end{pmatrix},$$
$$\mathcal{T}_2(1,2,-x_{5/2}^3) = \begin{pmatrix} 0 & -x_{5/2}^3 \\ 0 & x_3^2 \end{pmatrix}, \quad \mathcal{T}_2(1,2,-x_2^3) = \begin{pmatrix} 0 & -x_2^3 \\ 0 & x_{5/2}^2 \end{pmatrix}.$$

We also use the similar notation $\mathcal{E}_m(i,j,\pm x_k^r)$. Now the result for B_r is stated as

Theorem 6.3 ([136]). *For unrestricted T-system for B_r, the following formula is valid:*

$$T_m^{(a)}(u) = \det\big(\mathcal{T}_{2m-1}(1,1,x_{-m+1}^a) + \mathcal{E}_{2m-1}(1,2,x_{-m+r-a+\frac{1}{2}}^r)\big) \quad \textit{for } 1 \leq a < r,$$
$$T_m^{(r)}(u) = (-1)^{m(m-1)/2} \det\big(\mathcal{T}_m(1,2,-x_{-\frac{m}{2}+1}^{r-1}) + \mathcal{E}_m(1,1,x_{-\frac{m}{2}+\frac{1}{2}}^r)\big).$$

6.3. **Type C_r.** Here we introduce the infinite dimensional matrix $\mathcal{T}$ by

$$\mathcal{T}_{ij} = \begin{cases} x_{\frac{i+j}{2}-1}^{j-i+1} & \text{if } i-j \in \{1,0,\ldots,1-r\}, \\ -x_{\frac{i+j}{2}-1}^{i-j+2r+1} & \text{if } i-j \in \{-1-r,-2-r,\ldots,-1-2r\}, \\ 0 & \text{otherwise.} \end{cases} \tag{6.9}$$

For instance, for C_2, it reads

$$(\mathcal{T}_{ij})_{i,j\ge 1} = \begin{pmatrix} x^1_0 & x^2_{1/2} & 0 & -x^2_{3/2} & -x^1_2 & -1 & 0 & 0 & \\ 1 & x^1_1 & x^2_{3/2} & 0 & -x^2_{5/2} & -x^1_3 & -1 & 0 & \cdots \\ 0 & 1 & x^1_2 & x^2_{5/2} & 0 & -x^2_{7/2} & -x^1_4 & -1 & \\ 0 & 0 & 1 & x^1_3 & x^2_{7/2} & 0 & -x^2_{9/2} & -x^1_5 & \\ & & & & \vdots & & & & \ddots \end{pmatrix}.$$

We keep the notation (6.4) and $\mathcal{T}_m(i,j,\pm x^a_k)$ $(1 \le a \le r)$ as in Section 6.2. Note that $\mathcal{T}_m(1,2,-x^r_k)$ is an anti-symmetric matrix for any m.

Theorem 6.4 ([136])**.** *For unrestricted T-system for C_r, the following formula is valid:*

$$T^{(a)}_m(u) = \det \mathcal{T}_m(1,1,x^a_{-\frac{m}{2}+\frac{1}{2}}) \quad \textit{for } 1 \le a < r, \tag{6.10}$$

$$T^{(r)}_m(u) = (-1)^m \mathrm{pf}\, \mathcal{T}_{2m}(1,2,-x^r_{-m+1}). \tag{6.11}$$

As an additional result, we have the following relations.

$$T^{(r)}_m(u-\tfrac{1}{2})T^{(r)}_m(u+\tfrac{1}{2}) = \det \mathcal{T}_{2m}(1,1,x^r_{-m+\frac{1}{2}}), \tag{6.12}$$

$$T^{(r)}_m(u)T^{(r)}_{m+1}(u) = \det \mathcal{T}_{2m+1}(1,1,x^r_{-m}). \tag{6.13}$$

If one extends the definition of x^a_k (6.4) by $x^a_k + x^{2r+2-a}_k = 0$ in accordance with (9.31), then (6.10) is identical with the result (6.1) for A_{2r+1}.

As remarked in the end of Section 2.1, the T-systems for B_2 and C_2 are equivalent by the interchange $T^{(1)}_m(u) \leftrightarrow T^{(2)}_m(u)$. Therefore Theorems 6.3 and 6.4 supply these T-systems with two kinds of Jacobi-Trudi type formulas.

6.4. **Type $\boldsymbol{D_r}$.** Here we define the infinite dimensional matrices $\mathcal{T}$ and $\mathcal{E}$ by

$$\mathcal{T}_{ij} = \begin{cases} x^{\frac{j-i}{2}+1}_{\frac{i+j}{2}-1} & \text{if } i \in 2\mathbb{Z}+1 \text{ and } \frac{i-j}{2} \in \{1,0,\ldots,3-r\}, \\ -x^{r-1}_{\frac{i+j-1}{2}} & \text{if } i \in 2\mathbb{Z}+1 \text{ and } \frac{i-j}{2} = \frac{5}{2}-r, \\ -x^{r}_{\frac{i+j-3}{2}} & \text{if } i \in 2\mathbb{Z}+1 \text{ and } \frac{i-j}{2} = \frac{3}{2}-r, \\ -x^{\frac{i-j}{2}+2r-3}_{\frac{i+j}{2}-1} & \text{if } i \in 2\mathbb{Z}+1 \text{ and } \frac{i-j}{2} \in \{1-r,-r,\ldots,3-2r\}, \\ 0 & \text{otherwise.} \end{cases} \tag{6.14}$$

$$\mathcal{E}_{ij} = \begin{cases} \pm 1 & \text{if } i = j-2\pm 2 \text{ and } i \in 2\mathbb{Z}, \\ x^{r-1}_i & \text{if } i = j-3 \text{ and } i \in 2\mathbb{Z}, \\ x^r_{i-2} & \text{if } i = j-1 \text{ and } i \in 2\mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \tag{6.15}$$

For instance for D_4, they read

$$(\mathcal{T}_{ij})_{i,j\geq 1} = \begin{pmatrix} x_0^1 & 0 & x_1^2 & -x_2^3 & 0 & -x_2^4 & -x_3^2 & 0 & -x_4^1 & 0 & -1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & x_2^1 & 0 & x_3^2 & -x_4^3 & 0 & -x_4^4 & -x_5^2 & 0 & -x_6^1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ & & & & & & \vdots & & & & & \ddots \end{pmatrix},$$

$$(\mathcal{E}_{ij})_{i,j\geq 1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 1 & x_0^4 & 0 & x_2^3 & -1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & x_2^4 & 0 & x_4^3 & -1 & 0 & \\ & & & & & \vdots & & & & \end{pmatrix}.$$

We keep the notations (6.4), $\mathcal{T}_m(i,j,\pm x_k^a)$ $(1 \leq a \leq r-2)$ and $\mathcal{T}_m(i,j,-x_k^a), \mathcal{E}_m(i,j,x_k^a)$ $(a = r-1, r)$ as in Section 6.2.

Theorem 6.5 ([136]). *For unrestricted T-system for D_r, the following formula is valid:*

$$T_m^{(a)}(u) = \det\bigl(\mathcal{T}_{2m-1}(1,1,x_{-m+1}^a) + \mathcal{E}_{2m-1}(2,3,x_{-m-r+a+4}^r)\bigr), \ \textit{for } 1 \leq a \leq r-2, \tag{6.16}$$

$$T_m^{(r-1)}(u) = \mathrm{pf}\bigl(\mathcal{T}_{2m}(2,1,-x_{-m+1}^{r-1}) + \mathcal{E}_{2m}(1,2,x_{-m+1}^{r-1})\bigr), \tag{6.17}$$

$$T_m^{(r)}(u) = (-1)^m \mathrm{pf}\bigl(\mathcal{T}_{2m}(1,2,-x_{-m+1}^r) + \mathcal{E}_{2m}(2,1,x_{-m+1}^r)\bigr). \tag{6.18}$$

The matrices in (6.17) and (6.18) are indeed anti-symmetric. The following relations also hold.

$$T_m^{(r-1)}(u)T_m^{(r)}(u) = (-1)^m \det\bigl(\mathcal{T}_{2m}(1,1,-x_{-m+1}^{r-1}) + \mathcal{E}_{2m}(2,2,x_{-m+1}^r)\bigr),$$

$$T_m^{(r-1)}(u+1)T_m^{(r)}(u-1) = (-1)^m \det\bigl(\mathcal{T}_{2m}(1,1,-x_{-m}^r) + \mathcal{E}_{2m}(2,2,x_{-m+2}^{r-1})\bigr),$$

$$T_{m+1}^{(r-1)}(u)T_m^{(r)}(u-1) = (-1)^{m+1} \det\bigl(\mathcal{T}_{2m+1}(1,1,-x_{-m}^{r-1}) + \mathcal{E}_{2m+1}(2,2,x_{-m}^r)\bigr),$$

$$T_m^{(r-1)}(u+1)T_{m+1}^{(r)}(u) = (-1)^m \det\bigl(\mathcal{T}_{2m+1}(2,1,x_{-m+1}^{r-2}) + \mathcal{E}_{2m+1}(1,1,x_{-m}^r)\bigr).$$

Theorems 6.3–6.5 can be proved only by using (6.2) and the fact $(\mathrm{pf})^2 = \det$.

6.5. **Another Jacobi-Trudi type formula for B_r.** For B_r and D_r, a variant of the Jacobi-Trudi type formula is known which has a quite similar structure to the A_r case. Compared with the rather sparse matrices $\mathcal{T}$ and $\mathcal{E}$, the relevant matrices are dense and involve some auxiliary variables. Here we present the result for B_r. The D_r case is similar although slightly more involved.

Given $T_1^{(1)}(u), \dots, T_1^{(r)}(u)$, we introduce the auxiliary variable $T^a(u)$ for all $a \in \mathbb{Z}$ by

$$T^a(u) = \begin{cases} 0 & a < 0, \\ 1 & a = 0, \\ T_1^{(a)}(u) & 1 \leq a \leq r-1, \end{cases} \tag{6.19}$$

$$T^a(u) + T^{2r-1-a}(u) = T_1^{(r)}(u - r + a + \tfrac{1}{2})T_1^{(r)}(u + r - a - \tfrac{1}{2}) \quad \text{for all } a \in \mathbb{Z}. \tag{6.20}$$

Recall that $t_a = 1$ for $a \neq r$ and $t_r = 2$ for B_r according to (2.1).

Theorem 6.6 ([137])**.** *For unrestricted T-system for B_r, the following formula is valid:*

$$T^{(a)}_{t_a m}(u) = \det(T^{a+i-j}(u+i+j-m-1))_{1\le i,j\le m}, \quad \textit{for } 1 \le a \le r, \tag{6.21}$$

$$T^{(r)}_{2m+1}(u) = \begin{vmatrix} T^{(r)}_1(u-m) & T^{r-1}(u-m+\frac{1}{2}) & T^{r-2}(u-m+\frac{3}{2}) & \cdots & T^{r-m}(u-\frac{1}{2}) \\ T^{(r)}_1(u-m+2) & T^{r}(u-m+\frac{3}{2}) & T^{r-1}(u-m+\frac{5}{2}) & \cdots & T^{r-m+1}(u+\frac{1}{2}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T^{(r)}_1(u+m) & T^{r+m-1}(u+\frac{1}{2}) & T^{r+m-2}(u+\frac{3}{2}) & \cdots & T^{r}(u+m-\frac{1}{2}) \end{vmatrix}, \tag{6.22}$$

where the matrix (6.22) is of size $m+1$, its $(i+1,1)$ element is $T^{(r)}_1(u-m+2i)$ and the rest has the same pattern as (6.21) for $T^{(r)}_{2m+2}(u-\frac{1}{2})$.

6.6. **Bibliographical notes.** The formulas (6.1)–(6.3) for A_r in Theorem 6.1 first appeared in [59] before the T-system was formulated. There, transfer matrices more general than $T^{(a)}_m(u)$ were considered. Theorems 6.3–6.5 supplemented the determinant conjectures in [1] with Pfaffians. A result for D_r analogous to Theorem 6.6 is available in [138].

7. Tableau sum formula

7.1. **Type A_r.** Let $\boxed{1}_u, \ldots, \boxed{r+1}_u$ be variables depending on u. If we set $T^{(1)}_1(u) = \sum_{a=1}^{r+1} \boxed{a}_u$, then

$$T^{(1)}_1(u-1)T^{(1)}_1(u+1) = \sum_{a\le b} \boxed{a}_{u-1}\boxed{b}_{u+1} + \sum_{a>b} \begin{array}{l} \boxed{b}_{u+1} \\ \boxed{a}_{u-1} \end{array}, \tag{7.1}$$

where the both arrays of the boxes stand for the product. Comparing this with the T-system relation $T^{(1)}_1(u-1)T^{(1)}_1(u+1) = T^{(1)}_2(u) + T^{(2)}_1(u)$, one may identify $T^{(1)}_2(u)$ and $T^{(2)}_1(u)$ individually with the two terms in (7.1), and try to further establish similar formulas for higher $T^{(a)}_m(u)$. Such a procedure leads to a solution of the T-system expressed as a sum of tableaux. In fact, if one forgets the spectral parameter u in (7.1), it can be viewed as the identity among Schur functions corresponding to the irreducible decomposition of the A_r-modules:

$$\square \otimes \square = \square\square \oplus \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \tag{7.2}$$

In this sense the result presented in the sequel for A_r is a deformation of the classical tableau sum formula for the Schur functions [135].

Consider the Young diagram (m^a) of $a \times m$ rectangular shape. Let $\mathrm{Tab}(m^a)$ be the set of semistandard tableaux on (m^a) with numbers $\{1, 2, \ldots, r+1\}$. The inscribed numbers are strictly increasing to the bottom and non-decreasing to the right. For example when $r = 2$,

$$\mathrm{Tab}(2) = \{ \begin{array}{|c|c|}\hline 1&1\\ \hline\end{array},\ \begin{array}{|c|c|}\hline 1&2\\ \hline\end{array},\ \begin{array}{|c|c|}\hline 1&3\\ \hline\end{array},\ \begin{array}{|c|c|}\hline 2&2\\ \hline\end{array},\ \begin{array}{|c|c|}\hline 2&3\\ \hline\end{array},\ \begin{array}{|c|c|}\hline 3&3\\ \hline\end{array} \},$$

$$\mathrm{Tab}(2^2) = \left\{ \begin{array}{|c|c|}\hline 1&1\\ \hline 2&2\\ \hline\end{array},\ \begin{array}{|c|c|}\hline 1&1\\ \hline 2&3\\ \hline\end{array},\ \begin{array}{|c|c|}\hline 1&1\\ \hline 3&3\\ \hline\end{array},\ \begin{array}{|c|c|}\hline 1&2\\ \hline 2&3\\ \hline\end{array},\ \begin{array}{|c|c|}\hline 1&2\\ \hline 3&3\\ \hline\end{array},\ \begin{array}{|c|c|}\hline 2&2\\ \hline 3&3\\ \hline\end{array} \right\}.$$

Note that $\mathrm{Tab}(m^a)$ is empty for $a > r+1$. We define

$$T_u = \prod_{i=1}^{a}\prod_{j=1}^{m} \boxed{t_{ij}}_{u+a-m-2i+2j} \qquad \text{for } T = (t_{ij}) \in \mathrm{Tab}(m^a), \tag{7.3}$$

where t_{ij} denotes the entry of the box in the ith row and the jth column from the top left.

Theorem 7.1.

$$T^{(a)}_m(u) = \sum_{T\in \mathrm{Tab}(m^a)} T_u \quad (1 \le a \le r+1) \tag{7.4}$$

is a solution of the T-system for A_r (2.5).

We note that $T^{(r+1)}_m(u)$ here is not just 1 but non trivially chosen as (7.4) as opposed to the original definition of the T-system. However, $\mathrm{Tab}(m^{r+1})$ consists of a unique tableau, therefore (7.4) says that $T^{(r+1)}_m(u)$ is a monomial:

$$T^{(r+1)}_m(u) = \prod_{j=1}^{m} T^{(r+1)}_1(u-m-1+2j), \quad T^{(r+1)}_1(u) = \prod_{i=1}^{r+1} \boxed{i}_{u+r+2-2i}. \tag{7.5}$$

Thus the situation $T^{(r+1)}_m(u) = 1$ can be restored if the variables $\boxed{1}_u, \ldots, \boxed{r+1}_u$ are chosen so as to satisfy the simple relation $T^{(r+1)}_1(u) = 1$. Theorem 7.1 yields the q-characters by the special choice

$$\boxed{a}_u = z_a(u) := Y^{-1}_{a-1,q^{u+a}} Y_{a,q^{u+a-1}} \qquad (Y_{0,q^u} = Y_{r+1,q^u} = 1), \tag{7.6}$$

which indeed satisfies the condition $T^{(r+1)}_m(u) = 1$. The restriction (4.23)-(4.24) of the resulting q-character $T^{(a)}_m(u) = \chi_q(W^{(a)}_m(u))$ is given by

$$\mathrm{res}\, T^{(a)}_m(u) = \chi(V_{m\omega_a}) \tag{7.7}$$

in the notation of (4.25) since the $a \times m$ rectangle Young diagram corresponds to the highest weight $m\omega_a$.

In the rest of this section we shall present the tableau sum formulas for $\mathfrak{g} = B_r, C_r, D_r$ along the context of the q-characters $T^{(a)}_m(u) = \chi_q(W^{(a)}_m(u))$. The contents cover all the fundamental ones $T^{(1)}_1(u), \ldots, T^{(r)}_1(u)$, which is enough in principle to determine all the higher ones $T^{(a)}_m(u)$ due to Theorem 4.3. Some $T^{(a)}_m(u)$ allowing a relatively simple description will also be included.

7.2. **Type B_r.** Let us introduce the index set and a total order on it as

$$J = \{1, 2\ldots, r, 0, \overline{r}, \ldots, \overline{2}, \overline{1}\}, \quad 1 \prec \cdots \prec r \prec 0 \prec \overline{r} \prec \cdots \prec \overline{1}. \tag{7.8}$$

We introduce the variables corresponding to single box tableaux.

$$\begin{aligned}
z_a(u) &= Y_{a,q^{2u+2a-2}} Y^{-1}_{a-1,q^{2u+2a}} \qquad (1 \le a \le r-1),\\
z_r(u) &= Y_{r,q^{2u+2r-3}} Y_{r,q^{2u+2r-1}} Y^{-1}_{r-1,q^{2u+2r}},\\
z_0(u) &= Y_{r,q^{2u+2r-1}} Y_{r,q^{2u+2r-3}} Y^{-1}_{r,q^{2u+2r+1}} Y^{-1}_{r,q^{2u+2r-1}},\\
z_{\overline{r}}(u) &= Y_{r-1,q^{2u+2r-2}} Y^{-1}_{r,q^{2u+2r-1}} Y^{-1}_{r,q^{2u+2r+1}},\\
z_{\overline{a}}(u) &= Y_{a-1,q^{2u+4r-2a-2}} Y^{-1}_{a,q^{2u+4r-2a}} \qquad (1 \le a \le r-1),
\end{aligned} \tag{7.9}$$

where $Y_{0,q^k} = 1$. ($z_0(u)$ in p1427 of [139] contains a misprint.) Consider the Young diagram (m^a) of $a \times m$ rectangular shape. Let $\mathrm{Tab}(B_r, m^a)$ be the set of tableaux on (m^a) with entries from J. The letter $t_{i,j} \in J$ inscribed on the ith row and the jth column from the top left corner should satisfy the following conditions for any adjacent pair:

$$\begin{aligned} &t_{i,j} \preceq t_{i,j+1} \quad \text{and} \quad (t_{i,j}, t_{i,j+1}) \neq (0,0), \\ &t_{i,j} \prec t_{i+1,j} \quad \text{or} \quad (t_{i,j}, t_{i+1,j}) = (0,0). \end{aligned} \tag{7.10}$$

Given a tableau $T = (t_{i,j}) \in \mathrm{Tab}(B_r, m^a)$ we set

$$T_u = \prod_{i=1}^{a} \prod_{j=1}^{m} z_{t_{i,j}}(u + a - m + 2i + 2j). \tag{7.11}$$

This is an analogue of the A_r case (7.3).

Theorem 7.2 ([137, 68]). *The q-character $T^{(a)}_{t_a m}(u) = \chi_q(W^{(a)}_{t_a m}(u))$ is given by*

$$T^{(a)}_{t_a m}(u) = \sum_{T \in \mathit{Tab}(B_r, m^a)} T_u \qquad (1 \leq a \leq r). \tag{7.12}$$

Recall that t_a (2.1) is 1 except $t_r = 2$ for B_r. The formula (7.12) is related to (6.21) in a parallel way with the A_r case explained in the previous subsection. A similar result is available for the remaining case $T^{(r)}_{2m+1}(u)$ based on (6.22) [137]. Theorem 7.2 follows by combining the facts that the RHS and the $T^{(r)}_{2m+1}(u)$ satisfy the T-system [137], q-characters also satisfy the T-system [68], and the $T^{(a)}_m(u)$ is uniquely determined by the T-system and $T^{(a)}_1(u)\,(a \in I)$. See also [140].

Here we only give the formula for $T^{(r)}_1(u)$. It is known that the $U_q(B^{(1)}_r)$-module $W^{(r)}_1(u)$ is isomorphic as a $U_q(B_r)$-module to the spin representation of the latter. Its weights are multiplicity-free and naturally labeled with the arrays $(\sigma_1, \ldots, \sigma_r) \in \{\pm 1\}^r$. Accordingly we introduce

$$(\sigma_1, \ldots, \sigma_r)_u = \prod_{a=1}^{r} \left(Y_{a, q^{2u+2r-1-\rho_a}}\right)^{\frac{1}{2}(\sigma_a - \sigma_{a+1})}, \tag{7.13}$$

$$\rho_a = 2(\sigma_1 + \cdots + \sigma_{a-1}) + \frac{\sigma_a - \sigma_{a+1}}{t_a}, \quad \sigma_{r+1} = -\sigma_r. \tag{7.14}$$

Then we have

$$T^{(r)}_1(u) = \sum_{\sigma_1, \ldots, \sigma_r = \pm 1} (\sigma_1, \ldots, \sigma_r)_u. \tag{7.15}$$

For $r = 2$, $T^{(2)}_1(u) = \chi_q(W^{(2)}_1(u))$ has been written down in Example 4.5.

7.3. **Type C_r.** Let us introduce the index set and a total order on it as

$$J = \{1, 2 \ldots, r, \overline{r}, \ldots, \overline{2}, \overline{1}\}, \quad 1 \prec \cdots \prec r \prec \overline{r} \prec \cdots \prec \overline{1}. \tag{7.16}$$

For $1 \leq a \leq r$ we set

$$\begin{aligned} z_a(u) &= Y_{a, q^{2u+a-1}} Y^{-1}_{a-1, q^{2u+a}}, \\ z_{\overline{a}}(u) &= Y_{a-1, q^{2u+2r-a+2}} Y^{-1}_{a, q^{2u+2r-a+3}}, \end{aligned} \tag{7.17}$$

where $Y_{0,q^k} = 1$. Here we present the tableau sum formulas for $T^{(1)}_m(u)$ and $T^{(a)}_1(u)$. Consider the Young diagram (m) with length m one row shape. Let $\mathrm{Tab}(C_r,(m))$ be the set of tableaux on it with entries from J having the following form:

$$\boxed{i_1}\boxed{\;\cdots\;}\boxed{i_k}\underbrace{\boxed{\overline{r}}\boxed{r}\boxed{\cdots}\boxed{\overline{r}}\boxed{r}}_{2n}\boxed{\overline{j}_l}\boxed{\;\cdots\;}\boxed{\overline{j}_1}$$

$$1 \preceq i_1 \preceq \cdots \preceq i_k \preceq r, \quad \overline{r} \preceq \overline{j}_l \preceq \cdots \preceq \overline{j}_1 \preceq \overline{1}. \tag{7.18}$$

Here k, l and n are any nonnegative integers satisfying $k + 2n + l = m$. Let those tableaux be denoted simply by the array of entries as $(i_1, \ldots, \overline{j}_1) \in J^m$. Then we have

$$T^{(1)}_m(u) = \sum_{(i_1,\ldots,i_m)\in \mathrm{Tab}(C_r,(m))} \prod_{k=1}^{m} z_{i_k}\big(u + \frac{2k-m-1}{2}\big). \tag{7.19}$$

Consider the Young diagram (1^a) with length a one column shape. Let $\mathrm{Tab}(C_r,(1^a))$ be the set of tableaux on it with entries from J. The letter $i_k \in J$ inscribed on the kth row from the top should satisfy the conditions:

$$\begin{aligned} &i_1 \prec \cdots \prec i_a, \\ &r + k - l \geq c \ \text{ for any } k, l, c \text{ such that } \ i_k = c, \ i_l = \overline{c}. \end{aligned} \tag{7.20}$$

Denote such a tableau by the array $(i_1, \ldots, i_a) \in J^a$. Then we have

$$T^{(a)}_1(u) = \sum_{(i_1,\ldots,i_a)\in \mathrm{Tab}(C_r,(1^a))} \prod_{k=1}^{a} z_{i_k}\big(u + \frac{a+1-2k}{2}\big) \qquad (1 \leq a \leq r). \tag{7.21}$$

We note that $T^{(1)}_m(u)$ and $T^{(a)}_1(u)$ are the simplest cases in that the tableau rules can actually be described just by arrays without introducing a tableau.

7.4. **Type D_r.** Here we treat $T^{(1)}_m(u)$ and the fundamental q-characters $T^{(a)}_1(u)$. Let us introduce the index set and a partial order on it as

$$J = \{1, 2\ldots, r, \overline{r}, \ldots, \overline{2}, \overline{1}\}, \quad 1 \prec \cdots \prec r-1 \prec \begin{matrix} r \\ \overline{r} \end{matrix} \prec \overline{r-1} \prec \cdots \prec \overline{1}, \tag{7.22}$$

where no order is assumed between r and $\overline{r}$. For $i \in J$, define $z_i(u)$ by

$$\begin{aligned} &z_a(u) = Y_{a,q^{u+a-1}} Y^{-1}_{a-1,q^{u+a}} \qquad (1 \leq a \leq r-2), \\ &z_{r-1}(u) = Y_{r-1,q^{u+r-2}} Y_{r,q^{u+r-2}} Y^{-1}_{r-2,q^{u+r-1}}, \\ &z_r(u) = Y_{r,q^{u+r-2}} Y^{-1}_{r-1,q^{u+r}}, \\ &z_{\overline{r}}(u) = Y_{r-1,q^{u+r-2}} Y^{-1}_{r,q^{u+r}}, \\ &z_{\overline{r-1}}(u) = Y_{r-2,q^{u+r-1}} Y^{-1}_{r-1,q^{u+r}} Y^{-1}_{r,q^{u+r}}, \\ &z_{\overline{a}}(u) = Y_{a-1,q^{u+2r-a-2}} Y^{-1}_{a,q^{u+2r-a-1}} \qquad (1 \leq a \leq r-2), \end{aligned} \tag{7.23}$$

where $Y_{0,q^k} = 1$.

Let $\mathrm{Tab}(D_r,(m))$ be the set of one row tableaux $(i_1, \ldots, i_m) \in J^m$ obeying the condition:

$$\begin{aligned} &i_1 \prec \cdots \prec i_m, \\ &r \ \text{ and } \ \overline{r} \ \text{ do not appear simultaneously.} \end{aligned} \tag{7.24}$$

Then we have

$$T_m^{(1)}(u) = \sum_{(i_1,\ldots,i_m)\in \mathrm{Tab}(D_r,(m))} \prod_{k=1}^{m} z_{i_k}(u+2k-m-1). \tag{7.25}$$

For $1 \le a \le r-2$, let $\mathrm{Tab}(D_r,(1^a))$ be the set of one column tableaux $(i_1,\ldots,i_a) \in J^a$ obeying the condition:

$$i_k \prec i_{k+1} \ \text{ or } \ (i_k,i_{k+1}) = (r,\overline{r}) \ \text{ or } \ (i_k,i_{k+1}) = (\overline{r},r) \ \text{ for } 1 \le k \le a-1. \tag{7.26}$$

Then we have

$$T_1^{(a)}(u) = \sum_{(i_1,\ldots,i_a)\in \mathrm{Tab}(D_r,(1^a))} \prod_{k=1}^{a} z_{i_k}(u+a+1-2k) \qquad (1 \le a \le r-2). \tag{7.27}$$

It is known that the $U_q(D_r^{(1)})$-modules $W_1^{(r-1)}(u)$ and $W_1^{(r)}(u)$ are isomorphic as $U_q(D_r)$-modules to the spin representations of the latter. Their weights are multiplicity-free and naturally labeled with the arrays $(\sigma_1,\ldots,\sigma_r) \in \{\pm 1\}^r$. Accordingly we introduce

$$(\sigma_1,\ldots,\sigma_r)_u = \left(Y_{r,q^{u+r-1-\rho_r}}\right)^{\frac{1}{2}(\sigma_r+\sigma_{r-1})} \prod_{a=1}^{r-1} \left(Y_{a,q^{u+r-1-\rho_a}}\right)^{\frac{1}{2}(\sigma_a-\sigma_{a+1})}, \tag{7.28}$$

$$\rho_a = \begin{cases} \sigma_1 + \cdots + \sigma_{a-1} + \frac{\sigma_a - \sigma_{a+1}}{2} & 1 \le a \le r-1, \\ \sigma_1 + \cdots + \sigma_{r-2} + \frac{\sigma_r + \sigma_{r-1}}{2} & a = r. \end{cases} \tag{7.29}$$

It follows that

$$(\sigma_1,\ldots,\sigma_{r-1},-\sigma_r)_u = (\sigma_1,\ldots,\sigma_r)_u|_{Y_{r,q^k} \leftrightarrow Y_{r-1,q^k}}. \tag{7.30}$$

We have

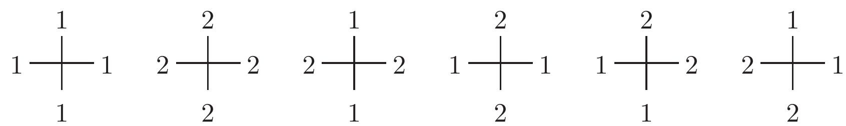
$$T_1^{(r-1)}(u) = \sum_{\substack{\sigma_1,\ldots,\sigma_r=\pm 1\\ \sigma_1\cdots\sigma_r=-1}} (\sigma_1,\ldots,\sigma_r)_u, \quad T_1^{(r)}(u) = \sum_{\substack{\sigma_1,\ldots,\sigma_r=\pm 1\\ \sigma_1\cdots\sigma_r=1}} (\sigma_1,\ldots,\sigma_r)_u. \tag{7.31}$$

7.5. **Bibliographical notes.** Tableau sums in 7.1 and 7.2 were respectively given in [59] and [137] in the context of analytic Bethe ansatz for more general skew shape Young diagrams. A uniform proof of the equality between the Jacobi-Trudi type determinant and the tableau sum is available in [141]. For type A_r, see also [142] for an account from the viewpoint of Macdonald's ninth variation of Schur functions [143]. The tableau sums in Sections 7.3 and 7.4 first appeared in the analytic Bethe ansatz [144]. The sums of the same structure are used in the deformed W-algebras [145]. Tableau constructions of higher $T_m^{(a)}(u)$ for C_r and D_r, which are significantly more involved than A_r and B_r, have been achieved in [146, 140]. In this section we have only treated the untwisted case $U_q(\hat{\mathfrak{g}})$. For tableau sum formulas for T-systems in twisted case, see [11, 12] and reference therein.

8. Analytic Bethe ansatz

Let $T_m^{(a)}(u)$ be the commuting transfer matrix of a solvable lattice model in the sense of Section 3. There is an empirical method called *analytic Bethe ansatz* to produce eigenvalues of $T_m^{(a)}(u)$ in many cases. Those eigenvalue formulas possess a specific "dressed vacuum form" which necessarily satisfy the T-system in Remark 2.7 with a nontrivial $g_m^{(a)}(u)$. Here we consider the Bethe equation and dressed vacuum forms for general $\mathfrak{g}$ and $T_m^{(a)}(u)$, and reformulate the conventional analytic Bethe ansatz via its connection with q-characters.

8.1. ***A_1* case.** Consider the 6 vertex model (3.1). Here we employ the normalization



$$[2+u]_{q^{1/2}} \qquad [2+u]_{q^{1/2}} \qquad [u]_{q^{1/2}} \qquad [u]_{q^{1/2}} \qquad z^{1/2}[2]_{q^{1/2}} \qquad z^{-1/2}[2]_{q^{1/2}}, \tag{8.1}$$

which is obtained by dividing (3.1) by $(zq)^{1/2}(1-q)$ and setting $z = q^u$. For the definition of the symbol $[u]_q$, see (3.18). Let $T_1(u)$ be the transfer matrix (3.11) with $m = 1$ and $w_j = q^{v_j}$. Its eigenvalue (denoted by the same symbol) is given by [2]

$$T_1(u) = \boxed{1}_u + \boxed{2}_u, \tag{8.2}$$

$$\boxed{1}_u = \phi(u+2)\frac{Q(u-1)}{Q(u+1)}, \quad \boxed{2}_u = \phi(u)\frac{Q(u+3)}{Q(u+1)}. \tag{8.3}$$

Here $\phi(u) = \prod_{j=1}^N [u - v_j]_{q^{1/2}}$ and $Q(u) = Q_1(u)$ is called Baxter's Q-function $Q(u) = \prod_{j=1}^n [u - u_j]_{q^{1/2}}$ with $u_1, \ldots, u_n$ determined from the Bethe equation

$$-\frac{\phi(u_j+1)}{\phi(u_j-1)} = \frac{Q(u_j+2)}{Q(u_j-2)} \qquad (1 \le j \le n). \tag{8.4}$$

Here, n is the number of down spins preserved under $T_1(u)$. The factors $\phi(u+2)$ and $\phi(u)$ in (8.3) are called vacuum parts in the sense that they are already present in the vacuum sector $n = 0$ where $Q(u) = 1$. In fact, the vector $11\ldots1$ is obviously the unique eigenvector with the vacuum eigenvalue:

$$\prod_{j=1}^N [u - v_j + 2]_{q^{1/2}} + \prod_{j=1}^N [u - v_j]_{q^{1/2}} = \phi(u+2) + \phi(u). \tag{8.5}$$

The factors involving Q-functions in (8.3) are called dress parts, and the eigenvalue formula of the form (8.2)–(8.3) is called a dressed vacuum form. The vacuum part is non-universal in that it is directly affected by the normalization of the Boltzmann weights (relevant R matrix) and also depends on the quantum space data such as inhomogeneity $\{v_j\}$ entering $\phi(u)$. On the other hand, the dress part encodes the structure of the auxiliary space essentially as we will see below.

The dressed vacuum form has an apparent pole at $u = -1 + u_j$ because of $Q(u_j) = 0$. The Bethe equation (8.4) tells that it is actually spurious provided that u_j is distinct from the other roots. This is compatible with the property that eigenvalues of the transfer matrix are regular functions of u if the local Boltzmann weights are so.

The analytic Bethe ansatz is a hypothesis that one can reverse these arguments to reproduce the eigenvalue formula from its characteristic properties bypassing the construction of eigenvectors. One starts with the ansatz dressed vacuum form with the prescribed vacuum part

$$T_1(u) = \phi(u+2)\frac{Q(u+a)}{Q(u+b)} + \phi(u)\frac{Q(u+c)}{Q(u+d)}. \tag{8.6}$$

Then a, b, c, d are determined by demanding that the pole-freeness is formally guaranteed by the Bethe equation (8.4) which one somehow admits from the onset. In the present example, this certainly fixes a, b, c, d uniquely as in (8.3). Further supplementary conditions may also be taken into account such as asymptotic behavior as $|u| \to \infty$ and the symmetry under complex conjugation, etc. It is not known whether such a procedure indeed leads to the unique and correct eigenvalue formula in general. Instead we shall propose in Section 8.2 a constructive way of producing the dressed vacuum form for general $U_q(\hat{\mathfrak{g}})$ by utilizing q-characters.

In the remainder of this subsection, we illustrate the simplest solution of the T-system for A_1 in the dressed vacuum form. Although the result is obtainable by specializing the tableau sum formula (7.3), we re-derive it here for later convenience. For simplicity $T^{(1)}_m(u)$ will be denoted by $T_m(u)$. Then the product of (8.2) is written as

$$T_1(u-1)T_1(u+1) = \boxed{1}_{u-1}\boxed{1}_{u+1} + \boxed{1}_{u-1}\boxed{2}_{u+1} + \boxed{2}_{u-1}\boxed{2}_{u+1} + \begin{array}{l}\boxed{1}_{u+1}\\ \boxed{2}_{u-1}\end{array}.$$

By (8.3), the last term becomes $\phi(u-1)\phi(u+3)$, which is independent of $Q(u)$. Identifying the other three terms with $T_2(u)$, one has

$$T_1(u-1)T_1(u+1) = T_2(u) + \phi(u-1)\phi(u+3),$$

which is an affinization of the identity (doublet)$^{\otimes 2}$ = (triplet) $\oplus$ (singlet) depicted as (7.2). It is easy to systematize this calculation to show that

$$T_m(u) = \sum_{1 \le i_1 \le \cdots \le i_m \le 2} \boxed{i_1}_{u-m+1}\boxed{i_2}_{u-m+3} \cdots \boxed{i_m}_{u+m-1} \tag{8.7}$$

is a solution of the unrestricted T-system for A_1 on the eigenvalues:

$$T_m(u-1)T_m(u+1) = T_{m-1}(u)T_{m+1}(u) + g_m(u), \tag{8.8}$$

$$g_m(u) = \prod_{k=0}^{m-1} \phi(u+2k-m)\phi(u+4+2k-m). \tag{8.9}$$

Explicitly, (8.7) reads as

$$T_m(u) = \Bigl(\prod_{k=1}^{m-1} \phi(u+m+1-2k)\Bigr) \sum_{j=0}^{m} \frac{Q(u-m)Q(u+m+2)\phi(u+m+1-2j)}{Q(u+m-2j)Q(u+m+2-2j)}. \tag{8.10}$$

The summands in (8.7) are naturally labeled with the semistandard tableaux of length m row shape (m) on numbers $\{1, 2\}$. Note that

$$g_m(u-1)g_m(u+1) = g_{m-1}(u)g_{m+1}(u) \tag{8.11}$$

is satisfied with $g_0(u) = 1$. Although the explicit form (8.10) is not particularly more illuminating than (8.7), one can easily check that it is formally pole-free in the same manner as before thanks to the Bethe equation (8.4). Another way of seeing this is of course by the Jacobi-Trudi type formula (6.1) with $r=1$ modified as $T^{(2)}_1(u) = g_1(u)$, e.g.,

$$T_3(u) = \begin{vmatrix} T_1(u-2) & g_1(u-1) & 0 \\ 1 & T_1(u) & g_1(u+1) \\ 0 & 1 & T_1(u+2) \end{vmatrix}.$$

Thus the pole-freeness of $T_m(u)$ is an obvious corollary of that for $T_1(u)$.

8.2. **Dressed vacuum form and q-characters.** The analytic Bethe ansatz is extended to the general $U_q(\hat{\mathfrak{g}})$ and further sharpened by a connection with the theory of q-characters. First we make a motive observation on the simplest example. Recall the q-character of $W^{(1)}_1(u)$, the "spin $\frac{1}{2}$ representation" of $U_q(A^{(1)}_1)$ in Example 4.4:

$$\chi_q(W^{(1)}_1(u)) = Y_z + Y^{-1}_{zq^2} \qquad (z = q^u). \tag{8.12}$$

On the other hand, the dressed vacuum form (8.2)–(8.3) of the 6-vertex model transfer matrix reads

$$T^{(1)}_1(u) = \phi(u+2)\frac{Q(u-1)}{Q(u+1)} + \phi(u)\frac{Q(u+3)}{Q(u+1)}. \tag{8.13}$$

Upon substitution

$$Y_{q^u} \to \frac{\eta(u-1)Q(u-1)}{\eta(u+1)Q(u+1)},$$

the q-character (8.12) becomes

$$\frac{\eta(u-1)}{\eta(u+1)}\frac{Q(u-1)}{Q(u+1)} + \frac{\eta(u+3)}{\eta(u+1)}\frac{Q(u+3)}{Q(u+1)}.$$

Thus the above substitution with the following overall renormalization

$$\phi(u+2)\frac{\eta(u+1)}{\eta(u-1)}\chi_q(W^{(1)}_1(u)) = \phi(u+2)\frac{Q(u-1)}{Q(u+1)} + \phi(u+2)\frac{\eta(u+3)}{\eta(u-1)}\frac{Q(u+3)}{Q(u+1)}$$

reproduces the dressed vacuum form (8.13) if $\eta(u)$ is assumed to obey the difference equation

$$\frac{\phi(u+1)}{\phi(u-1)} = \frac{\eta(u-2)}{\eta(u+2)}. \tag{8.14}$$

Note that this equation has the form of the Bethe equation (8.4):

$$-\frac{\phi(u_j+1)}{\phi(u_j-1)} = \frac{Q(u_j+2)}{Q(u_j-2)}$$

without the sign factor, and Q and u_j being replaced by η^{-1} and u, respectively. The same feature will be adopted in (8.19). The connection of (8.12) and (8.13) originates in the fact that the former is the q-character of $W^{(1)}_1(u)$ which is the auxiliary space of the transfer matrix relevant to the latter.

Now we generalize these observations to $U_q(\hat{\mathfrak{g}})$. Consider the trigonometric vertex model associated with $U_q(\hat{\mathfrak{g}})$ under the periodic boundary condition. Let $T^{(a)}_m(u)$ be the transfer matrix (3.44) with the auxiliary space $W^{(a)}_m(u)$ and the quantum space $W^{(r_1)}_{s_1}(v_1) \otimes \cdots \otimes W^{(r_N)}_{s_N}(v_N)$:

$$\begin{aligned} T^{(a)}_m(u) &= \mathrm{Tr}_{W^{(a)}_m(u)}\left(R^{(a,m;r_N,s_N)}_{0,N}(z/w_N)\cdots R^{(a,m;r_1,s_1)}_{0,1}(z/w_1)\right) \\ &\in \mathrm{End}(W^{(r_1)}_{s_1}(v_1) \otimes \cdots \otimes W^{(r_N)}_{s_N}(v_N)), \end{aligned} \tag{8.15}$$

where $z = q^{tu}, w_i = q^{tv_i}$. Due to the Yang-Baxter equation, they are commutative, i.e., $[T_m^{(a)}(u), T_n^{(b)}(v)] = 0$. The problem is to find their joint spectrum.

Let us construct a relevant dressed vacuum form $\Lambda_m^{(a)}(u)$ for $T_m^{(a)}(u)$. In the following, a simple identity

$$A_{a,z}|_{Y_{c,z} \to \frac{f_c(u-1/t_c)}{f_c(u+1/t_c)}} = \prod_{b=1}^{r} \frac{f_b(u - (\alpha_a|\alpha_b))}{f_b(u + (\alpha_a|\alpha_b))} \quad (z = q^{tu}) \tag{8.16}$$

for any functions $f_1, \ldots, f_r$ will be useful. See (2.1) and (4.26) for the definitions of t_a, t and $A_{a,z}$. First we introduce an "unnormalized" dressed vacuum form:

$$\tilde{\Lambda}_m^{(a)}(u) = \chi_q(W_m^{(a)}(u)) \ \text{ with substitution } \ Y_{c,q^{tv}} \to \frac{\eta_c(v - \frac{1}{t_c})}{\eta_c(v + \frac{1}{t_c})} \frac{Q_c(v - \frac{1}{t_c})}{Q_c(v + \frac{1}{t_c})}. \tag{8.17}$$

Let $\tilde{A}_{c,q^{tv}}$ be the result of the same substitution into $A_{c,q^{tv}}$. By the definition we have

$$\tilde{\Lambda}_m^{(a)}(u) = \frac{\eta_a(u - \frac{m}{t_a})}{\eta_a(u + \frac{m}{t_a})} \frac{Q_a(u - \frac{m}{t_a})}{Q_a(u + \frac{m}{t_a})} \left(1 + \sum_{c,v} \text{monomial in } \tilde{A}_{c,q^{tv}}^{-1}\right). \tag{8.18}$$

Here the factor $(\eta_a Q_a)/(\eta_a Q_a)$ is the top term specified by (4.22) and (8.17). The appearance of $\tilde{A}_{c,q^{tv}}^{-1}$ is due to Theorem 4.6 (1). As for the functions $\eta_1, \ldots, \eta_r$, we postulate, as the generalization of (8.14), the following difference equation

$$\prod_{\substack{k=1 \\ r_k = a}}^{N} \frac{\left[u - v_k + \frac{s_k}{t_a}\right]_{q^{t/2}}}{\left[u - v_k - \frac{s_k}{t_a}\right]_{q^{t/2}}} = \prod_{b=1}^{r} \frac{\eta_b(u - (\alpha_a|\alpha_b))}{\eta_b(u + (\alpha_a|\alpha_b))} \quad (1 \le a \le r), \tag{8.19}$$

where $[u]_p$ is defined in (3.18). Then using (8.16) and (8.19) we find

$$\begin{aligned} \tilde{A}_{a,q^{tu}} &= \prod_{b=1}^{r} \frac{\eta_b(u - (\alpha_a|\alpha_b)) Q_b(u - (\alpha_a|\alpha_b))}{\eta_b(u + (\alpha_a|\alpha_b)) Q_b(u + (\alpha_a|\alpha_b))} \\ &= \prod_{\substack{k=1 \\ r_k = a}}^{N} \frac{\left[u - v_k + \frac{s_k}{t_a}\right]_{q^{t/2}}}{\left[u - v_k - \frac{s_k}{t_a}\right]_{q^{t/2}}} \cdot \prod_{b=1}^{r} \frac{Q_b(u - (\alpha_a|\alpha_b))}{Q_b(u + (\alpha_a|\alpha_b))}. \end{aligned} \tag{8.20}$$

Next we adjust the overall normalization. Consider the R matrix on $W_m^{(a)}(u) \otimes W_s^{(b)}(v)$ and write its unique diagonal matrix element between the tensor product of the highest weight vectors as $\phi_{m,s}^{(a,b)}(u-v)$. Namely,

$$\phi_{m,s}^{(a,b)}(u-v) = \text{Boltzmann weight of the vertex} \quad \begin{array}{ccc} & s\omega_b & \\ m\omega_a & + & m\omega_a. \\ & s\omega_b & \end{array} \tag{8.21}$$

Now we define the normalized dressed vacuum form by

$$\begin{aligned} \Lambda_m^{(a)}(u) &= \left(\prod_{k=1}^{N} \phi_{m,s_k}^{(a,r_k)}(u - v_k)\right) \frac{\eta_a(u + \frac{m}{t_a})}{\eta_a(u - \frac{m}{t_a})} \tilde{\Lambda}_m^{(a)}(u) \\ &= \left(\prod_{k=1}^{N} \phi_{m,s_k}^{(a,r_k)}(u - v_k)\right) \frac{Q_a(u - \frac{m}{t_a})}{Q_a(u + \frac{m}{t_a})} \left(1 + \sum_{c,v} \text{monomial in } \tilde{A}_{c,q^{tv}}^{-1}\right). \end{aligned} \tag{8.22}$$

Besides the (in principle) known Boltzmann weights $\phi^{(a,b)}_{m,k}$, this only contains the Q-functions $Q_1, \ldots, Q_r$ and the LHS of (8.19).

Recall that the transfer matrices preserve the subspaces (sectors) of the quantum space specified by the weight. Let us parameterize the weight by the nonnegative integers $n_1, \ldots, n_r$ as

$$\sum_{k=1}^{N} s_k \omega_{r_k} - \sum_{a=1}^{r} n_a \alpha_a, \tag{8.23}$$

where $\omega_1, \ldots, \omega_r$ denote the fundamental weights of $\mathfrak{g}$ (2.2). Given n_a, we set

$$Q_a(u) = \prod_{j=1}^{n_a} [u - u_j^{(a)}]_{q^{t/2}} \tag{8.24}$$

by introducing the unknowns $\{u_j^{(a)} | 1 \le a \le r, 1 \le j \le n_a\}$.

Conjecture 8.1. *Let $T_m^{(a)}(u)$ (8.15) be the transfer matrix normalized as (8.21). Then its eigenvalues in the sector (8.23) are given by the dressed vacuum form $\Lambda_m^{(a)}(u)$ (8.22), (8.24) with the numbers $\{u_j^{(a)} | 1 \le a \le r, 1 \le j \le n_a\}$ satisfying the Bethe equation:*

$$\prod_{\substack{k=1\\ r_k=a}}^{N} \frac{\left[u_j^{(a)} - v_k + \frac{s_k}{t_a}\right]_{q^{t/2}}}{\left[u_j^{(a)} - v_k - \frac{s_k}{t_a}\right]_{q^{t/2}}} = -\prod_{b=1}^{r} \frac{Q_b(u_j^{(a)} + (\alpha_a|\alpha_b))}{Q_b(u_j^{(a)} - (\alpha_a|\alpha_b))}. \tag{8.25}$$

Practically the results in Section 7 serve as a large input to the prescription (8.17) to produce $\Lambda_m^{(a)}(u)$. The functions $Q_a(u)$ are called the (generalized) Baxter Q-functions. In view of Theorem 4.6 (2), we expect that their zeros, if in a generic position, do not cause a pole in $\Lambda_m^{(a)}(u)$ due to the Bethe equation.

Let $\mathcal{P}_a(\zeta)$ be the product of the ath Drinfeld polynomial (4.8) for each component in the quantum space $W_{s_1}^{(r_1)}(v_1) \otimes \cdots \otimes W_{s_N}^{(r_N)}(v_N)$:

$$\mathcal{P}_a(\zeta) = \prod_{\substack{k=1\\ r_k=a}}^{N} \prod_{i=1}^{s_k} (1 - \zeta q^{t(v_k + (s_k+1-2i)/t_a)}), \quad \deg \mathcal{P}_a = \sum_{\substack{k=1\\ r_k=a}}^{N} s_k. \tag{8.26}$$

We remark that the LHS of (8.19) is expressed as

$$\prod_{\substack{k=1\\ r_k=a}}^{N} \frac{\left[u - v_k + \frac{s_k}{t_a}\right]_{q^{t/2}}}{\left[u - v_k - \frac{s_k}{t_a}\right]_{q^{t/2}}} = q_a^{\deg \mathcal{P}_a} \frac{\mathcal{P}_a(\zeta q_a^{-1})}{\mathcal{P}_a(\zeta q_a)} \qquad (\zeta = q^{-tu}), \tag{8.27}$$

which further becomes the LHS of the Bethe equation (8.25) by the specialization $u = u_j^{(a)}$. This has formally the same form as (4.7). Note however that the quantum space $W_{s_1}^{(r_1)}(v_1) \otimes \cdots \otimes W_{s_N}^{(r_N)}(v_N)$ under consideration is not necessarily irreducible in general, and the above $\mathcal{P}_a(\zeta)$ is the ath Drinfeld polynomial of its irreducible quotient containing the tensor product of the highest weight vectors.

By the construction (8.17) and Theorem 4.8, the unnormalized dressed vacuum form $\tilde{\Lambda}_m^{(a)}(u)$ satisfies the unrestricted T-system for $\mathfrak{g}$. It follows that the normalized one $T_m^{(a)}(u) = \Lambda_m^{(a)}(u)$ (8.22) satisfies the modified T-system containing an extra factor $g_m^{(a)}(u)$ as (2.22):

$$T_m^{(a)}(u - \tfrac{1}{t_a}) T_m^{(a)}(u + \tfrac{1}{t_a}) = T_{m-1}^{(a)}(u) T_{m+1}^{(a)}(u) + g_m^{(a)}(u) M_m^{(a)}(u),$$

where the original T-system corresponds to $g_m^{(a)}(u) = 1$ as in (2.18). The scalar factor $g_m^{(a)}(u)$ has the properties:

(i)Apart from (a, m, u), it only depends on the quantum space data $W_{s_1}^{(r_1)}(v_1) \otimes \cdots \otimes W_{s_N}^{(r_N)}(v_N)$.

(ii) It satisfies the relation (2.23):

$$g_m^{(a)}(u - \tfrac{1}{t_a}) g_m^{(a)}(u + \tfrac{1}{t_a}) = g_{m-1}^{(a)}(u) g_{m+1}^{(a)}(u).$$

In fact this has been encountered for $\mathfrak{g} = A_1$ in (8.11). To derive these properties, note that the fusion construction implies that the diagonal element of the R-matrix (8.21) is factorized as $\phi_{m,s}^{(a,b)}(u) = \prod_{i=1}^m \phi_{1,s}^{(a,b)}(u + (m+1-2i)/t_a)$. Thus the first relation in (8.22) is written as

$$\tilde{\Lambda}_m^{(a)}(u) = \Lambda_m^{(a)}(u) \prod_{i=1}^m \gamma_a\Bigl(u + \frac{m+1-2i}{t_a}\Bigr), \tag{8.28}$$

$$\gamma_a(u) = \frac{\eta_a(u - \frac{1}{t_a})}{\eta_a(u + \frac{1}{t_a})} \prod_{k=1}^N \phi_{1,s_k}^{(a,r_k)}(u - v_k)^{-1}. \tag{8.29}$$

In view of (8.28), replace $T_m^{(a)}(u)$ in the original T-system with $T_m^{(a)}(u) \prod_{i=1}^m \gamma_a(u + (m+1-2i)/t_a)$. After removing the common factor, the result is indeed reduced to the form (2.22) with

$$g_m^{(a)}(u) = \prod_{i=1}^m g_1^{(a)}\Bigl(u + \frac{m+1-2i}{t_a}\Bigr), \tag{8.30}$$

$$g_1^{(a)}(u) = A_{a,z}^{-1}|_{Y_{c,z} \to \gamma_c(u)} \quad (z = q^{tu}). \tag{8.31}$$

The property (ii) directly follows from (8.30) without using the concrete form of $g_1^{(a)}(u)$. The property (i) is essentially due to the remark after (8.22). In fact, it is attributed to $g_1^{(a)}(u)$ (8.31). With regard to $\gamma_c(u)$ therein, $\phi_{1,s_k}^{(c,r_k)}(u - v_k)$ in (8.29) depends on the quantum space data only, and so does the contribution from η_c because of (8.16) and (8.19).

Remark 8.2. The transfer matrix (8.15) can be generalized by the "magnetic field" as $T_m^{(a)}(u) = \mathrm{Tr}_{W_m^{(a)}(u)}(e^{\mathcal{H}} R_{0,N}^{(a,m;r_N,s_N)}(z/w_N) \cdots R_{0,1}^{(a,m;r_1,s_1)}(z/w_1))$ without spoiling the commutatibity and the T-system. Here $\mathcal{H}$ is any element in the Cartan subalgebra of $U_q(\mathfrak{g})$ acting on the auxiliary space. The dressed vacuum form for such $T_m^{(a)}(u)$ is obtained by modifying the substitution (8.17) into $Y_{c,q^{tv}} \to e^{\omega_c(\mathcal{H})} \frac{\eta_c(v - \frac{1}{t_c}) Q_c(v - \frac{1}{t_c})}{\eta_c(v + \frac{1}{t_c}) Q_c(v + \frac{1}{t_c})}$. Accordingly $\tilde{A}_{a,q^{tu}}$ (8.20) and the LHS of the Bethe equation (8.25) get multiplied by the extra factor $e^{\alpha_a(\mathcal{H})}$.

8.3. **RSOS models.** We consider the spectrum of the transfer matrix $T_m^{(a)}(u)$ $(1 \le m \le t_a\ell)$ (3.50) for the trigonometric level ℓ RSOS models sketched in Section 3.7. ($T_{t_a\ell}^{(a)}(u)$ corresponds to a frozen model.) Conjecturally, it is covered by the dressed vacuum form in Remark 8.2 specialized along (i)–(iii) in the sequel.

(i) The parameter q entering through $[u]_{q^{t/2}}$ is set $q = \exp\left(\frac{\pi\sqrt{-1}}{t(\ell + h^\vee)}\right)$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$ (2.3).

(ii) The integers $n_1, \ldots, n_r$ entering (8.24) are fixed by demanding (8.23) be 0, which is possible thanks to (3.51).

(iii) The magnetic field is taken so that $\omega_c(\mathcal{H}) = \frac{2\pi\sqrt{-1}(\omega_c|\Lambda+\rho)}{\ell+h^\vee}$, where $\rho = \sum_{a\in I}\omega_a$ and Λ is an element from P_ℓ (3.45).

Introduce the specialized q-character $Q^{(a)}_m(\Lambda) := \chi_q(W^{(a)}_{t_a\ell}(u))|_{Y_{c,q^{tv}}\to e^{\omega_c(\mathcal{H})}}$, where Λ-dependence enters through the above $\mathcal{H}$. Then according to the conjecture in [1, (A.8)-(A.9)], the relation $\prod_{b\in I} Q^{(b)}_{t_b\ell}(\Lambda)^{C_{ab}} = 1$ holds. The quantity $Q^{(a)}_m = \dim_q \operatorname{res} W^{(a)}_{t_a\ell}$ in Section 14.6 is equal to $Q^{(a)}_m(0)$ in the notation here. The above relation is a generalization of $Q^{(a)}_{t_a\ell}(0) = 1$ in Section 14.6.

8.4. **Bibliographical notes.** The analytic Bethe ansatz was proposed in [54] by extracting the idea from Baxter's solution of the 8-vertex model [52]. It was applied systematically in [55, 144, 137] to a wide class of solvable vertex models. Formulation of the Bethe equation by root system goes back, for instance, to [147, 55]. A relation between dressed vacuum forms and q-characters similar to Section 8.2 has also been argued in [70, section 6].

9. Wronskian type (Casoratian) formula

Here we present the solution of the T-system for A_r and C_r in terms of Casoratian (difference analogue of Wronskian). It is most naturally done by introducing a difference analogue of L-operators in soliton theory. It also provides a Casoratian interpretation and generalization of the Baxter Q-functions. Our description is along the context of q-characters, hence the identification of the variables

$$Y_{a,q^{tu}} = \frac{Q_a(u-\frac{1}{t_a})}{Q_a(u+\frac{1}{t_a})} \tag{9.1}$$

is assumed. See (8.17). (t, t_a are defined in (2.1).) Resulting formulas can suitably be modified to fit transfer matrices with specific normalizations according to the argument in Section 8.2. We will also give analogous L-operators for B_r, D_r and $sl(r|s)$.

9.1. **Difference L operators.** We treat the A_r case first as an illustration. Let $D = e^{2\partial_u}$ be the shift operator $Df(u) = f(u+2)D$. Using $z_a(u)$ (7.6), we introduce the difference L operator:

$$L(u) = (1 - z_{r+1}(u)D)\cdots(1 - z_2(u)D)(1 - z_1(u)D). \tag{9.2}$$

Expanding the product, one identifies the coefficients with $m = 1$ case of (7.4) to find

$$L(u) = \sum_{a=0}^{r+1}(-1)^a T^{(a)}_1(u+a-1)D^a, \tag{9.3}$$

where $T^{(0)}_1 = T^{(r+1)}_1 = 1$. Thus $L(u)$ is a generating function of the fundamental q-characters $T^{(a)}_1(u) = \chi_q(W^{(a)}_1(u))$.

Define the action of the screening operator S_a (4.27) on difference operators by $S_a\cdot(\sum_i f_i(u)D^i) = \sum_i(S_a\cdot f_i(u))D^i$. Let us calculate $S_a\cdot L(u)$ by using the factorized form (9.2). According to the rule (4.27), S_a acts non trivially only on the variable $Y_{a,z}$. From (7.6), it is contained only in $z_a(u)$ and $z_{a+1}(u)$. The action on this part is calculated as

$$
\begin{aligned}
&S_a \cdot (1 - z_{a+1}(u)D)(1 - z_a(u)D)\\
&= S_a \cdot (1 - Y^{-1}_{a,q^{u+a+1}} Y_{a+1,q^{u+a}} D - Y^{-1}_{a-1,q^{u+a}} Y_{a,q^{u+a-1}} D + Y^{-1}_{a-1,q^{u+a+2}} Y_{a+1,q^{u+a}} D^2)\\
&= S_{a,q^{u+a+1}} Y^{-1}_{a,q^{u+a+1}} Y_{a+1,q^{u+a}} D - S_{a,q^{u+a-1}} Y^{-1}_{a-1,q^{u+a}} Y_{a,q^{u+a-1}} D = 0,
\end{aligned}
$$

where the last equality is due to (4.28) and (4.26):

$$S_{a,q^{u+a+1}} = A_{a,q^{u+a}} S_{a,q^{u+a-1}} = Y_{a,q^{u+a-1}} Y_{a,q^{u+a+1}} Y^{-1}_{a-1,q^{u+a}} Y^{-1}_{a+1,q^{u+a}} S_{a,q^{u+a-1}}.$$

In this way one gets

$$S_a \cdot L(u) = 0 \quad (1 \le a \le r). \tag{9.4}$$

In view of (9.3), this offers a simple way of checking $T^{(a)}_1(u) \in \bigcap_{b=1}^r \mathrm{Ker} S_b$ in agreement with Theorem 4.6 (2). When $r = 1$, the change of variables from $\{z_a(u)\}$ to $\{T^{(a)}_1(u)\}$ is a difference analogue of the Miura transformation $q = q(u) \to f = f(u) = q^2 - \partial_u q$ by

$$(\partial_u - q)(\partial_u + q) = \partial_u^2 - f.$$

With regard to the inverse

$$L(u)^{-1} = (1 - z_1(u)D)^{-1}(1 - z_2(u)D)^{-1} \cdots (1 - z_{r+1}(u)D)^{-1},$$

the simple expansion formula

$$L(u)^{-1} = \sum_{m \ge 0} T^{(1)}_m(u + m - 1) D^m \tag{9.5}$$

holds due to (7.4), confirming similarly that $T^{(1)}_m(u) \in \bigcap_{b=1}^r \mathrm{Ker} S_b$. The product of (9.3) and (9.5) leads to the two types of TT-relations:

$$
\begin{aligned}
&\sum_{0 \le a \le \min(r+1,m)} (-1)^a T^{(a)}_1(u + a) T^{(1)}_{m-a}(u + m + a) = \delta_{m0},\\
&\sum_{0 \le a \le \min(r+1,m)} (-1)^a T^{(a)}_1(u + m - a) T^{(1)}_{m-a}(u - a) = \delta_{m0}
\end{aligned}
$$

for $m \ge 0$.

9.2. **Casoratian formula.** Consider the linear difference equation on $w(u)$

$$L(u) w(u) = 0. \tag{9.6}$$

This is of order $r + 1$ with respect to D. Letting $\{w_1(u), \dots, w_{r+1}(u)\}$ be a basis of the solution, we denote the Casoratian by

$$C_u[i_1, \dots, i_k] = \det \begin{pmatrix} w_1(u + i_1) & \cdots & w_1(u + i_k) \\ \vdots & & \vdots \\ w_k(u + i_1) & \cdots & w_k(u + i_k) \end{pmatrix} \tag{9.7}$$

for $1 \le k \le r + 1$. Thus for example $C_{u+2}[i_1, \dots, i_k] = C_u[i_1 + 2, \dots, i_k + 2]$. By using (9.3), the relations $L(u) w_k(u) = 0$ with $k = 1, \dots, r + 1$ are expressed in the matrix form:

$$
\begin{pmatrix} w_1(u) \\ w_2(u) \\ \vdots \\ w_{r+1}(u) \end{pmatrix}
=
\begin{pmatrix} w_1(u+2) & w_1(u+4) & \cdots & w_1(u+2r+2) \\ w_2(u+2) & w_2(u+4) & \cdots & w_2(u+2r+2) \\ \vdots & & & \vdots \\ w_{r+1}(u+2) & w_{r+1}(u+4) & \cdots & w_{r+1}(u+2r+2) \end{pmatrix}
\begin{pmatrix} T^{(1)}_1(u) \\ (-1) T^{(2)}_1(u+1) \\ \vdots \\ (-1)^r T^{(r+1)}_1(u+r) \end{pmatrix},
$$

where $T_1^{(r+1)}(u) = 1$ in our normalization here (q-characters) as noted under (7.6). By Cramer's formula, we have

$$T_1^{(a)}(u+a-1) = \frac{C_u[0,\ldots,2a-2,2a+2,\ldots,2r+2]}{C_u[2,\ldots,2r+2]} \quad (0 \le a \le r+1), \tag{9.8}$$

where ... signifies that the omitted arrays are consecutive with difference 2. The relation $L(u)w_k(u) = 0$ means that $w_k(u+2r+2) = (-1)^r w_k(u)+$terms involving $w_k(u+2), \ldots, w_k(u+2r)$. It follows the periodicity

$$C_u[0,2,\ldots,2r] = C_{u+2}[0,2,\ldots,2r]. \tag{9.9}$$

Its actual value becomes important in physical applications, and the resulting relation on $C_u[0,2,\ldots,2r]$ is called the quantum Wronskian condition. See for example [148, 149].

The solution to the T-system for A_r that matches (9.8) is given by

$$T_m^{(a)}(u+a+m-2) = \frac{C_u[0,\ldots,2a-2,2a+2m,\ldots,2r+2m]}{C_u[0,\ldots,2r]} \quad (0 \le a \le r+1). \tag{9.10}$$

This satisfies the boundary conditions $T_m^{(0)}(u) = T_0^{(a)}(u) = 1$ and $T_{-1}^{(a)}(u) = 0$. In fact, if (9.10) is substituted into (2.5), the denominator can be removed as an overall factor owing to (9.9). Then (2.5) is identified with a simplest Plücker relation

$$\xi_m^{(a)}(u)\xi_m^{(a)}(u+2) - \xi_{m+1}^{(a)}(u)\xi_{m-1}^{(a)}(u+2) - \xi_m^{(a+1)}(u)\xi_m^{(a-1)}(u+2) = 0 \tag{9.11}$$

among the determinant $\xi_m^{(a)}(u) = C_u[0,\ldots,2a-2,2a+2m,\ldots,2r+2m]$.

The Casoratian formula (9.10) is a Yang-Baxterization (u-dependent generalization) of the Weyl character formula. To see this, recall the restriction map res (4.24). From (7.6) we have $\mathrm{res}\,(z_a(u)) = x_a$, where the latter is defined by $x_a = y_a/y_{a-1} = e^{\omega_a - \omega_{a-1}}$ with $\omega_0 = \omega_{r+1} = 0$. We extend res naturally to the difference L operator and the wave functions as

$$\mathrm{res}\,L(u) = (1 - x_{r+1}D)\cdots(1 - x_1 D), \quad \mathrm{res}\,(w_i(u)) = x_i^{-u/2}. \tag{9.12}$$

The latter is certainly annihilated by the former. By using $x_1 \cdots x_{r+1} = 1$, it is straightforward to see that the restriction of (9.10) becomes

$$\mathrm{res}\left(\frac{C_u[0,\ldots,2a-2,2a+2m,\ldots,2r+2m]}{C_u[0,\ldots,2r]}\right) = \frac{\det(x_i^{\lambda_j+r+1-j})_{1\le i,j\le r+1}}{\det(x_i^{r+1-j})_{1\le i,j\le r+1}}, \tag{9.13}$$

where (λ_j) corresponds to the $a \times m$ rectangular Young diagram, namely, $\lambda_j = m$ if $1 \le j \le a$ and $\lambda_j = 0$ otherwise. The RHS is the Weyl character formula of the Schur function for (λ_j) as is well known.

The Casoratian formula here and the tableau sum formula (Section 7.1) are connected by the following general fact.

Proposition 9.1 ([142]). *Let $C_u[i_1,\ldots,i_k]$ be as in (9.7). ($L(u)w_j(u) = 0$ is not assumed.) Given even integers $0 = i_0 < i_1 < \cdots < i_{N-1}$, let $\mu = (\mu_j)$ be the Young diagram with depth less than N specified by $\mu_j = \frac{i_{N-j}}{2} + j - N$. Take any $d \ge \mu_1$. Then*

$$\frac{C_u[0,i_1,i_2,\cdots,i_{N-1}]}{C_{u+2d}[0,2,\ldots,2N-2]} = \sum_{\mathcal{T}} \prod_{(\alpha,\beta)\in(d^N)/\mu} \tilde{x}_{\mathcal{T}(\alpha,\beta)}(u+2\alpha+2\beta-4),$$

where $\tilde{x}_j(u) = \frac{C_u[0,2,\ldots,2j-2]C_u[4,6,\ldots,2j]}{C_u[2,4,\ldots,2j]C_u[2,4,\ldots,2j-2]}$ *and the sum* $\sum_{\mathcal{T}}$ *extends over the semistandard tableaux on the skew Young diagram* $(d^N)/\mu$ [135] *on letters* $\{1,\ldots,N\}$. $\mathcal{T}(\alpha,\beta)$ *denotes the entry of* $\mathcal{T}$ *at the* α*th row and the* β*th column from the bottom left corner.*

According to Proposition 9.1, the RHS of (9.10) equals the sum over semistandard tableaux on $a \times m$ Young diagram on letters $\{1,\ldots,r+1\}$. The building block of the tableau variable $\tilde{x}_j(u)$ is the principal minors of the Casoratian (quantum Wronskian) $C_u[0,2,\ldots,2r]$. Combined with (9.6), they are identified with the Baxter Q-functions as we will see in the next subsection.

9.3. ***Q*-functions.** From the full L operator (9.2), we extract the partial ones by

$$L_j(u) = (1 - z_j(u)D)\cdots(1 - z_2(u)D)(1 - z_1(u)D) \quad (1 \le j \le r+1). \tag{9.14}$$

The original one corresponds to $L_{r+1}(u)$. By the definition we have

$$\mathrm{Ker}\, L_1(u) \subset \mathrm{Ker}\, L_2(u) \subset \cdots \subset \mathrm{Ker}\, L_{r+1}(u). \tag{9.15}$$

Choose the basis of $\mathrm{Ker}\, L_j(u)$ according to this flag structure as

$$\{w_1(u)\} \subset \{w_1(u), w_2(u)\} \subset \cdots \subset \{w_1(u),\ldots,w_{r+1}(u)\}. \tag{9.16}$$

As the simplest example, $w_1(u) \in \mathrm{Ker}\, L_1(u)$ is the condition $0 = (1 - z_1(u)D)w_1(u)$. In view of (7.6) and (9.7), this is the $j = 1$ case of

$$\big(1 - Y_{j,q^{u+j-1}}D\big)C_u[0,\ldots,2j-2] = 0 \quad (1 \le j \le r). \tag{9.17}$$

To derive this, note that a direct calculation using (7.6) leads to

$$L_j(u) = 1 + (-1)^j Y_{j,q^{u+j-1}} D^j + \text{terms involving}\, D,\ldots,D^{j-1}.$$

Therefore $L_j(u)w_k(u) = 0$ $(1 \le k \le j)$ implies

$$Y_{j,q^{u+j-1}} w_k(u+2j) = (-1)^{j-1} w_k(u) + \sum_{l=1}^{j-1} c_{j,l}(u) w_k(u+2l),$$

where $c_{j,l}(u)$ is independent of k. The second term in (9.17) is equal to $Y_{j,q^{u+j-1}} C_u[2,\ldots,2j-2,2j]$. Applying the above relation to the last column of this, we find the result is equal to $C_u[0,\ldots,2j-2]$, hence (9.17).

If we express the variable Y_{a,q^u} in q-characters in terms of Q-functions as in (9.1), the solution of the first order difference equation (9.17) is given by

$$C_u[0,\ldots,2j-2] = \sigma_j(u) Q_j(u+j-2) \quad (1 \le j \le r), \tag{9.18}$$

where $\sigma_j(u)$ is any variable satisfying $\sigma_j(u+2) = \sigma_j(u)$. In this way, the Q-functions are identified with the principal minors of the Casoratian $C_u[0,\ldots,2r]_u$ made of the wave functions $\{w_i(u)\}$ especially chosen along the scheme (9.16). The simplest case $j = 1$ of (9.18) is $w_1(u) = \sigma_1(u)Q_1(u-1)$. Thus $L(u)w_1(u) = 0$ is rephrased as

$$\sum_{a=0}^{r+1} (-1)^a T_1^{(a)}(u+a) Q_1(u+2a) = 0, \tag{9.19}$$

which is an example of TQ-relations.

9.4. **Bäcklund transformations.** Here we remove the boundary condition $T^{(a)}_0(u) = T^{(0)}_m(u) = 1$ and redefine $T^{(a)}_m(u)$ in (9.10) and $Q_j(u)$ in (9.18) as

$$T^{(a)}_m(u+a+m-2) = C_u[0,\dots,2a-2,2a+2m,\dots,2r+2m], \tag{9.20}$$
$$Q_a(u+a-1) = C_u[0,\dots,2a-2]. \tag{9.21}$$

These functions are special cases of more general ones:

$$\begin{aligned} & T^{(s,a)}_m(u+a+m-2) \\ & = \begin{vmatrix} w_1(u) & \cdots & w_1(u+2a-2) & w_1(u+2a+2m) & \cdots & w_1(u+2s+2m) \\ \vdots & & & & & \vdots \\ w_{s+1}(u) & \cdots & w_{s+1}(u+2a-2) & w_{s+1}(u+2a+2m) & \cdots & w_{s+1}(u+2s+2m) \end{vmatrix}, \\ & Q_{\{i_1,\dots,i_a\}}(u+a-1) = \begin{vmatrix} w_{i_1}(u) & \cdots & w_{i_1}(u+2a-2) \\ \vdots & & \vdots \\ w_{i_a}(u) & \cdots & w_{i_a}(u+2a-2) \end{vmatrix}, \end{aligned} \tag{9.22}$$

where $\cdots$ in determinants signify that u increases by 2. $T^{(s,a)}_m(u)$ is defined for $0 \le a \le s+1, 0 \le s \le r$ and $m \ge 0$. The set $\{i_1,\dots,i_a\}$ is any subset of $\{1,\dots,r+1\}$. By the definition, $T^{(r,a)}_m(u) = T^{(a)}_m(u)$ and $Q_{\{1,\dots,a\}}(u) = Q_a(u)$. These functions obey various relations as the consequence of identities among determinants. Let us mention a few of them that have analogy with soliton theory.

The symmetric group $\mathfrak{S}_{r+1}$ acts on the basis $w_1(u),\dots,w_{r+1}(u)$ as their permutations keeping $L(u)$ invariant. This can be viewed as Bäcklund transformations generating the functions $Q_{\{i_1,\dots,i_a\}}$ from $Q_1,\dots,Q_{r+1}$. Its generator, the transposition s_a of $w_a(u)$ and $w_{a+1}(u)$, acts trivially as $s_a(Q_b) = Q_b$ for $a > b$ and similarly as $s_a(Q_b) = -Q_b$ for $a < b$. The nontrivial case $s_a(Q_a) = Q_{\{1,\dots,a-1,a+1\}}$ satisfies the QQ-relation:

$$D(Q_a)s_a(Q_a) - Q_aDs_a(Q_a) + D(Q_{a-1})Q_{a+1} = 0, \tag{9.23}$$

where the first term denotes $Q_a(u+2)s_a(Q_a)(u)$ for instance. This is derived by applying the Jacobi identity (6.2) to the $a, a+1$ rows and $1, a+1$ columns for the determinant of Q_{a+1}.

With regard to $T^{(s,a)}_m(u)$, it is the T-function for $A_s (\subset A_r)$. Writing $T^{(s,a)}_m(u)$ and $T^{(s-1,a)}_m(u)$ simply as $T^{(a)}_m(u)$ and $\tilde{T}^{(a)}_m(u)$ respectively, one can derive

$$\begin{aligned} T^{(a)}_m(u)\tilde{T}^{(a-1)}_m(u-1) &= T^{(a-1)}_m(u-1)\tilde{T}^{(a)}_m(u) + T^{(a)}_{m-1}(u-1)\tilde{T}^{(a-1)}_{m+1}(u), \\ T^{(a)}_{m+1}(u-1)\tilde{T}^{(a)}_m(u) &= T^{(a)}_m(u)\tilde{T}^{(a)}_{m+1}(u) + T^{(a+1)}_m(u-1)\tilde{T}^{(a-1)}_{m+1}(u) \end{aligned} \tag{9.24}$$

from the Plücker relation. This is a Bäcklund transformation between T-functions associated with A_s and A_{s-1}. The T-system for $T^{(a)}_m(u)$ arises as a compatibility of the two linear equations on $\tilde{T}^{(a)}_m(u)$ [150]. For more examples, see [151, 152, 23, 24] and reference therein. It is an open problem to construct such a Lax representation of the T-system for general $\mathfrak{g}$.

9.5. **Type C_r.** Let D be the difference operator $Df(u) = f(u+1)D$. We use the variable $z_a(u)\,(a \in J)$ (7.17) which are related to the Q-functions by (9.1). We also introduce the variables $x_1(u), \ldots, x_{2r+2}(u)$ by

$$
\begin{aligned}
x_a(u) = z_a(u), \quad x_{2r+3-a}(u) = z_{\overline{a}}(u) \ \ (1 \le a \le r), \\
x_{r+1}(u) = -x_{r+2}(u) = \frac{Q_r(u+\frac{r-1}{2})Q_r(u+\frac{r+3}{2})}{Q_r(u+\frac{r+1}{2})^2}.
\end{aligned} \tag{9.25}
$$

Note that $x_{r+1}(u)$ and $x_{r+2}(u)$ are *not* contained in $\mathbb{Z}[Y^{\pm}_{a,z}]_{a\in I, z\in\mathbb{C}^\times}$. With the notation

$$
\prod_{1\le i\le k}^{\longrightarrow} X_i = X_1X_2\cdots X_k, \qquad \prod_{1\le i\le k}^{\longleftarrow} X_i = X_k\cdots X_2X_1, \tag{9.26}
$$

the difference L-operator is

$$
L(u) = \prod_{1\le a\le r}^{\longrightarrow}(1 - z_{\overline{a}}(u)D)\cdot(1 - z_{\overline{r}}(u)z_r(u+1)D^2)\cdot \prod_{1\le a\le r}^{\longleftarrow}(1 - z_a(u)D). \tag{9.27}
$$

One can easily check $S_a \cdot L(u) = 0$ as in type A. The middle quadratic operator can be factorized as

$$
\begin{aligned}
1 - Y_{r,q^{2u+r+1}}Y^{-1}_{r,q^{2u+r+3}}D^2 &= 1 - \frac{Q_r(u+\frac{r+5}{2})Q_r(u+\frac{r-1}{2})}{Q_r(u+\frac{r+1}{2})Q_r(u+\frac{r+3}{2})}D^2 \\
&= (1 \pm x_{r+2}(u)D)(1 \pm x_{r+1}(u)D).
\end{aligned}
$$

Thus (9.27) is expressed as

$$
L(u) = \prod_{1\le i\le 2r+2}^{\longleftarrow}(1 - x_i(u)D), \tag{9.28}
$$

which resembles curiously A_{2r+1} case rather than A_{2r-1}. The operator $L(u)$ generates each fundamental q-character "twice".

Theorem 9.2 ([139])**.**

$$
L(u) = \sum_{a=0}^{r}(-1)T_1^{(a)}(u+\frac{a-1}{2})D^a - \sum_{a=r+2}^{2r+2}(-1)^aT_1^{(2r+2-a)}(u+\frac{a-1}{2})D^a,
$$

where $T_1^{(0)} = 1$.

From Theorem 9.2 and (9.28), we obtain another tableau sum formula for the fundamental q-characters:

$$
T_1^{(a)}(u+\frac{a-1}{2}) = \sum_{1\le i_1\le\cdots\le i_a\le 2r+2}\prod_{k=1}^{a} x_{i_k}(u+a-k) \quad (1 \le a \le r). \tag{9.29}
$$

Although this is formally the same form as A_{2r+1} case (7.4), the variable $x_{r+2}(u)$ (9.25) is "negative" here. It is highly nontrivial that the cancellation due to the sign yields the previous formula (7.21) described by the rule (7.20), which constitutes a substantial part of the proof of Theorem 9.2. On the other hand it is easy to see

$$
L(u)^{-1} = \sum_{m\ge 0} T_m^{(1)}(u+\frac{m-1}{2})D^m \tag{9.30}
$$

from (7.19), (7.18) and (9.27).

The rest of this subsection will be brief as the content is more or less parallel with A_{2r+1} case. We formally extend the fundamental q-characters $T^{(a)}_1(u)$ to $1 \le a \le 2r+2$ by

$$T^{(a)}_1(u) + T^{(2r+2-a)}_1(u) = 0 \quad (0 \le a \le 2r+2). \tag{9.31}$$

Then Theorem 9.2 is rephrased as

$$L(u) = \sum_{a=0}^{2r+2} (-1)^a T^{(a)}_1(u + \frac{a-1}{2}) D^a. \tag{9.32}$$

We consider the difference equation $L(u)w(u) = 0$ and a basis of the solution $\{w_1(u), \ldots, w_{2r+2}(u)\}$. With the same notation $C_u[i_1, \ldots, i_k]$ as (9.7), we have the Casoratian formula

$$T^{(a)}_1(u + \frac{a-1}{2}) = \frac{C_u[0, \ldots, a-1, a+1, \ldots, 2r+2]}{C_u[1, \ldots, 2r+2]} \quad (0 \le a \le 2r+2), \tag{9.33}$$

where $\ldots$ signifies that the omitted arrays are consecutive with difference 1. The denominator possesses the periodicity

$$C_u[0, 1, \ldots, 2r+1] = -C_{u+1}[0, 1, \ldots, 2r+1], \tag{9.34}$$

which is a C_r analogue of the quantum Wronskian condition.

Set

$$\begin{aligned} \xi^{(a)}_m(u) &= C_u[0, \ldots, a-1, a+m, \ldots, 2r+1+m], \\ \xi(u) &= C_u[0, 1, \ldots, 2r+1]. \end{aligned} \tag{9.35}$$

The solution of the unrestricted T-system for C_r that matches (9.33) is given by

Theorem 9.3 ([139])**.** *The following is a solution of the T-system for C_r.*

$$\begin{aligned} &T^{(a)}_m(u + \frac{a+m-2}{2}) = (-1)^{m-1} \frac{\xi^{(a)}_m(u)}{\xi(u+1)} \quad (1 \le a \le r-1), \\ &T^{(r)}_m(u + \frac{r+2m-1}{2}) T^{(r)}_m(u + \frac{r+2m-3}{2}) = \frac{\xi^{(r)}_{2m}(u)}{\xi(u)}, \\ &T^{(r)}_m(u + \frac{r+2m-1}{2}) T^{(r)}_{m+1}(u + \frac{r+2m-1}{2}) = \frac{\xi^{(r)}_{2m+1}(u)}{\xi(u+1)}, \\ &T^{(r)}_m(u + \frac{r+2m-1}{2})^2 = \frac{\xi^{(r+1)}_{2m}(u)}{\xi(u)}. \end{aligned}$$

As for the first three, there is an alternative expression derived by using the identity $\xi^{(a)}_m(u) = (-1)^{a+m+r+1} \xi^{(2r+2-a)}_m(u + a - r - 1)$. See Proposition 4.3 in [139] for details.

9.6. **Type B_r and D_r.** Here we only give the L-operators and their expansions. Let D be the difference operator $Df(u) = f(u+2)D$. We use the variables $z_a(u)$ for B_r (7.9) and D_r (7.23) which are related to the Q-function by (9.1). The difference

L-operators are

$$B_r: \quad L(u) = \prod_{1\le a\le r}^{\longrightarrow} (1 - z_{\overline{a}}(u)D)\cdot(1 + z_0(u)D)^{-1}\cdot \prod_{1\le a\le r}^{\longleftarrow} (1 - z_a(u)D), \tag{9.36}$$

$$D_r: \quad L(u) = \prod_{1\le a\le r}^{\longrightarrow} (1 - z_{\overline{a}}(u)D)\cdot(1 - z_r(u)z_{\overline{r}}(u+2)D)^{-1}\cdot \prod_{1\le a\le r}^{\longleftarrow} (1 - z_a(u)D). \tag{9.37}$$

One can check $S_a \cdot L(u) = 0$ by expanding the middle factor into a power series in D. Introduce the expansion coefficients of $L(u)$ as

$$L(u) = \sum_{a\ge 0}(-1)^a T^a(u+a-1)D^a, \quad L(u)^{-1} = \sum_{m\ge 0} T_m(u+m-1)D^m. \tag{9.38}$$

They are related to the previous tableau constructions as follows:

$$T_m(u) = T^{(1)}_m(u) \ \ (7.12) \text{ for } B_r \text{ and } (7.25) \text{ for } D_r,$$
$$T^a(u) = T^{(a)}_{t_a}(u) \ \ (7.12) \text{ for } B_r, 1\le a\le r \text{ and } (7.27) \text{ for } D_r, 1\le a\le r-2.$$

With the convention $T^a(u) = 0$ for $a < 0$, the coefficient $T^a(u)$ beyond these upper bound is characterized by the following relations with the q-characters of spin representations:

$$B_r: \quad T^a(u) + T^{h^\vee - a}(u) = T^{(r)}_1(u + \frac{h^\vee}{2} - a)T^{(r)}_1(u - \frac{h^\vee}{2} + a), \tag{9.39}$$

$$\begin{aligned} D_r: \quad T^a(u) + T^{h^\vee - a}(u) &= T^{(r)}_1(u + \frac{h^\vee}{2} - a)T^{(r-\delta)}_1(u - \frac{h^\vee}{2} + a) \\ &\quad + T^{(r-1)}_1(u + \frac{h^\vee}{2} - a)T^{(r-1+\delta)}_1(u - \frac{h^\vee}{2} + a). \end{aligned} \tag{9.40}$$

Here $a \in \mathbb{Z}$ is arbitrary and $\delta = 0$ if $a \equiv r \mod 2$ and $\delta = 1$ otherwise. $h^\vee$ is the dual Coxeter number (2.3), i.e., $h^\vee = 2r-1$ for B_r and $h^\vee = 2r-2$ for D_r. In particular, one has $T^{r-1}(u) = T^{(r)}_1(u)T^{(r-1)}_1(u)$ for D_r.

9.7. **Type $sl(r|s)$.** There are two kinds of roots, odd and even for the graded algebra $sl(r|s)$. The choice of simple roots is not unique. The most standard one is called distinguished, where all roots but α_r is even. Here we follow [19] and set $\mathrm{I} = \{1, \cdots, r+s\} = \mathrm{I}_1 \cup \mathrm{I}_2$, $\mathrm{I}_1 = \{1, 2, \ldots, r\}$, $\mathrm{I}_2 = \{r+1, r+2, \ldots, r+s\}$, and assign the grading p_a by $p_a = 1\,(-1)$ if $a \in \mathrm{I}_1\,(\mathrm{I}_2)$. The Cartan matrix is expressed by the grading as

$$(\alpha_k|\alpha_j) = (p_k + p_{k+1})\delta_{kj} - p_{k+1}\delta_{k+1,j} - p_k\delta_{k,j+1}.$$

Now the analogue of (7.6) is

$$z_a(u) = Y^{-p_a}_{a-1,q^{u+s_a}} Y^{p_a}_{a,q^{u+s_{a-1}}} \quad (a \in \mathrm{I}),$$

where $s_a = \sum_{j=1}^{a} p_j$ and $Y_{0,q^u} = Y_{r+s,q^u} = 1$. Let D be the difference operator $Df(u) = f(u+2)D$. Then the analogue of (9.3) and (9.5) are given as

$$(1 + z_{r+s}(u)D)^{p_{r+s}} \cdots (1 + z_1(u)D)^{p_1} = \sum_{a=0}^{\infty} T_1^{(a)}(u + a - 1)D^a,$$

$$(1 - z_1(u)D)^{-p_1} \cdots (1 - z_{r+s}(u)D)^{-p_{r+s}} = \sum_{m=0}^{\infty} T_m^{(1)}(u + m - 1)D^m.$$

Example 9.4.
$sl(2|1)$, $p_1 = p_2 = -p_3 = 1$.

$$\begin{aligned}
T_1^{(1)}(u) &= Y_{1,z} + Y_{1,zq^2}^{-1} Y_{2,zq} - Y_{2,zq},\\
T_1^{(2)}(u) &= Y_{2,z} - Y_{1,zq}Y_{2,z} - Y_{1,zq^3}^{-1} Y_{2,z}Y_{2,zq^2} + Y_{2,z}Y_{2,zq^2},\\
T_1^{(3)}(u) &= -Y_{2,zq^{-1}}Y_{2,zq} + Y_{1,zq^2}Y_{2,zq^{-1}}Y_{2,zq} + Y_{1,zq^4}^{-1} Y_{2,zq^{-1}}Y_{2,zq}Y_{2,zq^3}\\
&\quad - Y_{2,zq^{-1}}Y_{2,zq}Y_{2,zq^3}.
\end{aligned}$$

$sl(2|1)$, $p_1 = -p_2 = p_3 = 1$.

$$\begin{aligned}
T_1^{(1)}(u) &= Y_{1,z} - Y_{1,z}Y_{2,zq}^{-1} + Y_{2,zq}^{-1},\\
T_1^{(2)}(u) &= -Y_{1,zq^{-1}}Y_{1,zq}Y_{2,z}^{-1} + Y_{1,zq}Y_{2,z}^{-1} + Y_{1,zq^{-1}}Y_{1,zq}Y_{2,zq^2}^{-1}Y_{2,z}^{-1} - Y_{1,zq}Y_{2,zq^2}^{-1}Y_{2,z}^{-1},\\
T_1^{(3)}(u) &= Y_{1,zq^{-2}}Y_{1,z}Y_{1,zq^2}Y_{2,zq^{-1}}^{-1}Y_{2,zq}^{-1} - Y_{1,z}Y_{1,zq^2}Y_{2,zq^{-1}}^{-1}Y_{2,zq}^{-1},\\
&\quad - Y_{1,zq^{-2}}Y_{1,z}Y_{1,zq^2}Y_{2,zq^{-1}}^{-1}Y_{2,zq}^{-1}Y_{2,zq^3}^{-1} + Y_{1,z}Y_{1,zq^2}Y_{2,zq^{-1}}^{-1}Y_{2,zq}^{-1}Y_{2,zq^3}^{-1}.
\end{aligned}$$

9.8. **Bibliographical notes.** The Casoratian solution (9.10) for A_r has been known in various context. For the T-system of transfer matrices, a slightly more general solution than (9.20) was given in eq.(2.25) in [150] containing $2r+2$ arbitrary functions. It does not satisfy the natural boundary condition $T_{-1}^{(a)}(u) = 0$ for fusion transfer matrices in general. As usual, such a "Dirichlet" condition halves the arbitrary functions to $w_1(u), \ldots, w_{r+1}(u)$, which brings one back to (9.20). Casoratian solutions are known also for the restricted T-systems for A_r [124] and C_r [16].

The L-operator for type A has been studied from the viewpoint of difference analogue of Drinfeld-Sokolov reduction [153]. The concrete forms for type BCD and their application to q-characters were given in [139]. Analogous difference L-operators for all the twisted cases except $E_6^{(2)}$ have been constructed in [154]. The results (9.39) and (9.40) are taken from Theorem 2.3 in [137] and Proposition 2.3 in [138], respectively.

10. T-system in ODE

T-system appears also in the connection problem of 1d Schrödinger equation, which is a typical example of the ODE (ordinary differential equations)/IM (integrable models) correspondence. As a comprehensible review on the ODE/IM correspondence is already available in [149], we only discuss the issue briefly in view of T-system. Wronskians appear naturally in the context of ODE. They will be shown to coincide with the analogous object, the Casoratian (9.7) in the difference equation in Section 9.

10.1. **Generalized Stokes multipliers - the 2nd order case.** As the simplest example, we consider the 1d Schrödinger equation on the real axis with a potential term:

$$\Big(-\frac{d^2}{dx^2}+x^{2M}\Big)\psi(x)=E\psi(x), \tag{10.1}$$

where $M\in\mathbb{Z}_{>0}$. The boundary condition $\psi(\pm\infty)=0$ is imposed. We find it convenient to extend x into the complex plane[19].

Since the Schrödinger equation has the irregular singularity at ∞, we expect a sudden change of $\psi(x)$ when crossing a boarder line of sectors defined below. This is called the Stokes phenomenon. The change is characterized by the Stokes multiplier τ_1. Below we will introduce a set of generalized Stokes multipliers $\{\tau_j\}_{j=1}^{2M}$ and show that they satisfy the level $2M$ restricted T-system for A_1.

First, let $\mathcal{S}_j$ be a sector in the complex plane defined by

$$\mathcal{S}_j=\Big\{x\Big|\,\Big|\arg x-\frac{j\pi}{M+1}\Big|<\frac{\pi}{2M+2}\Big\}.$$

The sector $\mathcal{S}_0$ thus includes the positive real axis. We then introduce a solution $\phi(x,E)$ to (10.1) which decays exponentially as x tends to ∞ inside $\mathcal{S}_0$ as

$$\phi(x,E)\sim\frac{x^{-M/2}}{\sqrt{2i}}\exp\Big(-\frac{x^{M+1}}{M+1}\Big),\qquad x\in\mathcal{S}_0. \tag{10.2}$$

This is referred to as the subdominant solution. There should be another solution to (10.1) which diverges exponentially in $\mathcal{S}_0$ as x tends to ∞. We call it dominant. It is also represented by ϕ. To see this, note the invariance of (10.1) under the simultaneous transformations $x\to q^{-1}x$ and $E\to Eq^2$, where $q=\exp(\frac{\pi i}{M+1})$. We call this "discrete rotational symmetry". We thus introduce $y_j=q^{j/2}\phi(q^{-j}x,q^{2j}E)$ so that $y_0=\phi$. The above observation tells that any y_j is a solution to (10.1). Moreover, we can show that the pair (y_j,y_{j+1}) forms the fundamental system of solutions (FSS) in $\mathcal{S}_j$. This is easily seen by introducing the Wronskian matrix Φ_j and the Wronskian $W[y_i,y_j]$:

$$\Phi_j=\begin{pmatrix}y_j & y_{j+1}\\ \partial y_j & \partial y_{j+1}\end{pmatrix},\qquad W[y_i,y_j]=\det\begin{pmatrix}y_i & y_j\\ \partial y_i & \partial y_j\end{pmatrix}.$$

By using the asymptotic form (10.2), one can check $W[y_j,y_{j+1}]=1$, hence the pair (y_j,y_{j+1}) is independent. Thus, y_0 (equals to ϕ) is the subdominant solution in $\mathcal{S}_0$, while y_1 is a dominant one.

We are interested in the relation among FSS in different sectors. Let us start from $\mathcal{S}_0$ and $\mathcal{S}_1$. Obviously y_2 must be represented by the linear combination of y_0 and y_1 as $y_2=a_0y_0+a_1y_1$. As $W[y_j,y_{j+1}]=1$ for any j, we find $a_0=-1$. The coefficient a_1 can be regarded as a function of E and we write it as $\tau_1(E)=a_1$, which is referred to as the Stokes multiplier. The result can be neatly represented in the matrix form

$$\Phi_0=\Phi_1\mathcal{M}_{1,0},\qquad \mathcal{M}_{1,0}=\begin{pmatrix}\tau_1(E) & 1\\ -1 & 0\end{pmatrix}.$$

The general adjacent FSS Φ_j and Φ_{j+1} are connected by $\Phi_j=\Phi_{j+1}\mathcal{M}_{j+1,j}$, and the "discrete rotational symmetry" leads to $\mathcal{M}_{j+1,j}=\mathcal{M}_{1,0}|_{E\to Eq^{2j}}$. We introduce

[19] For a general reference to ODE in the complex domain, we recommend [155].

the matrix connecting well separated sectors

$$\Phi_0 = \Phi_j \mathcal{M}_{j,0}. \tag{10.3}$$

By the definition, the recursion relation

$$\mathcal{M}_{j,0} = \mathcal{M}_{j,1}\mathcal{M}_{1,0} \tag{10.4}$$

holds. The solution to this takes the form

$$\mathcal{M}_{j,0} = \begin{pmatrix} \tau_j(E) & \tau_{j-1}(Eq^2) \\ -\tau_{j-1}(E) & -\tau_{j-2}(Eq^2) \end{pmatrix}. \tag{10.5}$$

Here τ_j is the function uniquely determined from τ_1 and the recursion relation

$$\tau_j(q^2E)\tau_1(E) = \tau_{j+1}(E) + \tau_{j-1}(q^4E) \tag{10.6}$$

with $\tau_0(E) = 1$. We set $\tau_{-1}(E) = 0$ so that this holds also at $j = 0$. In addition we have $\tau_{2M}(E) = 1, \tau_{2M+1}(E) = 0$ as after $360°$ rotation, FSS must come coincide with the original one times (-1). (cf. [155, (21.31)].) We call τ_j $(j \geq 2)$ generalized Stokes multipliers. The generalized Stokes multipliers satisfy the relation

$$\tau_j(E)\tau_j(Eq^2) = \tau_{j-1}(Eq^2)\tau_{j+1}(E) + 1. \tag{10.7}$$

This is equivalent to $\det \mathcal{M}_{j,0} = 1$. It is shown either by (10.3) or by induction on j using (10.6). See also the discussion in Section 10.3. Setting

$$T_j(u) = \tau_j(Eq^{-j-1}), \qquad \text{where } E = \exp\Big(\frac{\pi i u}{M+1}\Big),$$

we therefore have

Proposition 10.1. *$\{T_j(u)\}$ satisfy the level $2M$ restricted T-system for A_1*

$$T_j(u+1)T_j(u-1) = T_{j-1}(u)T_{j+1}(u) + 1 \qquad (j = 1, \cdots, 2M), \tag{10.8}$$

where $T_0(u) = 1$ and $T_{2M+1}(u) = 0$.

Example 10.2. By (10.3), (10.5) and $\det \mathcal{M}_{j,0} = 1$, one has

$$\tau_j(E) = W[y_0, y_{j+1}],$$

where the RHS is independent of x. The consistency of $\tau_{2M} = 1$ and $\tau_{2M+1} = 0$ with $y_{2M+1} = -y_{-1}$ and $y_{2M+2} = -y_0$ is reconfirmed. The relation (10.7) is also re-derived from the simple identity among Wronskians $[y_\alpha, y_\beta][y_\gamma, y_\delta] = [y_\alpha, y_\gamma][y_\beta, y_\delta] + [y_\alpha, y_\delta][y_\gamma, y_\beta]$ by the specialization $\alpha = 0, \beta = j+1, \gamma = 1, \delta = j+2$. Note $W[y_k, y_{k+1}] = 1$ for any k.

10.2. **Higher order ODE.** One can extend the observation on the second order ODE to higher order case corresponding to $\mathfrak{g} = A_r$ [156, 157, 158, 159]. Consider a natural generalization of (10.1):

$$(-1)^r \frac{d^{r+1}y}{dx^{r+1}} + x^\ell y = Ey = \lambda^{r+1} y. \tag{10.9}$$

Let $q = e^{i\theta}$ with $\theta = \frac{2\pi}{\ell+r+1}$. The sector $\mathcal{S}_k$ is now defined by $|\arg x - k\theta| \leq \frac{\theta}{2}$. We pay attention to the solution $\phi(x, \lambda)$ in $\mathcal{S}_0$ which decays most rapidly as $x \to \infty$ as

$$\phi(x, \lambda) \sim C x^{-r\ell/(2r+2)} \exp\Big(-\frac{x^\nu}{\nu}\Big), \qquad \nu = \frac{\ell + r + 1}{r+1}.$$

The normalization factor C will be determined later. As in the 2nd order ODE case, (10.9) is invariant under $x \to xq^{-1}, E \to Eq^{r+1}$. Thus in terms of λ, $y_k = q^{rk/2}\phi(xq^{-k}, \lambda q^k)$ is also a solution to (10.9) for any $k \in \mathbb{Z}$.

The FSS in $\mathcal{S}_k$ consists of $(y_k, \cdots, y_{k+r})$. It is convenient to introduce a Wronskian matrix

$$\Phi_k = \begin{pmatrix} y_k & y_{k+1} & \cdots & y_{k+r} \\ \vdots & & & \vdots \\ \partial^r y_k & \partial^r y_{k+1} & \cdots & \partial^r y_{k+r} \end{pmatrix}.$$

We write the determinant of a slightly more general matrix (for $m \le r$) as

$$W[y_{i_0}, y_{i_1}, \cdots, y_{i_m}] = \det \begin{pmatrix} y_{i_0} & y_{i_1} & \cdots & y_{i_m} \\ \vdots & & & \vdots \\ \partial^m y_{i_0} & \partial^m y_{i_1} & \cdots & \partial^m y_{i_m} \end{pmatrix}. \tag{10.10}$$

Due to (10.9), the Wronskians ($m = r$ cases) are independent of x. In particular, the normalization constant C can be fixed so that $\det \Phi_k = W[y_k, \cdots, y_{k+r}] = 1$ for any k. We introduce the connection matrix $\mathcal{M}_{k+1,k}$ by

$$\Phi_k = \Phi_{k+1}\mathcal{M}_{k+1,k}. \tag{10.11}$$

It has the form

$$\mathcal{M}_{k+1,k} = \begin{pmatrix} \tau_1^{(1)}(\lambda q^k) & 1 & 0 & 0 & \cdots & 0 \\ \tau_1^{(2)}(\lambda q^k) & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ \tau_1^{(r)}(\lambda q^k) & 0 & 0 & 0 & \cdots & 1 \\ \tau_1^{(r+1)}(\lambda q^k) & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

By using Cramer's formula, $\tau_1^{(a)}(\lambda q^k)$ is expressed as the Wronskian

$$\tau_1^{(a)}(\lambda q^k) = W[y_{k+1}, \cdots, y_{k+a-1}, y_k, y_{k+a+1}, \cdots, y_{k+r+1}].$$

Especially, one finds $\tau_1^{(r+1)}(\lambda q^k) = (-1)^r$. We further introduce the generalized Stokes multipliers $\tau_m^{(a)}(\lambda)$ for $m \ge 2$ by

$$\tau_m^{(a)}(\lambda) = W[y_1, y_2, \cdots y_{a-1}, y_0, y_{a+m}, y_{a+m+1} \cdots y_{r+m}]. \tag{10.12}$$

Note that m does not extend to infinity. Due to $y_{r+1+\ell} = (-)^r y_0$, one has $\tau_{\ell+1}^{(a)}(\lambda) = 0$. This causes a truncation analogous to the level restriction in quantum group at root of unity. It is elementary to prove

Proposition 10.3. *The generalized Stokes multipliers $\tau_m^{(a)}(\lambda)$ satisfy the level ℓ restricted T-system for A_r*

$$\tau_m^{(a)}(\lambda)\tau_m^{(a)}(\lambda q) = \tau_{m+1}^{(a)}(\lambda)\tau_{m-1}^{(a)}(\lambda q) + \tau_m^{(a+1)}(\lambda)\tau_m^{(a-1)}(\lambda q) \qquad (1 \le a \le r),$$

where the boundary conditions are modified as $\tau_m^{(0)}(\lambda) = 1, \tau_m^{(r+1)}(\lambda) = (-1)^r$ and $\tau_0^{(a)}(\lambda) = (-1)^{a-1}$.

Remark 10.4. One might expect that $\tau_m^{(a)}(\lambda)$ may appear in the generalized connection matrix $\mathcal{M}_{k+m,k}$ connecting Φ_k and Φ_{k+m} ($m \ge 2$). This is not the case. As the Schur functions, one can define generalized Stokes multipliers associated to (skew) Young tableaux of general shape. Entries of $\mathcal{M}_{k+m,k}$ are generally identified with such objects. Especially $(a, 1)$ component of $\mathcal{M}_{k+m,k}$ corresponds to the Young tableau of the hook shape of width m and height a.

10.3. **Wronskian-Casoratian duality.** The $(i+1,1)$ element from the matrix relation (10.11) with $k=0$ reads $\partial^i y_0 = \tau_1^{(1)}(\lambda)\partial^i y_1 + \cdots + \tau_1^{(r+1)}(\lambda)\partial^i y_{r+1}$. Remember that $y_k = q^{rk/2}\phi(xq^{-k}, \lambda q^k)$ involves x but $\tau_1^{(a)}(\lambda)$ does not. Thus one obtains an x-independent relation by setting $x=0$ as

$$\partial^i y_0|_{x=0} = \tau_1^{(1)}(\lambda)\partial^i y_1|_{x=0} + \cdots + \tau_1^{(r+1)}(\lambda)\partial^i y_{r+1}|_{x=0} \quad (0 \le i \le r). \tag{10.13}$$

In view of $y_k = q^{rk/2}\phi(xq^{-k}, \lambda q^k)$, this has the same form as the difference equation (TQ-relation) (9.6) with (9.3):

$$w(u) - T_1^{(1)}(u)w(u+2) + \cdots + (-1)^{r+1}T_1^{(r+1)}(u+r)w(u+2r+2) = 0. \tag{10.14}$$

In fact, under the formal (ODE/IM) correspondence between the Stokes multipliers and the transfer matrix eigenvalues

$$\tau_1^{(a)}(\lambda) = (-1)^{a-1}T_1^{(a)}(u+a-1) \quad (1 \le a \le r+1), \tag{10.15}$$

the identification $w(u+2j) = \partial^i y_j|_{x=0}$ provides a solution to (10.14) for any $0 \le i \le r$. The variables u and λ are related so that the shift $u \to u+2$ corresponds to $\lambda \to \lambda q$. Now we are entitled to substitute

$$w_i(u+2j) = \partial^{i-1}y_j|_{x=0} \qquad (1 \le i \le r+1) \tag{10.16}$$

into the Casoratian C_u (9.7). The result is the equality

$$W[y_{i_1}, \ldots, y_{i_k}]|_{x=0} = C_u[2i_1, \ldots, 2i_k], \tag{10.17}$$

which we call the Wronskian-Casoratian duality. One can remove "$|_{x=0}$" when $k=r+1$. Remember that in Section 9.1–9.3, a variety of generalizations of $T_1^{(a)}$ are expressed in terms of Casoratians C_u. The relations (10.15) and (10.17) enable us to import those results to establish a number of Wronskian formulas for the generalized Stokes multipliers. For example, the formula (9.10) leads to (10.12).

The Wronskian-Casoratian duality further provides the Stokes multipliers with dressed vacuum forms like the ones for A_r in Section 8. Recall that Proposition 9.1 expresses the Casoratians as the sums over semistandard tableaux like (skew) Schur functions. The variables attached to tableau letters are ratio of the principal minors of $C_u[0, 2, \ldots, 2r]$, namely $Q_a(u+a-1) = C_u[0, \ldots, 2a-2]$ (9.21), which are called Baxter's Q-functions. Via the Wronskian-Casoratian duality, this is translated to a dressed vacuum form for Stokes multipliers. The tableau variables are ratio of $W[y_{k+1}, y_{k+2}, \ldots, y_{k+a}]|_{x=0}$, which are to be identified with Baxter's Q-functions $Q_a(\lambda q^{a+k})$ in the present context.

As explained in Section 9.4 for Casoratians, the solutions $w_1, \ldots, w_{r+1}$ to (10.14) may be renumbered arbitrarily, and this freedom generates Bäcklund transformations among Q-functions. Even more generally, one may consider arbitrary linear combinations of (10.13) instead of (10.16) as

$$w_i(u+2j) = \sum_{n=0}^{r} A_{in}\partial^n y_j|_{x=0} \qquad (1 \le i \le r+1), \tag{10.18}$$

where $(A_{in})_{1\le i\le r+1, 0\le n\le r}$ is any invertible matrix. In the Wronskian language, this corresponds to identifying $Q_a(\lambda q^{a+k})$

$$\sum_{0\le n_1<\cdots<n_a\le r} \det(A_{i,n_j})_{1\le i,j\le a} \det\begin{pmatrix} \partial^{n_1}y_{k+1} & \partial^{n_1}y_{k+2} & \cdots & \partial^{n_1}y_{k+a} \\ \vdots & & & \vdots \\ \partial^{n_a}y_{k+1} & \partial^{n_a}y_{k+2} & \cdots & \partial^{n_a}y_{k+a} \end{pmatrix}$$

evaluated at $x = 0$. In this way the same Stokes multiplier acquires a variety of representations.

We note that in the simple cases like $\tau_1^{(1)}(\lambda)$, the recursion relation (see for example [156, 157])

$$\frac{[y_0, y_2, \cdots y_m]}{[y_1, \cdots, y_m]} = \frac{[y_0, y_2, \cdots y_{m-1}]}{[y_1, \cdots, y_{m-1}]} + \frac{[y_0, y_1, \cdots y_{m-1}][y_2, \cdots, y_m]}{[y_1, \cdots, y_m][y_1, \cdots, y_{m-1}]} \tag{10.19}$$

is handy to derive the dressed vacuum forms without recourse to Proposition 9.1 and the Wronskian-Casoratian duality (10.17).

10.4. **Bibliographical notes.** The functional relations have appeared in ODE in the context of asymptotic analysis [155] or of complex WKB method [160]. The connection to integrable models has been realized in [161] and the machineries of the latter have been applied since then [162, 163, 164]. The connection not only provides the information on Stokes multipliers but also solves the spectral problem of ODE. With an assumption on analyticity, one can transform (10.8) to the thermodynamic Bethe ansatz equation that describes a conformal field theory (CFT) in the ground state. It provides a quantitative tool to obtain the eigenvalues of (10.1). A more direct relation can be established between the spectral determinant associated to ODE and the vacuum expectation value of the Baxter's Q operator in CFT [163, 164].

It is tempting to consider Schrödinger operators with more general polynomial potentials. Although we can argue the algebraic part in an almost same manner, the problem with the analyticity defies most attempts up to now. The case with $V(x) = \alpha x^{M-1} + x^{2M}$ is exceptionally treated nicely [165]. The underlying model seems to possess $gl(2|1)$ symmetry. The fundamental reason why this symmetry appears remains to be clarified. This case seems interesting in its relation to $\mathcal{PT}$ symmetric quantum systems [166] and spontaneous breakdown of the symmetry [167]. The integro-differential systems corresponding to non exceptional classical Lie algebras in the similar sense are proposed in [168].

The role played by the excited states of CFT is studied in [169]. The corresponding Schrödinger operators with potentials possessing singularities are identified. A further argument from the viewpoint of the Langlands correspondence is given in [170].

In general, CFTs are realized as scaling limits of lattice models. Then one may wonder if there exists an ODE which corresponds to a lattice model on a finite system. This is investigated in [171, 172] for particular cases. As for generalizations related to massive deformations of CFT, see [173, 174].

11. Applications in gauge/string theories

The AdS/CFT correspondence is a huge subject in theoretical and mathematical physics. Here we pick just two topics rather briefly, planar AdS/CFT spectrum (Section 11.1–11.4) and area of minimal surface in AdS (Section 11.5–11.8), from the gauge and the string theory sides, respectively. These subjects have been growing rapidly during the last couple of years where some specific T and Y-systems have found notable applications.

11.1. **Planar AdS/CFT spectrum.** Recall the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence between the type IIB superstring on the curved space time $\mathrm{AdS}_5 \times \mathrm{S}^5$ and the large N conformal $\mathcal{N} = 4$ super Yang-Mills (SYM) gauge theory in four dimensions on the boundary of AdS_5 [175, 176, 177]. The correspondence implies that the energies of specific string states should coincide with anomalous scaling dimensions of local gauge invariant operators in the SYM. We call the sought common spectrum the *planar AdS/CFT spectrum*.

To be concrete, let us consider simplest examples from the SYM side, linear combinations of single trace scalar operators without derivatives

$$\textstyle\sum_{i_1 \ldots i_L} c^{i_1 \ldots i_L} \mathrm{Tr}\, \Phi_{i_1} \cdots \Phi_{i_L}, \tag{11.1}$$

where Φ_i $(i = 1, \ldots, 6)$ denote the six scalar fields of $\mathcal{N} = 4$ SYM in the adjoint representation of $SU(N)$. They contain important examples like chiral primary and BMN operators [178] as special cases and form an interesting sector that are mixed only among themselves at one-loop renormalization. In fact, the last property reduces the one-loop calculation of scaling dimensions of (11.1) to the diagonalization of the Wilson matrix $\left(\frac{\partial \ln Z}{\partial \ln \Lambda}\right)$ consisting of the wave function renormalization factors $Z = (Z_{ij})$, where Λ is the UV cutoff. This problem turns out rather remarkably identical with a periodic spin chain of length L associated with rational R matrix for $SO(6)$. Thus in the large L limit, one can evaluate, for example, the largest possible scaling dimension of (11.1) by the Bethe ansatz as [179]

$$L + \frac{\lambda L}{8\pi^2}\Bigl(\frac{\pi}{2} + \ln 2\Bigr) + \mathcal{O}(\lambda^2),$$

where $\lambda = g_{\mathrm{YM}}^2 N$ is the 't Hooft coupling. One sees how the bare dimension L (1st term) acquires the anomalous correction.

Although this is a one-loop perturbative approximation to the planar AdS/CFT spectrum in a very limited sector, the connection to the Bethe ansatz is a signal of the integrability of the full problem. In fact, this theme has been explored both from the gauge and string theory perspectives extensively by an enormous amount of works. We do not intend to cover them here but refer to the literatures that will be cited in the next subsection and [178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191] for example and reference therein. See also [192, 193] for earlier observations before AdS/CFT.

11.2. **T and Y-system for $\mathbf{AdS_5/CFT_4}$.** The planar AdS/CFT spectrum is accessible from the gauge theory side via an integrable long range quantum spin chain with $PSU(2,2|4)$ symmetry [194]. This is actually so at least asymptotically if the relevant quantum numbers like the bare scaling dimension are large enough. In the language of spin chains, such situations correspond to the thermodynamic limit where "impurities" (Bethe roots) are kept dilute.

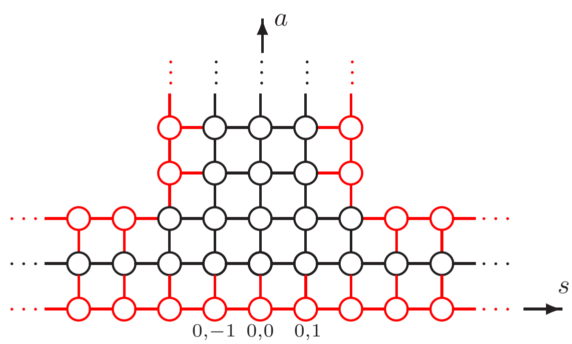
Complementally, the *exact* spectrum including "finite size effects" may be encoded in some T and Y-systems together with an appropriate, albeit highly elaborate, analyticity input[20]. A candidate for such a Y-system has been proposed in [195] with a companion T-system. The Y-system has been derived [196, 22, 197] along the ground state TBA equation associated with the asymptotic Bethe ansatz (ABA) equation [198, 194, 199] in the mirror form [200].

[20] Such features are illustrated along the elementary example of the XXZ chain in Section 16.

The underlying symmetry of the ABA equation is $PSU(2,2|4)$ [194]. Reflecting this fact, the Y-system in question contains two copies of the Y-systems for the subgroup $SU(2|2)$[21] denoted by $SU(2|2)_L$ and $SU(2|2)_R$. Apparently it takes the same form as type A case:

$$\frac{Y_{a,s}(u-\frac{i}{2})Y_{a,s}(u+\frac{i}{2})}{Y_{a+1,s}(u)Y_{a-1,s}(u)} = \frac{(1+Y_{a,s+1}(u))(1+Y_{a,s-1}(u))}{(1+Y_{a+1,s}(u))(1+Y_{a-1,s}(u))}. \tag{11.2}$$

A peculiarity here is that $Y_{a,s}(u)$ is defined for those (a,s) that correspond to the black nodes in the following T-shaped fat hook:



(11.3)

The relevant T-system is also formally of type A:

$$T_{a,s}(u-\tfrac{i}{2})T_{a,s}(u+\tfrac{i}{2}) = T_{a,s-1}(u)T_{a,s-1}(u) + T_{a-1,s}(u)T_{a+1,s}(u), \tag{11.4}$$

where this time (a,s) ranges over black as well as red nodes in (11.3). The relation to the Y-system $Y_{a,s}(u) = \frac{T_{a,s-1}(u)T_{a,s+1}(u)}{T_{a-1,s}(u)T_{a+1,s}(u)}$ is as usual. The diagram (11.3) is meant to capture the structure of the equations (11.2) and (11.4)[22].

Recall that the Y-system for $U_q(sl(2|2))$ in Section 2.6 involves the variables $Y^{(a)}_m$ with (a,m) ranging over $H_{2,2}$ (2.39) which is an L-shaped "thin" hook. This and its copy are embedded into (11.3) as $Y_{a,m}$ and $Y_{a,-m}$. The extra variables $Y_{a,0}(u)$ on the middle vertical array $(a,0)_{a\geq 1}$ are the carriers of the "momentum" (cf. (11.5)). The two wings $s<0$ and $s>0$ correspond to $SU(2|2)_L$ and $SU(2|2)_R$ mentioned earlier. The range $m \in \mathbb{Z}$ for the "fusion degree" or "string length" for $T_{a,m}$ and $Y_{a,m}$ is a natural convention in those systems equipped with doubled symmetry, e.g., the $O(4)$ nonlinear sigma model ($SU(2)$ principal chiral field) having the global $SU(2)_L \times SU(2)_R$ symmetry [201].

11.3. **Formula for planar AdS/CFT spectrum.** Now the planar AdS/CFT spectrum (with R-charge subtracted) is given in terms of the solutions to the Y-system in the previous subsection by the formula

$$\sum_{j=1}^{K_0} \epsilon_1(u_{0,j}) + \sum_{a\geq 1}\int_{-\infty}^{\infty}\frac{du}{2\pi i}\frac{\partial \epsilon^*_a(u)}{\partial u}\ln(1+Y^*_{a,0}(u)). \tag{11.5}$$

Here K_0 is specified from the sector in question (see (11.9)–(11.10)) and $\epsilon_a(u)$ is defined by $\epsilon_a(u) = a + \frac{2ig}{x(u+\frac{ia}{2})} - \frac{2ig}{x(u-\frac{ia}{2})}$ in terms of $x(u)$ satisfying $\frac{u}{g} = x(u) + x(u)^{-1}$ and $|x(u\pm\frac{ia}{2})| > 1$. The parameter g is related to the 't Hooft coupling λ

[21] It is essentially the Y-system for $U_q(sl(2|2))$ in Section 2.6.

[22] Another, yet more intrinsic way of encoding the Y-system together with the T-system is by the quiver in the cluster algebra formulation in Section 5.3.

by $\lambda = (4\pi g)^2$. The above choice of the branch is called physical kinematics. On the other hand, $\epsilon^*_a(u)$ with $a \geq 1$ is defined by the same formula but with another branch called mirror kinematics (cf. [197, 195, 191, 22]). The function $Y^*_{a,0}(u)$ is defined by the mirror kinematics. Finally, the rapidities $u_{0,j}$ are determined by the Bethe equation

$$Y_{1,0}(u_{0,j}) = -1 \qquad (j = 1, \dots, K_0). \tag{11.6}$$

This description of the planar AdS/CFT spectrum has been claimed exact for any 't Hooft coupling (i.e., to all loop orders) and operators of any *finite* L [195].

11.4. **Asymptotic Bethe ansatz.** To be consistent with the ABA equation [194], the Y-system (11.2) should split into the left and right wings in the limit $L \to \infty$. Compatibly with this, the middle series should behave as

$$Y_{a\geq 1,0}(u) \simeq \left(\frac{x(u - \frac{ia}{2})}{x(u + \frac{ia}{2})} \right)^L \frac{\phi(u - \frac{ia}{2})}{\phi(u + \frac{ia}{2})} T^L_{a,-1}(u) T^R_{a,1}(u), \tag{11.7}$$

where ϕ is a function obeying the relation (11.15). The last two factors represent the T-functions for the decoupled $SU(2|2)_L$ and $SU(2|2)_R$. They are constructed from the $a = 1$ case [19, 23] in a way analogous to (9.2), (9.3) and (9.5). Explicitly, the $a = 1$ case is given as the dressed vacuum form

$$\begin{aligned} T^{L,R}_{1,\mp 1}(u) =& \frac{R^{(+)}_0(u - \frac{i}{2})}{R^{(-)}_0(u - \frac{i}{2})} \Bigg(\frac{Q_{\pm 2}(u - i) Q_{\pm 3}(u + \frac{i}{2})}{Q_{\pm 2}(u) Q_{\pm 3}(u - \frac{i}{2})} + \frac{Q_{\pm 2}(u + i) Q_{\pm 1}(u - \frac{i}{2})}{Q_{\pm 2}(u) Q_{\pm 1}(u + \frac{i}{2})} \\ & - \frac{R^{(-)}_0(u - \frac{i}{2}) Q_{\pm 3}(u + \frac{i}{2})}{R^{(+)}_0(u - \frac{i}{2}) Q_{\pm 3}(u - \frac{i}{2})} - \frac{B^{(+)}_0(u + \frac{i}{2}) Q_{\pm 1}(u - \frac{i}{2})}{B^{(-)}_0(u + \frac{i}{2}) Q_{\pm 1}(u + \frac{i}{2})} \Bigg), \end{aligned} \tag{11.8}$$

where $Q_l(u) = \prod_{j=1}^{K_l} (u - u_{l,j})$. In addition we introduce[23]

$$R_l(u) = \prod_{j=1}^{K_l} \frac{x(u) - x(u_{l,j})}{\sqrt{x(u_{l,j})}}, \quad R^{(\pm)}_l(u) = \prod_{j=1}^{K_l} \frac{x(u) - x(u_{l,j} \mp \frac{i}{2})}{\sqrt{x(u_{l,j} \mp \frac{i}{2})}}, \tag{11.9}$$

$$B_l(u) = \prod_{j=1}^{K_l} \frac{x(u)^{-1} - x(u_{l,j})}{\sqrt{x(u_{l,j})}}, \quad B^{(\pm)}_l(u) = \prod_{j=1}^{K_l} \frac{x(u)^{-1} - x(u_{l,j} \mp \frac{i}{2})}{\sqrt{x(u_{l,j} \mp \frac{i}{2})}} \tag{11.10}$$

for $-3 \leq l \leq 3$. They are factorized pieces of $Q_l(u)$ in that

$$R_l(u) B_l(u) = (-g)^{-K_l} Q_l(u), \quad R^{(\pm)}_l(u) B^{(\pm)}_l(u) = (-g)^{-K_l} Q_l(u \pm \tfrac{i}{2}). \tag{11.11}$$

The numbers K_l specify the relevant sectors. As usual in the analytic Bethe ansatz (cf. Section 8), analyticity of $T^{L,R}_{1,\pm 1}(u)$ leads to the equations

$$1 = \frac{Q_{\pm 2}(u_{\pm 1,k} + \frac{i}{2}) B^{(-)}_0(u_{\pm 1,k})}{Q_{\pm 2}(u_{\pm 1,k} - \frac{i}{2}) B^{(+)}_0(u_{\pm 1,k})}, \quad 1 = \frac{Q_{\pm 2}(u_{\pm 3,k} + \frac{i}{2}) R^{(-)}_0(u_{\pm 3,k})}{Q_{\pm 2}(u_{\pm 3,k} - \frac{i}{2}) R^{(+)}_0(u_{\pm 3,k})}, \tag{11.12}$$

$$-1 = \frac{Q_{\pm 1}(u_{\pm 2,k} - \frac{i}{2}) Q_{\pm 2}(u_{\pm 2,k} + i) Q_{\pm 3}(u_{\pm 2,k} - \frac{i}{2})}{Q_{\pm 1}(u_{\pm 2,k} + \frac{i}{2}) Q_{\pm 2}(u_{\pm 2,k} - i) Q_{\pm 3}(u_{\pm 2,k} + \frac{i}{2})}. \tag{11.13}$$

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[23] The Bethe roots $u_{1,j}, u_{2,j}, u_{3,j}, u_{0,j}, u_{-3,j}, u_{-2,j}, u_{-1,j}$ and the T-functions $T^L_{1,-1}, T^R_{1,1}$ here denote $u_{1L,j}, u_{2L,j}, u_{3L,j}, u_{4,j}, u_{3R,j}, u_{2R,j}, u_{1R,j}$ and $T^L_{1,1}, T^R_{1,1}$ in [195], respectively. The notation for the Q-functions is also slightly modified accordingly. These Bethe roots further correspond to $u_{1,j}, u_{2,j}, u_{3,j}, u_{4,j}, u_{5,j}, u_{6,j}, u_{7,j}$ in [194].

In addition, the cyclicity of the single trace operator in SYM is to be reflected as the "zero momentum" condition $\prod_{j=1}^{K_0} \frac{x(u_{0,j}+\frac{i}{2})}{x(u_{0,j}-\frac{i}{2})} = 1$. Upon a convention adjustment, these relations coincide with the ABA equation in [194, section 5.1] except the most complicated one

$$-1 = \left(\frac{x(u_{0,k}-\frac{i}{2})}{x(u_{0,k}+\frac{i}{2})}\right)^L \frac{\big(B_0^{(+)}B_1B_{-1}R_3R_{-3}/R_0^{(+)}\big)(u_{0,k}+\frac{i}{2})}{\big(B_0^{(-)}B_1B_{-1}R_3R_{-3}/R_0^{(-)}\big)(u_{0,k}-\frac{i}{2})} S(u_{0,k})^2, \tag{11.14}$$

which involves the dressing factor σ [199] via $S(u) = \prod_{j=1}^{K_0} \sigma(x(u), x_{0,j})$. The ABA equation (11.14) is to be reproduced in the present scheme as the large L limit of the equation (11.6). In view of $T^{L,R}_{1,\mp 1}(u_{0,j}) = -\frac{Q_{\pm 3}(u_{0,j}+\frac{i}{2})}{Q_{\pm 3}(u_{0,j}-\frac{i}{2})}$ and (11.7), this amounts to postulating that ϕ therein should satisfy the difference equation

$$\frac{\phi(u-\frac{i}{2})}{\phi(u+\frac{i}{2})} = \frac{B_0^{(+)}B_1B_{-1}}{R_0^{(+)}B_3B_{-3}}(u+\tfrac{i}{2})\frac{R_0^{(-)}B_3B_{-3}}{B_0^{(-)}B_1B_{-1}}(u-\tfrac{i}{2})S(u)^2. \tag{11.15}$$

The asymptotics (11.7) with (11.15) specifies the large L solution of the Y-system.

With regard to the finite L effects, the above formulation reproduces wrapping corrections at weak coupling for twist two operators obtained by other methods such as the Lüscher formula. For instance in the case of the Konishi operator $\mathrm{Tr}(D^2Z^2 - DZDZ)$, one gets the scaling dimension from ABA as $E_{\mathrm{ABA}} = 4 + 12g^2 - 48g^4 + 336g^6 - (2820 + 288\zeta(3))g^8$. The above Y-system approach yields the result $E_{\mathrm{ABA}} + E_{\mathrm{wrapping}}$ with the correction $E_{\mathrm{wrapping}} = (324 + 864\zeta(3) - 1440\zeta(5))g^8$ starting at four-loop in agreement with [191].

11.5. **Area of minimal surface in AdS.** Now we turn to the second topic of this section. The T and Y-systems play an essential role in calculating the action of classical open string solutions, i.e., the area of minimal surface, in AdS space. Via the AdS/CFT correspondence, this yields the planar amplitudes of gluon scattering in $\mathcal{N} = 4$ SYM at strong coupling. The gluon momenta are incorporated in null polygonal configurations at the AdS boundary. The first important step in this problem is to linearize the equation of motion of the AdS sigma model (Section 11.5). Once this is achieved, the T and Y-systems come into the game naturally through the Stokes phenomena of the auxiliary linear problem around the irregular singularity at the boundary of the worldsheet (Section 11.6). This part is close in spirit to Section 10.1. Extra complication can happen when passing to the TBA-type nonlinear integral equations most typically due to the complex nature of the driving terms ("complex mass" appearing in asymptotics of Y-functions). They are determined by period integrals of the Riemann surface reflecting the null polygonal boundary and the cross ratios of gluon momenta. The regularized area is formally expressed in the same form as the free energy in the conventional TBA analysis (Section 11.8). Sections 11.5–11.8 are quick digest of these recent progress [202, 203, 204, 205] along a simple version of AdS_3.

The AdS_3 is given in terms of the global coordinate $\vec{Y} = (Y_{-1}, Y_0, Y_1, Y_2) \in \mathbb{R}^{2,2}$ as

$$\vec{Y}\cdot\vec{Y} := -Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 = -1. \tag{11.16}$$

General product $\vec{A}\cdot\vec{B}$ in $\mathbb{R}^{2,2}$ is defined similarly with the signature $-1,-1,1,1$. The equation of motion and the Virasoro constraint read

$$\partial\bar{\partial}\vec{Y}-(\partial\vec{Y}\cdot\bar{\partial}\vec{Y})\vec{Y}=0,\qquad \partial\vec{Y}\cdot\partial\vec{Y}=\bar{\partial}\vec{Y}\cdot\bar{\partial}\vec{Y}=0, \tag{11.17}$$

where $\partial=\frac{\partial}{\partial z}, \bar{\partial}=\frac{\partial}{\partial\bar{z}}$ and z is a complex coordinate parameterizing the worldsheet. This classical motion of strings in AdS_3 is integrable. In fact, it is transformed to a $\mathbb{Z}_2$-projected $SU(2)$ Hitchin system through a Pohlmeyer type reduction [206, 207]. To see this, introduce the new variables α and p by

$$e^{2\alpha(z,\bar{z})}=\frac{1}{2}\partial\vec{Y}\cdot\bar{\partial}\vec{Y},\quad N_a=\frac{1}{2}\epsilon_{abcd}Y^b\partial Y^c\bar{\partial}Y^d, \tag{11.18}$$

$$p=\frac{1}{2}\vec{N}\cdot\partial^2\vec{Y},\quad \bar{p}=-\frac{1}{2}\vec{N}\cdot\bar{\partial}^2\vec{Y}. \tag{11.19}$$

Note that $\vec{N}\cdot\vec{Y}=\vec{N}\cdot\partial\vec{Y}=\vec{N}\cdot\bar{\partial}\vec{Y}=0$ and $\vec{N}\cdot\vec{N}=1$. The variable $\alpha=\alpha(z,\bar{z})$ is real and $\vec{N}$ is pure imaginary. Moreover it can be shown from (11.16)-(11.19) that $p=p(z)$ is holomorphic. The area is given by $4\int d^2ze^{2\alpha}$. The α satisfies the sinh-Gordon equation modified with p as $\partial\bar{\partial}\alpha-e^{2\alpha}+|p(z)|^2e^{-2\alpha}=0$. As this fact indicates, the equations (11.17) are expressible as the flatness condition of the connections:

$$\partial B^L_{\bar{z}}-\bar{\partial}B^L_z+[B^L_z,B^L_{\bar{z}}]=0,\quad \partial B^R_{\bar{z}}-\bar{\partial}B^R_z+[B^R_z,B^R_{\bar{z}}]=0, \tag{11.20}$$

where the connections are given by

$$B^L_z=B_z(1),\quad B^L_{\bar{z}}=B_{\bar{z}}(1),\quad B^R_z=UB_z(i)U^{-1},\quad B^R_{\bar{z}}=UB_{\bar{z}}(i)U^{-1}, \tag{11.21}$$

$$B_z(\zeta)=\begin{pmatrix}\frac{1}{2}\partial\alpha & -\zeta^{-1}e^{\alpha}\\ -\zeta^{-1}e^{-\alpha}p(z) & -\frac{1}{2}\partial\alpha\end{pmatrix},\quad B_{\bar{z}}(\zeta)=\begin{pmatrix}-\frac{1}{2}\bar{\partial}\alpha & -\zeta e^{-\alpha}\bar{p}(\bar{z})\\ -\zeta e^{\alpha} & \frac{1}{2}\bar{\partial}\alpha\end{pmatrix}, \tag{11.22}$$

with $U=\begin{pmatrix}0 & e^{\pi i/4}\\ e^{3\pi i/4} & 0\end{pmatrix}$. Here ζ is the spectral parameter. Actually the relation $\partial B_{\bar{z}}(\zeta)-\bar{\partial}B_z(\zeta)+[B_z(\zeta),B_{\bar{z}}(\zeta)]=0$ including ζ is satisfied. Splitting the connection into ζ dependent part and the rest as $B_z(\zeta)=\mathcal{A}_z+\zeta^{-1}\Phi_z$ and $B_{\bar{z}}(\zeta)=\mathcal{A}_{\bar{z}}+\zeta\Phi_{\bar{z}}$, one finds that the flatness conditions form the Hitchin system with gauge field $\mathcal{A}$ and Higgs field Φ. The gauge group is $SU(2)$ but the system is $\mathbb{Z}_2$-projected in the sense that the above form (11.22) belongs to the invariant subspace under the involution $\mathcal{A}_z\to\sigma^3\mathcal{A}_z\sigma^3$, $\Phi_z\to-\sigma^3\Phi_z\sigma^3$ and similarly for $\mathcal{A}_{\bar{z}}$ and $\Phi_{\bar{z}}$. (σ^3 is a Pauli matrix.)

With each zero curvature condition in (11.20), there are associated a pair of auxiliary linear problems whose compatibility yields it. Thanks to the relations (11.21), one can combine and promote them into the ζ-dependent versions $(\partial+B_z(\zeta))\psi=0$ and $(\bar{\partial}+B_{\bar{z}}(\zeta))\psi=0$ or equivalently,

$$\left(d+\frac{\Phi_zdz}{\zeta}+\mathcal{A}+\zeta\Phi_{\bar{z}}d\bar{z}\right)\psi=0 \tag{11.23}$$

with $\mathcal{A}=\mathcal{A}_zdz+\mathcal{A}_{\bar{z}}d\bar{z}$ for $\psi=\psi(z,\bar{z};\zeta)$. A useful property is that if $\psi(\zeta)$ is a flat section with spectral parameter ζ, then so is $\sigma^3\psi(e^{\pi i}\zeta)$ by the $\mathbb{Z}_2$-symmetry.

Given two solutions ψ,ψ' to (11.23), define their $SL(2)$-invariant pairing as $\langle\psi,\psi'\rangle=\epsilon^{\alpha\beta}\psi_\alpha\psi'_\beta$, where $\psi=(\psi_1,\psi_2)^T$, etc. This is a constant function on the worldsheet playing the role analogous to Wronskians in Section 10. Let $\psi^L_a=(\psi^L_{1,a},\psi^L_{2,a})^T$ $(a=1,2)$ be the two solutions $\psi(z,\bar{z},\zeta\!=\!1)$ normalized as $\langle\psi^L_a,\psi^L_b\rangle=$

ϵ_{ab}. Fix also the solutions $\psi^R_{\dot a} = (\psi^R_{1,\dot a}, \psi^R_{2,\dot a})^T$ ($\dot a = 1, 2$) which are similarly normalized at $\zeta = i$. Then the original AdS_3 coordinate $\vec{Y} = (Y_{-1}, Y_0, Y_1, Y_2)$ is reproduced from the auxiliary linear problem by

$$\begin{pmatrix} Y_{-1} + Y_2 & Y_1 - Y_0 \\ Y_1 + Y_0 & Y_{-1} - Y_2 \end{pmatrix}_{a,\dot a} = \psi^L_{1,a}\psi^R_{1,\dot a} + \psi^L_{2,a}\psi^R_{2,\dot a}. \tag{11.24}$$

This substantially achieves the linearization of the problem.

11.6. **Stokes phenomena, T and Y-system.** Scattering amplitudes for $2n$ gluons correspond to open string solutions having polygonal shapes with $2n$ cusps at the AdS_3 boundary. This translates to the following boundary condition:

$$\alpha \to \frac{1}{4}\ln|p(z)|^2 \;\; (z \to \infty), \quad p(z) = z^{n-2} + \cdots \text{ (polynomial of degree } n-2). \tag{11.25}$$

We assume that n is odd for simplicity. From (11.22), solutions of the auxiliary linear problem (11.23) as $|z| \to \infty$ behave as

$$\psi \sim \begin{pmatrix} (\bar p/p)^{\frac{1}{8}} \\ \pm(p/\bar p)^{\frac{1}{8}} \end{pmatrix} \exp\left(\pm\frac{1}{\zeta}\int\sqrt{p}dz \pm \zeta\int\sqrt{\bar p}d\bar z\right). \tag{11.26}$$

Since $\exp(\frac{1}{\zeta}\int\sqrt{p}dz) \sim \exp(\frac{z^{n/2}}{\zeta})$ holds asymptotically, there are n Stokes sectors which are separated by n rays in the z plane. We label them consecutively anticlockwise.

Let $s_k(\zeta)$ be the small (subdominant in the terminology of Section 10) solution in the kth Stokes sector. Then we have the properties like $\sigma^3 s_k(e^{\pi i}\zeta) \propto s_{k+1}(\zeta)$, $s_k(e^{2\pi i}\zeta) \propto s_{k+2}(\zeta)$ and $\langle s_j, s_k\rangle(e^{\pi i}\zeta) = \langle s_{j+1}, s_{k+1}\rangle(\zeta)$. Fixing the small solution $s_1(\zeta)$ in the first Stokes sector, we define the others by $s_{k+1}(\zeta) = (\sigma^3)^k s_1(e^{k\pi i}\zeta)$.

Set $T_k(\zeta) = \langle s_0, s_{k+1}\rangle(e^{-\pi i(k+1)/2}\zeta)$ in the normalization $\langle s_i, s_{i+1}\rangle(\zeta) = 1$. Then from the simplest Plücker relation or Schouten identity $\langle s_i, s_j\rangle\langle s_k, s_l\rangle - \langle s_i, s_k\rangle\langle s_j, s_l\rangle + \langle s_i, s_l\rangle\langle s_j, s_k\rangle = 0$, one finds

$$T_k(e^{\frac{\pi i}{2}}\zeta)T_k(e^{-\frac{\pi i}{2}}\zeta) = T_{k-1}(\zeta)T_{k+1}(\zeta) + 1. \tag{11.27}$$

This is a version of the level $n-2$ restricted T-system for A_1 where the conditions $T_0(\zeta) = 1$ and $T_{n-1}(\zeta) = 0$ are imposed[24]. Setting further $Y_k(\zeta) = T_{k-1}(\zeta)T_{k+1}(\zeta)$ as usual, one gets the level $n-2$ restricted Y-system (for Y^{-1}-variables in (2.11))

$$Y_k(e^{\frac{\pi i}{2}}\zeta)Y_k(e^{-\frac{\pi i}{2}}\zeta) = (1 + Y_{k-1}(\zeta))(1 + Y_{k+1}(\zeta)) \tag{11.28}$$

with the boundary condition $Y_0(\zeta) = Y_{n-2}(\zeta) = 0$ in the k direction.

11.7. **Asymptotics, WKB and TBA.** As is well known, the relation (11.28) determines the Y-functions effectively only with the information on their analyticity. By the definition, $Y_k(\zeta)$'s are analytic away from $\zeta^{\pm 1} = 0$ where they possess essential singularities. One can deduce the asymptotic behavior around them using the WKB approximation regarding $\zeta^{\pm 1}$ as the Planck constant. For example when $\zeta \to 0$, the solutions of (11.23), after a simple similarity transformation making Φ_z into $\sqrt{p}\,\mathrm{diag}(1, -1)$, behave as $\exp(\pm\frac{1}{\zeta}\int\sqrt{p}\,dz)$ times constant vectors. Thus they are well approximated by performing the integral along the Stokes (steepest descent) lines defined by $\Im\mathrm{m}\,(\sqrt{p(z)}dz/\zeta) = 0$. At a generic point in the z plane, there is one Stokes line passing through it. Exceptions are zeros of $p(z)$ (turning

[24] The latter is a slightly weaker condition than $T_{n-2}(\zeta) = 1$ in the definition of Section 2.2.

points). From a single zero, there emanate three Stokes lines. They go toward infinity along certain directions corresponding to Stokes sectors or flow into another turning point. The family of these infinitely many non-crossing lines constitute the WKB foliations.

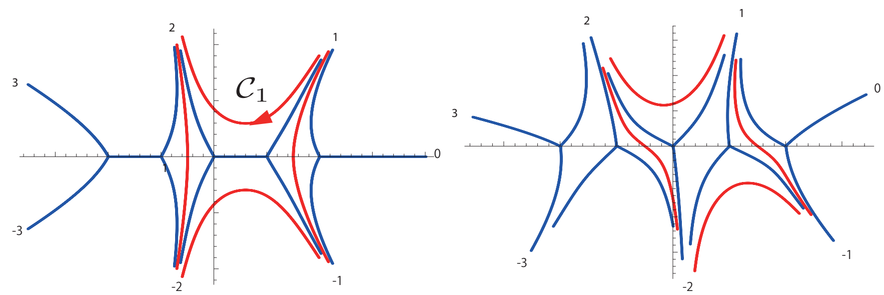

FIGURE 3. Example of Stokes lines for $p(z) = z(z^2-1)(z^2-4)$. The left and right figures correspond to $\arg(\zeta) = 0$ and $\frac{\pi}{3.1}$, respectively. Blue lines are those emanating from turning points. The number k specifies the Stokes sector where s_k is small. For example, $\langle s_1, s_2\rangle \sim \exp(-\frac{1}{\zeta}\int_{\mathcal{C}_1}\sqrt{p}dz)$. The integral $\int \sqrt{p}\,dz$ along the red lines anticlockwise yields asymptotics of $\ln Y_2(\zeta)$ as $\zeta \to 0$.

First consider the case in which the zeros of $p(z)$ are aligned on the real axis. Then one obtains the estimate like $\langle s_1, s_2\rangle \sim \exp(-\int_{C_1}\sqrt{p}\,dz/\zeta)$. Therefore the Y-variables (without the normalization constraint on s_i)

$$\begin{aligned} Y_{2k}(\zeta) &= \frac{\langle s_{-k}, s_k\rangle\langle s_{-k-1}, s_{k+1}\rangle}{\langle s_{-k-1}, s_{-k}\rangle\langle s_k, s_{k+1}\rangle}(\zeta), \\ Y_{2k+1}(\zeta) &= \frac{\langle s_{-k-1}, s_k\rangle\langle s_{-k-2}, s_{k+1}\rangle}{\langle s_{-k-2}, s_{-k-1}\rangle\langle s_k, s_{k+1}\rangle}(e^{\frac{\pi i}{2}}\zeta) \end{aligned} \tag{11.29}$$

have the asymptotics

$$\ln Y_{2k}(\zeta) \sim \frac{Z_{2k}}{\zeta} + \cdots, \quad \ln Y_{2k+1}(\zeta) \sim \frac{Z_{2k+1}}{i\zeta} + \cdots \quad (\zeta \to 0), \tag{11.30}$$

where $Z_k = -\oint_{\gamma_k}\sqrt{p}\,dz$ is the period integral along the cycle γ_k going around the kth and $(k+1)$th largest zeros of $p(z)$ (cf. Fig. 5 in [204]). The asymptotics as $\zeta \to \infty$ is similarly investigated. Together with the $\zeta \to 0$ case, the result is summarized as $\ln Y_k(e^\theta) = -m_k\cosh\theta + \cdots$ $(\theta \to \pm\infty)$, where $m_{2k} = -2Z_{2k}$ and $m_{2k+1} = 2iZ_{2k+1}$ are both positive. Now that the combination $\ln(Y_k(e^\theta)/e^{-m_k\cosh\theta})$ is analytic in the strip $|\Im\mathrm{m}\,\theta| \le \frac{\pi}{2}$ and decays as $|\theta| \to \infty$ within it, the standard argument leads to the integral equation:

$$\ln Y_k(e^\theta) = -m_k\cosh\theta + \int_{-\infty}^{\infty}\frac{\ln[(1+Y_{k-1}(e^{\theta'}))(1+Y_{k+1}(e^{\theta'}))]d\theta'}{2\pi\cosh(\theta-\theta')} \tag{11.31}$$

for $1 \le k \le n-3$ $(Y_0(\zeta) = Y_{n-2}(\zeta) = 0)$. Up to the driving (mass) term, this has the same form with the integral equation in TBA or QTM analyses associated with the level $n-2$ restricted Y-system for A_1. See for example (15.14) and (16.28).

So far, we have considered the case where the zeros of $p(z)$ are on the real axis. When they deviate from it, the T and Y-system remain unchanged. On the other hand, the asymptotics is modified as $\ln Y_k(\zeta) \sim -\frac{m_k}{2\zeta}\,(\zeta \to 0)$ and $\ln Y_k(\zeta) \sim -\frac{\bar{m}_k}{2}\zeta\,(\zeta \to \infty)$, where $m_k = |m_k|e^{i\varphi_k}$ is complex in general. Consequently, the integral equation (11.31) is replaced with

$$\ln \tilde{Y}_k(e^\theta) = -|m_k|\cosh\theta + \sum_{j=k\pm 1}\int_{-\infty}^{\infty}\frac{\ln(1+\tilde{Y}_j(e^{\theta'}))d\theta'}{2\pi\cosh(\theta-\theta'+i\varphi_k-i\varphi_j)}, \tag{11.32}$$

where $\tilde{Y}_k(e^\theta) = Y_k(e^{\theta+i\varphi_k})$. This holds for $|\varphi_k - \varphi_{k\pm1}| < \frac{\pi}{2}$. If the phases go beyond this range (so-called wall crossing), the integral equation acquires extra terms corresponding to the contributions of the poles from the convolution kernel. A simple illustration of such a situation has been given in [204, appendix B].

11.8. **Area and free energy.** The interesting part A of the area is given by[25]

$$A = 2\int d^2z\,\mathrm{Tr}(\Phi_z\Phi_{\bar{z}}) = i\int \sqrt{p}\,dz\wedge\Phi^{11}_{\bar{z}}d\bar{z} = -i\sum_{j,k=1}^{n-3} w_{jk}\oint_{\gamma_j}\sqrt{p}\,dz\oint_{\gamma_k}\Phi^{11}_{\bar{z}}d\bar{z}, \tag{11.33}$$

where the gauge $\Phi_z = \sqrt{p}\,\mathrm{diag}(1,-1)$ is taken and $\mathrm{Tr}\,\Phi_{\bar{z}} = 0$ is used. In the last equality we have dropped the contribution from infinity. The matrix (w_{jk}) is the inverse of the intersection forms[26] $(\langle\gamma_j,\gamma_k\rangle)$ specified by $\langle\gamma_{2k},\gamma_{2k\pm1}\rangle = 1$. Set $\hat{Y}_{2k}(\zeta) = Y_{2k}(\zeta)$ and $\hat{Y}_{2k+1}(\zeta) = Y_{2k+1}(e^{-\frac{\pi i}{2}}\zeta)$ somehow reconciling the shift in (11.29). The factor $\oint_{\gamma_k}\Phi^{11}_{\bar{z}}d\bar{z}$ in (11.33) also appears as the coefficient of $-\zeta$ in the small ζ expansion of $\ln\hat{Y}_k(\zeta)$ based on the perturbative solution of (11.23). On the other hand, the small $\zeta = e^\theta$ expansion of (11.32) gives

$$\ln\hat{Y}_k(\zeta) = \frac{Z_k}{\zeta} + \zeta\Big[\bar{Z}_k + \sum_j\frac{\langle\gamma_k,\gamma_j\rangle}{\pi i}\int\frac{d\zeta'}{\zeta'^2}\ln(1+\hat{Y}_j(\zeta'))\Big] + \cdots, \tag{11.34}$$

where the appearance of $\langle\gamma_k,\gamma_j\rangle$ is the effect of using $\hat{Y}_k(\zeta)$ rather than $Y_k(\zeta)$. Thus one can substitute $\oint_{\gamma_k}\Phi^{11}_{\bar{z}}d\bar{z}$ in (11.33) by $[\ldots]$ here times (-1). As the result the area is expressed as $A = A_{\mathrm{periods}} + A'_{\mathrm{free}}$ with

$$A_{\mathrm{periods}} = -i\sum_{j,k}w_{jk}Z_k\bar{Z}_j, \qquad A'_{\mathrm{free}} = -\frac{1}{\pi}\sum_k Z_k\int\frac{d\zeta}{\zeta^2}\ln(1+\hat{Y}_k(\zeta)). \tag{11.35}$$

Actually one should replace A'_{free} by the average A_{free} taking the contribution from large ζ into account. Thus the final result reads $A = A_{\mathrm{periods}} + A_{\mathrm{free}}$ with

$$A_{\mathrm{free}} = \sum_k|m_k|\int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\cosh\theta\ln(1+\tilde{Y}_k(e^\theta)) \tag{11.36}$$

in terms of $\tilde{Y}_k(e^\theta)$ defined after (11.32). This has the same form as the free energy in the conventional TBA. See for example (15.15).

To summarize, the symmetry aspects of the problem (AdS, Virasoro constraints, null-cusp boundary) are incorporated into the restricted T and Y-systems. Then, all the dynamical information (gluon momenta, Riemann surface, cycles) are remarkably integrated in the "complex mass" parameters $m_1,\ldots,m_{n-3}$.

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[25] Our Φ_z here is $\tilde{\Phi}_z$ in [204].

[26] The inverse exists under our assumption of n being odd. The intersection form $\langle\ ,\ \rangle$ here should not be confused with the $SL(2)$-invariant pairing of spinors.

11.9. **Bibliographical notes.** The subjects in this section are currently in the course of rapid development. For various aspects of the planar AdS/CFT spectrum, see the literatures given in the end of Section 11.1 and reference therein. We have only dealt with the limited issues related to T and Y-systems. The contents in Section 11.2–11.4 are mainly based on [195]. For numerical studies, it is important to formulate the analyticity more precisely and to derive the TBA (or other type of) integral equations including excited states. We refer to [196, 22, 197, 208] for this problem. Similar analyses have been made in [209, 210, 211] for the $\mathrm{AdS}_4/\mathrm{CFT}_3$ duality proposed recently [212].

Calculation of gluon scattering amplitudes at strong coupling using gauge/string duality was initiated in [202] and developed in a series of works [203, 213, 204, 205, 214, 215]. For classical integrability of AdS sigma models and their connection to Hitchin system, see also [216]. Auxiliary linear problem in Section 11.7 is a special case of that for general $SU(2)$ Hitchin system [217], where a number of aspects in the Riemann-Hilbert problem have been discussed including WKB triangulations, the Fock-Goncharov coordinates, the Kontsevich-Soibelman wall-crossing formula, TBA and so forth. The contents of Section 11.5–11.8 are mainly taken from [204]. We have treated n (number of gluons) odd case. For the case n even, see [214, 215]. In [215], further effect of operator insertion is studied, and the (slightly deformed) level 2 restricted Y-system for D_n has been obtained. For a similar appearance of the D type Y-system in A_1 related lattice models, see Remark 16.8. The generalized sinh-Gordon equation has also been studied in the context of generalized ODE/IM correspondence in [174].

12. Aspects as classical integrable system

Beside the quantum integrable systems, T and Y-systems also have interesting aspects as classical nonlinear difference equations. For instance, the T-system relation (2.5) is presented in the form

$$\tau_1\tau_{23} - \tau_2\tau_{31} + \tau_3\tau_{12} = 0 \tag{12.1}$$

with a suitable redefinition up to boundary condition. Here the indices signify a shift of the independent vector variable in the respective directions ($\tau_{ij} = \tau_{ji}$). This is a version of Hirota-Miwa equation on tau functions in the theory of discrete KP equations [218, 219, 220, 221]. A simplest account for its integrability is the Lax representation, namely, the compatibility of the linear system:

$$\psi_i - \psi_j = \frac{\tau\tau_{ij}}{\tau_i\tau_j}\psi \quad (i < j).$$

The Hirota-Miwa equation serves as a master equation generating a variety of soliton equations under suitable specializations and boundary conditions. See for instance [220, 222, 223]. Apart from this, there are numerous aspects in type A T-system, sometimes called octahedron recurrence, related to discrete geometry [224, 225, 226], Littlewood-Richardson rule [227], perfect matchings and partition functions on a network [228, 229] and so forth. For types other than A however, such results are relatively few.

Our presentation in this section is necessarily selective. In Section 12.1, we explain that the T-system for $\mathfrak{g}$ is a discretized Toda field equation that has decent continuous limits with a known Hamiltonian structure. In Section 12.2, a connection of the Y-system for A_∞ with discrete geometry is reviewed.

12.1. **Continuum limit.** We present a simple continuous limit of the T-system for general $\mathfrak{g}$ known as the lattice Toda field equation [230]. It is a difference-differential system containing continuous time and discrete space variables. Further continuous limit on the latter yields the Toda field equation on $1+1$ dimensional continuous space-time [231].

We begin by making a slight change of variables in the T-system as

$$T^{(a)}_m(u) = \tau_a(u + \frac{m}{t_a}, s + \varepsilon\frac{m}{t_a}) \quad (1 \le a \le r, u \in \mathbb{Z}/t, m \in \mathbb{Z}). \tag{12.2}$$

Here ε is a small parameter and s is going to be the continuous time variable soon. For the symbols t, t_a and root system data, see around (2.1). We substitute (12.2) into the T-system (2.22) $T^{(a)}_m(u - \frac{1}{t_a})T^{(a)}_m(u + \frac{1}{t_a}) - T^{(a)}_{m-1}(u)T^{(a)}_{m+1}(u) = g^{(a)}_m(u)M^{(a)}_m(u)$ with $m \in t_a\mathbb{Z}$. For each $\mathfrak{g}$ of rank r there are r such equations. (The case $m \notin t_a\mathbb{Z}$ leads to the same continuum limit as the one considered in the following.) For example, B_2 case reads

$$\tau_1(n-1,s)\tau_1(n+1,s) - \tau_1(n-1,s-\varepsilon)\tau_1(n+1,s+\varepsilon) = g_1\tau_2(n,s),$$
$$\tau_1(n-\tfrac{1}{2},s)\tau_1(n+\tfrac{1}{2},s) - \tau_1(n-\tfrac{1}{2},s-\tfrac{\varepsilon}{2})\tau_1(n+\tfrac{1}{2},s+\tfrac{\varepsilon}{2}) = g_2\tau_1(n-\tfrac{1}{2},s)\tau_1(n+\tfrac{1}{2},s),$$

where we have chosen $g_a = g^{(a)}_{t_a m}(u)$ to be a constant. We take the continuum limit in the time variable s keeping $n \in \mathbb{Z}/t$ as the coordinate of a one dimensional lattice without boundary. Namely, we replace g_a by $\varepsilon g_a/t_a$ and set $\varepsilon \to 0$. The result reads

$$D_s\tau_1(n-1)\cdot\tau_1(n+1) = g_1\tau_2(n),$$
$$D_s\tau_2(n-\frac{1}{2})\cdot\tau_2(n-\frac{1}{2}) = g_2\tau_1(n-\frac{1}{2})\tau_1(n+\frac{1}{2}).$$

Here we suppressed the time dependence as $\tau_a(n) = \tau_a(n,s)$, which we shall also do in the remainder of this subsection. D_s denotes the Hirota derivative:

$$D_s f\cdot g = \frac{\partial f}{\partial s}g - f\frac{\partial g}{\partial s}.$$

Similarly, the general $\mathfrak{g}$ case is given by

$$D_s\tau_a(n-\tfrac{1}{t_a})\cdot\tau_a(n+\tfrac{1}{t_a}) = g_a\mathcal{M}_a(n),$$
$$\mathcal{M}_a(n) := \prod_{b:C_{ab}=-1}\tau_b(n)\prod_{b:C_{ab}=-2}\tau_b(n-\tfrac{1}{2})\cdot\tau_b(n+\tfrac{1}{2})\prod_{b:C_{ab}=-3}\tau_b(n-\tfrac{2}{3})\tau_b(n)\tau_b(n+\tfrac{2}{3}), \tag{12.3}$$

where $n \in \mathbb{Z}/t$. We call this the lattice Toda field equation for $\mathfrak{g}$. In some case, it actually splits into disjoint sectors. For instance in types ADE, one has $t_a = t = 1$ for any $a \in I$, hence (12.3) closes among $\{\tau_a(n) | a \in I_{(-1)^n}\}$ or $\{\tau_a(n) | a \in I_{(-1)^{n+1}}\}$, where $I_\pm$ is the bipartite decomposition of the Dynkin diagram nodes $I = \{1,\ldots,r\} = I_+ \sqcup I_-$.

One can rewrite (12.3) in a form that looks more like Toda equation and explore its Hamiltonian structure. As an illustration, we first treat the A_1 case. Let us introduce the dynamical variables $x(n)$ and $\beta(n)$ by

$$x(n) = \frac{\partial}{\partial s}\ln\frac{\tau_1(n-1)}{\tau_1(n+1)}, \quad \beta(n) = \frac{x(n-1)}{x(n+1)} \quad (n \in Z). \tag{12.4}$$

The equation (12.3) for A_1 reads

$$\frac{\partial \tau_1(n-1)}{\partial s}\tau_1(n+1) - \tau_1(n-1)\frac{\partial \tau_1(n+1)}{\partial s} = g_1. \tag{12.5}$$

This allows us to rewrite (12.4) as

$$x(n) = \frac{g_1}{\tau_1(n-1)\tau_1(n+1)}, \quad \beta(n) = \frac{\tau_1(n+2)}{\tau_1(n-2)}. \tag{12.6}$$

From the expression of $x(n)$ in (12.4) and $\beta(n)$ in (12.6), one gets another form of the lattice Toda field equation for A_1:

$$\frac{\partial \ln \beta(n)}{\partial s} = -x(n-1) - x(n+1), \tag{12.7}$$

which is a discrete analogue of the Liouville equation. It is derived as the equation of motion

$$\frac{\partial \beta(n)}{\partial s} = \{\mathcal{H}, \beta(n)\}, \tag{12.8}$$

with the following Hamiltonian and Poisson bracket:

$$\mathcal{H} = \sum_{m\in\mathbb{Z}} x(m), \qquad \{x(m), x(n)\} = x(m)x(n)\,\mathrm{sgn}_2(n-m). \tag{12.9}$$

See (12.13) for the definition of $\mathrm{sgn}_2(n)$. We remark that (12.5), (12.7) and their relation explained in the above are difference-differential analogue of the T-system, Y-system and their transformation stated in Theorem 2.5 for A_1, respectively.

All these features are generalized to $\mathfrak{g}$ straightforwardly. The relevant dynamical variables are

$$x_a(n), \quad \beta_a(n) = \frac{x_a(n-\frac{1}{t_a})}{x_a(n+\frac{1}{t_a})} \qquad (a \in I,\ n \in \mathbb{Z}/t), \tag{12.10}$$

which are functions of the continuous time s. We keep the notation $I, t, t_a, C, (\alpha_a|\alpha_b)$ around (2.1) and set

$$B_{ab} = B_{ba} = \frac{t_b}{\max(t_a,t_b)}C_{ab} = \begin{cases} 2 & C_{ab}=2, \\ -1 & C_{ab}<0, \\ 0 & C_{ab}=0. \end{cases} \tag{12.11}$$

(B_{ab}) is the Cartan matrix for simply laced Dynkin diagram obtained by forgetting the multiplicity of oriented edges in that for $\mathfrak{g}$. We specify the Poisson bracket of $x_a(n)$ as

$$\{x_a(m), x_b(n)\} = \frac{1}{2}B_{ab}\, x_a(m)x_b(n)\,\mathrm{sgn}_{B_{ab}}\Big(\max(t_a,t_b)(n-m)\Big), \tag{12.12}$$

where $\mathrm{sgn}_k(v)$ with $k \in \{2,-1\}$ is the odd function of $v \in \mathbb{R}$ defined by[27]

$$\mathrm{sgn}_k(v) = \begin{cases} 1 & \text{if } v>0 \text{ and } v \in 2\mathbb{Z}+k, \\ -1 & \text{if } v<0 \text{ and } v \in 2\mathbb{Z}+k, \\ 0 & \text{otherwise.} \end{cases} \tag{12.13}$$

[27] $\mathrm{sgn}_0(v)$ is not necessary since the RHS of (12.12) contains the factor B_{ab}.

Consequently, the Poisson bracket concerning $\beta_a(n)$ becomes local in that it is non vanishing only with finitely many opponents.

$$\{x_a(m), \beta_b(n)\} = \begin{cases} -x_a(m)\beta_a(n)(\delta_{m,n+\frac{1}{t_a}} + \delta_{m,n-\frac{1}{t_a}}) & C_{ab} = 2, \\ x_a(m)\beta_b(n)\sum_{j=C_{ab}+1}^{-C_{ab}-1}\delta_{m+\frac{j}{t_a},n} & C_{ab} < 0, \\ 0 & C_{ab} = 0, \end{cases} \tag{12.14}$$

$$\{\beta_a(m), \beta_b(n)\} = \beta_a(m)\beta_b(n)(\delta_{m+(\alpha_a|\alpha_b),n} - \delta_{m-(\alpha_a|\alpha_b),n}). \tag{12.15}$$

In (12.14), the j-sum is taken with the condition $j \equiv C_{ab}+1 \mod 2$. The equation of motion with the Hamiltonian

$$\frac{\partial \beta_a(n)}{\partial s} = \{\mathcal{H}, \beta_a(n)\}, \quad \mathcal{H} = \sum_{a\in I, n\in\mathbb{Z}/t} x_a(n) \tag{12.16}$$

leads to the differential-difference system:

$$\begin{aligned} \frac{\partial \ln \beta_a(n)}{\partial s} &= -x_a(n - \frac{1}{t_a}) - x_a(n + \frac{1}{t_a}) \\ &+ \sum_{b:C_{ba}=-1} x_b(n) + \sum_{b:C_{ba}=-2}\Big(x_b(n-\frac{1}{2}) + x_b(n+\frac{1}{2})\Big) \\ &+ \sum_{b:C_{ba}=-3}\Big(x_b(n-\frac{2}{3}) + x_b(n) + x_b(n+\frac{2}{3})\Big). \end{aligned} \tag{12.17}$$

For $\mathfrak{g} = A_1$ this reduces to (12.7). The equation (12.17) with $x_a(n)$ and $\beta_a(n)$ related as (12.10) is another form of the lattice Toda field equation (12.3). In fact, the transformation between (12.3) and (12.17) is parallel with the A_1 case (12.4)–(12.7). Generalizing (12.4) we relate $x_a(n)$ and $\tau_a(n)$ by

$$x_a(n) = \frac{\partial}{\partial s}\ln\frac{\tau_a(n-\frac{1}{t_a})}{\tau_a(n+\frac{1}{t_a})} = \frac{g_a\mathcal{M}_a(n)}{\tau_a(n-\frac{1}{t_a})\tau_a(n+\frac{1}{t_a})}, \tag{12.18}$$

where the latter equality is due to the lattice Toda field equation (12.3). Substituting the latter form into (12.10), we find

$$\beta_a(n) = \prod_{b\in I}\frac{\tau_b(n+(\alpha_a|\alpha_b))}{\tau_b(n-(\alpha_a|\alpha_b))}. \tag{12.19}$$

This can also been derived from (8.16) by noting the same structure in $A^{-1}_{a,z=q^{tn}}$ (4.26) and $\mathcal{M}_a(n)/(\tau_a(n-\frac{1}{t_a})\tau_a(n+\frac{1}{t_a}))$ given by (12.3). Anyway, $\frac{\partial \ln\beta_a(n)}{\partial s}$ is expressed as a linear combination of $x_a(n)$ by using the first formula in (12.18). The result reproduces (12.17).

A further continuous limit on n can be taken by letting

$$x_a(n) \to 2\varepsilon\exp(\phi_a(z+\varepsilon n)), \quad \ln\beta_a(n) \to -\frac{2\varepsilon}{t_a}\phi'_a, \tag{12.20}$$

where $' = \frac{\partial}{\partial z}$. Then the limit $\varepsilon \to 0$ of (12.17) leads to a version of the Toda field equation for $\phi_a = \phi_a(z,s)$:

$$\frac{\partial^2\phi_a}{\partial z\partial s} = \sum_{b\in I} t_a t_b(\alpha_a|\alpha_b)e^{\phi_b}. \tag{12.21}$$

The case $\mathfrak{g} = A_1$ is the Liouville equation. Switching to ψ_a by $\phi_a = \sum_{b\in I} C_{ab}\psi_b - \ln t_a$, one may rewrite it in the form

$$\frac{\partial^2\psi_a}{\partial z\partial s} = \exp\Bigl(\sum_{b\in I} C_{ab}\psi_b\Bigr)$$

studied in [231]. An explicit construction of the general solution is known containing $2r$ arbitrary functions [231]. We see that (12.16) and (12.14) are lattice analogue of the Hamiltonian formulation of the Toda field equation:

$$\frac{\partial\phi'_a}{\partial s} = \{\mathcal{H}, \phi'_a\}, \quad \mathcal{H} = \sum_{a\in I}\int dz e^{\phi_a(z)}, \quad \{\phi_a(z), \phi'_b(z')\} = t_a t_b(\alpha_a|\alpha_b)\delta(z-z').$$

The Poisson structures (12.12)–(12.15) have an origin in the lattice analogue of the W-algebras going back to [232]. In particular, they may be deduced from the Poisson relations among appropriate constituent fields in the q-deformed W-algebra. See for example [233, 234, 153, 230, 235] and reference therein. Here we only mention, as an example, that (12.15) is a lattice analogue of the Poisson relation

$$\{A_a(z), A_b(w)\} = \Bigl(\delta\Bigl(q^{(\alpha_a|\alpha_b)}\frac{w}{z}\Bigr) - \delta\Bigl(q^{-(\alpha_a|\alpha_b)}\frac{z}{w}\Bigr)\Bigr) A_a(z)A_b(w)$$

among the fields $A_a(z)$ corresponding to the exponential simple root e^{α_a} whose counterpart in the theory of q-character has appeared in (4.26). See eq. (3.1) in [235] and also eq.(8.8) in [70] for the logarithmic form.

12.2. **Discrete geometry.** As we have seen in the previous subsection, continuous limits of T-system lead to Toda type differential equations. On the other hand, geometric origins of many differential equations of such kind have been known from the days of Darboux. Like the continuous case, it is natural to seek discrete geometry responsible for the integrability of discrete integrable equations. In fact, if we let such geometric objects speak of themselves, they would say "We exist, therefore it is integrable[28]". There are many results in this direction. See for example [236, 224, 225, 226, 237] and reference therein. In a sense they provide a most natural framework to set up Lax formalisms of the integrable difference equations from geometric points of view. Here we only include a simple exposition of the basic example [238, 239] connecting Y-system for A_∞ to a discrete analogue of the Laplace sequence of conjugate nets.

We begin by recalling the appearance of the Toda field equation in projective differential geometry. Consider a surface in the real projective space $\mathbb{P}^3$ which has the homogeneous coordinate vector $\mathbf{z} = \mathbf{z}(x,y) \in \mathbb{P}^3$. A local coordinate (x,y) of the surface is called a *conjugate net* if

$$\mathbf{z}_{xy} + a(x,y)\mathbf{z}_x + b(x,y)\mathbf{z}_y + c(x,y)\mathbf{z} = 0 \tag{12.22}$$

is valid for some functions a, b, c, where the indices mean the derivatives. Although $\mathbf{z}$ and $\mathbf{w}$ specify the same surface if they are related by $\mathbf{z} = \lambda\mathbf{w}$, the above equation is not invariant but changed into

$$\mathbf{w}_{xy} + \tilde{a}(x,y)\mathbf{w}_x + \tilde{b}(x,y)\mathbf{w}_y + \tilde{c}(x,y)\mathbf{w} = 0 \tag{12.23}$$

[28] V. V. Bazhanov, talk at Newton Institute, Cambridge, UK, March 2009.

with $\tilde{a} = a + (\ln\lambda)_y$, $\tilde{b} = b + (\ln\lambda)_x$, $\tilde{c} = c + a(\ln\lambda)_x + b(\ln\lambda)_y + \lambda_{xy}/\lambda$. A characteristic of a surface independent of the gauge λ is the Laplace invariant

$$h = a_x + ab - c, \quad k = b_y + ab - c, \tag{12.24}$$

satisfying $\tilde{h} = h$ and $\tilde{k} = k$. In what follows we consider the generic situation that they are non zero.

For the homogeneous coordinate vector $\mathbf{z}$ satisfying (12.22), the Laplace transformation $\mathcal{L}_\pm$ is defined by

$$\mathcal{L}_+(\mathbf{z}) = \mathbf{z}_y + a\mathbf{z}, \quad \mathcal{L}_-(\mathbf{z}) = \mathbf{z}_x + b\mathbf{z}. \tag{12.25}$$

This is compatible with the defining property (12.22) of the conjugate net in that $\mathcal{L}_+(\lambda\mathbf{w}) = \lambda(\mathbf{w}_y + \tilde{a}\mathbf{w})$ and $\mathcal{L}_-(\lambda\mathbf{w}) = \lambda(\mathbf{w}_x + \tilde{b}\mathbf{w})$ hold with $\tilde{a}$ and $\tilde{b}$ given in the above. Any component z of $\mathbf{z}$ transforms as $\mathcal{L}_- \circ \mathcal{L}_+(z) = hz$ and $\mathcal{L}_+ \circ \mathcal{L}_-(z) = kz$, meaning that $\mathcal{L}_+$ and $\mathcal{L}_-$ are inverse to each other as transformations in $\mathbb{P}^3$. The family of surfaces in $\mathbb{P}^3$ generated from $\mathbf{z}^{(0)} = \mathbf{z}$ as $\mathbf{z}^{(\pm n)} = (\mathcal{L}_\pm)^n(\mathbf{z})$ $(n \geq 1)$ is called a Laplace sequence. Denote by h_n, k_n the Laplace invariant associated with $\mathbf{z}^{(n)}$. It is easy to see that $\mathbf{z}^{(\pm 1)}$ satisfies (12.22) with a, b, c replaced by $a^{(\pm 1)}, b^{(\pm 1)}, c^{(\pm 1)}$ given by

$$\begin{aligned} &a^{(1)} = a - \frac{h_y}{h}, \quad b^{(1)} = b, \quad c^{(1)} = ab - h + h\Big(\frac{b}{h}\Big)_y, \\ &a^{(-1)} = a, \quad b^{(-1)} = b - \frac{k_x}{k}, \quad c^{(-1)} = ab - k + k\Big(\frac{a}{k}\Big)_x. \end{aligned} \tag{12.26}$$

Substituting this into (12.24), one can express $h_{\pm 1}$ and $k_{\pm 1}$ in terms of $h_0 = h$ and $k_0 = k$. The result tells that the sequence of Laplace invariants satisfy a Toda field equation for A_∞:

$$\frac{\partial^2 \ln h_n}{\partial x \partial y} = -h_{n-1} + 2h_n - h_{n+1}, \quad h_n = k_{n+1}. \tag{12.27}$$

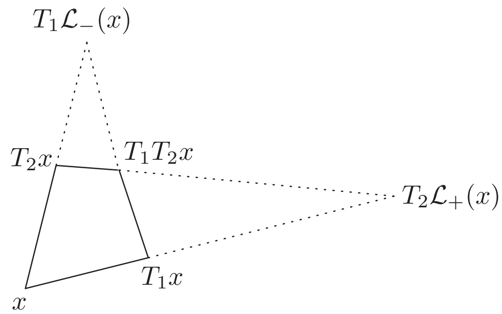
Now we move onto the discrete analogue of these constructions. The first step is to observe that (12.22) implies the four infinitesimally neighboring points are coplanar. This motivates us to introduce a map $\mathbf{x} : \mathbb{Z}^2 \to \mathbb{P}^3$ such that the 4 points $\mathbf{x}(n,m), \mathbf{x}(n+1,m), \mathbf{x}(n,m+1), \mathbf{x}(n+1,m+1)$ are coplanar for any $(n,m) \in \mathbb{Z}^2$. Such a map is called two dimensional *quadrilateral lattice*, which serves as a discrete analogue of the conjugate net. In the inhomogeneous coordinate of the projective space, a two dimensional quadrilateral lattice is represented by a map $x : \mathbb{Z}^2 \to \mathbb{R}^3$ satisfying the discrete analogue of (12.22) as follows:

$$\Delta_1\Delta_2 x = (T_1 A)\Delta_1 x + (T_2 B)\Delta_2 x. \tag{12.28}$$

Here $\Delta_i = T_i - 1$ and T_i changes n_i in any function $f(n_1, n_2)$ to $n_i + 1$. The functions A, B on $\mathbb{Z}^2$ are "gauge potentials" analogous to a, b in the continuum case. The Laplace transformation, denoted by the same symbol as before, reads

$$\mathcal{L}_+(x) = x - \frac{\Delta_1 x}{B}, \quad \mathcal{L}_-(x) = x - \frac{\Delta_2 x}{A}. \tag{12.29}$$

To see the geometric meaning of this, note that the four points x, T_1x, T_2x, T_1T_2x form a quadrilateral on a plane due to (12.28). The points $T_1\mathcal{L}_+(x)$ and $T_2\mathcal{L}_-(x)$ are intersections of the two lines extending the opposite sides of the quadrilateral.



As in (12.26), the postulate $\Delta_1\Delta_2\mathcal{L}_\pm(z) = T_1\mathcal{L}_\pm(A)\Delta_1\mathcal{L}_\pm(z)+T_2\mathcal{L}_\pm(B)\Delta_2\mathcal{L}_\pm(z)$ fixes the Laplace transformation of the gauge potentials as

$$\begin{aligned}\mathcal{L}_+(A) &= \frac{B}{T_2B}(1+T_1A)-1, \quad \mathcal{L}_+(B) = T_2^{-1}\left(\frac{T_1\mathcal{L}_+(A)}{\mathcal{L}_+(A)}(1+B)\right)-1,\\ \mathcal{L}_-(A) &= T_1^{-1}\left(\frac{T_2\mathcal{L}_-(B)}{\mathcal{L}_-(B)}(1+A)\right)-1, \quad \mathcal{L}_-(B) = \frac{A}{T_1A}(1+T_2B)-1.\end{aligned} \tag{12.30}$$

It follows that the Laplace transformation is invertible, i.e., $\mathcal{L}_+\circ\mathcal{L}_- = \mathcal{L}_-\circ\mathcal{L}_+ = \mathrm{id}$. Introduce the Laplace sequence as the continuous case by $x^{(0)} = x$ and $x^{(\pm n)} = (\mathcal{L}_\pm)^n(x)$ $(n \geq 1)$.

Now we are going to assign a cross ratio to each member of the Laplace sequence. For the four colinear points q_1, q_2, q_3, q_4 in $\mathbb{R}^3$, we define the cross ratio as

$$\mathrm{cr}(q_1,q_2,q_3,q_4) = \mathrm{cr}(q_2,q_1,q_4,q_3) = \frac{(q_3-q_1)(q_4-q_2)}{(q_3-q_2)(q_4-q_1)},$$

which is invariant under projective transformations. Define the sequence of cross ratio by

$$Y^{(n)} = -\mathrm{cr}(x^{(n)}, \mathcal{L}_+(x^{(n)}), T_1x^{(n)}, T_2\mathcal{L}_+(x^{(n)})) \quad (n \in \mathbb{Z}), \tag{12.31}$$

or equivalently, by setting $Y^{(0)} = Y$ and $Y^{(\pm n)} = (\mathcal{L}_\pm)^n(Y)(n \geq 1)$ with $Y^{(0)} = Y = -\mathrm{cr}(x, \mathcal{L}_+(x), T_1x, T_2\mathcal{L}_+(x))$. The four points in cr are colinear. By using (12.28)–(12.30) one can derive various formulas, e.g.,

$$\begin{aligned}Y &= \frac{T_2B-(1+T_1A)B}{(1+B)(1+T_1A)} = -\frac{\mathcal{L}_+(A)}{1+\mathcal{L}_+(A)}\frac{B}{1+B},\\ Y^{(-1)} &= -\mathrm{cr}(x, \mathcal{L}_-(x), T_2x, T_1\mathcal{L}_-(x)).\end{aligned}$$

The sequence $Y^{(n)}$ satisfies the functional relation [238, 239]

$$(T_1T_2Y^{(n)})Y^{(n)} = T_1\left(\frac{1+Y^{(n-1)}}{1+(Y^{(n)})^{-1}}\right)T_2\left(\frac{1+Y^{(n+1)}}{1+(Y^{(n)})^{-1}}\right). \tag{12.32}$$

With a suitable identification, this coincides with the Y-system for A_∞ (2.11)

$$Y_m^{(a)}(u-1)Y_m^{(a)}(u+1) = \frac{(1+Y_m^{(a-1)}(u))(1+Y_m^{(a+1)}(u))}{(1+Y_{m-1}^{(a)}(u)^{-1})(1+Y_{m+1}^{(a)}(u)^{-1})}$$

with no boundary conditions on a and m.

12.3. **Bibliographical notes.** The contents of Section 12.1 and Section 12.2 are mainly taken from [230, 235] and [238, 239], respectively.

13. Q-system and Fermionic formula

13.1. **Introduction.** Consider the T-system for $\mathfrak{g}$. If one formally forgets the spectral parameter u in $T^{(a)}_m(u)$, the resulting variable is conventionally denoted by $Q^{(a)}_m$ and the T-system reduces to the relation among them called *Q-system*. In the context of q-characters, $T^{(a)}_m(u)$ is the q-character $\chi_q(W^{(a)}_m(u))$ of the Kirillov-Reshetikhin module $W^{(a)}_m(u)$ (Theorem 4.8). Therefore

$$Q^{(a)}_m = \mathrm{res}\, T^{(a)}_m(u) \tag{13.1}$$

is the usual character of $\mathfrak{g}$ obtained by the restriction defined in (4.24). Consider an arbitrary product of $Q^{(a)}_m$'s and the two kinds of decompositions (we assume $\nu^{(a)}_m \in \mathbb{Z}_{\geq 0}$ for the time being)

$$\prod_{a,m}(Q^{(a)}_m)^{\nu^{(a)}_m} = \sum_\lambda b_\lambda\, \chi(V_\lambda) = \sum_\lambda c_\lambda\, e^\lambda. \tag{13.2}$$

Here $\chi(V_\lambda)$ denotes the (usual) character of the irreducible $\mathfrak{g}$-module V_λ with highest weight λ. The multiplicities b_λ of the irreducible representation V_λ (branching coefficients) and the multiplicities c_λ of weights λ (dimensions of weight spaces) are two basic quantities characterizing the decompositions. It turns out that analyses of the Q-system provide them with *Fermionic formulas* $b_\lambda = \mathcal{M}_\lambda$ and $c_\lambda = \mathcal{N}_\lambda$. They possess fascinating forms that symbolize the formal completeness of the *string hypothesis* in the Bethe ansatz at $q = 1$ and $q = 0$, respectively.

In Sections 13.2 and 13.3 we explain how $\mathcal{M}_\lambda$ and $\mathcal{N}_\lambda$ emerge from the Bethe ansatz along the simplest setting in $\mathfrak{g} = A_1$. Precise statements for A_1 are formulated in Section 13.4 and the proof by a unified perspective of the multivariable Lagrange inversion method is outlined in Section 13.5. All the essential ingredients are given by this point. In Section 13.6, we introduce the Q-system for $\mathfrak{g}$ and write down the associated Fermionic formulas $\mathcal{M}_\lambda$ and $\mathcal{N}_\lambda$. The main Theorem 13.11 in the general case is stated. In Section 13.7, the expansion of $Q^{(a)}_m$ into classical characters is given for non exceptional algebras A_r, B_r, C_r and D_r. There are a lot of further aspects which are beyond the scope of this review. They will be mentioned briefly in Section 13.8. For simplicity we restrict ourselves to untwisted affine Lie algebras in this section. Analogous results are also available in the twisted cases.

13.2. **Simplest example of $\boldsymbol{\mathcal{M}_\lambda}$.** Recall the Bethe equation (8.4) for the 6 vertex model. In the rational limit $q \to 1$, it takes the form

$$-\left(\frac{u_j + \sqrt{-1}}{u_j - \sqrt{-1}}\right)^L = \prod_{k=1}^n \frac{u_j - u_k + 2\sqrt{-1}}{u_j - u_k - 2\sqrt{-1}}, \tag{13.3}$$

where we have set all the inhomogeneity $w_j = 0$ and replaced u_j by $\sqrt{-1}u_j$. The string hypothesis [9] is that the roots $u_1, \ldots, u_n$ are arranged as (called originally "WellenKomlex" in [9])

$$\bigcup_{m\geq 1}\ \bigcup_{1\leq \alpha\leq N_m}\ \bigcup_{u_{m\alpha}\in\mathbb{R}} \{u_{m\alpha} + \sqrt{-1}(m+1-2i) + \epsilon_{m\alpha i} \mid 1 \leq i \leq m\} \tag{13.4}$$

for each partition $n = \sum_{m\geq 1} mN_m$ ($N_m \in \mathbb{Z}_{\geq 0}$). Here $\epsilon_{m\alpha i}$ stands for a small deviation. The m-tuple configuration (with negligible $\epsilon_{m\alpha i}$) is called a length m string with string center $u_{m\alpha}$. The N_m is the number of length m strings. The string hypothesis is not literally true as exemplified for instance when $n = 2$ and

$L > 21$ (cf. [240]). Nevertheless, a formal count of the number of solutions to (13.3) is done as follows [9, 241]. First one rewrites the Bethe equation into the one for the string centers. This is done by replacing u_j by a member of a string $u_{m\alpha} + \sqrt{-1}(m+1-2i) + \epsilon_{m\alpha i}$ and taking the product over $1 \le i \le m$. The resulting equation in the logarithmic form $\ln(\mathrm{LHS}/\mathrm{RHS}) \in 2\pi\sqrt{-1}\mathbb{Z}$ is cast, if $\epsilon_{m\alpha i}$ is negligible, into the form $f_m(u_{m\alpha}) \in \mathbb{Z}$ or $\mathbb{Z}+\frac{1}{2}$ $(1 \le \alpha \le N_m)$ which depends on m and the partition $\{N_m\}$. Explicitly, $f_m(u)$ is given by

$$f_m(u) = L\theta_{m,1}(u) - \sum_{k\ge 1}\sum_{\beta=1}^{N_k}(\theta_{m,k-1}+\theta_{m,k+1})(u-u_{k\beta}), \tag{13.5}$$

$$\theta_{m,k}(u) = \frac{1}{\pi}\sum_{\alpha=1}^{\min(m,k)} \tan^{-1}\left(\frac{u}{|m-k|+2\alpha-1}\right). \tag{13.6}$$

Let us employ the principal branch $-\frac{\pi}{2} \le \tan^{-1}(u) \le \frac{\pi}{2}$. Then from $\theta_{m,k}(\pm\infty) = \pm\min(m,k)/2$ and $(\theta_{m,k-1}+\theta_{m,k+1})(\pm\infty) = \pm(\min(m,k)-\delta_{m,k}/2)$, we get $f_m(\pm\infty) = \pm(P_m+N_m)/2$. Here P_m, called vacancy number, is given by

$$P_m = L - 2\sum_{k\ge 1}\min(m,k)N_k, \tag{13.7}$$

and will play a significant role in the sequel. The bold argument is then that if $P_m \ge 0$, the solutions $\{u_{m\alpha}\}$ (up to permutations of $u_{m1},\dots,u_{mN_m}$ for each m) are in one to one correspondence with the sequences $(I_1,\dots,I_{N_m}) \in (\mathbb{Z}+\frac{P_m+N_m+1}{2})^{N_m}$ such that $-f_m(\infty)+\frac{1}{2} \le I_1 < \cdots < I_{N_m} \le f_m(\infty)-\frac{1}{2}$. There are $\binom{P_m+N_m}{N_m}$ such sequences for each m. Accordingly if one admits the argument, the number of solutions is

$$\mathcal{M}_n = \sum_{\{N_m\}}\prod_{m\ge 1}\binom{P_m+N_m}{N_m}, \tag{13.8}$$

where the sum extends over all the partitions of n, namely those $N_m \ge 0$ satisfying $n = \sum_{m\ge 1} mN_m$. (We understand $\mathcal{M}_0 = 1$.)

What number should we expect for $\mathcal{M}_n$? The quantum space for the rational 6 vertex model is $(V_{\omega_1})^{\otimes L}$, where $V_{\omega_1} \simeq \mathbb{C}^2$ is the spin $\frac{1}{2}$ representation whose highest weight is the fundamental weight ω_1. As a result of the global $A_1 = sl_2$ symmetry, the Bethe vectors become by construction highest weight vectors in the quantum space [242]. The sector labeled by n carries the weight $(L-2n)\omega_1$. Thus for the Bethe's string hypothesis to be complete, one should have $\mathcal{M}_n = b_n$ for $0 \le n \le L/2$, where b_n is the branching coefficient in the irreducible decomposition $(V_{\omega_1})^{\otimes L} = \bigoplus_{0\le n\le L/2} b_n V_{(L-2n)\omega_1}$[29]. Explicitly, $b_n = \binom{L}{n} - \binom{L}{n-1}$. Note that the condition $0 \le n \le L/2$, and (13.7) imply that $P_1 \ge P_2 \ge \cdots \ge P_\infty = L-2n \ge 0$, which automatically guarantees the condition $P_m \ge 0$.

Example 13.1. For $L = 6$, one has $(V_{\omega_1})^{\otimes 6} = V_{6\omega_1} \oplus 5V_{4\omega_1} \oplus 9V_{2\omega_1} \oplus 5V_0$. Accordingly one can check $(\mathcal{M}_0,\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3) = (1,5,9,5)$. In fact, the nontrivial cases are

[29] This argument lacks the consideration on the associated Bethe vectors.

checked as

$$\mathcal{M}_1 = \underset{N_1=1}{\binom{4+1}{1}} = 5, \qquad \mathcal{M}_2 = \underset{N_2=1}{\binom{2+1}{1}} + \underset{N_1=2}{\binom{2+2}{2}} = 9,$$
$$\mathcal{M}_3 = \underset{N_3=1}{\binom{0+1}{1}} + \underset{N_1=N_2=1}{\binom{2+1}{1}\binom{0+1}{1}} + \underset{N_1=3}{\binom{0+3}{3}} = 5.$$

We postpone what can be proved mathematically in a more general setting to Section 13.4.

13.3. **Simplest example of $\mathcal{N}_{\boldsymbol{\lambda}}$.** Here we return to the trigonometric Bethe equation (8.4). After setting the inhomogeneity $w_j = 0$, $q = e^{-2\pi\hbar}$ and replacing u_j by $u_j/(\sqrt{-1}\hbar)$, it reads

$$\left(\frac{\sin\pi\big(u_j + \sqrt{-1}\hbar\big)}{\sin\pi\big(u_j - \sqrt{-1}\hbar\big)}\right)^L = -\prod_{k=1}^{n} \frac{\sin\pi\big(u_j - u_k + 2\sqrt{-1}\hbar\big)}{\sin\pi\big(u_j - u_k - 2\sqrt{-1}\hbar\big)}. \tag{13.9}$$

In this convention, the analogue of the string configuration (13.4) is

$$\bigcup_{m\geq 1} \bigcup_{1\leq\alpha\leq N_m} \bigcup_{u_{m\alpha}\in\mathbb{R}} \{u_{m\alpha} + \sqrt{-1}(m+1-2i)\hbar + \epsilon_{m\alpha i} \mid 1 \leq i \leq m\}, \tag{13.10}$$

where N_m is again the number of length m strings. Apart from $q = 1$ treated in the previous subsection, there is a point $q = 0$, i.e., the limit $\hbar \to \infty$ where one can make another formal but systematic counting of the string solutions [243]. Leaving the precise definitions and statements to [243], we just state here casually that at $q = 0$ the Bethe equation (13.9) becomes the following linear congruence equation on the string centers:

$$\sum_{k\geq 1}\sum_{\beta=1}^{N_k} A_{m\alpha,k\beta}u_{k\beta} \equiv \frac{P_m + N_m + 1}{2} \mod \mathbb{Z}. \tag{13.11}$$

Here the coefficient $A_{m\alpha,k\beta}$ is given by

$$A_{m\alpha,k\beta} = \delta_{mk}\delta_{\alpha\beta}(P_m + N_m) + 2\min(m,k) - \delta_{mk} \tag{13.12}$$

with the same P_m as in (13.7). The equation (13.11) is called the string center equation. The concrete form of its RHS will not matter in the counting problem considered in the sequel. Given a string pattern (N_m), one should actually regard the solutions to (13.11) as belonging to

$$(u_{k1}, u_{k2}, \ldots, u_{kN_k}) \in (\mathbb{R}/\mathbb{Z})^{N_k}/\mathfrak{S}_{N_k}$$

for each k, where $\mathfrak{S}_N$ denotes the degree N symmetric group. This is because the Bethe vector is a symmetric function of $e^{2\pi\sqrt{-1}u_{k1}}, \ldots, e^{2\pi\sqrt{-1}u_{kN_k}}$ for each k. We say that a solution $(u_{k\beta})$ to (13.11) is *off-diagonal* if $u_{k1}, u_{k2}, \ldots, u_{kN_k} \in \mathbb{R}/\mathbb{Z}$ are all distinct for each k. This definition is motivated by the fact that the Bethe vectors vanish unless the associated Bethe roots are all distinct [244].

For $0 \leq n \leq L/2$ we define

$$\mathcal{N}_n = \sum_{\{N_m\}} \sharp\{\text{off-diagonal solutions to the string center eq.(13.11)}\}, \tag{13.13}$$

where the sum is taken over $N_m \in \mathbb{Z}_{\geq 0}$ satisfying $n = \sum_{m\geq 1} mN_m$ as in (13.8). (We understand $\mathcal{N}_0 = 1$.)

Example 13.2. We derive $\mathcal{N}_n = \binom{L}{n}$ for $n = 1, 2$ as an illustration. When $n = 1$, the only possible string pattern (N_m) is $N_m = \delta_{m1}$. The equation (13.11) is just $Lu_{11} \equiv \text{const} \mod \mathbb{Z}$, hence there are $\mathcal{N}_1 = L$ off-diagonal solutions.

For $n = 2$ (hence $L \geq 4$), there are two possible string patterns (i) $N_m = \delta_{m2}$ and (ii) $N_m = 2\delta_{m1}$. In (i), eq.(13.11) is $Lu_{21} \equiv \text{const} \mod \mathbb{Z}$, which again yields L off-diagonal solutions. In (ii), eq.(13.11) reads in the matrix notation as

$$\begin{pmatrix} L-1 & 1 \\ 1 & L-1 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{12} \end{pmatrix} \equiv \vec{c} \qquad \mod \mathbb{Z}^2$$

for some $\vec{c}$. The number of solutions equals the determinant $L(L-2)$ of the coefficient matrix, which is positive by the assumption $L \geq 4$. However, they contain the collision $u_{11} = u_{12}$ L times which should be excluded from the off-diagonal solutions. Thus there are $(L(L-2) - L)/2$ off-diagonal solutions for (ii), where the division by 2 is due to the identification by $\mathfrak{S}_2$. Collecting the contributions from (i) and (ii), one gets $\mathcal{N}_2 = L + (L(L-2) - L)/2 = L(L-1)/2$ as desired.

It is possible to generalize the calculations in Example 13.2 by a systematic application of the inclusion-exclusion principle. The final result reads [243]

$$\begin{aligned} \mathcal{N}_n &= \sum_{\{N_m\}} \det_{m,k \in \mathcal{J}} (F_{m,k}) \prod_{m \in \mathcal{J}} \frac{1}{N_m} \binom{P_m + N_m - 1}{N_m - 1}, \\ F_{m,k} &= \delta_{mk} P_m + 2 \min(m,k) N_k, \end{aligned} \tag{13.14}$$

where $\mathcal{J} = \{j \in \mathbb{Z}_{\geq 1} \mid N_j \geq 1\}$ and P_m is defined by (13.7). Again the sum in (13.14) is taken in the same way as (13.13). As noted before Example 13.1, the assumption $0 \leq n \leq L/2$ implies $P_m \geq 0$ $(m \geq 1)$. By using this property it can be shown that $\det_{m,k \in \mathcal{J}}(F_{m,k}) > 0$ and the RHS of the first equality in (13.14) is a positive integer.

What number should we expect for $\mathcal{N}_n$? Unlike the rational case in the previous subsection, the 6 vertex model with $q \neq 1$ under the periodic boundary condition does not possess the global sl_2-symmetry. Thus for the string solutions (13.10) to be complete, one should have $\mathcal{N}_n = c_n$, where c_n is the weight multiplicity of the quantum space $(V_{\omega_1})^{\otimes L}$ with weight $(L-2n)\omega_1$[30]. Explicitly, $c_n = \binom{L}{n}$. This has been confirmed for $n = 1, 2$ in Example 13.2. The next case is checked as

$$\mathcal{N}_3 = \underset{N_3=1}{L} + \underset{N_1=N_2=1}{\begin{vmatrix} L-2 & 2 \\ 2 & L-2 \end{vmatrix}} + L \, \underset{N_1=3}{\frac{1}{3}\binom{L-6+2}{2}} = \frac{L(L-1)(L-2)}{6}.$$

One may wonder what happens for $n > L/2$ where c_n still makes sense. The answer will be given in the next subsection in a more general setting together with the analogous result for b_n. The only preliminary we mention here is that such considerations necessarily involve the situation $P_m < 0$ hence the binomial coefficients $\binom{X}{N}$ with $X < N$.

13.4. **Theorems for type A_1.** We have hitherto argued about three kinds of quantities

(i) Number of string solutions in the Bethe ansatz,
(ii) Fermionic forms $\mathcal{M}_n$ and $\mathcal{N}_n$,
(iii) Representation theoretical data b_n and c_n,

[30] The same remark as the previous footnote applies here.

especially without a much distinction between (i) and (ii). Here we redefine (ii) without recourse to (i) and formulate the theorems on the relations between (ii) and (iii). We treat the general spin case $\bigotimes_{m\geq 1}(V_{m\omega_1})^{\otimes \nu_m}$ and present the Fermionic character formulas. As power series formulas, they are actually valid for arbitrary $\nu_m \in \mathbb{C}$. The proof of the theorem, which will be outlined in the next subsection, does not lean on the string hypotheses but is solely derived from the Q-system. As such, it does not prove nor disprove the completeness of the string hypothesis.

Let Q_m ($\mathcal{Q}_m$) be the character (normalized character) of the irreducible $m+1$ dimensional representation $V_{m\omega_1}$. Namely,

$$Q_m = \chi(V_{m\omega_1}) = y^m + y^{m-2} + \cdots + y^{-m} = \frac{y^{m+1} - y^{-m-1}}{y - y^{-1}} \quad (y = e^{\omega_1}), \tag{13.15}$$

$$\mathcal{Q}_m = y^{-m} Q_m. \tag{13.16}$$

The Q_m is a simplified notation for the variable $Q^{(1)}_m$ (13.1) in the Q-system for A_1:

$$Q_m^2 = Q_{m-1}Q_{m+1} + 1. \tag{13.17}$$

See (13.41). The Q_m expressed as a function of Q_1 is the Chebyshev polynomial of the second kind. In Section 13.5, we will utilize the one adapted to the normalized character (13.16).

$$\frac{\mathcal{Q}_{m-1}\mathcal{Q}_{m+1}}{\mathcal{Q}_m^2} + y^{-2m}\mathcal{Q}_m^{-2} = 1. \tag{13.18}$$

Let $\nu_m \in \mathbb{C}\,(m \in \mathbb{Z}_{\geq 1})$ be arbitrary except that $\nu_m = 0$ for all but finitely many m. We define the branching coefficient b_n and the weight multiplicity c_n for all $n \in \mathbb{Z}_{\geq 0}$ by

$$\prod_{m\geq 1} (\mathcal{Q}_m)^{\nu_m} = \frac{\sum_{n\geq 0} b_n y^{-2n}}{1 - y^{-2}} = \sum_{n\geq 0} c_n y^{-2n}. \tag{13.19}$$

By the definition, the normalized character $\mathcal{Q}_m$ is a polynomial in y^{-2} with unit constant term. $(\mathcal{Q}_m)^{\nu_m}$ denotes its ν_mth power with unit constant term $1 + \nu_m(\mathcal{Q}_m - 1) + \frac{\nu_m(\nu_m-1)}{2}(\mathcal{Q}_m - 1)^2 + \cdots$, which is a polynomial or a power series in y^{-2} according as $\nu_m \in \mathbb{Z}_{\geq 0}$ or not. When $\nu_m \in \mathbb{Z}_{\geq 0}$ for any $m \geq 1$, this definition of b_n agrees with the one for the branching coefficient of $V_{(\sum_m m\nu_m - 2n)\omega_1}$ in $\bigotimes_{m\geq 1}(V_{m\omega_1})^{\otimes \nu_m}$ for $0 \leq n \leq \sum_m m\nu_m/2$. The above b_n is an extension of this by $b_n = -b_{-n+1+\sum_m m\nu_m}$, which is the skew symmetry under the Weyl group.

As for the Fermionic forms, we redefine $\mathcal{M}_n$ (13.8) and $\mathcal{N}_n$ (13.14) by replacing P_m (13.7) and the binomial coefficient therein with the generalized ones[31]:

$$P_m = \sum_{k\geq 1} \min(m,k)(\nu_k - 2N_k), \tag{13.20}$$

$$\binom{X}{N} = \frac{\prod_{i=1}^{N}(X - i + 1)}{N!} \qquad (X \in \mathbb{C},\, N \in \mathbb{Z}_{\geq 0}). \tag{13.21}$$

The sum over $\{N_m \,|\, m \in \mathbb{Z}_{\geq 1}\}$ is taken in the same way as (13.8) and (13.14). Namely, it is the finite sum over those $N_m \in \mathbb{Z}_{\geq 0}$ satisfying $\sum_{m\geq 1} mN_m = n$. There is *no* condition like $P_m \geq 0$ which does not make sense in the general setting

[31] In Sections 13.2 and 13.3, the symbol $\binom{X}{N}$ was used only for $0 \leq N \leq X$.

$\nu_m \in \mathbb{C}$ under consideration. The generalized binomial (13.21) is non zero except the N points $X = 0, 1, \ldots, N-1$, and appears in the expansion

$$(1-x)^{-\beta-1} = \sum_{N=0}^{\infty} \binom{\beta+N}{N} x^N, \tag{13.22}$$

for any $\beta \in \mathbb{C}$. With these definitions we have

Theorem 13.3 ([241, 243])**.** *The equalities (1) $\mathcal{M}_n = b_n$ and (2) $\mathcal{N}_n = c_n$ hold for all $n \in \mathbb{Z}_{\geq 0}$. Namely, the following power series formulas hold.*

$$\prod_{m\geq 1} (\mathcal{Q}_m)^{\nu_m} = \frac{\sum_{n\geq 0} \mathcal{M}_n y^{-2n}}{1-y^{-2}} = \sum_{n\geq 0} \mathcal{N}_n y^{-2n}. \tag{13.23}$$

The formulas (1) and (2) are due to [241] and [243], respectively. The theorem reproduces the observations in Sections 13.2 and 13.3 in the special case $\nu_m = L\delta_{m1}$ and $0 \leq n \leq L/2$, where $P_m \geq 0$ for any $m \geq 1$ automatically holds. However, even for this simple choice $\nu_m = L\delta_{m1}$, it further claims infinitely many nontrivial identities including $\mathcal{M}_n = 0$ for $n \geq L+2$ and $\mathcal{N}_n = 0$ and $n \geq L+1$.

Example 13.4. Assume that $\nu_m = 0$ for $m \geq 4$. Then LHS of (13.23) is $(1 + y^{-2})^{\nu_1}(1 + y^{-2} + y^{-4})^{\nu_2}(1 + y^{-2} + y^{-4} + y^{-6})^{\nu_3}$. Setting $\gamma_m = \sum_{k=1}^{3} \min(m,k)\nu_k$, we write down $\mathcal{M}_n$ (13.8) and $\mathcal{N}_n$ (13.14) for $n = 1, 2, 3$.

$$\begin{aligned}
&\mathcal{M}_1 = \gamma_1 - 1, \qquad \mathcal{M}_2 = \gamma_2 - 3 + \frac{1}{2}(\gamma_1 - 2)(\gamma_1 - 3),\\
&\mathcal{M}_3 = (\gamma_3 - 5) + (\gamma_1 - 3)(\gamma_2 - 5) + \frac{1}{6}(\gamma_1 - 3)(\gamma_1 - 4)(\gamma_1 - 5),\\
&\mathcal{N}_1 = \gamma_1, \qquad\qquad \mathcal{N}_2 = \gamma_2 + \frac{1}{2}\gamma_1(\gamma_1 - 3),\\
&\mathcal{N}_3 = \gamma_3 + \begin{vmatrix} \gamma_1 - 2 & 2 \\ 2 & \gamma_2 - 2 \end{vmatrix} + \frac{1}{6}\gamma_1(\gamma_1 - 4)(\gamma_1 - 5).
\end{aligned}$$

One can directly check these coefficients in the power series expansions (13.23). For instance in the simplest case $\nu_m = 0$ hence $\gamma_m = 0$ for all $m \geq 1$, all these coefficients vanish except $\mathcal{M}_1 = -1$ as they should.

In the case $\nu_m \in \mathbb{Z}_{\geq 0}\,(m \geq 1)$, P_m in (13.20) can be a nonnegative integer for some $\{N_m\}$. Then it makes sense to introduce the following variant of $\mathcal{M}_n$:

$$\overline{\mathcal{M}}_n = \sum_{\{N_m\}}^{+} \prod_{m\geq 1} \binom{P_m + N_m}{N_m}, \tag{13.24}$$

where P_m and $\binom{X}{N}$ are again specified by (13.20) and (13.21) as for $\mathcal{M}_n$. The only difference from it is that the sum $\sum_{\{N_m\}}^{+}$ extends over those $N_m \in \mathbb{Z}_{\geq 0}$ satisfying $n = \sum_{m\geq 1} mN_m$ with the extra condition $P_m \geq 0$ if $N_m \geq 1$.

Given $\{\nu_m\}$, n and $\{N_m\}$ satisfying $\sum_{m\geq 1} mN_m = n$, let m_0 be the maximal m such that $N_m \geq 1$. Then we have $P_{m_0} = \sum_{k\geq 1} \min(m,k)\nu_k - 2n \leq \sum_{k\geq 1} k\nu_k - 2n$. Thus we see $\overline{\mathcal{M}}_n = 0$ if $n > \frac{1}{2}\sum_{k\geq 1} k\nu_k$.

Theorem 13.5 ([245, 246])**.** *For any $\nu_m \in \mathbb{Z}_{\geq 0}$, the equality $\overline{\mathcal{M}}_n = b_n$ holds for $0 \leq n \leq \frac{1}{2}\sum_{m\geq 1} m\nu_m$.*

As remarked after Theorem 13.3, Theorem 13.5 is equivalent to Theorem 13.3 (1) in the the special case $\nu_m = L\delta_{m1}$ and $0 \le n \le L/2$. In general, they imply that the contributions to $\mathcal{N}_n$ involving $P_m < 0$ cancel out.

Example 13.6. Take $\nu_m = 2\delta_{m3}$ in Example 13.4. Then $(\gamma_1, \gamma_2, \gamma_3) = (2, 4, 6)$. The three terms in $\mathcal{M}_3$ correspond to choosing non zero N_m as $N_3 = 1$, $N_1 = N_2 = 1$ and $N_1 = 3$. The relevant P_m's are $P_3 = 0$, $P_1 = P_2 = -2$ and $P_1 = -4$, respectively. Thus $\overline{\mathcal{M}}_3$ is given by the first term only $\gamma_3 - 5 = 1$. This coincides with $\mathcal{M}_3$ since the other two terms cancel.

13.5. **Multivariable Lagrange inversion.** Here we outline the proof of Theorem 13.3. We describe an essential step of deriving (13.23) from (13.18) in a generalized setting applicable to $\mathfrak{g}$ case [247].

Let H denote a finite index set. Let $w = (w_i)_{i\in H}$ and $v = (v_i)_{i\in H}$ be complex multivariables, and let $G = (G_{ij})_{i,j\in H}$ be a complex square matrix of size $|H|$. We consider a holomorphic map $\mathcal{D} \to \mathbb{C}^H$, $v \mapsto w(v)$ with

$$w_i(v) = v_i \prod_{j\in H} (1 - v_j)^{-G_{ij}}, \tag{13.25}$$

where $\mathcal{D}$ is some neighborhood of $v = 0$ in $\mathbb{C}^H$. The Jacobian $(\partial w/\partial v)(v)$ is 1 at $v = 0$, so that the map $w(v)$ is bijective around $v = w = 0$. Let $v(w)$ be the inverse map around $v = w = 0$. Inverting (13.25), we obtain the following functional equation for $v_i(w)$'s:

$$v_i(w) = w_i \prod_{j\in H} (1 - v_j(w))^{G_{ij}}. \tag{13.26}$$

By introducing new functions

$$\mathcal{Q}_i(w) = 1 - v_i(w), \tag{13.27}$$

the equation (13.26) is written as

$$\mathcal{Q}_i(w) + w_i \prod_{j\in H} \mathcal{Q}_j(w)^{G_{ij}} = 1. \tag{13.28}$$

From now on, we regard (13.28) as equations for a family $(\mathcal{Q}_i(w))_{i\in H}$ of power series of $w = (w_i)_{i\in H}$ with the unit constant terms. The procedure from (13.25) to (13.28) can be reversed, therefore the power series expansion of $\mathcal{Q}_i(w)$ in (13.27) gives the unique family $(\mathcal{Q}_i(w))_{i\in H}$ of power series of w with the unit constant terms which satisfies (13.28).

We define (finite) Q-system to be the following equations for a family $(\mathcal{Q}_i(w))_{i\in H}$ of power series of w with the unit constant terms:

$$\prod_{j\in H} \mathcal{Q}_j(w)^{D_{ij}} + w_i \prod_{j\in H} \mathcal{Q}_j(w)^{G_{ij}} = 1 \quad (i \in H), \tag{13.29}$$

where $D = (D_{ij})_{i,j\in H}$ and $G = (G_{ij})_{i,j\in H}$ are arbitrary complex matrices with $\det D \neq 0$. The equation (13.28), which is the special case of (13.29) with $D = I$ (I: the identity matrix), is called a standard Q-system. By setting $\mathcal{Q}'_i(w) = \prod_{j\in H} \mathcal{Q}_j(w)^{D_{ij}}$, (13.29) is always transformed to the standard one (13.28) with G replaced by $G' = GD^{-1}$ and vice versa. Therefore, the Q-system (13.29) also has the unique solution.

Given the Q-system (13.29) and $\nu = (\nu_i)_{i\in H} \in \mathbb{C}^H$, we define two power series of w,

$$\mathcal{M}^\nu(w) = \sum_N \mathcal{M}(\nu, N) w^N, \quad \mathcal{N}^\nu(w) = \sum_N \mathcal{N}(\nu, N) w^N, \tag{13.30}$$

where $w^N = \prod_{i\in H} w_i^{N_i}$ and the sums run over $N = (N_i)_{i\in H} \in (\mathbb{Z}_{\geq 0})^H$. The coefficients are given by

$$\mathcal{M}(\nu, N) = \prod_{i\in H(N)} \binom{P_i + N_i}{N_i}, \tag{13.31}$$

$$\mathcal{N}(\nu, N) = \Big(\det_{H(N)} F_{ij}\Big) \prod_{i\in H(N)} \frac{1}{N_i} \binom{P_i + N_i - 1}{N_i - 1}, \tag{13.32}$$

where the binomial is defined by (13.21) and we have set $H(N) = \{\, i \in H \mid N_i \neq 0 \,\}$,

$$P_i = P_i(\nu, N) := -\sum_{j\in H} \nu_j (D^{-1})_{ji} - \sum_{j\in H} N_j (GD^{-1})_{ji}, \tag{13.33}$$

$$F_{ij} = F_{ij}(\nu, N) := \delta_{ij} P_j + (GD^{-1})_{ij} N_j. \tag{13.34}$$

$\det_{H(N)}$ is a shorthand notation for $\det_{i,j\in H(N)}$. In (13.31) and (13.32), $\det_\emptyset$ and $\prod_\emptyset$ mean 1, therefore $\mathcal{M}^\nu(w)$ and $\mathcal{N}^\nu(w)$ are power series with the unit constant terms. See [247, section 2] for the convergence radius. Note a similarity to (13.8) and (13.14).

Theorem 13.7 ([247])**.** *Let $(\mathcal{Q}_i(w))_{i\in H}$ be the unique solution of (13.29). For $\nu = (\nu_i)_{i\in H} \in \mathbb{C}^H$, the following formulas are valid:*

$$\prod_{i\in H} \mathcal{Q}_i(w)^{\nu_i} = \frac{\mathcal{M}^\nu(w)}{\mathcal{M}^0(w)} = \mathcal{N}^\nu(w). \tag{13.35}$$

$\mathcal{Q}_i(w)$ itself is obtained by setting $\nu_j = \delta_{ij}$.

Example 13.8. Let $|H| = 1$. Then, (13.29) is an equation for a single power series $Q(w)$,

$$\mathcal{Q}(w)^D + w\mathcal{Q}(w)^G = 1,$$

where $D \neq 0$ and G are complex numbers, Theorem 13.7 tells

$$\mathcal{Q}(w)^\nu = \mathcal{N}^\nu(w) = \frac{\nu}{D} \sum_{N=0}^{\infty} \frac{\Gamma((\nu + NG)/D)(-w)^N}{\Gamma((\nu + NG)/D - N + 1)N!}.$$

This power series formula is well known and have a very long history since Lambert (e.g. [248, pp. 306–307]).

As noted before, the Q-system (13.29) is bijectively transformed to the standard one (13.28). Under the corresponding changes $D \to I$, $\nu_i \to \sum_{j\in H} \nu_j (D^{-1})_{ji}$ and $G \to GD^{-1}$, the quantities (13.33) and (13.34) remain invariant, hence so are $\mathcal{M}(\nu, N)$ and $\mathcal{M}(\nu, N)$. Thus we have only to prove Theorem 13.7 for the standard case $D = I$, where $\mathcal{Q}_i(w)$ is described by (13.25)–(13.27). Therefore Theorem 13.7 follows from

Proposition 13.9 ([247] Proposition 2.8)**.** *Let $v = v(w)$ be the inverse map of (13.25). Let $\mathcal{M}^\nu(w)$ and $\mathcal{N}^\nu(w)$ be those for $D = I$ in (13.33) and (13.34). Then, the power series expansions*

$$\det_H\Bigl(\frac{w_j}{v_i}\frac{\partial v_i}{\partial w_j}(w)\Bigr)\prod_{i\in H}(1-v_i(w))^{\nu_i-1} = \mathcal{M}^\nu(w), \tag{13.36}$$

$$\prod_{i\in H}(1-v_i(w))^{\nu_i} = \mathcal{N}^\nu(w) \tag{13.37}$$

hold around $w = 0$.

This is a particularly nice example of the multivariable Lagrange inversion formula (e.g. [249]), where all the calculations can be carried through by a multivariable residue analysis.

Proof. The first formula (13.36). We evaluate the coefficient for w^N in the LHS of (13.36) as follows:

$$\begin{aligned}
&\mathrm{Res}_{w=0}\frac{\partial v}{\partial w}(w)\prod_{i\in H}\Bigl\{(1-v_i(w))^{\nu_i-1}(v_i(w))^{-1}(w_i)^{1-N_i-1}\Bigr\}dw\\
=&\,\mathrm{Res}_{v=0}\prod_{i\in H}\Bigl\{(1-v_i)^{\nu_i-1}(v_i)^{-1}\Bigl(v_i\prod_{j\in H}(1-v_j)^{-G_{ij}}\Bigr)^{-N_i}\Bigr\}dv\\
=&\,\mathrm{Res}_{v=0}\prod_{i\in H}\Bigl\{(1-v_i)^{-P_i(\nu,N)-1}(v_i)^{-N_i-1}\Bigr\}dv = \prod_{i\in H}\binom{P_i(\nu,N)+N_i}{N_i} = \mathcal{M}(\nu,N),
\end{aligned}$$

where we used (13.22) to get the last line. Thus, (13.36) is proved.

The second formula (13.37). By a simple calculation, we have

$$\det_H\Bigl(\frac{v_j}{w_i}\frac{\partial w_i}{\partial v_j}(v)\Bigr)\prod_{i\in H}(1-v_i) = \det_H\Bigl(\delta_{ij}+(-\delta_{ij}+G_{ij})v_i\Bigr) = \sum_{J\subset H}d_J\prod_{i\in J}v_i, \tag{13.38}$$

where $d_J := \det_J(-\delta_{ij}+G_{ij})$, and the sum is taken over all the subsets J of H. Therefore, the LHS of (13.37) is written as

$$\det_H\Bigl(\frac{w_j}{v_i}\frac{\partial v_i}{\partial w_j}(w)\Bigr)\sum_{J\subset H}d_J\prod_{i\in H}\Bigl\{(1-v_i(w))^{\nu_i-1}v_i(w)^{\theta(i\in J)}\Bigr\}. \tag{13.39}$$

By a similar residue calculation as above, the coefficient for w^N of (13.39) is evaluated as ($\theta(\text{true}) = 1$ and $\theta(\text{false}) = 0$)

$$\begin{aligned}
&\sum_{J\subset H}d_J\mathrm{Res}_{v=0}\prod_{i\in H}\Bigl\{(1-v_i)^{-P_i(\nu,N)-1}(v_i)^{-N_i+\theta(i\in J)-1}\Bigr\}dv\\
=&\sum_{J\subset H(N)}d_J\prod_{i\in H(N)}\binom{P_i(\nu,N)+N_i-\theta(i\in J)}{N_i-\theta(i\in J)}\\
=&\Bigl(\sum_{J\subset H(N)}d_J\prod_{i\in J}N_i\prod_{i\in H(N)\setminus J}(P_i+N_i)\Bigr)\prod_{i\in H(N)}\frac{1}{N_i}\binom{P_i+N_i-1}{N_i-1}\\
=&\det_{H(N)}\Bigl(\delta_{ij}(P_j+N_j)+(-\delta_{ij}+G_{ij})N_j\Bigr)\prod_{i\in H(N)}\frac{1}{N_i}\binom{P_i+N_i-1}{N_i-1}\\
=&\,\mathcal{N}(\nu,N).
\end{aligned}$$

This completes the proof of Theorem 13.7. What is left to prove Theorem 13.3 from it? Comparing the Q-systems (13.29) and (13.18) and also P_m in (13.33) and (13.20), we see that Theorem 13.3 formally corresponds to taking

$$\begin{aligned}
&H = \mathbb{Z}_{\geq 1}, \quad w_i = y^{-2i}, \\
&(D^{-1})_{ij} = -\min(i,j), \quad D_{ij} = \delta_{i,j+1} + \delta_{i,j-1} - 2\delta_{ij}, \quad G_{ij} = -2\delta_{ij}
\end{aligned} \tag{13.40}$$

in Theorem 13.7, and claiming $\mathcal{M}^0(w) = 1 - y^{-2}$ thereunder. Since we started with the assumption that H is a finite set, it is nontrivial how to make sense of these choices and claims. We refer to [247] for a proper treatment of such an infinite ($|H| = \infty$) Q-system as a projective limit of the finite Q-systems. According a result therein, Theorem 13.3 is shown, among other things, from the convergence property: the limit $\lim_{m\to\infty} \mathcal{Q}_m(w_i = y^{-2i})$ exists in $\mathbb{C}[[y^{-2}]]$.

13.6. **Q-system and theorems for $\mathfrak{g}$.** Here we present the Q-system and analogue of Theorem 13.3 and Theorem 13.5 for general $\mathfrak{g}$. We use the notations in Section 2.1 such as I, t, t_a, $C = (C_{ab})$, α_a and ω_a. The unrestricted Q-system for $\mathfrak{g}$ is the following relations among the variables $\{Q^{(a)}_m \mid a \in I, m \geq 1\}$, where $Q^{(0)}_m = Q^{(a)}_0 = 1$ if they occur in the RHS.

For simply laced $\mathfrak{g}$,

$$(Q^{(a)}_m)^2 = Q^{(a)}_{m-1}Q^{(a)}_{m+1} + \prod_{b\in I: C_{ab}=-1} Q^{(b)}_m. \tag{13.41}$$

For $\mathfrak{g} = B_r$,

$$\begin{aligned}
(Q^{(a)}_m)^2 &= Q^{(a)}_{m-1}Q^{(a)}_{m+1} + Q^{(a-1)}_m Q^{(a+1)}_m \quad (1 \leq a \leq r-2), \\
(Q^{(r-1)}_m)^2 &= Q^{(r-1)}_{m-1}Q^{(r-1)}_{m+1} + Q^{(r-2)}_m Q^{(r)}_{2m}, \\
(Q^{(r)}_{2m})^2 &= Q^{(r)}_{2m-1}Q^{(r)}_{2m+1} + (Q^{(r-1)}_m)^2, \\
(Q^{(r)}_{2m+1})^2 &= Q^{(r)}_{2m}Q^{(r)}_{2m+2} + Q^{(r-1)}_m Q^{(r-1)}_{m+1}.
\end{aligned} \tag{13.42}$$

For $\mathfrak{g} = C_r$,

$$\begin{aligned}
(Q^{(a)}_m)^2 &= Q^{(a)}_{m-1}Q^{(a)}_{m+1} + Q^{(a-1)}_m Q^{(a+1)}_m \quad (1 \leq a \leq r-2), \\
(Q^{(r-1)}_{2m})^2 &= Q^{(r-1)}_{2m-1}Q^{(r-1)}_{2m+1} + Q^{(r-2)}_{2m}(Q^{(r)}_m)^2, \\
(Q^{(r-1)}_{2m+1})^2 &= Q^{(r-1)}_{2m}Q^{(r-1)}_{2m+2} + Q^{(r-2)}_{2m+1}Q^{(r)}_m Q^{(r)}_{m+1}, \\
(Q^{(r)}_m)^2 &= Q^{(r)}_{m-1}Q^{(r)}_{m+1} + Q^{(r-1)}_{2m}.
\end{aligned} \tag{13.43}$$

For $\mathfrak{g} = F_4$,

$$\begin{aligned}
(Q^{(1)}_m)^2 &= Q^{(1)}_{m-1}Q^{(1)}_{m+1} + Q^{(2)}_m, \\
(Q^{(2)}_m)^2 &= Q^{(2)}_{m-1}Q^{(2)}_{m+1} + Q^{(1)}_m Q^{(3)}_{2m}, \\
(Q^{(3)}_{2m})^2 &= Q^{(3)}_{2m-1}Q^{(3)}_{2m+1} + (Q^{(2)}_m)^2 Q^{(4)}_{2m}, \\
(Q^{(3)}_{2m+1})^2 &= Q^{(3)}_{2m}Q^{(3)}_{2m+2} + Q^{(2)}_m Q^{(2)}_{m+1} Q^{(4)}_{2m+1}, \\
(Q^{(4)}_m)^2 &= Q^{(4)}_{m-1}Q^{(4)}_{m+1} + Q^{(3)}_m.
\end{aligned}$$

For $\mathfrak{g} = G_2$,

$$\begin{aligned}
(Q^{(1)}_m)^2 &= Q^{(1)}_{m-1}Q^{(1)}_{m+1} + Q^{(2)}_{3m},\\
(Q^{(2)}_{3m})^2 &= Q^{(2)}_{3m-1}Q^{(2)}_{3m+1} + (Q^{(1)}_m)^3,\\
(Q^{(2)}_{3m+1})^2 &= Q^{(2)}_{3m}Q^{(2)}_{3m+2} + (Q^{(1)}_m)^2Q^{(1)}_{m+1},\\
(Q^{(2)}_{3m+2})^2 &= Q^{(2)}_{3m+1}Q^{(2)}_{3m+3} + Q^{(1)}_m(Q^{(1)}_{m+1})^2.
\end{aligned} \tag{13.44}$$

These relations are uniformly written as

$$(Q^{(a)}_m)^2 = Q^{(a)}_{m-1}Q^{(a)}_{m+1} + (Q^{(a)}_m)^2 \prod_{(b,k)\in H}(Q^{(b)}_k)^{G_{am,bk}}, \tag{13.45}$$

by using the notations (13.48) and (13.51). We shall introduce the restricted Q-system in Section 14.5.

As mentioned around (13.1), these relations follow from the T-systems by forgetting the spectral parameter u. Recall that $\mathrm{res}\,\chi_q(W^{(a)}_m(u))$ denotes the classical character of the Kirillov-Reshetikhin module $W^{(a)}_m(u)$. See (4.24) for the definition of res. Since res removes the dependence on u, we will simply write as $\mathrm{res}\,\chi_q(W^{(a)}_m)$ in the sequel. The following is a corollary of Theorem 4.8.

Proposition 13.10. *The substitution $Q^{(a)}_m = \mathrm{res}\,\chi_q(W^{(a)}_m)$ satisfies the unrestricted Q-system.*

From now on, we understand the symbol $Q^{(a)}_m$ as representing $\mathrm{res}\,\chi_q(W^{(a)}_m)$. By Theorem 4.6 (1), the normalized character

$$\mathcal{Q}^{(a)}_m = e^{-m\omega_a}Q^{(a)}_m \tag{13.46}$$

is a polynomial in $e^{-\alpha_1},\dots,e^{-\alpha_r}$ with unit constant term and coefficients from $\mathbb{Z}_{\geq 0}$. In terms of $\mathcal{Q}^{(a)}_m$, the Q-system is expressed as

$$\prod_{(b,k)\in H}(\mathcal{Q}^{(b)}_k)^{D_{am,bk}} + e^{-m\alpha_a}\prod_{(b,k)\in H}(\mathcal{Q}^{(b)}_k)^{G_{am,bk}} = 1 \tag{13.47}$$

for $(a,m)\in H$. Here H, $D_{am,bk}$ and $G_{am,bk}$ are defined by

$$H = \{(a,m)|\, a\in I, m\in\mathbb{Z}_{\geq 1}\}, \tag{13.48}$$

$$D_{am,bk} = -\delta_{ab}(2\delta_{mk} - \delta_{m,k+1} - \delta_{m,k-1}), \tag{13.49}$$

$$(D^{-1})_{am,bk} = -\delta_{ab}\min(m,k). \tag{13.50}$$

$$G_{am,bk} = \begin{cases} -C_{ba}(\delta_{m,2k-1} + 2\delta_{m,2k} + \delta_{m,2k+1}) & t_a/t_b = 2,\\ -C_{ba}(\delta_{m,3k-2} + 2\delta_{m,3k-1} + 3\delta_{m,3k} & t_a/t_b = 3,\\ \qquad +2\delta_{m,3k+1} + \delta_{m,3k+2}) & \\ -C_{ab}\delta_{t_b m, t_a k} & \text{otherwise}. \end{cases} \tag{13.51}$$

For $\mathfrak{g} = A_1$, the data H, D, G here reduce to (13.40) hence (13.47) to (13.18). By an analysis parallel with A_1 case, one can establish the power series formulas involving Fermionic forms. They are read off (13.30)–(13.34) by formally replacing the single indices by double ones as $i \to (a,m)$, $j \to (b,k)$, etc. To be concrete, let $\nu = (\nu^{(a)}_m)_{(a,m)\in H} \in \mathbb{C}^H$, where $\nu^{(a)}_m = 0$ for all but finitely many (a,m). For

$N = (N_m^{(a)})_{(a,m)\in H} \in (\mathbb{Z}_{\geq 0})^H$, we define

$$\mathcal{M}(\nu, N) = \prod_{(a,m)\in H(N)} \binom{P_m^{(a)} + N_m^{(a)}}{N_m^{(a)}}, \tag{13.52}$$

$$\mathcal{N}(\nu, N) = \Bigl(\det_{H(N)} F_{am,bk}\Bigr) \prod_{(a,m)\in H(N)} \frac{1}{N_m^{(a)}} \binom{P_m^{(a)} + N_m^{(a)} - 1}{N_m^{(a)} - 1}, \tag{13.53}$$

where the binomial is the generalized one (13.21). We have set $H(N) = \{\,(a,m) \in H \mid N_m^{(a)} \neq 0\,\}$ and $\det_{H(N)}$ denotes $\det_{(a,m),(b,k)\in H(N)}$. Define further

$$P_m^{(a)} = \sum_{(b,k)\in H} \min(m,k)\nu_k^{(b)} - \sum_{(b,k)\in H} (\alpha_a|\alpha_b) \min(t_b m, t_a k) N_k^{(b)}, \tag{13.54}$$

$$F_{am,bk} = \delta_{ab}\delta_{mk} P_m^{(a)} + (\alpha_a|\alpha_b) \min(t_b m, t_a k) N_k^{(b)}. \tag{13.55}$$

With these definitions we have

Theorem 13.11 ([81, 250, 80, 247, 68])**.** *The following power series formulas are valid:*

$$\begin{aligned}\prod_{(a,m)\in H} (\mathcal{Q}_m^{(a)})^{\nu_m^{(a)}} &= \frac{\sum_N \mathcal{M}(\nu, N) e^{-\sum_{(a,m)\in H} m N_m^{(a)} \alpha_a}}{\prod_{\alpha\in\Delta_+}(1 - e^{-\alpha})} \\ &= \sum_N \mathcal{N}(\nu, N) e^{-\sum_{(a,m)\in H} m N_m^{(a)} \alpha_a},\end{aligned} \tag{13.56}$$

where the sums run over $N = (N_m^{(a)})_{(a,m)\in H} \in (\mathbb{Z}_{\geq 0})^H$ without any constraints. The symbol Δ_+ denotes the set of positive roots of $\mathfrak{g}$.

See Section 13.8 how this theorem was established by integrating many works.

Let us turn to the special case $\nu_m^{(a)} \in \mathbb{Z}_{\geq 0}$ for any $(a,m) \in H$. Then the power series (13.56) actually truncates to a polynomial, and Theorem 13.11 implies the Fermionic formulas for the branching coefficient b_λ and the weight multiplicity c_λ in (13.2). To write them down, we introduce

$$\mathcal{M}_\lambda = \sum_N \mathcal{M}(\nu, N), \quad \mathcal{N}_\lambda = \sum_N \mathcal{N}(\nu, N) \qquad (\lambda \in \sum_{a=1}^r \mathbb{Z}\omega_a), \tag{13.57}$$

where the sums run over $N = (N_m^{(a)})_{(a,m)\in H} \in (\mathbb{Z}_{\geq 0})^H$ satisfying the weight condition

$$\lambda = \sum_{(a,m)\in H} m\nu_m^{(a)}\omega_a - \sum_{(a,m)\in H} m N_m^{(a)} \alpha_a. \tag{13.58}$$

Then the following is a corollary of Theorem 13.11.

$$\prod_{a,m} (Q_m^{(a)})^{\nu_m^{(a)}} = \sum_\lambda b_\lambda\, \chi(V_\lambda), \quad b_\lambda = \mathcal{M}_\lambda \qquad \text{for } \lambda \in \sum_{a=1}^r \mathbb{Z}_{\geq 0}\, \omega_a, \tag{13.59}$$

$$\prod_{a,m} (Q_m^{(a)})^{\nu_m^{(a)}} = \sum_\lambda c_\lambda\, e^\lambda, \qquad c_\lambda = \mathcal{N}_\lambda \qquad \text{for } \lambda \in \sum_{a=1}^r \mathbb{Z}\, \omega_a. \tag{13.60}$$

As the generalization of (13.24), we further introduce

$$\overline{\mathcal{M}}_\lambda = \sum_N{}^+ \mathcal{M}(\nu, N), \tag{13.61}$$

where the sum $\sum_N^+$ extends over $N = (N_m^{(a)})_{(a,m)\in H} \in (\mathbb{Z}_{\geq 0})^H$ satisfying (13.58) and the extra condition that $P_m \geq 0$ whenever $N_m \geq 1$. Then the following is the $\mathfrak{g}$ version of Theorem 13.5.

Theorem 13.12 ([251, 245, 246, 252])**.** *For $\lambda \in \sum_{a=1}^r \mathbb{Z}_{\geq 0}\,\omega_a$, the equality $b_\lambda = \overline{\mathcal{M}}_\lambda$ is valid.*

13.7. **$Q_m^{(a)}$ as a classical character.** Here we present the expansion of $Q_m^{(a)}$ into classical characters. Such an example has already been given in (4.25) for the rank 2 algebras $\mathfrak{g} = A_2, B_2, C_2$ and G_2. Here are a few examples from E_8:

$$\begin{aligned}
Q_1^{(1)} &= \chi(V_{\omega_1}) + \chi(V_0), \qquad Q_2^{(1)} = \chi(V_{2\omega_1}) + \chi(V_{\omega_1}) + \chi(V_0),\\
Q_1^{(2)} &= \chi(V_{\omega_2}) + 2\chi(V_{\omega_1}) + \chi(V_{\omega_7}) + \chi(V_0),\\
Q_1^{(3)} &= \chi(V_{\omega_3}) + 2\chi(V_{\omega_8}) + 4\chi(V_{\omega_7}) + \chi(V_{\omega_1+\omega_7})\\
&\quad + 3\chi(V_{\omega_2}) + \chi(V_{2\omega_1}) + 4\chi(V_{\omega_1}) + 2\chi(V_0),
\end{aligned}$$

which satisfy a Q-system relation $(Q_1^{(1)})^2 = Q_2^{(1)} + Q_1^{(2)}$ for instance. In general from (13.59) and (13.58), the expansion takes the form

$$Q_m^{(a)} = \chi(V_{m\omega_a}) + \overbrace{\sum_{\lambda < m\omega_a} b_\lambda \chi(V_\lambda)}^{\text{called "children"}}, \tag{13.62}$$

where b_λ is obtained by specializing $\nu_m^{(a)}$ in (13.59) or Theorem 13.12. As we see in the above example, the description of the children is complicated in general for $\mathfrak{g}$ of exceptional types. However, for non exceptional $\mathfrak{g}$, they can be described by simple combinatorial rules given below. For simplicity we write $\chi(V_\lambda)$ as $\chi(\lambda)$.

For $\mathfrak{g} = A_r$, there is no children.

$$Q_m^{(a)} = \chi(m\omega_a). \tag{13.63}$$

To check the relation $(Q_m^{(a)})^2 = Q_{m-1}^{(a)}Q_{m+1}^{(a)} + Q_m^{(a-1)}Q_m^{(a+1)}$ is an easy exercise on Schur functions. It is customary to depict the weights $m_1\omega_1 + \cdots + m_r\omega_r$ ($m_i \in \mathbb{Z}_{\geq 0}$) as a Young diagram. The rule is to regard each ω_a as a depth a column. Thus (13.63) is represented as the $a \times m$ rectangle Young diagram. As we will see, in the other non exceptional algebras, the children for most $Q_m^{(a)}$ are described by removals of dominos from the $a \times m$ rectangle.

For $\mathfrak{g} = C_r$, we have

$$Q_m^{(a)} = \begin{cases} \chi(k_1\omega_1 + \cdots + k_a\omega_a) & 1 \leq a \leq r-1,\\ \chi(m\omega_r) & a = r, \end{cases} \tag{13.64}$$

where the sum is taken over nonnegative integers $k_1, \ldots, k_a$ that satisfy $k_1 + \cdots + k_a \leq m, k_j \equiv m\delta_{ja} \mod 2$ for all $1 \leq j \leq a$. The summands correspond to the removals of horizontal dominos (shape 1×2 Young diagram).

For $\mathfrak{g} = B_r$ and D_r, we have

$$\begin{aligned}
Q_m^{(a)} &= \sum \chi(k_{a_0}\omega_{a_0} + \cdots + k_{a-2}\omega_{a-2} + \cdots + k_a\omega_a) \quad 1 \leq a \leq r',\\
&\quad r' = r \text{ for } B_r,\ r' = r-2 \text{ for } D_r, \quad a_0 \equiv a \mod 2, \ \ a_0 = 0 \text{ or } 1,\\
Q_m^{(a)} &= \chi(m\omega_a) \quad a = r-1, r \quad \text{for } D_r.
\end{aligned} \tag{13.65}$$

Here $\omega_0 = 0$. The sum extends over non-negative integers $k_{a_0}, k_{a_0+2}, \ldots, k_a$ obeying the constraint $t_a(k_{a_0} + k_{a_0+2} + \cdots + k_{a-2}) + k_a = m$. The summands correspond to the removals of vertical dominos (shape 2×1 Young diagram).

13.8. **Bibliographical notes and further aspects.** The Q-system[32] for $\mathfrak{g}$ first appeared in [81, 92]. In [81], it was claimed that (in a nowadays terminology) $Q^{(a)}_m = \mathrm{res}\chi_q(W^{(a)}_m)$ satisfies the Q-system, and the generalization of Bethe's Fermionic formula $b_\lambda = \overline{\mathcal{M}}_\lambda$ (Theorem 13.12) holds. These assertions became known as the Kirillov-Reshetikhin conjecture. Together with the closely related formulas $b_\lambda = \mathcal{M}_\lambda$, $c_\lambda = \mathcal{N}_\lambda$ and Theorem 13.11, they have now been established by the integration of numerous works since then. Here we shall only mention the literatures that are most relevant to our presentation in this section. More detailed accounts are available in [247, section 5.7] and [12, section 1].

The method of multivariable residue analysis was initiated in [241, 253] for A_1, A_r and extended to $\mathfrak{g}$ in [250]. The main conclusion from this approach is that the Fermionic formula $b_\lambda = \mathcal{M}_\lambda$ follows from the Q-system and a convergence property of $\mathcal{Q}^{(a)}_m$ as $m \to \infty$. It was found in [243, 80] that these properties also lead to another version of the Fermionic formula $c_\lambda = \mathcal{N}_\lambda$. The two stories $b_\lambda = \mathcal{M}_\lambda$ ("XXX type") and $c_\lambda = \mathcal{N}_\lambda$ ("XXZ type") were put in a unified perspective by a version of multivariable Lagrange inversion [247] with a proper passage from the finite to infinite Q-systems. Last but a crucial input that $\mathrm{res}\,\chi_q(W^{(a)}_m)$ actually satisfies the Q-system for any $\mathfrak{g}$ was proved as a corollary of Theorem 4.8 [67, 68] together with the convergence property [68, Theorem 3.3(2)]. Thus, Theorem 13.3 (1) and (2) for A_1 are due to [241] and [243], respectively. Its $\mathfrak{g}$ version, Theorem 13.11, is an outcome of [81, 250, 80, 247, 68].

The identity $b_\lambda = \overline{\mathcal{M}}_\lambda$ (Theorem 13.12) has been proved by combinatorial methods in [251, 245, 246] for A_r. Thanks to $b_\lambda = \mathcal{M}_\lambda$, it suffice to show $\overline{\mathcal{M}}_\lambda = \mathcal{M}_\lambda$ for dominant λ. A uniform proof of the latter for all $\mathfrak{g}$ is given in [252] by a generating function method.

The expansion of $Q^{(a)}_m$ into classical characters as in Section 13.7 also has a long history going back to [147]. By many works e.g., [82, 65, 86, 250], such formulas have been established for all $Q^{(a)}_m$'s for A_r, B_r, C_r, D_r and many ones from $E_{6,7,8}, F_4$ and G_2.

We close with a few remarks on further aspects which have not been discussed in this section.

(i) The series $\mathcal{M}(w)$ (13.30) has an interpretation of the grand partition function of the ideal gas with the Haldane exclusion statistics [254]. The finite Q-system (13.29) appeared in [254] as the thermal equilibrium condition for the distribution functions of the same system[33]. See also [255] for another interpretation. The one variable case (Example 13.8) also appeared in [256] as the thermal equilibrium condition for the distribution function of the Calogero-Sutherland model. As an application of our second formula in Theorem 13.7, we can quickly reproduce the "cluster expansion formula" in [257, eq. (129)]. Setting $D = I$ in (13.30)–(13.34),

[32]They are named so in [1] after the notation $Q^{(a)}_m$ due to [81, 92], which was adopted to mean "quantum character". (A. N. Kirillov, private communication.)

[33]For the translation, substitute $w_i = \mathcal{Q}_i/(1-\mathcal{Q}_i)$ in eq. (10) in [254].

we have

$$\begin{aligned}\ln \mathcal{Q}_i(w) &= \Big[\frac{\partial}{\partial \nu_i}\mathcal{N}^{\nu}(w)\Big]_{\nu=0}\\ &= \sum_N \det_{\substack{H(N)\\ j,k\neq i}} F_{jk}(0,N) \prod_{j\in H(N)} \frac{1}{N_j}\binom{P_j(0,N)+N_j-1}{N_j-1} w^N,\end{aligned} \tag{13.66}$$

where $\{Q_i(w)\}_{i\in H}$ is the solution of (13.28). The Sutherland-Wu equation also plays an important role for the CFT spectra. See [258] and the references therein.

(ii) There are decent q-analogues of b_λ and c_λ by using the crystal base of $U_q(\hat{\mathfrak{g}})$ [259]. A typical one for A_r is the Kostka-Foulkes polynomial [135]. Correspondingly, there is a q-analogue of the Fermionic formula $b_\lambda = \overline{\mathcal{M}}_\lambda$ known as "$X = M$ conjecture" [250, 260], which has been solved for A_r [245, 246] and some other cases. There is also a conjectural q-analogue of $\overline{\mathcal{M}}_\lambda = \mathcal{M}_\lambda$ [250, eq. (4.21)]. These formulas have the level restricted versions and are related to RSOS models and CFT characters. For a historical survey, see [260, section1] and [261].

(iii) The Q-system, Theorem 13.11 and the expansion formula as in Section 13.7 have been generalized to twisted quantum affine Lie algebras $U_q(X_N^{(\kappa)})$ [260, 247, 262, 12].

14. Y-system and thermodynamic Bethe ansatz

In this section we explain how the level ℓ restricted Y-system for $\mathfrak{g}$ (2.11)–(2.15) emerges from the thermodynamic Bethe ansatz (TBA) equation associated with $U_q(\hat{\mathfrak{g}})$ at $q = \exp(\frac{\pi\sqrt{-1}}{t(\ell+h^\vee)})$. (See (2.1) and (2.3) for t and $h^\vee$.) The TBA equation is relevant to level ℓ RSOS models and quoted from Section 15. We also introduce the constant Y-system and explain its relation to the Q-system in the both unrestricted and level restricted versions. Conjecturally, the level restricted Q-system allows a solution via a specialization of characters to the q-dimension with q being the root of unity. They play important roles in the dilogarithm identity related to conformal field theory and the TBA analysis of RSOS models. We use the notation

$$\ell_a = t_a\ell, \quad L = \ell + h^\vee, \tag{14.1}$$

$$H_\ell = \{(a,m) \mid a\in I, 1\le m\le \ell_a - 1, m\in\mathbb{Z}\}, \tag{14.2}$$

where t_a is defined in (2.1) and $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$ (2.3). The set H_ℓ is the level truncation of H (13.48). We will further use

$$t_{ab} = \max(t_a, t_b), \tag{14.3}$$

$$N_{ab} = 2\delta_{ab} - B_{ab}, \quad B_{ab} = B_{ba} = \frac{t_b}{t_{ab}}C_{ab} = \begin{cases} 2 & C_{ab}=2,\\ -1 & C_{ab}<0,\\ 0 & C_{ab}=0.\end{cases} \tag{14.4}$$

This B_{ab} is the same as (12.11).

14.1. Y-system for ADE and deformed Cartan matrices.

For simplicity we first deal with the simply laced algebras $\mathfrak{g} = A_r, D_r$ and $E_{6,7,8}$. In Section 15.1, we obtain the TBA equation for level ℓ ($\ell\in\mathbb{Z}_{\ge 2}$) critical RSOS model in (15.14). It is the following nonlinear integral equation on the functions $\{\epsilon^{(a)}_m(u) \mid (a,m)\in$

$H_\ell, u \in \mathbb{R}\}$:

$$\frac{\epsilon\beta\gamma\delta_{pa}\delta_{sm}}{4\cosh(\pi u/2)} = \beta\epsilon^{(a)}_m(u) + \int_{-\infty}^{\infty} dv \frac{\ln\left[\frac{\prod_{b\in I}\left(1+\exp(-\beta\epsilon^{(b)}_m(v))\right)^{N_{ab}}}{\left(1+\exp(\beta\epsilon^{(a)}_{m-1}(v))\right)\left(1+\exp(\beta\epsilon^{(a)}_{m+1}(v))\right)}\right]}{4\cosh(\pi(u-v)/2)}. \quad (14.5)$$

Here $\beta, \gamma > 0$, $\epsilon = \pm 1$ and $(p, s) \in H_\ell$ are model parameters specifying the temperature, normalization of energy, two critical regimes and representation $W^{(p)}_s$ (fusion type) with which the model is associated, respectively. The physical meaning of $\epsilon^{(a)}_m(u)$ is the pseudo energy defined by $\exp(-\beta\epsilon^{(a)}_m(u)) = \rho^{(a)}_m(u)/\sigma^{(a)}_m(u)$ in terms of the color a length m string density $\rho^{(a)}_m(u)$ and hole density $\sigma^{(a)}_m(u)$. More details can be found in Section 15.1, but we do not need those background here.

We assume that (14.5) can be analytically continued off the real axis of u until $|\Im\mathrm{m}\, u| \le 1$. Setting $u \to u \pm i \mp 0i$, take the sum of the resulting two equations. The LHS vanishes and the RHS is evaluated by means of

$$\frac{1}{4\cosh\frac{\pi}{2}(u-v+i-0i)} + \frac{1}{4\cosh\frac{\pi}{2}(u-v-i+0i)} = \delta(u-v) \quad (14.6)$$

as the convolution kernel. By introducing the variable $Y^{(a)}_m(u) = \exp(-\beta\epsilon^{(a)}_m(u))$, the Boltzmann factor of the pseudo energy, the result is the logarithm of

$$Y^{(a)}_m(u-i)Y^{(a)}_m(u+i) = \frac{\prod_{b\in I}(1+Y^{(b)}_m(u))^{N_{ab}}}{(1+Y^{(a)}_{m-1}(u)^{-1})(1+Y^{(a)}_{m+1}(u)^{-1})}. \quad (14.7)$$

This is the Y-system for $\mathfrak{g} = A_r, D_r$ and $E_{6,7,8}$ (2.11) in the convention that $Y^{(a)}_m(u+k)$ there becomes $Y^{(a)}_m(u+ik)$. It is level ℓ restricted since only $Y^{(a)}_m(u)$ with $(a, m) \in H_\ell$ are present.

Notice that the LHS of (14.5) that had carried the model dependent informations β, γ, ϵ and (p, s) disappeared all together. In this sense, the Y-system is a universal feature of all the physical systems described by the TBA equation (14.5) whose LHS is any $2i$-antiperiodic function of u. Put it differently, the LHS encodes the specific properties in each model that are coupled as a driving term to the universal structure (Y-system).

Let us observe another aspect of the Y-system (14.7). It is written as

$$\frac{(1+Y^{(a)}_m(u-i)^{-1})(1+Y^{(a)}_m(u+i)^{-1})}{(1+Y^{(a)}_{m-1}(u)^{-1})(1+Y^{(a)}_{m+1}(u)^{-1})} = \frac{(1+Y^{(a)}_m(u-i))(1+Y^{(a)}_m(u+i))}{\prod_{b\in I}(1+Y^{(b)}_m(u))^{N_{ab}}}. \quad (14.8)$$

The LHS and RHS of (14.7) possess parallel structures related to $A_{\ell-1}$ and $\mathfrak{g}$, respectively. In the Fourier space they are encoded in the deformed Cartan matrices with indices corresponding to the length m and the color a, respectively. To see it, define the Fourier transformation $\hat{f} = \hat{f}(x)$ of $f = f(u)$ by

$$f(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(x)e^{iux}dx, \quad \hat{f}(x) = \int_{-\infty}^{\infty} f(u)e^{-iux}du. \quad (14.9)$$

If we formally interpret the multiplication with e^{cx} ($c \in \mathbb{R}$) in the Fourier (x) space as the difference operator $u \to u - ic$ in the "real" (u) space, the logarithm of the RHS of (14.8) is assigned with the Fourier component $\sum_{b\in I}\hat{\mathcal{M}}_{ab}(x)\widehat{\ln}\big(1+Y^{(b)}_m\big)$, where

$$\hat{\mathcal{M}}_{ab}(x) = 2\delta_{ab}\cosh x - N_{ab} \quad \text{(for ADE)} \tag{14.10}$$

is the deformed Cartan matrix of $\mathfrak{g}$. Actually the Fourier transformation of the TBA equation (14.5) contains $\sum_{b\in I}\frac{\hat{\mathcal{M}}_{ab}(x)}{2\cosh x}\widehat{\ln}\big(1+Y_m^{(b)}\big)$ so that the identity (14.6) works in the real space. Parallel remarks apply to the LHS of (14.8).

We call the functions like $\hat{\mathcal{M}}_{ab}(x)$ TBA kernels as they emerge in the TBA calculation (Section 15.1) and play important roles as building blocks of integral kernels in the TBA equation.

14.2. **TBA kernels.** Here we summarize the definitions and useful properties of the TBA kernels for general $\mathfrak{g}$. In place of (14.10), we redefine $\hat{\mathcal{M}}_{ab}(x)$ and introduce $\hat{\mathcal{K}}_a^{mn}(x)$ as

$$\hat{\mathcal{M}}_{ab}(x) = 2\delta_{ab}\cosh(\frac{x}{t_a}) - N_{ab} = B_{ab} + 2\delta_{ab}\Big(\cosh(\frac{x}{t_a}) - 1\Big), \tag{14.11}$$

$$\hat{\mathcal{K}}_a^{mn}(x) = \delta_{mn} - \frac{\delta_{m,n-1}+\delta_{m,n+1}}{2\cosh\big(\frac{x}{t_a}\big)}. \tag{14.12}$$

For $(a,m),(b,k)\in H_\ell$, we further introduce

$$\hat{\mathcal{A}}_{ab}^{mk}(x) = \frac{\sinh\big(\min(\frac{m}{t_a},\frac{k}{t_b})x\big)\sinh\big((\ell-\max(\frac{m}{t_a},\frac{k}{t_b}))x\big)}{\sinh(\frac{x}{t_{ab}})\sinh(\ell x)}, \tag{14.13}$$

$$\hat{\mathcal{K}}_{ab}^{mk}(x) = \hat{\mathcal{A}}_{ab}^{mk}(x)\hat{\mathcal{M}}_{ab}(x), \tag{14.14}$$

$$\hat{\mathcal{J}}_{ab}^{mk}(x) = \sum_{n=1}^{\ell_a-1}\hat{\mathcal{K}}_a^{mn}(x)\hat{\mathcal{K}}_{ab}^{nk}(x) = \frac{\hat{\mathcal{M}}_{ab}(x)\hat{\mathcal{P}}_{ab}^{mk}(x)}{2\cosh\big(\frac{x}{t_a}\big)}, \tag{14.15}$$

$$\begin{aligned}\hat{\mathcal{P}}_{ab}^{mk}(x) &= 2\cosh(\frac{x}{t_a})\sum_{n=1}^{\ell_a-1}\hat{\mathcal{K}}_a^{mn}(x)\hat{\mathcal{A}}_{ab}^{nk}(x) \\ &= \frac{\sinh(\frac{x}{t_a})}{\sinh(\frac{x}{t_{ab}})}\delta_{t_b m, t_a k} + \sum_{j=1}^{t_b-t_a}\frac{\sinh(\frac{jx}{t_b})}{\sinh(\frac{x}{t_b})}\big(\delta_{\frac{t_b}{t_a}(m+1)-j,\,k} + \delta_{\frac{t_b}{t_a}(m-1)+j,\,k}\big).\end{aligned} \tag{14.16}$$

The sum $\sum_{j=1}^{t_b-t_a}$ in (14.16) is to be understood as zero if $t_a \geq t_b$. Since the expression (14.16) does not contain ℓ, we can and do extend the definition of $\hat{\mathcal{J}}_{ab}^{mk}(x)$ and $\hat{\mathcal{P}}_{ab}^{mk}(x)$ to all the nonnegative integers $m,k\geq 0$. The inverse Fourier transform $\mathcal{J}_{ab}^{mk}(u)$ is an even function of u but $\mathcal{J}_{ab}^{mk}(u)\neq\mathcal{J}_{ba}^{km}(u)$ in general as opposed to $\hat{\mathcal{A}}_{ab}^{mk}(x) = \hat{\mathcal{A}}_{ba}^{km}(x)$ and $\hat{\mathcal{K}}_{ab}^{mk}(x) = \hat{\mathcal{K}}_{ba}^{km}(x)$. The $\hat{\mathcal{K}}_a^{mn}(x)$ in (14.12) should be distinguished from $\hat{\mathcal{K}}_{aa}^{mn}(x)$ in (14.14). The following relations are easily checked:

$$2\cosh(\frac{x}{t_a})\sum_{n=1}^{\ell_a-1}\hat{\mathcal{A}}_{aa}^{mn}(x)\hat{\mathcal{K}}_a^{nk}(x) = \delta_{mk}, \tag{14.17}$$

$$2\cosh(\frac{x}{t_a})\sum_{n=1}^{\ell_a-1}\hat{\mathcal{A}}_{aa}^{mn}(x)\hat{\mathcal{J}}_{ab}^{nk}(x) = \hat{\mathcal{K}}_{ab}^{mk}(x), \tag{14.18}$$

$$\hat{\mathcal{J}}_{ab}^{mk}(x) = \delta_{ab}\delta_{mk} - \frac{N_{ab}\hat{\mathcal{P}}_{ab}^{mk}(x)}{2\cosh\big(\frac{x}{t_a}\big)}. \tag{14.19}$$

All the TBA kernels (14.11)–(14.16) are deduced from $\hat{\mathcal{A}}^{mn}_{ab}(x)$ and $\hat{\mathcal{M}}_{ab}(x)$ by using these relations. The basic ones $\hat{\mathcal{A}}^{mn}_{ab}(x)$ and $\hat{\mathcal{M}}_{ab}(x)$ are obtained as

$$\int_{-\infty}^{\infty} due^{-iux}\frac{\partial}{\partial u}\Theta^m_a\bigl(u,\frac{s}{t_a}\bigr) = \hat{\mathcal{A}}^{ms}_{aa}(x)|_{\ell\to L}, \tag{14.20}$$

$$\int_{-\infty}^{\infty} due^{-iux}\frac{\partial}{\partial u}\Theta^{mk}_{ab}(u,(\alpha_a|\alpha_b)) = -\delta_{ab}\delta_{mk} + \hat{\mathcal{M}}_{ab}(x)\hat{\mathcal{A}}^{mk}_{ab}(x)|_{\ell\to L}, \tag{14.21}$$

where $\Theta^m_a\bigl(u,\frac{s}{t_a}\bigr)$ (15.3) and $\Theta^{mk}_{ab}(u,(\alpha_a|\alpha_b))$ (15.4) are the logarithm of the LHS and the RHS of the Bethe equation under the string hypothesis, respectively. See (15.1)–(15.4).

When $\mathfrak{g}$ is simply laced, the TBA kernels simplify as

$$\hat{\mathcal{A}}^{mk}_{ab}(x) = \frac{\sinh\bigl(\min(m,k)x\bigr)\sinh\bigl((\ell-\max(m,k))x\bigr)}{\sinh x\sinh(\ell x)} \tag{14.22}$$

$$\hat{\mathcal{J}}^{mk}_{ab}(x) = \frac{\hat{\mathcal{M}}_{ab}(x)\delta_{mk}}{2\cosh x} = \left(\delta_{ab} - \frac{N_{ab}}{2\cosh x}\right)\delta_{mk}, \tag{14.23}$$

$$\hat{\mathcal{P}}^{mk}_{ab}(x) = \delta_{mk}. \tag{14.24}$$

14.3. **Y-system for $\mathfrak{g}$ from TBA equation.** Let us derive the level ℓ restricted Y-system for general $\mathfrak{g}$ from the TBA equation. We quote the latter obtained in (15.13) with the notation $Y^{(a)}_m(u) = \exp(-\beta\epsilon^{(a)}_m(u))$:

$$\frac{\epsilon\beta\gamma\delta_{pa}\delta_{sm}}{4t_p^{-1}\cosh(t_p\pi u/2)} = -\ln Y^{(a)}_m(u) - \int_{-\infty}^{\infty} dv\frac{\ln\Bigl[\bigl(1+Y^{(a)}_{m-1}(v)^{-1}\bigr)\bigl(1+Y^{(a)}_{m+1}(v)^{-1}\bigr)\Bigr]}{4t_a^{-1}\cosh(t_a\pi(u-v)/2)}$$
$$+\sum_{(b,k)\in H_\ell} N_{ab}\int_{-\infty}^{\infty} dv\frac{\Bigl[\mathcal{P}^{mk}_{ab} * \ln\bigl(1+Y^{(b)}_k\bigr)\Bigr](v)}{4t_a^{-1}\cosh(t_a\pi(u-v)/2)}. \tag{14.25}$$

$\mathcal{P}^{mk}_{ab}$ is defined via its Fourier component (14.16) and $*$ denotes the convolution

$$(f_1 * f_2)(u) = \int_{-\infty}^{\infty} dv f_1(u-v)f_2(v). \tag{14.26}$$

As the simply laced case, we assume that (14.25) can be analytically continued off the real axis of u until $|\Im\mathrm{m}\, u| \le t_a^{-1}$. Then the sum after the shifts $u \to u \pm t_a^{-1}i \mp 0i$ eliminates the LHS, giving

$$\ln\Bigl[Y^{(a)}_m(u-\tfrac{i}{t_a})Y^{(a)}_m(u+\tfrac{i}{t_a})\Bigr] = -\ln\Bigl[\bigl(1+Y^{(a)}_{m-1}(u)^{-1}\bigr)\bigl(1+Y^{(a)}_{m+1}(u)^{-1}\bigr)\Bigr]$$
$$+\sum_{(b,k)\in H_\ell} N_{ab}\Bigl[\mathcal{P}^{mk}_{ab} * \ln\bigl(1+Y^{(b)}_k\bigr)\Bigr](u). \tag{14.27}$$

For simply laced algebras, $P^{mk}_{ab}(u) = \delta_{mk}\delta(u)$ by (14.24), and we are done. To illustrate the general case, take $\mathfrak{g} = G_2$ with $(a,b) = (1,2)$ as an example. Then

$(t_a, t_b) = (1, 3)$ and (14.16) reads

$$\begin{aligned}
\hat{\mathcal{P}}^{mk}_{ab}(x) = \hat{\mathcal{P}}^{mk}_{12}(x) &= (e^{\frac{2x}{3}} + 1 + e^{-\frac{2x}{3}})\delta_{3m,k} + \delta_{3m-2,k} + \delta_{3m+2,k} \\
&\quad + (e^{\frac{x}{3}} + e^{-\frac{x}{3}})(\delta_{3m-1,k} + \delta_{3m+1,k}), \\
\mathcal{P}^{mk}_{12}(u) &= (\delta(u - \tfrac{2i}{3}) + \delta(u) + \delta(u + \tfrac{2i}{3}))\delta_{3m,k} + \delta(u)(\delta_{3m-2,k} + \delta_{3m+2,k}) \\
&\quad + (\delta(u - \tfrac{i}{3}) + \delta(u + \tfrac{i}{3}))(\delta_{3m-1,k} + \delta_{3m+1,k}).
\end{aligned}$$

If $\ln(1+Y^{(2)}_k(v))$ is analytic in the strip $|\Im\mathrm{m}\, v| \le \frac{2}{3}$[34] and decays rapidly as $|\Re\mathrm{e}\, v| \to \infty$, one can shift the convolution integral $\int dv \mathcal{P}^{mk}_{12}(u-v)\ln(1+Y^{(2)}_k(v))$ off the real axis of v to pick the support of delta functions. In this way the last term in (14.27) gives the logarithm of

$$\begin{aligned}
&(1+Y^{(2)}_{3m}\big(u-\tfrac{2i}{3}\big))(1+Y^{(2)}_{3m}(u))(1+Y^{(2)}_{3m}\big(u+\tfrac{2i}{3}\big))(1+Y^{(2)}_{3m-2}(u))(1+Y^{(2)}_{3m+2}(u)) \\
&\times(1+Y^{(2)}_{3m-1}\big(u-\tfrac{i}{3}\big))(1+Y^{(2)}_{3m-1}\big(u+\tfrac{i}{3}\big))(1+Y^{(2)}_{3m+1}\big(u-\tfrac{i}{3}\big))(1+Y^{(2)}_{3m+1}\big(u+\tfrac{i}{3}\big)).
\end{aligned}$$

This is the numerator of the RHS in the first relation of the Y-system for G_2 (2.15).

The general case is similar and (14.27) gives rise to the logarithmic form of the (restricted) Y-system for $\mathfrak{g}$. On account of (14.16), in general it suffices to assume that $\ln(1+Y^{(a)}_m(u))$ is analytic in the strip $|\Im\mathrm{m}\, u| \le \frac{t_a-1}{t_a}$ and decays rapidly as $|\Re\mathrm{e}\, u| \to \infty$.

If the analyticity argument can be left out, the Y-system is deduced more quickly from the TBA kernels in the Fourier space. In fact, one can start with the TBA equation (15.12) without the LHS[35]:

$$\sum_{n=1}^{\ell_a-1} \hat{\mathcal{K}}^{mn}_a(x)\,\widehat{\ln}\big(1+(Y^{(a)}_n)^{-1}\big) = \sum_{(b,k)\in H_\ell} \hat{\mathcal{J}}^{mk}_{ab}(x)\,\widehat{\ln}\big(1+Y^{(b)}_k\big). \tag{14.28}$$

Multiply with $2\cosh(\frac{x}{t_a})$ and use (14.12) and (14.19) to rearrange it slightly as

$$\begin{aligned}
2\cosh\Big(\frac{x}{t_a}\Big)\,\widehat{\ln} Y^{(a)}_m = &\sum_{(b,k)\in H_\ell} N_{ab}\hat{\mathcal{P}}^{mk}_{ab}(x)\,\widehat{\ln}\big(1+Y^{(b)}_k\big) \\
&- \widehat{\ln}\Big[\big(1+(Y^{(a)}_{m-1})^{-1}\big)\big(1+(Y^{(a)}_{m+1})^{-1}\big)\Big].
\end{aligned} \tag{14.29}$$

This is the Y-system if $\cosh(\frac{x}{t_a})$ and $\hat{\mathcal{P}}^{mk}_{ab}(x)$ (14.16) are regarded as the difference operators as mentioned after (14.9).

We have demonstrated that the Y-system is a difference equation whose structure is governed by the TBA kernels. On the other hand, recall that Theorem 2.5 offers another route to obtain the Y-system by invoking its connection to the T-system. It is yet to be understood why the two "characterizations" of the Y-system coincide.

14.4. **Constant Y-system.** In either unrestricted or level ℓ restricted Y-system, one can discard the dependence of $Y^{(a)}_m(u)$ on u. The resulting algebraic equation on $Y^{(a)}_m = Y^{(a)}_m(u)$ is called the unrestricted or level ℓ restricted constant Y-system[36].

---

[34] Actually $|\Im\mathrm{m}\, v| \le \frac{2}{3}$ for $\ln(1+Y^{(2)}_{3m}(v))$ and $|\Im\mathrm{m}\, v| \le \frac{1}{3}$ for $\ln(1+Y^{(2)}_{3m\pm1}(v))$ suffice.

[35] According to our previous argument, it is actually more proper to suppress the LHS after multiplying $2\cosh(\frac{x}{t_a})$.

[36] The level ℓ restricted constant Y-system here is the same with the one introduced in Section 5.1.

The unrestricted constant Y-system for $\mathfrak{g}$ is the set of algebraic equations on $\{Y_m^{(a)} \mid (a,m) \in H\}$. ($H$ is defined in (13.48).)
For simply laced $\mathfrak{g}$, it has the form

$$(Y_m^{(a)})^2 = \frac{\prod_{b\in I: C_{ab}=-1}(1+Y_m^{(b)})}{(1+(Y_{m-1}^{(a)})^{-1})(1+(Y_{m+1}^{(a)})^{-1})}, \tag{14.30}$$

where $(Y_0^{(a)})^{-1}=0$. See (2.11). The nonsimply laced case is similarly written down from (2.12)-(2.15).
For $\mathfrak{g}=B_r$,

$$\begin{aligned}
(Y_m^{(a)})^2 &= \frac{(1+Y_m^{(a-1)})(1+Y_m^{(a+1)})}{(1+(Y_{m-1}^{(a)})^{-1})(1+(Y_{m+1}^{(a)})^{-1})} \qquad (1\le a\le r-2),\\
(Y_m^{(r-1)})^2 &= \frac{(1+Y_m^{(r-2)})(1+Y_{2m-1}^{(r)})(1+Y_{2m}^{(r)})^2(1+Y_{2m+1}^{(r)})}{(1+(Y_{m-1}^{(r-1)})^{-1})(1+(Y_{m+1}^{(r-1)})^{-1})},\\
(Y_{2m}^{(r)})^2 &= \frac{1+Y_m^{(r-1)}}{(1+(Y_{2m-1}^{(r)})^{-1})(1+(Y_{2m+1}^{(r)})^{-1})},\\
(Y_{2m+1}^{(r)})^2 &= \frac{1}{(1+(Y_{2m}^{(r)})^{-1})(1+(Y_{2m+2}^{(r)})^{-1})}.
\end{aligned} \tag{14.31}$$

For $\mathfrak{g}=C_r$,

$$\begin{aligned}
(Y_m^{(a)})^2 &= \frac{(1+Y_m^{(a-1)})(1+Y_m^{(a+1)})}{(1+(Y_{m-1}^{(a)})^{-1})(1+(Y_{m+1}^{(a)})^{-1})} \qquad (1\le a\le r-2),\\
(Y_{2m}^{(r-1)})^2 &= \frac{(1+Y_{2m}^{(r-2)})(1+Y_m^{(r)})}{(1+(Y_{2m-1}^{(r-1)})^{-1})(1+(Y_{2m+1}^{(r-1)})^{-1})},\\
(Y_{2m+1}^{(r-1)})^2 &= \frac{1+Y_{2m+1}^{(r-2)}}{(1+(Y_{2m}^{(r-1)})^{-1})(1+(Y_{2m+2}^{(r-1)})^{-1})},\\
(Y_m^{(r)})^2 &= \frac{(1+Y_{2m-1}^{(r-1)})(1+Y_{2m}^{(r-1)})^2(1+Y_{2m+1}^{(r-1)})}{(1+(Y_{m-1}^{(r)})^{-1})(1+(Y_{m+1}^{(r)})^{-1})}.
\end{aligned} \tag{14.32}$$

For $\mathfrak{g} = F_4$,

$$
\begin{aligned}
(Y_m^{(1)})^2 &= \frac{1 + Y_m^{(2)}}{(1 + (Y_{m-1}^{(1)})^{-1})(1 + (Y_{m+1}^{(1)})^{-1})}, \\
(Y_m^{(2)})^2 &= \frac{(1 + Y_m^{(1)})(1 + Y_{2m-1}^{(3)})(1 + Y_{2m}^{(3)})^2(1 + Y_{2m+1}^{(3)})}{(1 + (Y_{m-1}^{(2)})^{-1})(1 + (Y_{m+1}^{(2)})^{-1})}, \\
(Y_{2m}^{(3)})^2 &= \frac{(1 + Y_m^{(2)})(1 + Y_{2m}^{(4)})}{(1 + (Y_{2m-1}^{(3)})^{-1})(1 + (Y_{2m+1}^{(3)})^{-1})}, \\
(Y_{2m+1}^{(3)})^2 &= \frac{1 + Y_{2m+1}^{(4)}}{(1 + (Y_{2m}^{(3)})^{-1})(1 + (Y_{2m+2}^{(3)})^{-1})}, \\
(Y_m^{(4)})^2 &= \frac{1 + Y_m^{(3)}}{(1 + (Y_{m-1}^{(4)})^{-1})(1 + (Y_{m+1}^{(4)})^{-1})}.
\end{aligned}
\tag{14.33}
$$

For $\mathfrak{g} = G_2$,

$$
\begin{aligned}
(Y_m^{(1)})^2 &= \frac{(1 + Y_{3m-2}^{(2)})(1 + Y_{3m-1}^{(2)})^2(1 + Y_{3m}^{(2)})^3(1 + Y_{3m+1}^{(2)})^2(1 + Y_{3m+2}^{(2)})}{(1 + (Y_{m-1}^{(1)})^{-1})(1 + (Y_{m+1}^{(1)})^{-1})}, \\
(Y_{3m}^{(2)})^2 &= \frac{1 + Y_m^{(1)}}{(1 + (Y_{3m-1}^{(2)})^{-1})(1 + (Y_{3m+1}^{(2)})^{-1})}, \\
(Y_{3m+1}^{(2)})^2 &= \frac{1}{(1 + (Y_{3m}^{(2)})^{-1})(1 + (Y_{3m+2}^{(2)})^{-1})}, \\
(Y_{3m+2}^{(2)})^2 &= \frac{1}{(1 + (Y_{3m+1}^{(2)})^{-1})(1 + (Y_{3m+3}^{(2)})^{-1})}.
\end{aligned}
\tag{14.34}
$$

The level ℓ restricted constant Y-system for $\mathfrak{g}$ is obtained from (14.30)-(14.34) by setting $(Y_{t_a \ell}^{(a)})^{-1} = 0$ and naturally restricting the variables $\{Y_m^{(a)} \mid (a,m) \in H\}$ to $\{Y_m^{(a)} \mid (a,m) \in H_\ell\}$. ($H_\ell$ is defined in (14.2).)

For the TBA analysis, it is useful to recognize that the level ℓ restricted constant Y-system is expressed in terms of the 0th Fourier component ($x = 0$) of the TBA kernels. We prepare the notations for them.

$$
\bar{C}_{mn}^a = 2\hat{\mathcal{K}}_a^{mn}(0), \quad (\bar{C}_{mn}^a)_{1 \le m,n \le \ell_a - 1} = \text{Cartan matrix of } A_{\ell_a - 1}, \tag{14.35}
$$

$$
K_{ab}^{mk} = \hat{\mathcal{K}}_{ab}^{mk}(0) = \Big(\min(t_b m, t_a k) - \frac{mk}{\ell}\Big)(\alpha_a|\alpha_b), \tag{14.36}
$$

$$
P_{ab}^{mk} = \hat{\mathcal{P}}_{ab}^{mk}(0) = \frac{t_{ab}}{t_a}\delta_{t_b m, t_a k} + \sum_{j=1}^{t_b - t_a} j\big(\delta_{\frac{t_b}{t_a}(m+1)-j,\, k} + \delta_{\frac{t_b}{t_a}(m-1)+j,\, k}\big), \tag{14.37}
$$

$$
J_{ba}^{km} = \hat{\mathcal{J}}_{ba}^{km}(0) = \frac{1}{2}\sum_{n=1}^{\ell_b - 1} \bar{C}_{kn}^b K_{ab}^{mn} = \delta_{ab}\delta_{mk} - \frac{1}{2}N_{ab}P_{ba}^{km} = -\frac{1}{2}G_{am,bk}, \tag{14.38}
$$

where (14.14) – (14.19) are used. $G_{am,bk}$ is defined in (13.51). The sum $\sum_{j=1}^{t_b - t_a}$ in (14.37) is to be understood as zero if $t_a \ge t_b$ as in (14.16). Note that $K_{ab}^{mk} = K_{ba}^{km}$

but $P_{ab}^{mk} \neq P_{ba}^{km}$ and $J_{ab}^{mk} \neq J_{ba}^{km}$ in general. We have $P_{ab}^{mk} \in \mathbb{Z}$. From (14.35), the specialization $x = 0$ of (14.17) gives

$$\sum_{n=1}^{\ell_a - 1} \hat{\mathcal{A}}_{aa}^{mn}(0) \bar{C}_{nk}^{a} = \delta_{mk}. \tag{14.39}$$

Using N_{ab} and P_{ab}^{mk} in the above, the level ℓ restricted constant Y-system is expressed uniformly for all $\mathfrak{g}$ as

$$(Y_m^{(a)})^2 = \frac{\prod_{(b,k)\in H_\ell} (1 + Y_k^{(b)})^{N_{ab} P_{ab}^{mk}}}{(1 + (Y_{m-1}^{(a)})^{-1})(1 + (Y_{m+1}^{(a)})^{-1})} \qquad ((a,m) \in H_\ell), \tag{14.40}$$

where $(Y_0^{(a)})^{-1} = 0$. This is easily seen from (14.29). The unrestricted version is similarly presented by replacing H_ℓ here with H.

The level ℓ restricted constant Y-system is expressed in several guises.

$$\sum_{n=1}^{\ell_a - 1} \hat{\mathcal{K}}_a^{mn}(0) \ln\bigl(1 + (Y_n^{(a)})^{-1}\bigr) = \sum_{(b,k)\in H_\ell} \hat{\mathcal{J}}_{ab}^{mk}(0) \ln\bigl(1 + Y_k^{(b)}\bigr), \tag{14.41}$$

$$f_m^{(a)} = \prod_{(b,k)\in H_\ell} (1 - f_k^{(b)})^{K_{ab}^{mk}}, \qquad \text{where } \ f_m^{(a)} = \frac{Y_m^{(a)}}{1 + Y_m^{(a)}}. \tag{14.42}$$

The form (14.41) directly follows from (14.28) and shows up naturally as the TBA equation in a certain asymptotic limit. See (15.18). On the other hand, (14.42) is deduced from (14.35) and (14.38). It is related to the conjectural q-series formula [106] for the string function $c_\lambda^{\ell\Lambda_0}(q)$ [10] of the level ℓ vacuum module of $\hat{\mathfrak{g}}$ up to a power of q:

$$\prod_{j=1}^{\infty} (1 - q^j)^{-\mathrm{rank}\,\mathfrak{g}} \sum_{\{N_m^{(a)}\}} \frac{q^{\frac{1}{2}\sum_{(a,m),(b,k)\in H_\ell} K_{ab}^{mk} N_m^{(a)} N_k^{(b)}}}{\prod_{(a,m)\in H_\ell} (1-q)(1-q^2)\cdots(1-q^{N_m^{(a)}})}. \tag{14.43}$$

The outer sum is over $N_m^{(a)} \in \mathbb{Z}_{\geq 0}$ such that $\sum_{(a,m)\in H_\ell} m N_m^{(a)} \alpha_a \equiv \lambda \mod \ell \sum_{a\in I} \mathbb{Z}\, t_a \alpha_a$. In fact, the crude approximation of the extremum condition on the summand is

$$q^{\sum_{(b,k)} K_{ab}^{mk} N_k^{(b)}} = 1 - q^{N_m^{(a)}},$$

which is cast into (14.42) upon setting $q^{N_m^{(a)}} = 1 - f_m^{(a)}$.

The level ℓ restricted constant Y-system is the set of $|H_\ell|$ algebraic equations on the same number of unknowns $\{Y_m^{(a)} \mid (a,m) \in H_\ell\}$. With regard to its solution, the uniqueness of the positive real one (Theorem 5.1) is fundamental. The concrete construction of the solution is a subject of the subsequent sections 14.5 and 14.6.

14.5. **Relation with Q-system.** Recall that the unrestricted Q-system for $\mathfrak{g}$ (13.45) is

$$(Q_m^{(a)})^2 = Q_{m-1}^{(a)} Q_{m+1}^{(a)} + (Q_m^{(a)})^2 \prod_{(b,k)\in H} (Q_k^{(b)})^{-2J_{km}^{ba}}, \tag{14.44}$$

where we have replaced the notation of the power $G_{am,bk}$ by (14.38). Given $\ell \in \mathbb{Z}_{\geq 1}$, we define the level ℓ restricted Q-system for $\mathfrak{g}$ to be the relations obtained from

(14.44) by restricting the variables $Q^{(a)}_m$ to those with $(a,m) \in H_\ell$ by imposing $Q^{(a)}_{\ell_a} = 1$. Thus it reads

$$(Q^{(a)}_m)^2 = Q^{(a)}_{m-1}Q^{(a)}_{m+1} + (Q^{(a)}_m)^2 \prod_{(b,k)\in H_\ell} (Q^{(b)}_k)^{-2J^{ba}_{km}} \quad \text{for } (a,m) \in H_\ell. \tag{14.45}$$

Proposition 14.1. *Suppose $Q^{(a)}_m$ satisfies the level ℓ restricted Q-system for $\mathfrak{g}$. Then*

$$Y^{(a)}_m = \frac{(Q^{(a)}_m)^2 \prod_{(b,k)\in H_\ell}(Q^{(b)}_k)^{-2J^{km}_{ba}}}{Q^{(a)}_{m-1}Q^{(a)}_{m+1}} \tag{14.46}$$

is a solution of the level ℓ restricted constant Y-system for $\mathfrak{g}$. The same holds between the unrestricted Q-system and the unrestricted constant Y-system if the product $\prod_{(b,k)\in H_\ell}$ in (14.46) is replaced by $\prod_{b\in I, k\geq 1}$.

This is a corollary (constant version) of Theorem 2.5. For instance in the restricted case, it can also be verified directly by noting

$$1 + (Y^{(a)}_m)^{-1} = \prod_{(b,n)\in H_\ell} (Q^{(b)}_k)^{2J^{nm}_{ba}}, \quad 1 + Y^{(b)}_k = \prod_{n=1}^{\ell_b - 1} (Q^{(b)}_n)^{\bar{C}^b_{kn}}, \tag{14.47}$$

where $\bar{C}^b_{kn}$ is defined by (14.35). By virtue of (14.42), the assertion is reduced to $2J^{nm}_{ba} = \sum_{n=1}^{\ell_b-1} \bar{C}^b_{kn} K^{mk}_{ab}$, which indeed holds by (14.38). For $\mathfrak{g}$ simply laced, (14.46) reads

$$Y^{(a)}_m = \frac{\prod_{b\in I: C_{ab}=-1} Q^{(b)}_m}{Q^{(a)}_{m-1}Q^{(a)}_{m+1}}. \tag{14.48}$$

14.6. $\boldsymbol{Q^{(a)}_m}$ **at root of unity.** We fix the level $\ell \in \mathbb{Z}_{\geq 1}$. Let $\chi(V_\omega)$ be the character of the irreducible finite dimensional representation V_ω of $\mathfrak{g}$ with highest weight $\omega \in \sum_{a\in I} \mathbb{Z}_{\geq 0}\,\omega_a$. We introduce the following specialization of $\chi(V_\omega)$:

$$\dim_q V_\omega = \prod_{\alpha\in\Delta_+} \frac{\sin\frac{\pi(\alpha|\omega+\rho)}{\ell+h^\vee}}{\sin\frac{\pi(\alpha|\rho)}{\ell+h^\vee}}, \tag{14.49}$$

where $h^\vee$ is the dual Coxeter number (2.3), Δ_+ is the set of positive roots of $\mathfrak{g}$ and $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha = \sum_{a\in I}\omega_a$. The quantity $\prod_{\alpha\in\Delta_+}\frac{[(\alpha|\omega+\rho)]_{q^t}}{[(\alpha|\rho)]_{q^t}}$ is a q-analogue of the dimension of V_ω. Thus (14.49) is the q-dimension at the root of unity $q = \exp(\frac{\pi\sqrt{-1}}{t(\ell+h^\vee)})$.

By Proposition 13.10, we know that the classical character of the Kirillov-Reshetikhin module $Q^{(a)}_m = \mathrm{res}\,\chi_q(W^{(a)}_m)$ satisfies the unrestricted Q-system. As shown in (13.62) and (13.59), $\mathrm{res}\,\chi_q(W^{(a)}_m)$ is a linear combination of various $\chi(V_\omega)$'s. The specialization of $\mathrm{res}\,\chi_q(W^{(a)}_m)$ to the q-dimension will be denoted by $\dim_q \mathrm{res}\, W^{(a)}_m$. By the definition, $Q^{(a)}_m = \dim_q \mathrm{res}\, W^{(a)}_m$ still satisfies the unrestricted Q-system. Furthermore, it seems to match the level truncation as follows.

Conjecture 14.2. $Q_m^{(a)} = \dim_q \operatorname{res} W_m^{(a)}$ *satisfies the level ℓ restricted Q-system. More strongly, the following properties hold for any $a \in I$.*

$$Q_m^{(a)} = Q_{\ell_a - m}^{(a)} \quad \text{for } 0 \le m \le \ell_a, \tag{14.50}$$

$$Q_m^{(a)} < Q_{m+1}^{(a)} \quad \text{for } 0 \le m < [\ell_a/2], \tag{14.51}$$

$$Q_{\ell_a+j}^{(a)} = 0 \qquad \text{for } 1 \le j \le t_a h^\vee - 1, \tag{14.52}$$

where $[\ell_a/2]$ is the largest integer not exceeding $\ell_a/2$ (not q-integer).

Remark 14.3. Conjecture 14.2 implies $Q_m^{(a)} > 0$ for all $(a,m) \in H_\ell$. Thus $Y_m^{(a)}$ constructed by (14.46) with the substitution $Q_m^{(a)} = \dim_q \operatorname{res} W_m^{(a)}$ is real positive for all $(a,m) \in H_\ell$ Therefore it must coincide with the unique solution characterized in Theorem 5.1.

We note that (14.50) implies $Q_{\ell_a}^{(a)} = Q_0^{(a)} = 1$, therefore $j = 1$ case of (14.52) as well because of the Q-system relation $(Q_{\ell_a}^{(a)})^2 = Q_{\ell_a-1}^{(a)} Q_{\ell_a+1}^{(a)} + \prod_{b(\neq a)} (Q_{\ell_b}^{(b)})^{-C_{ab}}$ and the fact that $Q_{\ell_a-1}^{(a)} \neq 0$ by (14.51).

Example 14.4. For $\mathfrak{g} = A_r$, one has $Q_m^{(a)} = \dim_q \operatorname{res} W_m^{(a)} = \dim_q V_{m\omega_a}$ from (13.63). Thus

$$Q_m^{(a)} = \prod_{i=1}^{a} \prod_{j=1}^{r+1-a} \frac{\sin \frac{\pi(m+i+j-1)}{\ell+r+1}}{\sin \frac{\pi(i+j-1)}{\ell+r+1}}. \tag{14.53}$$

The property (14.50) and $Q_m^{(a)} > 0$ for $(a,m) \in H_\ell$ are easily checked. Substitution of this into (14.48) gives the real positive solution of the level ℓ restricted constant Y-system:

$$Y_m^{(a)} = \frac{\sin \frac{\pi a}{\ell+r+1} \sin \frac{\pi(r+1-a)}{\ell+r+1}}{\sin \frac{\pi m}{\ell+r+1} \sin \frac{\pi(\ell-m)}{\ell+r+1}}, \quad 1 + Y_m^{(a)} = \frac{\sin \frac{\pi(a+m)}{\ell+r+1} \sin \frac{\pi(a+\ell-m)}{\ell+r+1}}{\sin \frac{\pi m}{\ell+r+1} \sin \frac{\pi(\ell-m)}{\ell+r+1}}. \tag{14.54}$$

Obviously $(Y_0^{(a)})^{-1} = (Y_\ell^{(a)})^{-1} = 0$ and $Y_m^{(a)} > 0$ hold for $(a,m) \in H_\ell$. When $r = 1$, this reduces to $Y_m^{(1)}$ in Example 5.3.

One of the most remarkable features of the level ℓ restricted constant Y-system and Q-system is their connection with the dilogarithm identity (5.5) in Theorem 5.2. The LHS emerges from the TBA analysis (Section 15). The $Y_m^{(a)}$ in the dilogarithm is characterized by the Y-system as in Theorem 5.1 or constructed by the Q-system as in Remark 14.3.

14.7. **Bibliographical notes.** The idea of converting TBA equations into difference equations (Y-system) as described in this section was put into practice by [3] for factorized scattering theories describing integrable perturbations of conformal field theories. The TBA equation treated there corresponds to the simply laced $\mathfrak{g}$ with level $\ell = 2$ in the terminology here up to the driving term. There are numerous Y-systems or related nonlinear integral equations in the similar TBA approaches to various integrable field theories, e.g., [5, 263, 264, 265, 266, 267, 268]. The Y-systems considered here appear as typical building blocks in these theories in many cases.

There are also exotic variants and applications of Y-systems related to Takahashi-Suzuki's continued fraction TBA [269] in the context of polymers [270], the sine-Gordon model [271] and the T-system for XXZ model [272]. Intricate examples of T and Y-systems are also worked out for the dilute A_L models [273].

With regard to the Q-system, there are conjectures concerning more general specialization than $\dim_q$ and related dilogarithm sum rules. See [1, appendix A], [134, appendix D], [4] and [101, section 1.4].

15. TBA analysis of RSOS models

We digest the TBA analysis of the $U_q(\hat{\mathfrak{g}})$ Bethe equation, which is a natural candidate for the level ℓ critical restricted solid-on-solid (RSOS) model associated with the representation $W^{(p)}_s$ of $U_q(\hat{\mathfrak{g}})$ ($\ell \in \mathbb{Z}_{\geq 1}$, $(p,s) \in H_\ell$ (14.2)). Basic features of the model have been sketched in Section 3.3. The derivation of high temperature entropy and central charges in two critical regimes is outlined. The level ℓ restricted Q-system, constant Y-system and the dilogarithm identity described in Sections 5.1 and 14.4–14.6 play a fundamental role.

We make a uniform treatment for general $\mathfrak{g}$ elucidating the origin of the Y-system. The results cover vertex models formally as the limit $\ell \to \infty$. The TBA equation (15.13) also applies to a number of situations in other contexts, most notably, integrable perturbations of conformal field theories (cf. Section 14.7) with a suitable modification of the LHS.

Apart from the relatively well known results in the ADE case, a curious aspect in nonsimply laced $\mathfrak{g}$ is that the central charges in one of the regimes correspond to the Goddard-Kent-Olive construction of Virasoro modules [274] involving the embeddings

$$B^{(1)}_r \hookrightarrow D^{(1)}_{r+1}, \quad C^{(1)}_r \hookrightarrow A^{(1)}_{2r-1}, \quad F^{(1)}_4 \hookrightarrow E^{(1)}_6, \quad G^{(1)}_2 \hookrightarrow B^{(1)}_3.$$

See (15.28)–(15.34). These results have stimulated notable developments in crystal basis theory of quantum groups [260]. The content of this section is based on [59] for ADE case and [17] for general $\mathfrak{g}$.

15.1. **TBA equation.**

We keep the notations t, t_a, α_a, C in (2.1)–(2.2) and L, ℓ_a, H_ℓ in (14.1)–(14.2). The Bethe equation is the following for the unknowns $\{u^{(a)}_j \mid a \in I, 1 \leq j \leq n_a\}$:

$$\left(\frac{\sinh \frac{\pi}{2L}\bigl(u^{(a)}_j - \sqrt{-1}\frac{s}{t_p}\delta_{ap}\bigr)}{\sinh \frac{\pi}{2L}\bigl(u^{(a)}_j + \sqrt{-1}\frac{s}{t_p}\delta_{ap}\bigr)}\right)^N = \Omega_a \prod_{b=1}^{r}\prod_{k=1}^{n_b} \frac{\sinh \frac{\pi}{2L}\bigl(u^{(a)}_j - u^{(b)}_k - \sqrt{-1}(\alpha_a|\alpha_b)\bigr)}{\sinh \frac{\pi}{2L}\bigl(u^{(a)}_j - u^{(b)}_k + \sqrt{-1}(\alpha_a|\alpha_b)\bigr)}. \tag{15.1}$$

Here $n_a = Ns(C^{-1})_{ap}$ as in (3.51) with $(r_i, s_i) = (p, s)$ for all i, and Ω_a is a root of unity without which (15.1) is essentially the same as the Bethe equation for the vertex model (8.25) at $q = \exp(\frac{\pi\sqrt{-1}}{tL})$[37]. The Bethe equation (15.1) is indeed valid [59] for $U_q(A^{(1)}_r)$ RSOS model [43].

It is a well known mystery that the TBA analysis yields supposedly correct results in the end despite that it involves arguments that can hardly be justified mathematically[38]. Our arguments in the sequel are no exception.

[37] $\Omega_a = e^{-\alpha_a(\mathcal{H})}$ in the notation of (iii) in Section 8.3.

[38] A more reliable derivation based on T-system is given in Section 16.3.

We employ a string hypothesis. Suppose that $\{u_j^{(a)} | a \in I, 1 \le j \le n_a\}$ is approximately grouped as the union of $\{u_{m,i}^{(a)} + \sqrt{-1}t_a^{-1}(m+1-2n) | 1 \le n \le m, 1 \le i \le N_m^{(a)}, u_{m,i}^{(a)} \in \mathbb{R}\}$ and the rest. Here $u_{m,i}^{(a)}$ is the center of a color a length m string and $N_m^{(a)}$ is the number of such strings. Then the hypothesis is that $\lim_{N\to\infty} \sum_{m=1}^{\ell_a} mN_m^{(a)}/n_a = 1$ for all $a \in I$. It means that for color a, only those strings with length $\le \ell_a$ contribute to the thermodynamic quantities. This is a peculiar feature in the RSOS model and one of the most significant effects of the phase factor Ω_a. Substituting the string forms into (15.1) and taking product over the internal coordinate of strings, one gets

$$N\delta_{ap}\Theta_a^m\big(u_{m,i}^{(a)}, \frac{s}{t_a}\big) = I_{m,i}^{(a)} + \sum_{\substack{b\in I \\ 1\le k\le \ell_b}} \sum_{j=1}^{N_k^{(b)}} \Theta_{ab}^{mk}(u_{m,i}^{(a)} - u_{k,j}^{(b)}, (\alpha_a|\alpha_b)). \qquad (15.2)$$

Here $I_{m,i}^{(a)} \in \mathbb{Z} + \text{constant}$, and $\Theta_a^m, \Theta_{ab}^{mk}$ are defined by

$$\Theta_a^m(u,\Delta) = \frac{1}{2\pi\sqrt{-1}} \sum_{n=1}^{m} \ln \frac{\sinh\frac{\pi}{2L}(u + \sqrt{-1}t_a^{-1}(m+1-2n) - \sqrt{-1}\Delta)}{\sinh\frac{\pi}{2L}(u + \sqrt{-1}t_a^{-1}(m+1-2n) + \sqrt{-1}\Delta)}, \qquad (15.3)$$

$$\Theta_{ab}^{mk}(u,\Delta) = \Theta_{ba}^{km}(u,\Delta) = \sum_{j=1}^{k} \Theta_a^m(u + \sqrt{-1}t_b^{-1}(k+1-2j), \Delta). \qquad (15.4)$$

One assumes that each solution satisfying $u_{m,1}^{(a)} < u_{m,2}^{(a)} < \cdots < u_{m,N_m^{(a)}}^{(a)}$ corresponds to an array such that $I_{m,1}^{(a)} < I_{m,2}^{(a)} < \cdots < I_{m,N_m^{(a)}}^{(a)}$, and introduce the string density $\rho_m^{(a)}(u)$ and the hole density $\sigma_m^{(a)}(u)$ for $u \sim u_{m,i}^{(a)}$ with large enough N by

$$\rho_m^{(a)}(u) = \frac{1}{N(u_{m,i}^{(a)} - u_{m,i-1}^{(a)})}, \quad \sigma_m^{(a)}(u) = \frac{I_{m,i}^{(a)} - I_{m,i-1}^{(a)} - 1}{N(u_{m,i}^{(a)} - u_{m,i-1}^{(a)})}. \qquad (15.5)$$

Then (15.2) is converted into an integral equation. A little inspection of it shows a characteristic property $\sigma_{\ell_a}^{(a)}(u) = 0$, which enables one to eliminate the density of the "longest strings" $\rho_{\ell_a}^{(a)}(u)$. For such calculations, it is convenient to work in the Fourier components. We attach $\hat{}$ to them. See (14.9). We shall flexibly present formulas either in the Fourier or original variables. By means of the basic formulas (14.20) and (14.21), the resulting integral equation is expressed in the Fourier space as[39]

$$\delta_{pa}\hat{\mathcal{A}}_{pa}^{sm}(x) = \hat{\sigma}_m^{(a)}(x) + \sum_{(b,k)\in H_\ell} \hat{\mathcal{K}}_{ab}^{mk}(x)\hat{\rho}_k^{(b)}(x) \quad \text{for } (a,m) \in H_\ell. \qquad (15.6)$$

The "TBA kernels" $\mathcal{A}_{ab}^{mk}(x)$, $\mathcal{K}_{ab}^{mk}(x)$ etc., and their useful properties are summarized in Section 14.2. By (14.17) and (14.15), (15.6) is also written as

$$\frac{\delta_{ap}\delta_{sm}}{2\cosh(\frac{x}{t_a})} = \sum_{n=1}^{\ell_a-1} \hat{\mathcal{K}}_a^{mn}(x)\hat{\sigma}_n^{(a)}(x) + \sum_{(b,k)\in H_\ell} \hat{\mathcal{J}}_{ab}^{mk}(x)\hat{\rho}_k^{(b)}(x) \qquad (15.7)$$

[39]The replacement $\ell \to L$ in (14.20) and (14.21) has become unnecessary here due to the elimination of $\rho_{\ell_a}^{(a)}(u)$.

for $(a,m) \in H_\ell$. The equation (15.6) or equivalently (15.7) is the Bethe equation for the string and hole densities.

We will actually consider the thermodynamics of the "quantum spin" chain associated with the row to row transfer matrix $T^{(p)}_s(u)$ of the RSOS model. We chose its Hamiltonian density $\mathcal{H}$ as

$$\mathcal{H} = -\frac{\epsilon\gamma}{N}\frac{\partial}{\partial u}\ln T^{(p)}_s(u)|_{u=u_0} \quad (\epsilon = \pm 1), \tag{15.8}$$

where $\gamma > 0$ is a normalization constant and $\epsilon = \pm 1$ specifies the two critical regimes in the RSOS model. The point u_0 is such that $T^{(p)}_s(u_0)$ becomes a cyclic shift (generator of momentum) up to an overall multiple, i.e., (3.50) becomes (scalar) $\prod_{i=1}^{N}\delta_{\lambda_i,\mu_{i-1}}\delta_{\alpha_i,\beta_{i-1}}$. In view of Section 8.3, it is natual to assume that the spectrum $\mathcal{E}$ of $\mathcal{H}$ is obtained from the derivative of the top term $Q_p(u-\frac{s}{t_p})/Q_p(u+\frac{s}{t_p})$ therein up to an overall factor independent of the Bethe roots. Thus up to an additve constant we get[40]

$$\begin{aligned}\mathcal{E} &= \frac{\epsilon\gamma}{N}\sum_{m=1}^{\ell_p}\sum_{i=1}^{N^{(p)}_m}\frac{\partial}{\partial u}\Theta^m_p\Big(u,\frac{s}{t_p}\Big)\Big|_{u=u^{(p)}_{m,i}}\\ &\simeq \epsilon\gamma\sum_{m=1}^{\ell_p}\int_{-\infty}^{\infty}du\,\frac{\partial}{\partial u}\Theta^m_p\Big(u,\frac{s}{t_p}\Big)\rho^{(p)}_m(u) = \frac{\epsilon\gamma}{2\pi}\sum_{m=1}^{\ell_p-1}\hat{\mathcal{A}}^{sm}_{pp}\hat{\rho}^{(p)}_m + \epsilon\mathcal{E}_0,\end{aligned} \tag{15.9}$$

where in the last step $\rho^{(p)}_{\ell_p}(u)$ is eliminated as was done for (15.6). $\mathcal{E}_0$ is a constant whose concrete form ([17, (2.20)]) is irrelevant in the sequel. On the other hand, the eigenvalues of the momentum density $\mathcal{P}$ is directly related to the top term itself, and is given as

$$\mathcal{P} = \frac{2\pi}{N}\sum_{m=1}^{\ell_p}\sum_{i=1}^{N^{(p)}_m}\Theta^m_p\Big(u^{(p)}_{m,i},\frac{s}{t_p}\Big) \simeq 2\pi\sum_{m=1}^{\ell_p}\int_{-\infty}^{\infty}du\,\Theta^m_p\Big(u,\frac{s}{t_p}\Big)\rho^{(p)}_m(u). \tag{15.10}$$

The Yang-Yang type entropy density $\mathcal{S}$ [6] responsible for the arrangement of strings and holes is

$$\begin{aligned}\mathcal{S} = \sum_{(a,m)\in H_\ell}\int_{-\infty}^{\infty}du\big(&(\rho^{(a)}_m(u)+\sigma^{(a)}_m(u))\ln(\rho^{(a)}_m(u)+\sigma^{(a)}_m(u))\\ &-\rho^{(a)}_m(u)\ln\rho^{(a)}_m(u)-\sigma^{(a)}_m(u)\ln\sigma^{(a)}_m(u)\big).\end{aligned} \tag{15.11}$$

The thermal equilibrium condition at temperature $T=\beta^{-1}$ is obtained by demanding that the free energy density $\mathcal{F} = \mathcal{E} - T\mathcal{S}$ be extremum with respect to $\rho^{(a)}_m(u)$, namely $\delta\mathcal{F}/\delta\rho^{(a)}_m(u) = 0$, under the constraint (15.6). Setting $\sigma^{(a)}_m(u)/\rho^{(a)}_m(u) = \exp(\beta\epsilon^{(a)}_m(u))$, the result reads ($(a,m)\in H_\ell$)

$$\begin{aligned}\frac{\epsilon\beta\gamma\delta_{pa}\delta_{sm}}{4t_p^{-1}\cosh(t_p\pi u/2)} &= \sum_{n=1}^{\ell_a-1}\int_{-\infty}^{\infty}dv\mathcal{K}^{mn}_a(u-v)\ln\big(1+\exp(\beta\epsilon^{(a)}_n(v))\big)\\ &\quad-\sum_{(b,k)\in H_\ell}\int_{-\infty}^{\infty}dv\mathcal{J}^{mk}_{ab}(u-v)\ln\big(1+\exp(-\beta\epsilon^{(b)}_k(v))\big).\end{aligned} \tag{15.12}$$

[40] The sign (-1) in (15.8) is absent here since $T^{(p)}_s(u)$ is related to $\frac{\partial}{\partial v}\Theta^m_p(v,s/t_p)\Big|_{v=\sqrt{-1}u}$.

The nonlinear integral equation (15.12) is an example of the TBA equation, which serves as the basis in studying thermodynamic quantities. By using (14.12) and (14.19) it can be slightly rearranged as

$$\frac{\epsilon\beta\gamma\delta_{pa}\delta_{sm}}{4t_p^{-1}\cosh(t_p\pi u/2)} = \beta\epsilon_m^{(a)}(u) - \int_{-\infty}^{\infty} dv \frac{\ln\Big[\big(1+\exp(\beta\epsilon_{m-1}^{(a)}(v))\big)\big(1+\exp(\beta\epsilon_{m+1}^{(a)}(v))\big)\Big]}{4t_a^{-1}\cosh(t_a\pi(u-v)/2)}$$
$$+ \sum_{(b,k)\in H_\ell} N_{ab}\int_{-\infty}^{\infty} dv \frac{\Big[\mathcal{P}_{ab}^{mk} * \ln\big(1+\exp(-\beta\epsilon_k^{(b)})\big)\Big](v)}{4t_a^{-1}\cosh(t_a\pi(u-v)/2)}. \tag{15.13}$$

When $\mathfrak{g}$ is simply laced, one has $\mathcal{P}_{ab}^{mk}(u) = \delta_{mk}\delta(u)$ from (14.24) and (14.9). Therefore (15.13) simplifies considerably to

$$\frac{\epsilon\beta\gamma\delta_{pa}\delta_{sm}}{4\cosh(\pi u/2)} = \beta\epsilon_m^{(a)}(u) - \int_{-\infty}^{\infty} dv \frac{\ln\left[\frac{\left(1+\exp(\beta\epsilon_{m-1}^{(a)}(v))\right)\left(1+\exp(\beta\epsilon_{m+1}^{(a)}(v))\right)}{\prod_{b\in I}\left(1+\exp(-\beta\epsilon_m^{(b)}(v))\right)^{N_{ab}}}\right]}{4\cosh(\pi(u-v)/2)}. \tag{15.14}$$

15.2. **High temperature entropy.** The free energy density is expressed as

$$\mathcal{F} = \epsilon\mathcal{E}_0 - T\sum_{m=1}^{\ell_p-1}\int_{-\infty}^{\infty} du\, \mathcal{A}_{pp}^{sm}(u)\ln\big(1+\exp(-\beta\epsilon_m^{(p)}(u))\big) \tag{15.15}$$

by means of (15.12), (14.17) and (14.18). Let us evaluate the high temperature limit of the entropy density

$$\mathcal{S}_{\text{high}} = -\lim_{T\to\infty}\frac{\mathcal{F}}{T}. \tag{15.16}$$

When $T\to\infty$, the leading part of the asymptotic of $\epsilon_m^{(a)}(u)$ is expected to become independent of u. Thus we set $Y_m^{(a)} = \exp(-\beta\epsilon_m^{(a)}(u))$ to be a constant and obtain from (15.15) that

$$\mathcal{S}_{\text{high}} = \sum_{m=1}^{\ell_p-1}\hat{\mathcal{A}}_{pp}^{sm}(0)\ln\big(1+Y_m^{(p)}\big). \tag{15.17}$$

Here $\hat{\mathcal{A}}_{pp}^{sm}(0)$ is the 0th Fourier component of $\mathcal{A}_{pp}^{sm}(u)$ given by (14.13). Similarly the TBA equation (15.12) tends to

$$\sum_{n=1}^{\ell_a-1}\hat{\mathcal{K}}_a^{mn}(0)\ln\big(1+Y_n^{(a)-1}\big) = \sum_{(b,k)\in H_\ell}\hat{\mathcal{J}}_{ab}^{mk}(0)\ln\big(1+Y_k^{(b)}\big). \tag{15.18}$$

This is the logarithmic form of the level ℓ restricted constant Y-system (14.41). Thus we employ the solution $Q_m^{(a)} = \dim_q \operatorname{res} W_m^{(a)}$ explained in Remark 14.3 constructed from the q-dimension at a root of unity (14.49). Substituting the latter formula in (14.47) into (15.17) and applying (14.39), we find

$$\mathcal{S}_{\text{high}} = \ln Q_s^{(p)}. \tag{15.19}$$

This is consistent with the dimension of the space of states $\mathcal{H}(N)$ of the RSOS spin chain (3.49). Namely, (15.19) implies

$$\lim_{N\to\infty} (\dim \mathcal{H}(N))^{1/N} = \dim_q \operatorname{res} W_s^{(p)}, \tag{15.20}$$

which agrees with (3.54).

15.3. **Central charges.** The central charge c of the underlying conformal field theory is extracted from the low temperature asymptotics of the entropy as $\mathcal{S}_{\text{low}} \simeq \frac{\pi c T}{3 v_F}$ [275, 276], where v_F is the Fermi velocity of the low lying massless excitations. In each regime $\epsilon = \pm 1$, the result is expressed as

$$c = \epsilon \frac{6}{\pi^2} \sum_{(a,m)\in H_\ell} \Big(L(f_m^{(a)}(\infty)) - L(f_m^{(a)}(-\infty)) \Big), \tag{15.21}$$

where $L(x)$ is the Rogers dilogarithm (5.1). The number $f_m^{(a)}(\infty)$ is the positive real solution of $\ln f_m^{(a)}(\infty) = \sum_{(b,k)\in H_\ell} K_{ab}^{mk} \ln(1 - f_k^{(b)}(\infty))$ in the both regimes $\epsilon = \pm 1$, where K_{ab}^{mk} is the 0th Fourier component of $\mathcal{K}_{ab}^{mk}$ (14.36). By Theorem 5.1, $f_m^{(a)}(\infty)$ equals $f_m^{(a)}$ in (14.42) constructed from the unique real positive solution of the level ℓ restricted constant Y-system for $\mathfrak{g}$.

One the other hand, the numbers $f_m^{(a)}(-\infty)$ are to satisfy formally the same equation $\ln f_m^{(a)}(-\infty) = \sum_{(b,k)\in H_\ell} K_{ab}^{mk} \ln(1 - f_k^{(b)}(-\infty))$ but with extra condition $f_m^{(a)}(-\infty) = (1-\epsilon)/2$ for $(a,m) \in H_\ell^\epsilon$ in the regime $\epsilon = \pm 1$. Here the subset $H_\ell^\pm$ of H_ℓ is specified as

$$H_\ell^+ = \{(p,m) \mid 1 \le m \le \ell_p - 1\}, \tag{15.22}$$

$$H_\ell^- = \begin{cases} \{(a, \frac{s t_a}{t_p}) \mid a \in I\} & \frac{s}{t_p} \in \mathbb{Z}, \\ H(p,s) \cap H_\ell & \frac{s}{t_p} \notin \mathbb{Z}, \end{cases} \tag{15.23}$$

$$\begin{aligned} H(p,s) = \{ & \big(a, \frac{s-s_0}{t_p}\big), \big(a, \frac{s-s_0}{t_p} + 1\big) \mid a \in I, t_a = 1\} \\ & \cup \{(a, s-s_0), (a,s), (a, s-s_0+t_p) \mid a \in I, t_a = t_p\}, \\ s \equiv s_0 & \mod t_p, \quad 1 \le s_0 \le t_p - 1. \end{aligned}$$

Consequently, the equations governing the remaining $f_m^{(a)}(-\infty)$'s are split into the subsets corresponding to the complement $H_\ell \setminus H_\ell^\epsilon$. Their solutions are obtained by restricted constant Y-system associated with various subalgebras of $\mathfrak{g}$ and levels. The detail can be found in [17, section 3]. In any case, the dilogarithm identity (5.5) suffices to evaluate the sum (15.21). Below we list the results using the RHS of (5.5)

$$\mathcal{L}(\mathfrak{g}, \ell) = \frac{\ell \dim \mathfrak{g}}{\ell + h^\vee} - \operatorname{rank} \mathfrak{g} \tag{15.24}$$

as the building block.

Regime $\epsilon = +1$.

$\mathfrak{g} = A_r$,

$$c = \mathcal{L}(A_r, \ell) - \mathcal{L}(A_{p-1}, \ell) - \mathcal{L}(A_{r-p}, \ell) \quad 1 \le p \le r.$$

$\mathfrak{g} = B_r$,

$$\begin{aligned} c &= \mathcal{L}(B_r,\ell) - \mathcal{L}(A_{p-1},\ell) - \mathcal{L}(B_{r-p},\ell) && 1 \le p \le r-2, \\ &= \mathcal{L}(B_r,\ell) - \mathcal{L}(A_{p-1},\ell) - \mathcal{L}(A_{r-p},2\ell) && p = r-1, r. \end{aligned}$$

$\mathfrak{g} = C_r$,

$$c = \mathcal{L}(C_r,\ell) - \mathcal{L}(A_{p-1},2\ell) - \mathcal{L}(C_{r-p},\ell) \quad 1 \le p \le r.$$

$\mathfrak{g} = D_r$,

$$\begin{aligned} c &= \mathcal{L}(D_r,\ell) - \mathcal{L}(A_{p-1},\ell) - \mathcal{L}(D_{r-p},\ell) && 1 \le p \le r-2, \\ &= \mathcal{L}(D_r,\ell) - \mathcal{L}(A_{r-1},\ell) && p = r-1, r. \end{aligned}$$

$\mathfrak{g} = E_6$,

$$\begin{aligned} c &= \mathcal{L}(E_6,\ell) - \mathcal{L}(D_5,\ell) && p = 1, 6, \\ &= \mathcal{L}(E_6,\ell) - \mathcal{L}(A_1,\ell) - \mathcal{L}(A_4,\ell) && p = 2, 5, \\ &= \mathcal{L}(E_6,\ell) - 2\mathcal{L}(A_2,\ell) - \mathcal{L}(A_1,\ell) && p = 3, \\ &= \mathcal{L}(E_6,\ell) - \mathcal{L}(A_5,\ell) && p = 4. \end{aligned}$$

$\mathfrak{g} = E_7$,

$$\begin{aligned} c &= \mathcal{L}(E_7,\ell) - \mathcal{L}(D_6,\ell) && p = 1, \\ &= \mathcal{L}(E_7,\ell) - \mathcal{L}(A_1,\ell) - \mathcal{L}(A_5,\ell) && p = 2, \\ &= \mathcal{L}(E_7,\ell) - \mathcal{L}(A_1,\ell) - \mathcal{L}(A_2,\ell) - \mathcal{L}(A_3,\ell) && p = 3, \\ &= \mathcal{L}(E_7,\ell) - \mathcal{L}(A_4,\ell) - \mathcal{L}(A_2,\ell) && p = 4, \\ &= \mathcal{L}(E_7,\ell) - \mathcal{L}(A_1,\ell) - \mathcal{L}(D_5,\ell) && p = 5, \\ &= \mathcal{L}(E_7,\ell) - \mathcal{L}(E_6,\ell) && p = 6, \\ &= \mathcal{L}(E_7,\ell) - \mathcal{L}(A_6,\ell) && p = 7. \end{aligned}$$

$\mathfrak{g} = E_8$,

$$\begin{aligned} c &= \mathcal{L}(E_8,\ell) - \mathcal{L}(E_7,\ell) && p = 1, \\ &= \mathcal{L}(E_8,\ell) - \mathcal{L}(A_1,\ell) - \mathcal{L}(E_6,\ell) && p = 2, \\ &= \mathcal{L}(E_8,\ell) - \mathcal{L}(A_2,\ell) - \mathcal{L}(D_5,\ell) && p = 3, \\ &= \mathcal{L}(E_8,\ell) - \mathcal{L}(A_3,\ell) - \mathcal{L}(A_4,\ell) && p = 4, \\ &= \mathcal{L}(E_8,\ell) - \mathcal{L}(A_4,\ell) - \mathcal{L}(A_2,\ell) - \mathcal{L}(A_1,\ell) && p = 5, \\ &= \mathcal{L}(E_8,\ell) - \mathcal{L}(A_6,\ell) - \mathcal{L}(A_1,\ell) && p = 6, \\ &= \mathcal{L}(E_8,\ell) - \mathcal{L}(D_7,\ell) && p = 7, \\ &= \mathcal{L}(E_8,\ell) - \mathcal{L}(A_7,\ell) && p = 8. \end{aligned}$$

$\mathfrak{g} = F_4$,

$$\begin{aligned} c &= \mathcal{L}(F_4,\ell) - \mathcal{L}(C_3,\ell) && p = 1, \\ &= \mathcal{L}(F_4,\ell) - \mathcal{L}(A_{p-1},\ell) - \mathcal{L}(A_{4-p},2\ell) && p = 2, 3, \\ &= \mathcal{L}(F_4,\ell) - \mathcal{L}(B_3,\ell) && p = 4. \end{aligned}$$

$\mathfrak{g} = G_2$,

$$\begin{aligned} c &= \mathcal{L}(G_2, \ell) - \mathcal{L}(A_1, 3\ell) \quad p = 1, \\ &= \mathcal{L}(G_2, \ell) - \mathcal{L}(A_1, \ell) \quad\ \ p = 2. \end{aligned}$$

Regime $\epsilon = -1$. If $\frac{s}{t_p} \in \mathbb{Z}$, the central charge is given by

$$c = \mathcal{L}\left(\mathfrak{g}, \frac{s}{t_p}\right) + \mathcal{L}\left(\mathfrak{g}, \ell - \frac{s}{t_p}\right) - \mathcal{L}\left(\mathfrak{g}, \ell\right) + \operatorname{rank}\mathfrak{g}. \tag{15.25}$$

This is the value corresponding to the coset pair

$$\begin{array}{lccccc} & \hat{\mathfrak{g}} & \oplus & \hat{\mathfrak{g}} & \supset & \hat{\mathfrak{g}} \\ \text{level} & \ell - \dfrac{s}{t_p} & & \dfrac{s}{t_p} & & \ell. \end{array} \tag{15.26}$$

The situation $\frac{s}{t_p} \notin \mathbb{Z}$ can take place in nonsimply laced algebras. The central charges for such cases are given as follows.

$\mathfrak{g} = B_r$ ($p = r$, $1 \leq s \leq 2\ell - 1$, $s \in 2\mathbb{Z} + 1$),

$$c = \mathcal{L}\left(B_r, \frac{s-1}{2}\right) + \mathcal{L}\left(B_r, \ell - \frac{s+1}{2}\right) - \mathcal{L}\left(B_r, \ell\right) + 2r + 1. \tag{15.27}$$

This value corresponds to the following coset pair via the embedding $B_r^{(1)} \hookrightarrow D_{r+1}^{(1)}$.

$$\begin{array}{lccccccc} & B_r^{(1)} & \oplus & B_r^{(1)} & \oplus & D_{r+1}^{(1)} & \supset & B_r^{(1)} \\ \text{level} & \ell - \dfrac{s+1}{2} & & \dfrac{s-1}{2} & & 1 & & \ell. \end{array} \tag{15.28}$$

$\mathfrak{g} = C_r$ ($1 \leq p \leq r - 1$, $1 \leq s \leq 2\ell - 1$, $s \in 2\mathbb{Z} + 1$),

$$c = \mathcal{L}\left(C_r, \frac{s-1}{2}\right) + \mathcal{L}\left(C_r, \ell - \frac{s+1}{2}\right) - \mathcal{L}\left(C_r, \ell\right) + 3r - 1. \tag{15.29}$$

This value corresponds to the following coset pair via the embedding $C_r^{(1)} \hookrightarrow A_{2r-1}^{(1)}$.

$$\begin{array}{lccccccc} & C_r^{(1)} & \oplus & C_r^{(1)} & \oplus & A_{2r-1}^{(1)} & \supset & C_r^{(1)} \\ \text{level} & \ell - \dfrac{s+1}{2} & & \dfrac{s-1}{2} & & 1 & & \ell. \end{array} \tag{15.30}$$

$\mathfrak{g} = F_4$ ($p = 3, 4$, $1 \leq s \leq 2\ell - 1$, $s \in 2\mathbb{Z} + 1$),

$$c = \mathcal{L}\left(F_4, \frac{s-1}{2}\right) + \mathcal{L}\left(F_4, \ell - \frac{s+1}{2}\right) - \mathcal{L}\left(F_4, \ell\right) + 10. \tag{15.31}$$

This value corresponds to the following coset pair via the embedding $F_4^{(1)} \hookrightarrow E_6^{(1)}$.

$$\begin{array}{lccccccc} & F_4^{(1)} & \oplus & F_4^{(1)} & \oplus & E_6^{(1)} & \supset & F_4^{(1)} \\ \text{level} & \ell - \dfrac{s+1}{2} & & \dfrac{s-1}{2} & & 1 & & \ell. \end{array} \tag{15.32}$$

$\mathfrak{g} = G_2$ ($p = 2$, $1 \leq s \leq 3\ell - 1$, $s \equiv s_0 \mod 3$, $s_0 = 1, 2$),

$$c = \mathcal{L}\left(G_2, \frac{s-s_0}{3}\right) + \mathcal{L}\left(G_2, \ell - \frac{s-s_0}{3} - 1\right) + \mathcal{L}\left(A_1, 2\right) - \mathcal{L}\left(G_2, \ell\right) + 5. \tag{15.33}$$

This value corresponds to the following coset pair via the embedding $G_2^{(1)} \hookrightarrow B_3^{(1)}$.

$$\begin{array}{cccccc} & G_2^{(1)} & \oplus & G_2^{(1)} \oplus B_3^{(1)} & \supset G_2^{(1)} \\ \text{level} & \ell - \dfrac{s-s_0}{3} - 1 & & \dfrac{s-s_0}{3} \quad 1 & \ell. \end{array} \tag{15.34}$$

In (15.27), (15.29), (15.31), (15.33), the contributions $2r+1, 3r-1, 10, 5$ other than the dilogarithm $\mathcal{L}$ are equal to $|H(p,s)|$ in (15.23).

These values of the central charges and coset pairs are consistent with the analyses of RSOS models [35, 56, 277] by Baxter's corner transfer matrix method [2]. For A_r level ℓ, the central charges in regime $\epsilon = +1$ and $\epsilon = -1$ are transformed to each other via the interchange $(r-1, \ell, p, s) \leftrightarrow (\ell, r-1, s, p)$, which is a manifestation of the level-rank duality [56, 59, 278].

So far we have considered the N site RSOS chain with the homogeneous quantum space, namely the one corresponding to $(W_s^{(p)})^{\otimes N}$ in the dual picture of vertex models. One can extend the whole analysis to the inhomogeneous case corresponding to $(W_{s_1}^{(p_1)} \otimes \cdots \otimes W_{s_k}^{(p_k)})^{\otimes N}$. Then the LHS of (15.12) becomes non vanishing for $(a,m) = (p_1, s_1), \ldots, (p_k, s_k)$, and H_ℓ^ϵ in (15.22) and (15.23) gets replaced by $\cup_{i=1}^k (H_\ell^\epsilon \text{ for } (p_i, s_i))$. As the result, a broad list of central charges are realized, e.g., the coset pair $(\hat{\mathfrak{g}})^{\oplus k+1} \supset \hat{\mathfrak{g}}$ for ADE case in the regime $\epsilon = -1$. For more details see [17, section 4.2]. Such a generalization has also been consistently incorporated into the crystal basis theory of one dimensional configuration sums [260, section 3.2].

16. T-system in use

Here we present various applications of the T and Y-systems to solvable lattice models.

16.1. **Correlation lengths of vertex models.** The correlation length ξ is the simplest quantity to characterize ordered states. It is evaluated from the energy gap, which needs a lengthy calculation in the Bethe ansatz approach. As an application of the T-system for transfer matrices, we will demonstrate a quick derivation of ξ [279, 134] based on the "periodicity at level 0".

We consider the vertex models associated with quantum affine algebra $U_q(\hat{\mathfrak{g}})$. The row transfer matrix $T_m^{(a)}(u)$ is given by (3.44). We employ the parameterization $q = \mathrm{e}^{-\lambda/t}$ with $\lambda > 0$, where $t = 1, 2, 3$ is defined in (2.1). To simplify the argument, we consider the homogeneous case $(r_i, s_i, w_i) = (p, s, 1)$ for all i, thus $T_m^{(a)}(u)$ acts on the quantum space $W_s^{(p)}(0)^{\otimes N}$. We assume that $t_p = 1$ and the system size N is even. Possible vertex configurations and the Boltzmann weights are explicitly given in (3.1) for $U_q(A_1^{(1)})$ for instance. The vertex weights associated to $U_q(\hat{\mathfrak{g}})$ with $\mathfrak{g}$ other than A_1 have also been written down explicitly in some cases [49, 48]. Based on the concrete example from the $U_q(A_1^{(1)})$ case, we assume that there is a range of the spectral parameter u in which the model is in anti-ferroelectric order in the sense that those features explained below are realized[41]. For a more account, see [134, section 2.1].

In the ordered regime, the ground state and the first excited state are almost degenerate. The relevant energy gap is thus given by the energy difference between

[41] In the parameterization (3.1) for $U_q(A_1^{(1)})$ case, the range is $-1 < u < 0$. We assume the same range for general $U_q(\hat{\mathfrak{g}})$ leaving the precise Boltzmann weights corresponding to it unspecified.

the ground state and the 2nd excited state(s). Let T_{ground} and T_{2nd} be the corresponding eigenvalues of the transfer matrix. Consequently, $1/\xi = \ln(T_{\text{ground}}/T_{\text{2nd}})$. We will show that ξ is given as

$$\xi = -\frac{1}{\ln k}, \tag{16.1}$$

where $k\,(0 < k < 1)$ is determined by the data $U_q(\mathfrak{g})$ as

$$\frac{K'(k)}{K(k)} = \frac{\lambda h^\vee}{\pi},$$

where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$ (2.3) as before. $K(k)\,(K'(k))$ stands for the complete elliptic integral of the first (second) kind with modulus k.

Recall that the unrestricted T-system for $\mathfrak{g}$ (2.22) has the form

$$T^{(a)}_m(u - \tfrac{1}{t_a})T^{(a)}_m(u + \tfrac{1}{t_a}) = T^{(a)}_{m-1}(u)T^{(a)}_{m+1}(u) + g^{(a)}_m(u)M^{(a)}_m(u),$$

where the scalar function $g^{(a)}_m(u)$ depends on the normalization of vertex weights. The factor $M^{(a)}_m(u)$ is a product of $T^{(b)}_k$'s. We assume $m \in t_a\mathbb{Z}_{>0}$ and denote the eigenvalues of $T^{(a)}_m(u)$ also by the same symbol. For the ground state in the antiferroelectric regime, the second term in the RHS is exponentially larger than the first. So it is a good approximation to drop the first term in the RHS. The same is true for the second excited state(s). Let $L^{(a)}_m(u)$ be the ratio of the eigenvalues

$$L^{(a)}_m(u) = (T^{(a)}_m(u))_{\text{2nd}}/(T^{(a)}_m(u))_{\text{ground}}.$$

Then the above argument implies that it satisfies

$$L^{(a)}_m(u - \tfrac{1}{t_a})L^{(a)}_m(u + \tfrac{1}{t_a}) = M^{(a)}_m(u)|_{\forall T^{(b)}_k(v) \to L^{(b)}_k(v)}. \tag{16.2}$$

This is regarded as the level zero restricted T-system. From (2.4)–(2.10), one can check that it closes among those $L^{(a)}_m(u)$'s with $m \in t_a\mathbb{Z}_{>0}$. Moreover it enforces the following periodicity. (See also (3.55).)

Proposition 16.1 ([16], Theorem 8.8). *Suppose that $L^{(a)}_m(u)$ satisfies the bulk T-system (16.2). Then the relation*

$$L^{(a)}_m(u)L^{(\omega(a))}_m(u + h^\vee) = 1$$

is valid for $m \in t_a\mathbb{Z}_{>0}$. Here ω is the involution on the index set I such that $\omega(a) = a$ except for *the following cases (see Fig 1)*[42]*:*

$$\begin{aligned} &\mathfrak{g} = A_r, && \omega(a) = r + 1 - a, \\ &\mathfrak{g} = D_r \ (r\text{: odd}), && \omega(r-1) = r,\ \omega(r) = r - 1, \\ &\mathfrak{g} = E_6, && \omega(1) = 6,\ \omega(2) = 5,\ \omega(5) = 2,\ \omega(6) = 1. \end{aligned}$$

In particular, $L^{(a)}_m(u) = L^{(a)}_m(u + 2h^\vee)$ holds.

See also [134, appendix A] for some manipulation leading to the above result. Below we only consider a such that $\omega(a) = a$. Obviously $L^{(a)}_m(u)$ has another periodicity in the imaginary direction

$$L^{(a)}_m(u) = L^{(a)}_m(u + \frac{2\pi i}{\lambda})$$

[42] For $\mathfrak{g} = D_r$ (r: even), we set $\omega(a) = a$ for any $a \in I$.

because the vertex weights are rational functions of $z = q^{tu} = e^{-\lambda u}$. We thus conclude that $L^{(a)}_m(u)$ is doubly periodic. Introduce two further functions h_1, h_2 by

$$h_1(u,u_0) = \sqrt{k}\,\mathrm{sn}\Big(\frac{i\lambda K(k)}{\pi}(u-u_0)\Big),$$
$$h_2(u,u_0) = \sqrt{k}\,\mathrm{sn}\Big(\frac{i\lambda K(k)}{\pi}(u-u_0+h^\vee)\Big).$$

These are meromorphic, $2h^\vee$-periodic, $\frac{2\pi i}{\lambda}$-anti-periodic functions of u and satisfy

$$h_j(u,u_0)h_j(u+h^\vee,u_0) = 1 \quad (j=1,2).$$

We note also that $h_1(u,u_0)(h_2(u,u_0))$ has one simple zero (pole) and no poles (zeros) in the rectangle $\Omega := [0,h^\vee)\times[0,2\pi i/\lambda)$ for $u-u_0 \in \Omega$. We denote by $\{u_z\}$, $\{u_p\}$ the set of zeros[43] and poles of $L^{(a)}_m(u)$ in Ω, respectively. The ratio defined below is analytic and non-zero for $0 \le \Re\mathrm{e}\, u < h^\vee$,

$$h(u) = \frac{L^{(a)}_m(u)}{\prod_{u_z} h_1(u,u_z)\prod_{u_p} h_2(u,u_p)}.$$

Furthermore we have

$$h(u)h(u+h^\vee) = 1. \tag{16.3}$$

The Liouville theorem and (16.3) claim that $h(u) = \pm 1$. We thus obtain the representation

$$L^{(a)}_m(u) = \pm\prod_{u_z}\sqrt{k}\,\mathrm{sn}\Big(\frac{i\lambda K(k)}{\pi}(u-u_z)\Big)\prod_{u_p}\sqrt{k}\,\mathrm{sn}\Big(\frac{i\lambda K(k)}{\pi}(u-u_p+h^\vee)\Big).$$

The lower excited states are described by only two zeros. The above expression is then simplified to

$$L^{(a)}_m(u) = \mathcal{L}^{(a)}_m(u;u_1,u_2) := \pm k\,\mathrm{sn}\Big(\frac{i\lambda K(k)}{\pi}(u-u_1)\Big)\,\mathrm{sn}\Big(\frac{i\lambda K(k)}{\pi}(u-u_2)\Big). \tag{16.4}$$

The locations of these zeros label the excitations. The energy levels are almost degenerate with slight change in the locations of zeros. Thus, we observe the band structure of second excited states. The correlation function $G(R)$ must sum up all the contributions from the band [280] as

$$G(R) - G(\infty) \simeq \int du_1 \int du_2\,\rho(u_1,u_2)\Big(\mathcal{L}^{(a)}_m(u;u_1,u_2)\Big)^R.$$

By $\rho(u_1,u_2)$ we mean some weight function whose explicit form is not necessary for our argument. Substitution of (16.4) to the above leads to

$$G(R) - G(\infty) \simeq \mathrm{const}\cdot k^R,$$

showing (16.1).

[43]In the Bethe ansatz, these zeros show up as "holes".

16.2. **Finite size corrections.** Evaluation of finite size corrections to the energy spectra of the Hamiltonian or the free energy provides information on the critical behavior such as central charges and scaling dimensions [276, 275, 281]. Numerical approaches often suffer from the smallness of system size and other technical problems such as logarithmic corrections. The evaluation of finite size corrections is a non trivial problem even for integrable models. The Bethe equation is highly transcendental and it simplifies only in the thermodynamics limit to an integral equation. For an arbitrary given system size, it is not possible in general to find the exact locations of the Bethe roots. Nevertheless, there are successful results in deriving finite size corrections based on clever manipulations of Bethe equations [282, 283, 284, 285]. Here we demonstrate yet another method utilizing the T-system in place of the Bethe equation following [286, 18].

As a concrete example we treat a level ℓ critical RSOS model associated with $A_1^{(1)}$ in Section 3.3–3.6 ($\ell \in \mathbb{Z}_{\geq 2}$). Local states on lattice sites range over $\{1, 2, \ldots, \ell + 1\}$. We consider the fusion model in which any neighboring pair of local states is s-admissible ($1 \leq s \leq \ell - 1$). See (3.34) and (3.35) for the definition of the admissibility. The transfer matrix $T_s(u)$ is defined by (3.38) with m, s_i and v_i replaced by s, s and 0, respectively. We assume the system size N is even and treat the range $-2 \leq u \leq 0$ (referred to as the regime III/IV critical line [34]) for simplicity. We set

$$q = \mathrm{e}^{i\lambda}, \qquad \lambda = \frac{\pi}{\ell + 2},$$

in the RSOS Boltzmann weights according to (3.33).

Although we are concerned with such an isotropic model, the key in our approach is to embed it in a family of models in which the admissibility (fusion degree) conditions in the horizontal and vertical directions can be different. We consider the level ℓ fusion RSOS model in which neighboring states in the horizontal direction are s-admissible while those in the vertical direction are m-admissible. The corresponding transfer matrix is denoted by $T_m(u)$ and depicted in (3.38) with $s_i = s$ and $v_i = 0$. The evaluation of the finite size correction to the largest eigenvalue of $T_s(u)$ utilizing the restricted T-system among $\{T_j(u)\}$ will be the main issue in the sequel.

First we need to fix the normalizations. Let $W_{1,s}$ be the RSOS Boltzmann weights obtained by the s-fold fusion in the horizontal direction (cf.(3.24)). Our normalization is such that

$$W_{1,s}\left(\begin{matrix} a+s-1 & a-1 \\ a+s & a \end{matrix} \middle| \, u \right) = \frac{[u+s+1]_{q^{1/2}}}{[2]_{q^{1/2}}}.$$

See (3.33) for the symbol $[u]_{q^{1/2}}$. From now on we use $x = (u+1)i$ as the spectral parameter, and $T_m(u)$ will also be written as $T_m(x)$. We furthermore define the normalized transfer matrices by $\tilde{T}_0(x) = 1$ and

$$\tilde{T}_m(x) = \begin{cases} T_m(x) & 1 \leq m \leq s, \\ \frac{T_m(x)}{\prod_{j=1}^{m-s} \phi(x + (m-s+1-2j)i)} & s+1 \leq m \leq \ell, \end{cases}$$

where we have introduced

$$\phi(x) = \Big(\frac{\sinh \frac{\lambda x}{2}}{\sin \lambda} \Big)^N.$$

Thanks to these normalizations $\tilde{T}_j(x)$ is of degree $N\min(j,s)$ in $[ix+\cdots]_{q^{1/2}}$ for $1 \le j \le \ell$. One then obtains the level ℓ restricted T-system for $\mathfrak{g} = A_1$

$$\tilde{T}_j(x-i)\tilde{T}_j(x+i) = f_j(x)\tilde{T}_{j-1}(x)\tilde{T}_{j+1}(x) + g_j(x) \qquad (1 \le j \le \ell-1). \tag{16.5}$$

Here the scalar factors are given by $f_j(x) = \phi(x)^{\delta_{js}}$ and

$$g_j(x) = \prod_{k=0}^{\min(j,s)-1} \phi(x+(s+j-2k)i)\phi(x-(s+j-2k)i).$$

Numerical calculations for small system sizes suggest the following analyticity of $\tilde{T}_j(x)$.

Assumption 16.2. $\tilde{T}_j(x)\,(1 \le j \le \ell)$ is analytic and non zero in the strip $|\Im\mathrm{m}\,x| \le 1$.

We then construct $Y_j(x)\,(1 \le j \le \ell-1)$ by[44]

$$Y_j(x) = \frac{f_j(x)\tilde{T}_{j-1}(x)\tilde{T}_{j+1}(x)}{g_j(x)}. \tag{16.6}$$

This leads to the Y-system

$$Y_j(x-i)Y_j(x+i) = (1+Y_{j-1}(x))(1+Y_{j+1}(x)) \qquad (1 \le j \le \ell-1), \tag{16.7}$$

where $Y_0(x) = Y_\ell(x) = 0$. The assumption on $T_j(x)$ is inherited to the analyticity of $Y_j(x)$ except for $Y_s(x)$: $Y_s(x)$ has order N zero at the origin due to $f_s(x)$. We thus define the modified Y by

$$\tilde{Y}_j(x) = \frac{Y_j(x)}{(\tanh\frac{\pi}{4}x)^{N\delta_{js}}}. \tag{16.8}$$

Then the above assumption is rephrased as follows.

Assumption 16.3. $\tilde{Y}_j(x)\,(1 \le j \le \ell-1)$ is analytic and non zero in the strip $|\Im\mathrm{m}\,x| \le 1$. Also, $1+Y_j(x)$ is analytic and non zero in the strip $|\Im\mathrm{m}\,x| \le \epsilon$ for small positive ϵ.

Y and $\tilde{Y}$ satisfy

$$\tilde{Y}_j(x-i)\tilde{Y}_j(x+i) = (1+Y_{j-1}(x))(1+Y_{j+1}(x)), \tag{16.9}$$

where a simple identity $\tanh\frac{\pi}{4}(x-i)\tanh\frac{\pi}{4}(x+i) = -1$ is used. With the above analyticity assumption, one can apply the Fourier transformation to the logarithmic derivative of the Y-system[45]. After solving it with respect to the logarithmic derivative of $\ln Y_j$, the inverse Fourier transformation followed by an integration converts the Y-system into the coupled integral equation $(1 \le j \le \ell-1)$:

$$\ln Y_j(x) = \delta_{js}\ln\tanh^N\frac{\pi x}{4} + \int_{-\infty}^{\infty} K(x-x')\ln[(1+Y_{j-1}(x'))(1+Y_{j+1}(x'))]\frac{dx'}{2\pi}, \tag{16.10}$$

$$K(x) = \frac{\pi}{2\cosh\frac{\pi x}{2}}. \tag{16.11}$$

[44] We employ the inverse of (2.24) to make the resulting integral equation suitable for numerical investigations.

[45] The derivative here is not essential. It is done just in order to ensure the convergence.

The integration constant turns out to be zero due to the asymptotic values

$$Y_j(\infty) = \frac{\sin(j\vartheta)\sin((j+2)\vartheta)}{\sin^2\vartheta} =: \iota(j,\vartheta) \tag{16.12}$$

with $\vartheta = \frac{\pi}{\ell+2}$. Up to the driving term, (16.10) coincides with the thermodynamic Bethe ansatz (TBA) equation (15.14) for $\mathfrak{g} = A_1$ although they originate from completely different contexts. The asymptotic value (16.12) is an example of solutions to the constant Y-system. See Example 5.3 and Example 14.4.

Once $Y_j(x)$ is obtained from (16.10), the quantity $T_s(x)$ in question can be evaluated by using the relation

$$T_s(x-i)T_s(x+i) = g_s(x)(1+Y_s(x)). \tag{16.13}$$

Note $\tilde{T}_s(x) = T_s(x)$. As numerical data tells $|Y_s(x)| \ll 1$, the bulk contribution $T_s^{\mathrm{bulk}}(x)$ is determined by $T_s^{\mathrm{bulk}}(x-i)T_s^{\mathrm{bulk}}(x+i) = g_s(x)$. To separate the bulk part and finite size correction, let $T_s(x) = T_s^{\mathrm{bulk}}(x)T_s^{\mathrm{finite}}(x)$. Then (16.13) yields

$$\begin{aligned}\ln T_s^{\mathrm{bulk}}(x) &= -N\int_{-\infty}^{\infty}\frac{\sinh sk\cosh(\ell+1-s)k}{k\sinh 2k\sinh(\ell+2)k}\mathrm{e}^{-ikx}dk,\\ \ln T_s^{\mathrm{finite}}(x) &= \int_{-\infty}^{\infty}K(x-x')\ln(1+Y_s(x'))\frac{dx'}{2\pi}.\end{aligned}$$

So far, all the relations are valid for arbitrary even N. We now proceed to the evaluation of $\ln T_s^{\mathrm{finite}}(x)$ in the large N limit for $x \sim O(1)$. The main contribution to the integrals in (16.10) comes from $x' \sim \pm\frac{2}{\pi}\ln 2N$. Thus it is convenient to introduce

$$y_j^{\pm}(\theta) := \lim_{N\to\infty} Y_j\Big(\pm\frac{2}{\pi}(\theta+\ln 2N)\Big).$$

The evenness of the original Y_j as a function of x implies $y_j^+(\theta) = y_j^-(\theta)$. We then arrive at simpler expressions for N sufficiently large:

$$\begin{aligned}\ln y_j^{\epsilon}(\theta) &= -\delta_{js}\mathrm{e}^{-\theta} + \int_{-\infty}^{\infty}K_\theta(\theta-\theta')\ln[(1+y_{j-1}^{\epsilon}(\theta'))(1+y_{j+1}^{\epsilon}(\theta'))]\frac{d\theta'}{2\pi},\\ \ln T_s^{\mathrm{finite}}\Big(\frac{2\theta}{\pi}\Big) &= \frac{2\cosh\theta}{N}\int_{-\infty}^{\infty}\mathrm{e}^{-\theta'}\ln(1+y_s^+(\theta'))\frac{d\theta'}{2\pi},\end{aligned}$$

where $K_\theta(\theta) := \frac{2}{\pi}K(\frac{2}{\pi}\theta) = \frac{1}{\cosh\theta}$. The first equation exactly coincides with the TBA equation in the low temperature limit. Thus the dilogarithm trick (cf. [18, section 3.3], [134, section 3.2]) is naturally applied to evaluate $\ln T_s^{\mathrm{finite}}(x)$. The final result of the finite size correction to the largest eigenvalue of $T_s(x)$ is given by

$$\begin{aligned}\ln T_s^{\mathrm{finite}}\Big(\frac{2\theta}{\pi}\Big) &\simeq \frac{\cosh\theta}{2\pi N}\sum_{j=1}^{\ell-1}\int_{y_j^+(-\infty)}^{y_j^+(\infty)}\Big(\frac{\ln(1+y)}{y} - \frac{\ln y}{1+y}\Big)dy\\ &= \frac{\cosh\theta}{\pi N}\sum_{j=1}^{\ell-1}\Big(L_+(y_j^+(\infty)) - L_+(y_j^+(-\infty))\Big)\\ &= \frac{\pi\cosh\theta}{6N}\Big(\frac{3s}{s+2} - \frac{6s}{(\ell+2)(\ell+2-s)}\Big) =: \frac{\pi\cosh\theta}{6N}c.\end{aligned} \tag{16.14}$$

Here $L_+(y)$ is related to the Rogers dilogarithm $L(y)$ in (5.1) by

$$L_+(y) = L\Big(\frac{y}{1+y}\Big) = L(1) - L\Big(\frac{1}{1+y}\Big).$$

We have also used $y_j^+(\infty) = Y_j(\infty) = \iota(j, \frac{\pi}{\ell+2})$ as in (16.12) while

$$y_j^+(-\infty) = \begin{cases} \iota(j, \frac{\pi}{s+2}) & 1 \le j \le s-1, \\ \iota(j-s, \frac{\pi}{\ell+2-s}) & s \le j \le \ell-1. \end{cases}$$

Then the dilogarithm identity (5.7) is applied. The quantity c in the last expression in (16.14) is regarded as the central charge [275]. This value agrees with the TBA result (15.25) obtained from the low temperature specific heat with $\mathfrak{g} = A_1$ and $p = 1, t_p = 1$.

The above argument can be generalized to calculate the finite size correction in excited states with suitable modifications. The major difference from the ground state case is that Assumption 16.3 does not hold any longer. Instead, we assume the following for low lying excited states.

Assumption 16.4. There are finitely many zeros $\{z_\alpha^{(j)}\}$ of $\tilde{T}_j(x)$ in the strip $|\Im\mathrm{m}\, x| \le 1$.

Letting the zeros of $\tilde{T}_j(x)$ in the strip be $\{z_\alpha^{(j)}\}$, we modify (16.8) as

$$Y_j(x) = \tilde{Y}_j(x)(\tanh \frac{\pi}{4}x)^{N\delta_{js}} \prod_\alpha \tanh \frac{\pi}{4}(x - z_\alpha^{(j-1)}) \prod_{\alpha'} \tanh \frac{\pi}{4}(x - z_{\alpha'}^{(j+1)}),$$

which still satisfies (16.9). Then it is straightforward to derive the following equation valid for arbitrary N

$$\begin{aligned} \ln Y_j(x) &= D_j + \delta_{js} \ln \tanh^N \frac{\pi}{4}x \\ &\quad + \sum_\alpha \ln \tanh \frac{\pi}{4}(x - z_\alpha^{(j-1)}) + \sum_{\alpha'} \ln \tanh \frac{\pi}{4}(x - z_{\alpha'}^{(j+1)}) \\ &\quad + \int_{-\infty}^{\infty} K(x-x') \ln[(1+Y_{j-1}(x'))(1+Y_{j+1}(x'))] \frac{dx'}{2\pi}. \end{aligned} \tag{16.15}$$

The integration constant D_j takes account of the branch of $\ln\tanh$ and it must be fixed case by case. For low lying excitations in the thermodynamic limit, it is reasonable to assume $|z_\alpha^{(j)}| \gg 1$. Thus we employ the parameterization

$$z_\alpha^{(j)} = \begin{cases} \frac{2}{\pi}(\theta_{\alpha,+}^{(j)} + \ln 2N) & \text{for } z_\alpha^{(j)} \gg 1 \quad (1 \le \alpha \le n_+^{(j)}), \\ -\frac{2}{\pi}(\theta_{\alpha,-}^{(j)} + \ln 2N) & \text{for } z_\alpha^{(j)} \ll -1 \quad (1 \le \alpha \le n_-^{(j)}), \end{cases}$$

where $n_\pm^{(j)}$ denotes the number of $z_\alpha^{(j)}$ near $\pm\frac{2}{\pi}\ln 2N$. Then (16.15) is reduced in the limit $N \to \infty$ to

$$\begin{aligned} \ln y_j^\epsilon(\theta) &= D_j^\epsilon - \delta_{js} e^{-\theta} + \sum_\alpha \ln \tanh \frac{1}{2}(\theta - \theta_{\alpha,\epsilon}^{(j-1)}) + \sum_{\alpha'} \ln \tanh \frac{1}{2}(\theta - \theta_{\alpha',\epsilon}^{(j+1)}) \\ &\quad + \int_{-\infty}^{\infty} K_\theta(\theta-\theta') \ln(1+y_{j-1}^\epsilon(\theta'))(1+y_{j+1}^\epsilon(\theta')) \frac{d\theta'}{2\pi}. \end{aligned} \tag{16.16}$$

The constants $D_j^\pm$ can be in general different and depend on $n_\pm^{(j)}$, etc.

The subsidiary conditions $T_j(z_\alpha^{(j)}) = 0$ must also be satisfied. This is rephrased as $Y_j(z_\alpha^{(j)} + i) = -1$ or equivalently

$$\ln y_j^\epsilon(\theta_{\alpha,\epsilon}^{(j)} + \frac{\pi}{2}i) = (2I_{\alpha,\epsilon}^{(j)} + 1)\pi i$$

in terms of the branch cut integers $\{I^{(j)}_{\alpha,\pm}\}$. Thanks to (16.16), this is rewritten as

$$\begin{aligned}
&-\int_{-\infty}^{\infty} \frac{1}{\sinh(\theta^{(j)}_{\alpha,\epsilon}-\theta-i\epsilon')} \ln[(1+y^{\epsilon}_{j-1}(\theta))(1+y^{\epsilon}_{j+1}(\theta))]\frac{d\theta}{2\pi} \\
&\quad = (2I^{(j)}_{\alpha,\epsilon}+1)\pi + iD^{\epsilon}_j - \delta_{js}\mathrm{e}^{-\theta^{(j)}_{\alpha,\epsilon}} \\
&\quad + i\sum_{\alpha'} \ln\tanh\Big(\frac{\theta^{(j)}_{\alpha,\epsilon}-\theta^{(j-1)}_{\alpha',\epsilon}}{2}+\frac{\pi}{4}i\Big) + i\sum_{\alpha'} \ln\tanh\Big(\frac{\theta^{(j)}_{\alpha,\epsilon}-\theta^{(j+1)}_{\alpha',\epsilon}}{2}+\frac{\pi}{4}i\Big),
\end{aligned} \tag{16.17}$$

where $\epsilon' > 0$ is infinitesimally small. The finite part of the eigenvalue is now given by

$$\ln T^{\mathrm{finite}}_s\Big(\frac{2\theta}{\pi}\Big) = \sum_{\epsilon=\pm}\frac{\mathrm{e}^{\epsilon\theta}}{N}\Big(-\sum_{\alpha}\mathrm{e}^{-\theta^{(s)}_{\alpha,\epsilon}} + \int_{-\infty}^{\infty}\mathrm{e}^{-\theta}\ln(1+y^{\epsilon}_s(\theta))\frac{d\theta}{2\pi}\Big).$$

Although the expressions are more involved than the ground state case, one can still apply the dilogarithm trick to evaluate the above. In particular, (16.17) and the elementary relations $(\ln\tanh\frac{x}{2})' = 1/\sinh x$ and $\ln\tanh(x+\frac{\pi i}{4})+\ln\tanh(-x+\frac{\pi i}{4}) = \pi i$ are useful. The final result reads

$$\begin{aligned}
\ln T^{\mathrm{finite}}_s\Big(\frac{2\theta}{\pi}\Big) = \sum_{\epsilon=\pm}\frac{\mathrm{e}^{\epsilon\theta}}{2\pi N}\sum_{j=1}^{\ell-1}\Big(&L_+(y^{\epsilon}_j(\infty)) - L_+(y^{\epsilon}_j(-\infty)) + \frac{1}{2}D^{\epsilon}_j\ln\frac{1+y^{\epsilon}_j(\infty)}{1+y^{\epsilon}_j(-\infty)} \\
&- 2\pi n^{(j)}_{\epsilon} iD^{\epsilon}_j - 2\pi^2\sum_{\alpha=1}^{n^{(j)}_{\epsilon}}(2I^{(j)}_{\alpha,\epsilon}+1)\Big).
\end{aligned} \tag{16.18}$$

The above derivation is based on the first principle. However it lacks a general prescription to determine the integration constants and to choose the branch cut integers. With regard to this, an interesting observation has been made in [18, 287]. It is possible to absorb the additional driving terms in (16.15) to integrals by adopting deformed contours $\mathcal{L}_j$ as

$$\begin{aligned}
\ln Y_j(x) &= D_j + \delta_{js}\ln\tanh^N\frac{\pi}{4}x \\
&+ \int_{\mathcal{L}_{j-1}} K(x-x')\ln(1+Y_{j-1}(x')) + \int_{\mathcal{L}_{j+1}} K(x-x')\ln(1+Y_{j+1}(x'))\frac{dx'}{2\pi}.
\end{aligned}$$

Then the evaluation of the finite size correction goes parallel to the case of the largest eigenstate. The differences lie in the asymptotic values of $y^{\epsilon}_j(x)$ and the non trivial homotopy in the integration contours of $\mathcal{L}_j$. The authors of [18, 287] have found empirical rules for the choice of homotopy and integration constants to reproduce known scaling dimensions from conformal field theories.

We have seen that the T-system provides an efficient tool in the analysis of finite size corrections. It enables one to analytically calculate the central charge (16.14) in the ground state. The scaling dimensions of relevant operators can also be obtained by use of the result in excited states (16.18). The above calculation of the finite size correction of the largest eigenvalue has been generalized to RSOS models associated with $\mathfrak{g}$ in [134, section 3] up to analyticity argument on auxiliary functions. To check the validity of the analyticity assumption is an important future problem.

16.3. **Quantum transfer matrix approach.** According to Matsubara, finite size corrections and low temperature asymptotics are dual pictures of the same physical characteristics of a two dimensional system on an infinite cylinder of circumference $N = \beta$. Here N is the system size in the former picture and β is the inverse temperature in the latter. Our analyses of the $U_q(A_1^{(1)})$ RSOS model in Section 15 and Section 16.2 have been done along these two points of view. What is remarkable there is that beyond the formal coincidence of the two pictures, the two entirely different approaches end up with essentially the same integral equation of TBA type. One then expects a framework to treat the finite temperature problem in the same manner as the finite size corrections without recourse to string hypothesis. As we will see in the sequel, the Quantum Transfer Matrix (QTM) approach [288] offers such a scheme. For a further detail, see the recent reviews [289, 290].

QTM utilizes the equivalence between $d+1$ dimensional classical models and d dimensional quantum system [291]. To be concrete, we argue along the 1d spin 1/2 XXZ model as a prototypical integrable lattice system.

$$\mathcal{H} = \frac{J}{4}\sum_{j=1}^{N}\Big(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta(\sigma_j^z\sigma_{j+1}^z + 1)\Big) = \sum_{j=1}^{N}\hat{h}_{j,j+1}, \tag{16.19}$$

where σ^a $(a = x, y, z)$ are the Pauli matrices. The periodic boundary condition implies $\sigma_{N+1}^a = \sigma_1^a$. The anisotropy is parameterized as $\Delta = \cos\lambda$. The Hamiltonian acts on "the physical space" $V_{\rm phys} := \bigotimes_{j=1}^N V_j$ where V_j denotes the jth copy of $\mathbb{C}^2 = \mathbb{C}\mathbf{e}_+ \oplus \mathbb{C}\mathbf{e}_-$. The main subject here is to calculate the partition function exactly

$$Z_{\rm 1d}(\beta, N) = \mathrm{Tr}_{V_{\rm phys}} \mathrm{e}^{-\beta\mathcal{H}}.$$

It would be nice if this task can be done for any finite N, although we do not have a satisfactory progress at present. We thus concentrate on the evaluation of the free energy per site in the thermodynamic limit

$$f = -\lim_{N\to\infty}\frac{1}{\beta N}\ln Z_{\rm 1d}(\beta, N).$$

We introduce the six vertex model on the 2d square lattice. Let $R(u,v)$ be the $U_q(A_1^{(1)})$ R matrix (in a convention different from (3.1)):

$$R(u,v) = \begin{pmatrix} a(u,v) & & & \\ & b(u,v) & c(u,v) & \\ & c^{-1}(u,v) & b(u,v) & \\ & & & a(u,v) \end{pmatrix}$$

$$a(u,v) = \frac{[2+u-v]_{q^{1/2}}}{[2]_{q^{1/2}}}, \qquad b(u,v) = \frac{[u-v]_{q^{1/2}}}{[2]_{q^{1/2}}},$$

$$c(u,v) = q^{-\frac{u-v}{2}}, \qquad q = \mathrm{e}^{i\lambda}.$$

Define the matrix element $R_{\beta\delta}^{\alpha\gamma}$ by

$$R(u,v) = \sum_{\alpha,\beta,\gamma,\delta=1,2} R_{\beta\delta}^{\alpha\gamma}(u,v) E_{\alpha,\beta}\otimes E_{\gamma,\delta}.$$

The index 1(2) refers to $\mathbf{e}_+(\mathbf{e}_-)$ in Fig. 4. The arrows are assigned in order to distinguish this R matrix from other R matrices that will appear below. By $R_{j,j+1}(u,v)$ we mean the R matrix acting non trivially only on the tensor product

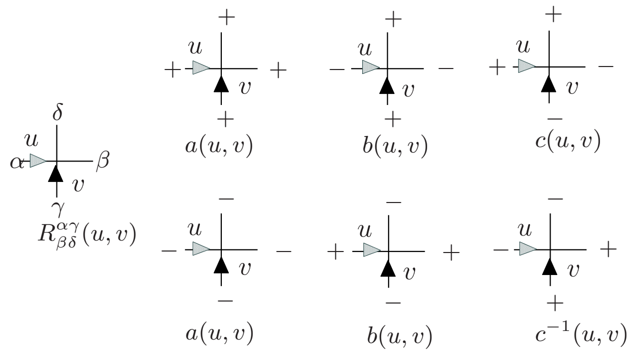


FIGURE 4. A graphic representation for $R^{\alpha\gamma}_{\beta\delta}(u,v)$. The spectral parameter u (v) is associated to horizontal (vertical) lines.

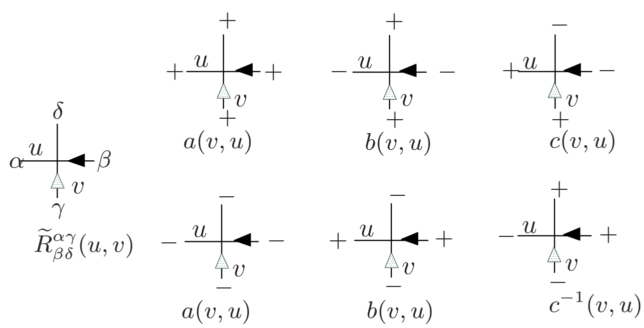


FIGURE 5. A graphic representation for $\widetilde{R}^{\alpha\gamma}_{\beta\delta}(u,v)$. The spectral parameter u (v) is associated to horizontal (vertical) lines.

$V_j(u) \otimes V_{j+1}(v)$. We introduce the row to row (RTR) transfer matrix $T_{\mathrm{RTR}}(u) \in \mathrm{End}(V_{\mathrm{phys}})$ by

$$T_{\mathrm{RTR}}(u) = \mathrm{Tr}_a \left(R_{a,N}(u,0)R_{a,N-1}(u,0)\cdots R_{a,1}(u,0)\right), \tag{16.20}$$

where the subscript "a" stands for the auxiliary space. With the lattice translation e^{iP} shifting the sites by one, the Baxter-Lüscher formula [52]

$$T_{\mathrm{RTR}}(u) = \mathrm{e}^{iP}\big(1 + \frac{\lambda u}{J\sin\lambda}\,\mathcal{H} + O(u^2)\big) \tag{16.21}$$

holds. With a rotated R matrix $\widetilde{R}^{\alpha\gamma}_{\beta\delta}(u,v) = R^{\gamma\beta}_{\delta\alpha}(v,u)$ (Fig. 5), we introduce a rotated transfer matrix $\widetilde{T}_{\mathrm{RTR}}(u) \in \mathrm{End}(V_{\mathrm{phys}})$ by

$$\widetilde{T}_{\mathrm{RTR}}(u) = \mathrm{Tr}_a \big(\widetilde{R}_{a,N}(-u,0)\widetilde{R}_{a,N-1}(-u,0)\cdots \widetilde{R}_{a,1}(-u,0)\big).$$

The expansion analogous to (16.21) holds as $\widetilde{T}_{\mathrm{RTR}}(u) = \mathrm{e}^{-iP}\big(1+\frac{\lambda u}{J\sin\lambda}\,\mathcal{H}+O(u^2)\big)$. We thus obtain an important identity

$$Z_{\mathrm{1d}}(\beta,N) = \mathrm{Tr}_{V_{\mathrm{phys}}}\mathrm{e}^{-\beta\mathcal{H}} = \lim_{M\to\infty}\mathrm{Tr}_{V_{\mathrm{phys}}}\Big(T_{\mathrm{double}}(u=u_M)^{\frac{M}{2}}\Big), \tag{16.22}$$

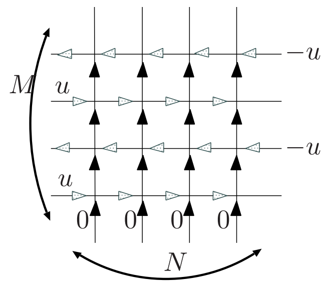


FIGURE 6. Fictitious two dimensional system

where $T_{\rm double}(u) := T_{\rm RTR}(u)\widetilde{T}_{\rm RTR}(u)$ and

$$u_M = -\frac{\beta J \sin\lambda}{M\lambda}. \tag{16.23}$$

The RHS of (16.22) can be interpreted as a partition function of a 2d classical system defined on $M \times N$ sites (Fig. 6)

$$Z_{\rm 1d}(\beta, N) = \lim_{M\to\infty} Z_{\rm 2d\,classical}(M, N, u_M).$$

This embodies the equivalence between $d+1$ dimensional classical models and d dimensional quantum system for $d = 1$. Since the spectra of $T_{\rm double}(u)$ is gapless, we still need a trick to evaluate $Z_{\rm 2d\,classical}(M, N, u_M)$.

We follow the observation in [288] and consider the transfer matrix propagating in the horizontal direction, that is, $T'_{\rm QTM}(u = u_M)$ which acts on a virtual space of size M. It was shown that this transfer matrix possesses a gap between the largest (Λ_0) and the other eigenvalues Λ_j ($j \geq 1$). This is a crucial benefit, as one only has to consider the largest eigenvalue to evaluate the free energy in the thermodynamic limit

$$\begin{aligned}
&\lim_{N\to\infty} Z^{\frac{1}{N}}_{\rm 2d\,classical}(M, N, u_M) = \lim_{N\to\infty}\Big(\mathrm{Tr}\, T'_{\rm QTM}(u = u_M)^N\Big)^{\frac{1}{N}} \\
&= \lim_{N\to\infty}(\Lambda_0^N + \Lambda_1^N + \cdots)^{\frac{1}{N}} = \lim_{N\to\infty}\Lambda_0\Big(1 + (\frac{\Lambda_1}{\Lambda_0})^N + \cdots\Big)^{\frac{1}{N}} \simeq \lim_{N\to\infty}\Lambda_0.
\end{aligned}$$

Although we have made use of the integrability for simplicity in the above argument, the same conclusion can be proved in a more general setting.

Theorem 16.5 ([288])**.** *Let Λ_0 be the largest eigenvalue of $T_{\rm QTM}$. Then the free energy per site is given by*

$$f = -\frac{1}{\beta}\lim_{M\to\infty}\ln\Lambda_0. \tag{16.24}$$

Two problems are still to be overcome. First we must evaluate the largest eigenvalue of $T'_{\rm QTM}(u_M)$ in which interaction depends on the fictitious system size M. Second we must take the "Trotter limit" $M \to \infty$. Both of these are highly nontrivial. Nevertheless we stress the above formulation makes it clear why the finite size correction and the finite temperature problem can be treated in the same way. To disentangle the difficulties, we introduce a slight generalization, a commuting QTM $T_{\rm QTM}(x, u)$, by assigning the parameter ix in the "horizontal" direction [292].

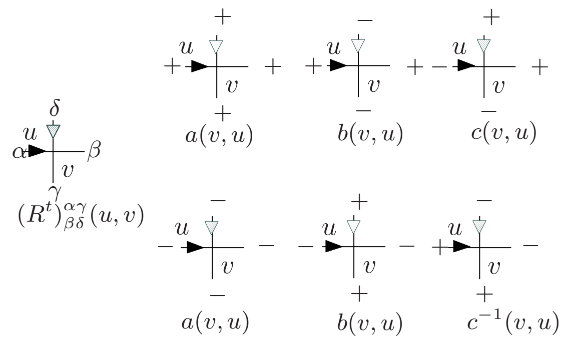


FIGURE 7. A graphic representation for $(R^t)^{\alpha\gamma}_{\beta\delta}(u,v)$. The spectral parameter u (v) is associated to horizontal (vertical) lines.

We let the transposed R matrix $R^t_{j,k}(u,v)$ [293] be $(R^t)^{\alpha\gamma}_{\beta\delta}(u,v) = R^{\delta\alpha}_{\gamma\beta}(v,u)$. See Fig. 7. Then $T_{\mathrm{QTM}}(x,u)$ is defined by

$$T_{\mathrm{QTM}}(x,u) = \mathrm{Tr}_a\left(R_{aM}(ix,-u)R^t_{a,M-1}(ix,u)\cdots R_{a2}(ix,-u)R^t_{a1}(ix,u)\right). \quad (16.25)$$

The parameter u will always be set to u_M (16.23), thus we drop its dependence hereafter. It is the new parameter x that will play the role of a spectral parameter instead. By this we mean that two QTMs with different values of x are intertwined by the same R matrix

$$R_{a,a'}(ix,iy)\mathcal{T}_a(x)\otimes\mathcal{T}_{a'}(x') = \mathcal{T}_a(x')\otimes\mathcal{T}_{a'}(x)R_{a,a'}(ix,iy).$$

Here $\mathcal{T}_a(x)$ denotes the monodromy matrix associated to $T_{\mathrm{QTM}}(x,u_M)$. The proof is elementary. Now we are able to introduce the fusion hierarchy of commuting transfer matrices $T_j(x)$ which contains $T_{\mathrm{QTM}}(x,u_M)$ as the first member. (The u_M-dependence will be suppressed.) By the construction, they satisfy the T-system

$$T_j(x-i)T_j(x+i) = T_{j-1}(x)T_{j+1}(x) + g_j(x),$$

where $g_j(x) = T_0(x+(j+1)i)T_0(x-(j+1)i)$ with

$$T_0(x) = \phi(x+(1+u_M)i)\phi(x-(1+u_M)i), \qquad \phi(x) = \left(\frac{\sinh\frac{\lambda x}{2}}{\sin\lambda}\right)^{\frac{M}{2}}. \quad (16.26)$$

As in Section 16.2, we need assumptions on the analyticity of $T_j(x)$. For simplicity we consider the case $\lambda \to 0$ for a moment. Then the numerical analysis suggests

Conjecture 16.6. *The zeros of $T_j(x)$ are distributed almost on the line $|\Im\mathrm{m}\, x| = j+1$.*

We set $Y_j(x) = T_{j-1}(x)T_{j+1}(x)/g_j(x)$ and introduce its modification

$$\tilde{Y}_j(x) = \frac{Y_j(x)}{\left(\tanh\frac{\pi}{4}(x-(1+u_M)i)\tanh\frac{\pi}{4}(x+(1+u_M)i)\right)^{\frac{M}{2}}}. \quad (16.27)$$

Note that u_M is a small negative quantity. Then the conjecture is translated to

Conjecture 16.7. *$\tilde{Y}_j(x)$ is analytic and non zero in the strip $|\Im\mathrm{m}\, x| \le 1$ and $1+Y_{j+1}(x)$ is analytic and non zero in the strip $|\Im\mathrm{m}\, x| \le \epsilon$ for small ϵ.*

This immediately leads to the integral equation

$$\ln Y_j(x) = \delta_{j1}\frac{1}{2}\ln\Big[\tanh^M\frac{\pi}{4}(x-(1+u_M)i)\tanh^M\frac{\pi}{4}(x+(1+u_M)i)\Big] + \int_{-\infty}^{\infty} K(x-x')\ln[(1+Y_{j-1}(x'))(1+Y_{j+1}(x'))]\frac{dx'}{2\pi}, \tag{16.28}$$

where $K(x)$ is defined in (16.11). The M enters only in the first line in (16.28). Therefore the Trotter limit $M\to\infty$ can be taken *analytically*, giving

$$\ln Y_j(x) = \delta_{j1}D(x) + \int_{-\infty}^{\infty} K(x-x')\ln[(1+Y_{j-1}(x'))(1+Y_{j+1}(x'))]\frac{dx'}{2\pi} \quad (j\ge 1). \tag{16.29}$$

where $D(x)$ in the driving term is given by

$$D(x) = -\frac{\beta\pi J\sin\lambda}{2\lambda\cosh\frac{\pi}{2}x}. \tag{16.30}$$

These are nothing but the Gaudin-Takahashi equations for the anti-ferromagnetic Heisenberg model. Also, they coincide with (16.10) up to the driving term. The free energy per site is obtained from the solution to the above equations as

$$f = -\frac{1}{\beta}\int_{-\infty}^{\infty} K(x')\ln(1+Y_1(x'))\frac{dx'}{2\pi}.$$

Summarizing, we have seen that T-system plays the central role for the quantitative studies on both finite size system and finite temperature system. A wider range of the parameter $0<\lambda\le\frac{\pi}{2}$ is treated in [272] under the restriction that the continued fractional expansion of π/λ terminates at a finite stage. A suitably chosen subset of the fusion QTMs are shown to satisfy a closed set of functional relations and it successfully recovers the well known Takahashi-Suzuki continued fraction TBA equation [269] without using string hypothesis. See [272] for details.

16.4. **Simplified TBA equations.** We continue our discussion on the XXZ spin chain at finite temperatures. We retain the definitions of the symbols such as $\phi(x), T_j(x), u_M$, etc. in the previous subsection. The TBA equation is a coupled set of integral equations with (finitely or infinitely) many unknown functions $Y_j(x)$. It is known that equations change their forms drastically according to a small change in coupling constant λ [269]. On the other hand, we expect only small changes in physical quantities. Thus one may hope alternative formulations that are more stable against the change in λ. Here we present one such approach which also originates from the T-system. It is sometimes referred to as a simplified TBA equation [294].

The idea is complementary to the QTM method where one pays attention to the zeros of $T_j(x)$. In the simplified TBA, one is concerned with singularities of a renormalized $T_j(x)$. The latter is defined by

$$\tilde{T}_j(x) = \frac{T_j(x)}{\phi(x+(j+1+u_M)i)\phi(x-(j+1+u_M)i)}, \tag{16.31}$$

where $\phi(x)$ is defined in (16.26). Note $\tilde{T}_j(x)$ possesses poles of order $M/2$ at $x \sim \pm(j+1)i$. Accordingly, the first equation of the T-system reads

$$\tilde{T}_1(x+i)\tilde{T}_1(x-i) = \tilde{T}_2(x) + b_1^{(M)}(x), \tag{16.32}$$

$$b_1^{(M)}(x) = \frac{\phi(x+(1-u_M)i)\phi(x-(1-u_M)i)}{\phi(x+(1+u_M)i)\phi(x-(1+u_M)i)}. \tag{16.33}$$

Let $\tau_j(x)$ be $\tilde{T}_j(x)$ after the Trotter limit

$$\tau_j(x) = \lim_{M\to\infty} \tilde{T}_j(x).$$

Then $\tau_1(x)$ develops singularity at $x = \pm 2i$. By construction, it is periodic under $x \to x + 2p_0 i$, where $p_0 = \pi/\lambda$. We thus assume the expansion

$$\tau_1(x) = 2 + \sum_{n\in\mathbb{Z}}\sum_{j=1}^{\infty} \frac{c_j}{(x-2i-2p_0 ni)^j} + \sum_{n\in\mathbb{Z}}\sum_{j=1}^{\infty} \frac{\bar{c}_j}{(x+2i-2p_0 ni)^j}. \tag{16.34}$$

We utilize the T-system and information on the locations of singularities to fix c_j and $\bar{c}_j$. Rewrite the Trotter limit of (16.32) as

$$\tau_1(x+i) = \frac{b_1(x)}{\tau_1(x-i)} + \frac{\tau_2(x)}{\tau_1(x-i)}, \tag{16.35}$$

$$b_1(x) = \lim_{M\to\infty} b_1^{(M)}(x) = \exp\Big(\frac{\beta J \sin^2\lambda}{\cosh\lambda x - \cos\lambda}\Big). \tag{16.36}$$

The LHS possesses the singularities at $x = i, -3i$, while only the first term in the RHS possesses singularity at $x = i$. Consequently we have

$$c_j = \oint_{y=i} \frac{b_1(y)}{\tau_1(y-i)}(y-i)^{j-1}\frac{dy}{2\pi i} = \oint_{y=0} \frac{b_1(y+i)}{\tau_1(y)} y^{j-1}\frac{dy}{2\pi i}.$$

The contour for the first integral is a small circle centered at $y = i$ and the same circle centered at $y = 0$ for the second. Similarly, by rewriting (16.32) in the form $\tau_1(x-i) = \frac{b_1(x)}{\tau_1(x+i)} + \frac{\tau_2(x)}{\tau_1(x+i)}$, one finds

$$\bar{c}_j = \oint_{y=0} \frac{b_1(y-i)}{\tau_1(y)} y^{j-1}\frac{dy}{2\pi i}.$$

By substituting the expressions for $c_j, \bar{c}_j$ into (16.34) and performing the summation over j and n, we arrive at the closed integral equation involving $\tau_1(x)$ only:

$$\tau_1(x) = 2 + \frac{\lambda}{4\pi i}\Big(\oint_{y=0} b_1(y+i)\coth\frac{\lambda}{2}(x-y-2i)\frac{dy}{\tau_1(y)} + \oint_{y=0} b_1(y-i)\coth\frac{\lambda}{2}(x-y+2i)\frac{dy}{\tau_1(y)}\Big).$$

Once the above equation is solved, the free energy is given by $f = -\frac{1}{\beta}\ln\tau_1(0)$.

It turned out the new equation works efficiently to produce the high temperature expansion. One assumes $\tau_1(x)$ in the form,

$$\tau_1(x) = \exp\Big(\sum_{n=0}^{\infty} a_n(x)(\beta J)^n\Big).$$

Then the coefficients $a_n(x)$ can be iteratively determined.

The simplified TBA equations are applied in many different contexts and they successfully provide high temperature data of the models [295, 296]. The derivation of the simplified TBA equations requires less information on the analyticity. Therefore it is quite efficient when the analytic property is difficult to investigate. The non-compact case is such an example. See [190] for a recent application to certain sectors of $\mathcal{N} = 4$ super Yang-Mills theory.

There is however a price to pay. Any eigenvalue of $T_j(x)$ satisfies the same equation after renormalization. Therefore the equation itself can not select the right answer. Rather, one has to know a priori the right goal to be achieved and start from a sufficiently near point to the goal in numerical approaches. The convergence becomes also problematic in the low temperature regime and one needs to apply, e.g., the Padé approximation to improve the accuracy.

16.5. **Hybrid equations.** There is yet further approach to the finite size and the finite temperature problems [293, 297, 298]. It also makes use of a finite set of unknown functions and different types of integral equations from those derived in the previous sections. Following [299], we refer to it as NLIE (NonLinear Integral Equation)[46] just in order to distinguish it from the other nonlinear integral equations discussed hitherto. It turns out that a hybridization of TBA and NLIE is possible [300]. The hybrid approach is especially efficient in dealing with thermodynamics of higher spin XXZ models as explained below.

We treat the integrable spin $s/2$ XXZ model whose Hamiltonian $\mathcal{H}$ is obtained from the fusion R matrix in Section 3.1 as

$$\mathcal{H} = \sum_{i=1}^{N} h_{i,i+1}, \qquad h_{i,i+1} \propto \frac{d}{du} P R^{(k,k)}(q^u)|_{u=0},$$

where P is the transposition. A simple generalization of the argument in Section 16.3 tells that the free energy per site is obtained from the largest value of QTM $T_s(x = 0)$ consisting of the R matrix acting on $V_s \otimes V_s$. As before we set

$$q = \mathrm{e}^{i\lambda}, \qquad \lambda = \frac{\pi}{p_0}$$

and assume $s \leq p_0 - 1$. As in Section 16.3, we introduce the auxiliary QTM $T_j(x)$. This time, we prepare only a finitely many ones $\{T_j(x)\}_{j=1}^{\ell}$, where the integer ℓ is *arbitrary* as far as it is in the range

$$s \leq \ell \leq 2p_0 - s - 2. \tag{16.37}$$

With a suitable normalization, we have the T-system

$$T_j(x+i)T_j(x-i) = f_j(x)T_{j-1}(x)T_{j+1}(x) + g_j(x) \quad (1 \leq j \leq s-1), \tag{16.38}$$

$$g_j(x) := \prod_{m=0}^{\min(j,s)-1} \Phi(x - (s+j-2m)i)\Phi(x + (s+j-2m)i),$$

$$\Phi(x) := \Big([x + (1+u)i]_{q^{\frac{1}{2}}} [x - (1+u)i]_{q^{\frac{1}{2}}}\Big)^{M/2},$$

[46] The equation first appeared in the context of finite size problem in the XXZ model [285]. The simplest case is sometimes referred to as the DDV equation in the context of integrable field theories.

where $f_j(x) = \Phi(x)^{\delta_{js}}$. This looks formally the same as (16.5), although the meaning of ℓ is different here. As usual we set $Y_j(x) = f_j(x)T_{j-1}(x)T_{j+1}(x)/g_j(x)$ and define its slight modification generalizing (16.27) as

$$\tilde{Y}_j(x) = \frac{Y_j(x)}{\left(\tanh\frac{\pi}{4}(x+(1+u)i)\tanh\frac{\pi}{4}(x-(1+u)i)\right)^{\frac{M}{2}\delta_{js}}}.$$

Then, the modified Y-system (16.9) holds for $1 \le j \le \ell - 2$.

In addition we introduce the auxiliary functions $\mathfrak{b}(x), \bar{\mathfrak{b}}(x)$. They are defined by the combination of the terms appearing in the dressed vacuum form of $T_\ell(x)$. For general n, the dressed vacuum form reads $T_n(x) = \sum_{m=1}^{n+1} \lambda_m^{(n)}(x)$, where

$$\lambda_m^{(n)}(x) = \Phi_m^{(n)}(x)\frac{Q(x+(n+1)i)Q(x-(n+1)i)}{Q(x+(2m-n-1)i)Q(x+(2m-n-3)i)},$$
$$\Phi_m^{(n)}(x) = \frac{\prod_{r=0}^{s-1}\Phi(x+(2m-n-s-1+2r)i)}{\prod_{r=1}^{\max(s-n,0)}\Phi(x-(s+1-n-2r)i)}.$$

Then the auxiliary functions are defined by

$$\mathfrak{b}(x) = \frac{\lambda_1^{(\ell)}(x+i)+\cdots+\lambda_\ell^{(\ell)}(x+i)}{\lambda_{\ell+1}^{(\ell)}(x+i)} \qquad (-1 \le \Im\mathrm{m}\, x < 0),$$
$$\bar{\mathfrak{b}}(x) = \frac{\lambda_2^{(\ell)}(x-i)+\cdots+\lambda_{\ell+1}^{(\ell)}(x-i)}{\lambda_1^{(\ell)}(x-i)} \qquad (0 < \Im\mathrm{m}\, x \le 1),$$

which are assumed to be analytic and non zero in the strips indicated in the parentheses for the largest eigenvalue of the QTM $T_s(x)$. We also introduce

$$\mathfrak{B}(x) = 1 + \mathfrak{b}(x), \qquad \bar{\mathfrak{B}}(x) = 1 + \bar{\mathfrak{b}}(x)$$

in each analytic strips. There are nice relations among them, e.g.,

$$Y_{\ell-1}(x-i)Y_{\ell-1}(x+i) = (1+Y_{\ell-2}(x))\mathfrak{B}(x)\bar{\mathfrak{B}}(x),$$
$$\mathfrak{b}(x) = \frac{\Phi(x)^{\delta_{\ell s}}}{\prod_{r=1}^{s}\Phi(x+(\ell-s+2r)i)}\frac{Q(x+(\ell+2)i)}{Q(x-\ell i)}T_{\ell-1}(x),$$
$$\bar{\mathfrak{b}}(x) = \frac{\Phi(x)^{\delta_{\ell s}}}{\prod_{r=1}^{s}\Phi(x-(\ell-s+2r)i)}\frac{Q(x-(\ell+2)i)}{Q(x+\ell i)}T_{\ell-1}(x),$$

which can be easily checked by using the definitions.

By use of the analyticity assumptions, it is straightforward to derive the following equations after the limit $M \to \infty$.

$$\ln Y_j(x) = \delta_{js} D(x) + \int_{-\infty}^{\infty} K(x-x') \ln[(1+Y_{j+1}(x'))(1+Y_{j-1}(x'))] \frac{dx'}{2\pi}, \quad 1 \le j \le \ell-2, \tag{16.39}$$

$$\ln Y_{\ell-1}(x) = \delta_{\ell-1,s} D(x) + \int_{-\infty}^{\infty} K(x-x') \ln(1+Y_{\ell-2}(x')) \frac{dx'}{2\pi} + \int_{C_-} K(x-x') \ln \mathfrak{B}(x') \frac{dx'}{2\pi} + \int_{C+} K(x-x') \ln \bar{\mathfrak{B}}(x') \frac{dx'}{2\pi}, \tag{16.40}$$

$$\ln \mathfrak{b}(x) = \delta_{\ell s} D(x) + \int_{-\infty}^{\infty} K(x-x') \ln(1+Y_{\ell-1}(x')) \frac{dx'}{2\pi} + \int_{C_-} F(x-x') \ln \mathfrak{B}(x') \frac{dx'}{2\pi} - \int_{C_+} F(x-x'+2i) \ln \bar{\mathfrak{B}}(x') \frac{dx'}{2\pi} \qquad x \in C_-, \tag{16.41}$$

$$\ln \bar{\mathfrak{b}}(x) = \delta_{\ell s} D(x) + \int_{-\infty}^{\infty} K(x-x') \ln(1+Y_{\ell-1}(x')) \frac{dx'}{2\pi} + \int_{C_+} F(x-x') \ln \bar{\mathfrak{B}}(x') \frac{dx'}{2\pi} - \int_{C_-} F(x-x'-2i) \ln \mathfrak{B}(x') \frac{dx'}{2\pi} \qquad x \in C_+, \tag{16.42}$$

where $C_+(C_-)$ is a contour just above (below) the real axis. The kernel $K(x)$ is given in (16.11) and F is related to the spinon S matrix

$$F(x) = \int_{-\infty}^{\infty} \frac{\sinh(p_0 - \ell - 1)k}{2\cosh k \sinh k(p_0 - \ell)} \mathrm{e}^{-ikx} dk.$$

The integration constants are found to be zero by comparing asymptotic values of the both sides and $D(x)$ is defined in (16.30).

Obviously (16.39) is a reminiscence of the TBA type equation (16.29), while (16.41) and (16.42) resemble NLIE were it not for the $\ln(1+Y_{\ell-1})$ term. In this sense we call the above equations hybrid. They fix the values of $Y_s(x)$. The functional relations similar to (16.13) and the trick mentioned around (16.13) then yield the evaluation of the free energy per site.

Remark 16.8. The number ℓ is arbitrary under the condition (16.37). This is quite different from "genuine" TBA equations at special λ [269, 272], where the number of equations is completely determined by λ. When $\lambda \to 0$, we can formally put $\ell = \infty$, which recovers the usual TBA equation in the rational limit as argued in Section 16.3 for $s = 1$. For $s = 1$, one can make $F(x)$ null by choosing $p_0 = \ell + 1$. The resulting system reproduces the known TBA equation corresponding to the level 2 restricted Y-system for $D_{\ell+1}$ for the XXZ chain. See [272, eq.(4.10)-eq.(4.12)] for example. For arbitrary $s \in \mathbb{Z}_{\ge 1}$, the choice $\ell = s$ recovers the result in [300].

The above equations are numerically stable and yield a quick convergence to the unique solution. They are efficient in the analysis of the low temperature regime. It is also known that with a suitable modification, one can derive the equations for excited states. We again have to pay the price. The systematic algorithm to construct the auxiliary functions is still lacking except for $\mathfrak{g} = A_1$ discussed here. This remains as an interesting future problem.

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