

# On nonclassical symmetries of generalized Huxley equations

N.M. Ivanova<sup>†,‡</sup> and C. Sophocleous<sup>‡</sup>

<sup>†</sup> *Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivska Str., 01601 Kyiv, Ukraine  
e-mail: ivanova@imath.kiev.ua*

<sup>‡</sup> *Department of Mathematics and Statistics, University of Cyprus, CY 1678 Nicosia, Cyprus  
e-mail: christod@ucy.ac.cy*

Nonclassical symmetries of a class of generalized Huxley equations of form  $u_t = u_{xx} + k(x)u^2(1 - u)$  are found. More precisely, for the class under consideration we completely classify reduction operators with  $\tau = 1$  and give a wide number of examples of equations admitting reduction operators with  $\tau = 0$ .

## 1 Introduction

Consider reaction-diffusion equation of form

$$u_t = u_{xx} + k(x)u^2(1 - u), \quad (1)$$

where  $k(x) \neq 0$ . This equation models many phenomena that occur in different areas of mathematical physics and biology. In particular, it can be used to describe the spread of a recessive advantageous allele through a population in which there are only two possible alleles at the locus in question. Equation (1) is interesting also in the area of nerve axon potentials [18]. Case  $k = \text{const}$  is the famous Huxley equation. For more details about application see [3, 4] and references therein.

Bluman and Cole [2] introduced a new method for finding group-invariant (called also similarity) solutions of partial differential equations. The method was called by the authors “nonclassical” to emphasize the difference between it and the “classical” Lie reduction method described, e.g., in [14, 15]. A precise and rigorous definition of nonclassical invariance was firstly formulated in [10] as “a generalization of the Lie definition of invariance” (see also [21]). Later operators satisfying the nonclassical invariance criterion were also called, by different authors, nonclassical symmetries, conditional symmetries,  $Q$ -conditional symmetries and reduction operators [8, 9, 13]. The necessary definitions, including ones of equivalence of reduction operators, and relevant statements on this subject are collected in [17, 19].

Bradshaw-Hajek *at al* [3, 4] started studying class (1) from the symmetry point of view. More precisely, they found some cases of equations (1) admitting Lie and/or nonclassical symmetries. Complete classification of Lie symmetries of class (1) is performed in [12]. Conditional symmetries of Huxley and Burgers–Huxley equations having nontrivial intersection with class (1) are investigated in [1, 5–7, 11]. The present paper is a step towards to the complete classification of nonclassical symmetries of class (1).

Reduction operators of equations (1) have the general form  $Q = \tau\partial_t + \xi\partial_x + \eta\partial_u$ , where  $\tau$ ,  $\xi$  and  $\eta$  are functions of  $t$ ,  $x$  and  $u$ , and  $(\tau, \xi) \neq (0, 0)$ . We consider two cases:

1.  $\tau \neq 0$ . Without loss of generality  $\tau = 1$ .
2.  $\tau = 0$ ,  $\xi \neq 0$ . Without loss of generality  $\xi = 1$ .

We present a complete classification for the case 1, while for the case 2 we found several examples.

## 2 Equivalence transformations and Lie symmetries

Since classification of nonclassical symmetries is impossible without detailed knowledge of Lie invariance properties, we review [12] the equivalence group and results of the group classification of class (1). The complete equivalence group  $G^\sim$  of class (1) contains only scaling and translation transformations of independent variables  $t$  and  $x$ . More precisely it consists of transformations

$$\tilde{t} = \varepsilon_1^2 t + \varepsilon_2, \quad \tilde{x} = \varepsilon_1 x + \varepsilon_3, \quad \tilde{u} = u, \quad \tilde{k} = \varepsilon_1^{-2} k,$$

where  $\varepsilon_i$ ,  $i = 1, 2, 3$  are arbitrary constants,  $\varepsilon_1 \neq 0$ .

**Theorem 1.** *There exists three  $G^\sim$ -inequivalent cases of equations from class (1) admitting nontrivial Lie invariance algebras (the values of  $k$  are given together with the corresponding maximal Lie invariance algebras,  $c = \text{const}$ ) (see [12]):*

- 1 :  $\forall k, \quad \langle \partial_t \rangle;$
- 2 :  $k = c, \quad \langle \partial_t, \partial_x \rangle;$
- 3 :  $k = cx^{-2}, \quad \langle \partial_t, 2t\partial_t + x\partial_x \rangle.$

In the following section we search for nonclassical symmetries which are not equivalent to the above Lie symmetries.

## 3 Nonclassical symmetries

Now let us recall the definition of nonclassical symmetry (or conditional symmetry, or reduction operator).

Reduction operators (nonclassical symmetries,  $Q$ -conditional symmetries) of a differential equation  $\mathcal{L}$  of form  $L(t, x, u_{(r)}) = 0$  have the general form

$$Q = \tau\partial_t + \xi\partial_x + \eta\partial_u,$$

where  $\tau$ ,  $\xi$  and  $\eta$  are functions of  $t$ ,  $x$  and  $u$ , and  $(\tau, \xi) \neq (0, 0)$ . Here  $u_{(r)}$  denotes the set of all the derivatives of the function  $u$  with respect to  $t$  and  $x$  of order not greater than  $r$ , including  $u$  as the derivative of order zero.

The first-order differential function  $Q[u] := \eta(t, x, u) - \tau(t, x, u)u_t - \xi(t, x, u)u_x$  is called the *characteristic* of the operator  $Q$ . The characteristic PDE  $Q[u] = 0$  is called also the *invariant surface condition*. Denote the manifold defined by the set of all the differential consequences of the characteristic equation  $Q[u] = 0$  in the jet space  $J^{(r)}$  by  $\mathcal{Q}^{(r)}$ .

**Definition 1.** The differential equation  $\mathcal{L}$  of form  $L(t, x, u_{(r)}) = 0$  is called *conditionally (non-classically) invariant* with respect to the operator  $Q$  if the relation  $Q_{(r)}L(t, x, u_{(r)})|_{\mathcal{L} \cap \mathcal{Q}^{(r)}} = 0$  holds, which is called the *conditional invariance criterion*. Then  $Q$  is called an operator of *conditional symmetry* (or  $Q$ -conditional symmetry, nonclassical symmetry, reduction operator etc) of the equation  $\mathcal{L}$ .

In Definition 1 the symbol  $Q_{(r)}$  stands for the standard  $r$ -th prolongation of the operator  $Q$  [14, 15].

The classical (Lie) symmetries are, in fact, partial cases of nonclassical symmetries. Therefore, below we solve the problem on finding only pure nonclassical symmetries which are not equivalent to classical ones. Moreover, our approach is based on application of the notion of equivalence of nonclassical symmetries with respect to a transformation group (see, e.g., [17]). For more details, necessary definitions and properties of nonclassical symmetries we refer the reader to [16, 17, 19, 21].

Since (1) is an evolution equation, there exist two principally different cases of finding  $Q$ :  
1.  $\tau \neq 0$  and 2.  $\tau = 0$ .

First we consider the case with  $\tau \neq 0$ . Here without loss of generality we can assume that  $\tau = 1$ . The results are summarized in the following theorem.

**Theorem 2.** *All possible cases of equations (1) admitting nonclassical symmetries with  $\tau = 1$  are exhausted by the following ones:*

1.  $k = c \tan^2 x$ :  $Q = \partial_t - \cot x \partial_x$ ,
2.  $k = c \tanh^2 x$ :  $Q = \partial_t - \coth x \partial_x$ ,
3.  $k = c \coth^2 x$ :  $Q = \partial_t - \tanh x \partial_x$ ,
4.  $k = cx^2$ :  $Q = \partial_t - \frac{1}{x} \partial_x$ ,
5.  $k = \frac{c^2}{2}$  ( $c > 0$ ):  $Q = \partial_t \pm \frac{c}{2}(3u - 1) \partial_x - \frac{3c^2}{4} u^2 (u - 1) \partial_u$
6.  $k = 2x^{-2}$ :  $Q = \partial_t + \frac{3}{x}(u - 1) \partial_x - \frac{3}{x^2} u (u - 1)^2 \partial_u$ ,

where  $c$  is an arbitrary constant.

*Proof.* We search for reduction operator (operator of nonclassical ( $Q$ -conditional) symmetry) in form  $Q = \partial_t + \xi(t, x, u) \partial_x + \eta(t, x, u) \partial_u$ . Then the system of determining equations for the coefficients of operator  $Q$  has the form

$$\begin{aligned} \xi_{uu} &= 0, \\ 2\xi\xi_u - 2\xi_{xu} + \eta_{uu} &= 0, \\ 2\xi\xi_x - 2\eta\xi_u - 3k\xi_u u^3 + 3k\xi_u u^2 + 2\eta_{xu} - \xi_{xx} + \xi_t &= 0, \\ -k\eta_u u^2(1 - u) + 2k\xi_x u^2(1 - u) + \eta_{xx} - 2\xi_x \eta & \\ = \eta_t - k_x \xi u^2(1 - u) - 2k\eta u + 3k\eta u^2. & \end{aligned}$$

From the first equation we obtain immediately that

$$\xi = \phi(t, x)u + \psi(t, x).$$

Substituting it to the second equation we derive

$$\eta = -\frac{1}{3}\phi^2 u^3 - \phi\psi u^2 + \phi_x u^2 + A(t, x)u + B(t, x).$$

Then, splitting the rest of determining equations with respect to different powers of  $u$  implies the following system of equations for coefficients  $\phi$ ,  $\psi$ ,  $A$  and  $B$ .

$$\begin{aligned} \frac{2}{3}\phi^3 - 3k\phi &= 0, \quad -4\phi\phi_x + 2\phi^2\psi + 3k\phi = 0, \\ -2\phi_x\psi + \phi_t - 2\phi\psi_x - 2\phi A + 3\phi_{xx} &= 0, \\ 2\psi\psi_x - 2\phi B + 2A_x - \psi_{xx} + \psi_t &= 0, \\ -\frac{1}{3}\phi^2\phi_x + \frac{1}{3}k\phi^2 + k\phi\psi &= 3k\phi_x + k_x\phi, \\ \frac{2}{3}\phi^2\psi_x - \frac{2}{3}\phi\phi_{xx} - \frac{8}{3}\phi_x^2 - 2k\psi_x - 2kA + 2\phi\phi_x\psi & \\ = -\frac{2}{3}\phi\phi_t - 2k\phi_x - k_x\phi + k_x\psi, & \\ -2\phi_x A + \phi_{xxx} + 2\phi\psi\psi_x + kA - \phi_{xx}\psi - 4\phi_x\psi_x - \phi\psi_{xx} + 2k\psi_x & \\ = \phi_{tx} - \phi_t\psi - \phi\psi_t - k_x\psi + 3kB, & \\ A_{xx} - 2\psi_x A = A_t + 2\phi_x B - 2kB, & \\ -2\psi_x B + B_{xx} = B_t. & \end{aligned} \tag{2}$$

Now from the first equation of (2) it is obvious that either (i)  $\phi = 0$  or (ii)  $\phi_t = 0$ ,  $k = \frac{2}{9}\phi^2$ . Consider separately these two possibilities.

*Case (i).*  $\phi(t, x) = 0$ . System (2) is read now like

$$\begin{aligned} 2A_x - \psi_{xx} + 2\psi\psi_x + \psi_t &= 0, \\ -2k\psi_x - 2kA - \psi k_x &= 0, \\ 2k\psi_x + kA + k_x\psi - 3kB &= 0, \\ A_{xx} - 2A\psi_x + 2kB - A_t &= 0, \\ B_{xx} - 2\psi_x B - B_t &= 0. \end{aligned}$$

From the second and third equations we deduce that  $A = -3B$ . After substituting this to the previous system we obtain

$$\begin{aligned} 2\psi\psi_x - 6B_x - \psi_{xx} + \psi_t &= 0, \\ k_x\psi + 2k\psi_x - 6kB &= 0, \\ 6\psi_x B - 3B_{xx} + 3B_t + 2kB &= 0, \\ B_{xx} - 2B\psi_x - B_t &= 0. \end{aligned}$$

It follows from the last two equations that  $B = 0$ . Then the rest of the determining equations is read like

$$2\psi\psi_x - \psi_{xx} + \psi_t = 0, \quad k_x\psi + 2k\psi_x = 0.$$

General solution of this system is

$$\begin{aligned} k &= c, \quad \psi = \text{const}, \\ k &= \frac{c}{(ax+b)^2}, \quad \psi = \frac{ax+b}{2ta+m}, \quad \text{and} \\ k &= \frac{c}{\psi^2}, \quad \psi' = \psi^2 + a. \end{aligned}$$

Nonclassical symmetry operator obtained from the first two branches of the solution of the above system are equivalent to the usual Lie symmetry. The third branch (up to equivalence transformations of scaling and translations of  $x$ ) gives cases 1–4 of the theorem.

*Case (ii).*  $\phi_t = 0$ ,  $k = \frac{2}{9}\phi^2$ . Substituting this to system (2) we obtain easily that  $\psi_t = A_t = B_t = 0$ . Then, the rest of the system (2) has the form

$$\begin{aligned} -4\phi_x + 2\phi\psi + \frac{2}{3}\phi^2 &= 0, \\ -2\phi_x\psi - 2\psi_x\phi - 2\phi A + 3\phi_{xx} &= 0, \\ 2\psi\psi_x - 2\phi B + 2A_x - \psi_{xx} &= 0, \\ \frac{2}{9}\phi^2\psi_x - \frac{2}{3}\phi\phi_{xx} - \frac{8}{3}\phi_x^2 - \frac{4}{9}\phi^2 A + \frac{14}{9}\phi\psi\phi_x &= -\frac{8}{9}\phi^2\phi_x, \\ \phi_{xxx} + \frac{4}{9}\phi^2\psi_x - \frac{2}{3}\phi^2 B + \frac{2}{9}\phi^2 A - \phi\psi_{xx} - \phi_{xx}\psi - 4\phi_x\psi_x - 2\phi_x A + 2\phi\psi\psi_x & \\ = -\frac{4}{9}\phi\phi_x\psi - \phi\psi_t, & \\ A_{xx} - 2\psi_x A = -\frac{4}{9}\phi^2 B + 2\phi_x B + A_t, & \\ B_{xx} = 2\psi_x B + B_t. & \end{aligned}$$

It follows then that

$$B = \frac{9(2\psi_x A - A_{xx})}{2(2\phi^2 - 9\phi_x)}, \quad \psi = \frac{6\phi_x - \phi^2}{3\phi}, \quad A = \frac{4\phi\phi_x - 3\phi_{xx}}{6\phi}.$$

Substituting these values into the above system we obtain a system of 4 differential equations for one function  $\phi$  only that has first order differential consequence of form  $3\phi_x^2 + \phi^2\phi_x = 0$ . It is not difficult to show that general solution of this constraint  $\phi = \frac{3}{x+c}$ ,  $\phi = c$  satisfies the whole system for  $\phi$ . These two values of  $\phi$  (taken up to equivalence transformations) give respectively cases 6 and 5 of the theorem.  $\square$

**Note 1.** Cases 1, 2 and 4 with  $c > 0$  were known in [3,4], constant coefficient case 5 with  $c = 2$  can be found in, e.g., [11], while 1,2 and 4 with  $c < 0$ , 3 and 6 are new.

Now we turn into the case 2. That is, we consider nonclassical symmetry operator with  $\tau = 0$ . Without loss of generality we assume it to be of the form

$$Q = \partial_x + \eta(t, x, u)\partial_u.$$

Any operator nonclassical symmetry of the above form satisfies the following equation

$$-\eta_{xx} - 2\eta\eta_{xu} - \eta^2\eta_{uu} + k\eta_u u^2 - k\eta_u u^3 + \eta_t + k_x u^3 - k_x u^2 - 2k\eta u + 3k\eta u^2 = 0. \quad (3)$$

The corresponding invariant surface condition is  $u_x = \eta$ . Eliminating  $u_x$  and  $u_{xx}$ , equation (1) reads

$$u_t = \eta\eta_u + \eta_x + k(x)u^2(1 - u). \quad (4)$$

Using a solution of (3), we can find  $u$  by integrating invariance surface condition and then substituting in (4) to derive a solution of (1).

Now as it was shown in [16,20] for more general case of  $(1+n)$ -dimensional evolution equations, integration of equation (3) is, in some sense, equivalent to integration of the initial equation (1). However, since it contains bigger number of unknown variables it is possible to construct certain partial solutions. Thus, for example, we have succeeded to find all ( $G^\sim$ -inequivalent) partial solutions of equation (3) of the form

$$\eta(x, t, u) = \sum_{p=-m}^n \phi_p(x, t)u^p,$$

where  $m$  and  $n$  are positive integers and  $\phi_p(x, t)$  unknown functions. We find the following results:

1.  $k = 2B^2$ ,  $\eta = B(x)u^2 - \tan x u$ , where

$$-4BB' + 4B'\tan x - B'' + 2B + 2B^2\tan x = 0.$$

For example, this equation has a solution of form  $B = \tan x$ .

2.  $k = 2B^2$ ,  $\eta = B(x)u^2 + \tanh x u$ , where

$$4BB' + 4B'\tanh x + B'' + 2B + 2B^2\tanh x = 0.$$

In particular, this equation has a solution of form  $B = -\tanh x$ .

3.  $k = 2B^2$ ,  $\eta = B(x)u^2 + \coth xu$ , where

$$4BB' + 4B' \coth x + B'' + 2B + 2B^2 \coth x = 0.$$

In particular, this equation has a solution of form  $B = -\coth x$ .

4.  $k = 2B^2$ ,  $\eta = B(x)u^2 + \frac{u}{x}$ , where

$$4xBB' + 4B' + xB'' + 2B^2 = 0.$$

A solution of this equation is  $B = -\frac{1}{x}$ .

5.  $k = \frac{2}{x^2}$ ,  $\eta = \frac{1}{x}(u^2 - 1)$ .

6.  $k = \frac{1}{2x^2}$ ,  $\eta = \frac{1}{2x}u^2$ .

7.  $k = 2 \tan^2 2x$ ,  $\eta = -u^2 \tan 2x$ .

8.  $k = 2 \tanh^2 2x$ ,  $\eta = u^2 \tanh 2x$ .

More detailed investigation of conditional symmetries and construction of associate similarity solutions of equations from class (1) will be the subject of a forthcoming paper.

## Acknowledgements

This research was supported by Cyprus Research Promotion Foundation (project number ΠΡΟΣΕΛΚΥΣΗ/ΠΡΟΝΕ/0308/01).

## References

- [1] Arrigo D.J., Hill J.M. and Broadbridge P., Nonclassical symmetry reductions of the linear diffusion equation with a nonlinear source, *IMA J. Appl. Math.*, 1994, V.52, 1–24.
- [2] Bluman G.W. and Cole J.D., The general similarity solution of the heat equation, *J. Math. Mech.*, 1969, V.18, 1025–1042.
- [3] Bradshaw-Hajek B., Reaction-diffusion equations for population genetics, PhD thesis, School of Mathematics and Applied Statistics, University of Wollongong, 2004. <http://ro.uow.edu.au/theses/201>
- [4] Bradshaw-Hajek B.H., Edwards M.P., Broadbridge P. and Williams G.H., Nonclassical symmetry solutions for reaction-diffusion equations with explicit spatial dependence, *Nonlin. Analysis*, 2007, V.67, 2541–2552.
- [5] Clarkson P.A. and Mansfield E.L., Symmetry reductions and exact solutions of a class of nonlinear heat equations, *Physica D*, 1994, V.70, 250–288.
- [6] Estevez P.G., Non-classical symmetries and the singular manifold method: the Burgers and the Burgers–Huxley equations, *J. Phys. A*, 1994, V.27, 2113–2127.
- [7] Estevez P.G., Gordoia P.R., Nonclassical symmetries and the singular manifold method: theory and six examples, *Stud. Appl. Math.*, 1995, V.95, 73–113.
- [8] Fushchich W.I. and Serov N.I., Conditional invariance and exact solutions of a nonlinear acoustics equation, *Dokl. Akad. Nauk Ukrain. SSR, Ser. A*, 1988, no.10, 27–31 (in Russian)
- [9] Fushchich W.I., Shtelen W.M., Serov M.I. and Popovych R.O.,  $Q$ -conditional symmetry of the linear heat equation, *Proc. Acad. of Sci. Ukraine*, 1992, no.12, 28–33.
- [10] Fushchich W.I. and Tsyfra I.M., On a reduction and solutions of the nonlinear wave equations with broken symmetry, *J. Phys. A: Math. Gen.*, 1987, V.20, L45–L48.

- [11] Hydon, P.E., *Symmetry Methods for Differential Equations*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2000.
- [12] Ivanova N.M., On Lie symmetries of a class of reaction-diffusion equations, *Proceedings of 4th Int. Workshop "Group Analysis of Differential Equations and Integrable Systems"*, (Protaras, Cyprus, 2008), 84–86.
- [13] Levi D. and Winternitz P., Non-classical symmetry reduction: example of the Boussinesq equation, *J. Phys. A: Math. Gen.*, 1989, V.22, 2915–2924.
- [14] Olver P., *Applications of Lie groups to differential equations*, Springer-Verlag, New York, 1986.
- [15] Ovsiannikov L.V., *Group analysis of differential equations*, New York, Academic Press, 1982.
- [16] Popovych R.O., On a class of  $Q$ -conditional symmetries and solutions of evolution equations, in *Symmetry and Analytic Methods in Mathematical Physics, Proceedings of Institute of Mathematics*, Kyiv, 1998, **19**, 194–199 (in Ukrainian).
- [17] Popovych R.O., Vaneeva O.O. and Ivanova N.M., Potential nonclassical symmetries and solutions of fast diffusion equation, *Phys. Lett. A*, 2007, V.362, 166–173 (math-ph/0506067).
- [18] Scott A.C., The electrophysics of a nerve fiber, *Reviews of Modern Physics*, 1975, 487–533.
- [19] Vaneeva O.O., Popovych R.O. and Sophocleous C., Enhanced group analysis and exact solutions of variable coefficient semilinear diffusion equations with a power source, *Acta Appl. Math.*, 2009, V.106, 1–46; arXiv:0708.3457.
- [20] Zhdanov R.Z. and Lahno V.I., Conditional symmetry of a porous medium equation, *Physica D*, 1998, **122**, 178–186.
- [21] Zhdanov R.Z., Tsyfra I.M. and Popovych R.O., A precise definition of reduction of partial differential equations, *J. Math. Anal. Appl.*, 1999, V.238, 101–123 (math-ph/0207023).