arXiv:1011.1614v1 [nlin.SI] 7 Nov 2010

The Gould-Hopper Polynomials in the Novikov-Veselov equation

Jen-Hsu Chang
Department of Computer Science and Information Engineering,
National Defense University,
Tauyuan, Taiwan

November 9, 2010

Abstract

We use the Gould-Hopper (GH) polynomials to investigate the Novikov-Veselov (NV) equation. The root dynamics of the σ-flow in the NV equation is studied using the GH polynomials and then the Lax pair is found. In particulr, when $N = 3, 4, 5,$, one can get the Gold-fish model. The singular rational solutions of the NV equation are also constructed via the Pfaffian of the GH polynomials. The asymptotic behavior is discussed.

PACS number: 02.30.Ik
Keywords: Gould-Hopper Polynomials, σ-flows, Lax Equation, Singular Rational Solutions

1 Introduction

The Novikov-Veselov equation [3, 11, 25, 32] is defined by (U and t is real)

$$\begin{aligned} U_t &= \partial_z^3 U + \bar{\partial}_z U + 3\partial_z(VU) + 3\bar{\partial}_z(\bar{V}U), \\ \bar{\partial}_z V &= \partial_z U. \end{aligned} \tag{1}$$

When $z = \bar{z} = x$, we get the famous KdV equation ($U = V = \bar{V}$)

$$U_t = 2U_{xxx} + 12UU_x.$$

The equation (1) can be represented as the form of Manakov's triad [21]

$$H_t = [A, H] + BH,$$

where H is the two-dimension Schrodinger operator

$$H = \partial_z \bar{\partial}_z + U$$

and

$$A = \partial_z^3 + V\partial_z + \bar{\partial}_z^3 + \bar{V}\bar{\partial}_z, \quad B = V_z + \bar{V}_{\bar{z}}.$$

It is equivalent to the linear representation

$$H\psi = 0, \quad \partial_t \psi = A\psi. \tag{2}$$

We see that the Novikov-Veselov equation (1) preserves a class of the purely potential self-adjoint operators H. Here the pure potential means H has no external electric and magnetic fields. The periodic inverse spectral problem for the two-dimensional Schrodinger operator H was investigated in terms of the Riemann surfaces with some group of involutions and the corresponding Prym Θ-functions [6, 13, 17, 14, 24, 29]. On the other hand, it is known that the Novikov-Veselov hierarchy is a special reduction of the two-component BKP hierarchy [26, 31](and references therein). In [26], the authors showed that the Drinfeld-Sokolov hierarchy of D-type is a reduction of the two-component BKP hierarchy using two different types of pseudo-differential operators, which is different from Shiota's point of view [29]. Finally, it is worthwhile to notice that the Novikov-Veselov equation (1) is a special reduction of the Davey-Stewartson equation [18, 19].

Let $H\psi = H\omega = 0$. Then via the Moutard transformation [1, 22, 23]

$$\begin{aligned} U(z,\bar{z}) &\longrightarrow \hat{U}(z,\bar{z}) = U(z,\bar{z}) + 2\partial\bar{\partial}\ln[i\int(\psi\partial\omega - \omega\partial\psi)dz - (\psi\bar{\partial}\omega - \omega\bar{\partial}\psi)d\bar{z}] \\ \psi &\longrightarrow \theta = \frac{i}{\omega}\int(\psi\partial\omega - \omega\partial\psi)dz - (\psi\bar{\partial}\omega - \omega\bar{\partial}\psi)d\bar{z}, \end{aligned}$$

one can construct a new Schrodinger operator $\hat{H} = \partial_z\bar{\partial}_z + \hat{U}$ and $\hat{H}\frac{1}{\theta} = 0$. The extended Moutard transformation was established such that $\hat{U}(t,z,\bar{z})$ and $\hat{V}(t,z,\bar{z})$ defined by [15, 27]

$$\begin{aligned}\hat{U}(t,z,\bar{z}) &= U(t,z,\bar{z}) + 2\partial\bar{\partial}\ln iW(\psi,\omega),\\ \hat{V}(t,z,\bar{z}) &= V(t,z,\bar{z}) + 2\partial\partial\ln iW,\end{aligned}$$

where the skew product W is defined by

$$\begin{aligned}W(\psi,\omega) &= \int(\psi\partial\omega-\omega\partial\psi)dz-(\psi\bar{\partial}\omega-\omega\bar{\partial}\psi)d\bar{z}+[\psi\partial^3\omega-\omega\partial^3\psi+\omega\bar{\partial}^3-\psi\bar{\partial}^3\omega\\ &+ 2(\partial^2\psi\partial\omega-\partial\psi\partial^2\omega)-2(\bar{\partial}^2\psi\bar{\partial}\omega-\bar{\partial}\psi\bar{\partial}^2\omega)+3V(\psi\partial\omega-\omega\partial\psi)\\ &- 3\bar{V}(\psi\bar{\partial}\omega-\omega\bar{\partial}\psi)]dt, \end{aligned}\tag{3}$$

will also satisfy the Novikov-Veselov equation. In particular, we can use $U = V = 0$ as the seed solution. Then $H = \partial\bar{\partial}$. Let us consider the holomorphic function $P(z,t)$ and satisfy

$$\frac{\partial P}{\partial t} = \frac{\partial^3 P}{\partial z^3}. \tag{4}$$

Then we have

Theorem 1.1 *[30]*
Let $P_1(t,z)$ and $P_2(t,z)$ be holomorphic functions of z and satisfy (4). One defines $\omega_1 = \mathcal{P}_1 + \bar{\mathcal{P}}_1$ and $\omega_2 = \mathcal{P}_2 + \bar{\mathcal{P}}_2$. Then

$$U(t,z,\bar{z}) = 2\partial\bar{\partial}\ln iW(\mathcal{P}_1,\mathcal{P}_2), \tag{5}$$

$$V(t,z,\bar{z}) = 2\partial\partial\ln iW(\mathcal{P}_1,\mathcal{P}_2), \tag{6}$$

where the skew product W is

$$\begin{aligned}W(\mathcal{P}_1,\mathcal{P}_2) &= \mathcal{P}_1\bar{\mathcal{P}}_2-\mathcal{P}_2\bar{\mathcal{P}}_1+\int[(\mathcal{P}_1'\mathcal{P}_2-\mathcal{P}_1\mathcal{P}_2')dz+(\bar{\mathcal{P}}_1\bar{\mathcal{P}}_2{}'-\bar{\mathcal{P}}_1{}'\bar{\mathcal{P}}_2)d\bar{z}]\\ &+ \int[\mathcal{P}_1'''\mathcal{P}_2-\mathcal{P}_1\mathcal{P}_2'''+2(\mathcal{P}_1'\mathcal{P}_2''-\mathcal{P}_1''\mathcal{P}_2')+\bar{\mathcal{P}}_1\bar{\mathcal{P}}_2{}'''-\bar{\mathcal{P}}_1{}'''\bar{\mathcal{P}}_2\\ &+ 2(\bar{\mathcal{P}}_1{}''\bar{\mathcal{P}}_2{}'-\bar{\mathcal{P}}_1{}'\bar{\mathcal{P}}_2{}'')]dt,\end{aligned}\tag{7}$$

is a solution of Novikov-Veselov equation. In particular, if $P_1(t,z)$ and $P_2(t,z)$ are polynomials, then the solution is rational in $z, \bar{z}, t$.

In [2, 7, 8, 12], the rational solutions and line solitons of the Novikov-Veselov equation (1) are constructed by the d-bar dressing method. To get these kinds of solutions, the scattering datum have to be delta-type and the reality of U also puts some extra constraints on them. In these cases, the W-function of (7) can be expressed as a determinant of some matrix.

To study the dispersion relation (4), Taimanov and Tsarev introduced the σ-flows for polynimial-type solutions [30]

$$\mathbb{P}_N(t,z) = z^N + \sigma_1 z^{N-1} + \sigma_2 z^{N-2} + \cdots + \sigma_{N-1} z + \sigma_N.$$

Then the flow (4) generates a linear flow

$$\dot{\sigma}_k = (N-K+3)(N-k+2)(N-k+1)\sigma_{k-3}, \quad k = 1,2,3\cdots N. \tag{8}$$

it can be seen that σ_1, σ_2 are conserved quantities. Indeed, $\sigma_1, \sigma_2, \cdots, \sigma_N$ are the elementary symmetric polynomials in the roots $q_1, q_2, \cdots, q_N$ of $\mathbb{P}_N(z)$:

$$\begin{aligned} \sigma_1(\vec{q}) &= -\sum_{i=1}^{N} q_i, \qquad \sigma_2(\vec{q}) = \sum_{i<j} q_i q_j, \\ \sigma_3(\vec{q}) &= -\sum_{i<j<k} q_i q_j q_k, \cdots, \sigma_N(\vec{q}) = (-1)^N q_1 q_2 \cdots q_N. \end{aligned} \tag{9}$$

The integrable (even linear) evolution of $\vec{\sigma} = (\sigma_1, \sigma_2, \cdots, \sigma_N)$ induces a dynamical system on the symmetric product $S^N C$ of the complex roots . We call such a dynamical system on $S^N C$ a σ-system.

From (7), we see that given two solutions $P_1(t,z)$ and $P_2(t,z)$, by a substitution of $e^{i\lambda_1} P_1(t,z)$ and $e^{i\lambda_2} P_2(t,z)$, where λ_1 and λ_2 are real-valued constants, into (7), one obtains a solution of the Novikov-Veselov solution. Therefore, to each pair of holomorphic solutions of (4), we can get an $(S^1 \times S^1)$-family of solutions to the Novikiv-Veselov equation [30].

The paper is organized as follows. In section 2, we describe the Gould-Hopper polynomials using the generating function and establish the recursive relation. In section 3, one studies the root dynamics of σ-flow and the Lax pair is constructed. Also, the asymptotic behavior is discussed. In section 4, the singular rational solutions are found using the Gould-Hopper polynomials and their asymptotic behavior is investigated. Section 5 is devoted to the concluding remarks.

2 Gould-Hopper Polynomials

In this section, we introduce the Gould-Hopper polynomials and use them to get the solutions of (4). To investigate the polynomial-type solutions of (4), inspired by the work in [10], one utilizes the Gould-Hopper polynomials [9]. The generating function of the Gould-Hopper polynomials $P_N(t,z)$ is

$$e^{\lambda z + \lambda^3 t} = \sum_{N=0}^{\infty} P_N(t,z)\lambda^N.$$

Indeed, the Gould-Hopper polynomials $P_N(t,z)$ has the operator representation

$$P_N(t,z) = e^{t\partial_z^3} z^N = [1 + t\partial_z^3 + \frac{t^2\partial_z^6}{2!} + \frac{t^3\partial_z^9}{3!} + \frac{t^4\partial_z^{12}}{4!} + \cdots]z^N.$$

We remark that in general the Gould-Hopper polynomials are defined by $P_N^{(m)}(t,z) = e^{t\partial_z^m} z^N$. Here we take $m = 3$.

One notices that the Gould-Hopper polynomials $P_N(t,z)$ are characterized by (4) and $P_N(0,z) = z^N$. For example,

$$\begin{aligned}
P_0 &= 1, \quad P_1 = z, \quad P_2 = z^2, \quad P_3 = z^3 + 6t, \quad P_4 = z^4 + 24tz,\\
P_5 &= z^5 + 60tz, \quad P_6 = z^6 + 120tz^3 + 360t^2\\
P_7 &= z^7 + 210tz^4 + 2520z, \quad P_8 = z^8 + 336tz^5 + 10080t^2z^2\\
P_9 &= z^9 + 504tz^6 + 30240t^2z^3 + 60480t^3\\
P_{10} &= z^{10} + 720tz^7 + 75600t^2z^4 + 604800t^3z
\end{aligned}$$

Actually, we have

$$\begin{aligned}
P_N(t,z) &= N! \sum_{k=0}^{[N/3]} \frac{t^k z^{N-3k}}{k!(N-3k)!}\\
\frac{dP_N(t,z)}{dz} &= NP_{N-1}(t,z) \qquad (10)
\end{aligned}$$

From the operation calculus, one has

$$(z + 3t\partial_z^2)P_{N-1}(t,z) = P_N(t,z), \quad N \geq 1.$$

Hence we yield the recursive relation

$$P_N(t,z) = zP_{N-1}(t,z) + 3t(N-1)(N-2)P_{N-3}(t,z). \qquad (11)$$

We can see that if we consider the equation (4) with the initial data of analytical function

$$P(0,z) = \sum_{N=0}^{\infty} \alpha_N z^N,$$

then the formal solution is

$$P(t,z) = e^{t\partial_z^3} \sum_{N=0}^{\infty} \alpha_N z^N = \sum_{N=0}^{\infty} \alpha_N P_N(t,z). \qquad (12)$$

The successive operations of the operator $(z + 3t\partial_z^2)$ on the solution (12) can help us construct more solutions of (4). For example, if $P(0,z) = \sin z$, then we have,

$$\begin{aligned}
e^{t\partial_z^3} \sin z &= e^{t\partial_z^3} \sum_{N=0}^{\infty} \frac{(-1)^N}{(2N+1)!} z^{2N+1} = \sum_{N=0}^{\infty} \frac{(-1)^N}{(2N+1)!} P_{2N+1}(t,z)\\
&= \sin(z-t).
\end{aligned}$$

The last equation uses the fact $e^{t\partial_z^3}e^z = e^{z+t}$. Hence

$$(z + 3t\partial_z^2)^N \sin(z - t), \quad N = 0, 1, 2, 3, 4, \cdots$$

are also solutions of (4).

Remark: Let's define

$$\varphi(\lambda) = e^{\lambda z + \lambda^3 t} - e^{-\lambda z - \lambda^3 t} = 2sinh(\lambda z + \lambda^3 t).$$

Then $\varphi(\lambda)$ satisfies

$$\varphi(\lambda)_{zz} = \lambda^2 \varphi(\lambda), \quad \varphi(\lambda)_t = \varphi(\lambda)_{zzz}.$$

On expanding

$$\varphi(\lambda) = \sum_{i=0}^{\infty} \phi_i \lambda^{2i+1},$$

one has

$$\phi_{0,zz} = 0, \quad \phi_{i+1,zz} = \phi_i, \quad \phi_{i,t} = \phi_{i,zzz}, \quad i \geq 0.$$

Actually,

$$\phi_i = \sum_{k=0}^{[\frac{2i+1}{3}]} \frac{1}{k!(2i + 1 - 3k)!} z^{2i+1-3k} t^k, \quad i \geq 0.$$

It is known that ϕ_i can be used to construct the Wronskian solutions of the KdV equation. The details can be found in [20].

3 Root Dynamics of σ-flows

In this section, one uses the Gould-Hopper polynomials to study the root dynamics of the σ-flows (8).

Let's write $\mathbb{P}_N(t, z)$ as

$$\mathbb{P}_N(t, z) = (z - q_1(t))(z - q_2(t)) \cdots (z - q_N(t)).$$

Then from the equation (4), one gets the root dynamics

$$\dot{q}_j = -6 \sum_{m<n, \quad j \neq m,n}^{N} \frac{1}{(q_j - q_m)(q_j - q_n).} \tag{13}$$

For example, when N=3, we have

$$\begin{aligned}
\dot{q}_1 &= -6 \frac{1}{(q_1 - q_2)(q_1 - q_3)} \\
\dot{q}_2 &= -6 \frac{1}{(q_2 - q_1)(q_2 - q_3)} \\
\dot{q}_3 &= -6 \frac{1}{(q_3 - q_1)(q_3 - q_2)}
\end{aligned}$$

For N=4, we get

$$\begin{aligned}
\dot{q_1} &= -6[\frac{1}{(q_1-q_2)(q_1-q_3)} + \frac{1}{(q_1-q_3)(q_1-q_4)} + \frac{1}{(q_1-q_2)(q_1-q_4)}] \\
\dot{q_2} &= -6[\frac{1}{(q_2-q_1)(q_2-q_3)} + \frac{1}{(q_2-q_3)(q_2-q_4)} + \frac{1}{(q_2-q_1)(q_2-q_4)}] \\
\dot{q_3} &= -6[\frac{1}{(q_3-q_1)(q_3-q_2)} + \frac{1}{(q_3-q_1)(q_3-q_4)} + \frac{1}{(q_3-q_2)(q_3-q_4)}] \\
\dot{q_4} &= -6[\frac{1}{(q_4-q_2)(q_4-q_3)} + \frac{1}{(q_4-q_1)(q_4-q_2)} + \frac{1}{(q_4-q_1)(q_1-q_3)}]
\end{aligned}$$

We notice that since σ_1 and σ_2 are conserved quantities, one knows that

$$\sum_{i=1}^{N} q_i, \qquad \sum_{i=1}^{N} q_i^2$$

are conserved densities of (13).

Now, we can investigate the properties of the root dynamics (13) by the Gould-Hopper polynomials:

- Initial Value Problem : The root dynamics of σ-flow can be solved by

$$\begin{aligned}
\mathbb{P}_N(t,z) &= (z-q_1(t))(z-q_2(t))\cdots(z-q_N(t)) \\
&= P_N(t,z) + C_1 P_{N-1}(t,z) + \cdots + C_N P_0(t,z), \qquad (14)
\end{aligned}$$

  where the constants $C_1, C_2, \cdots, C_{N-1}, C_N$ are determined by the initial values of $q_1(0), q_2(0), \cdots, q_N(0)$, that is,

$$\begin{aligned}
C_1 &= -\sum_{i=1}^{N} q_i(0), \qquad C_2 = \sum_{i<j} q_i(0)q_j(0), \\
C_3 &= -\sum_{i<j<k} q_i(0)q_j(0)q_k(0), \quad \cdots, \\
C_N &= (-1)^N q_1(0)q_2(0)\cdots q_N(0).
\end{aligned}$$

  Therefore, it is seen that the solutions $q_1(t), q_2(t), \cdots, q_N(t)$ can be obtained algebraically.

- Lax pair:
  Firstly, we study the root dynamics of the Gould-Hopper polynomials, which correspond to the initial values $q_1(0) = q_2(0) = \cdots = q_N(0) = 0$.

  Let's define the $N \times N$ matrix by

$$X(t) = \begin{cases} a_{i,i+1} = 1, & \text{if } i = 1,2,3,\cdots,N\ ; \\ a_{i,i-2} = -3t(i-1)(i-2), & \text{if } i = 3,4\cdots,N-1; \\ 0, & \text{otherwise.} \end{cases} \qquad (15)$$

Then from the recursive relation (11), one knows that

$$P_N(t,z) = det(X(t) - zI_N).$$

For example, when $N = 3$,

$$X(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6t & 0 & 0 \end{pmatrix};$$

N=4,

$$X(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -6t & 0 & 0 & 1 \\ 0 & -18t & 0 & 0 \end{pmatrix};$$

N=5,

$$X(t) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -6t & 0 & 0 & 1 & 0 \\ 0 & -18t & 0 & 0 & 1 \\ 0 & 0 & -36t & 0 & 0 \end{pmatrix}.$$

We can write $X(t)$ as

$$X(t) = R(t)QR^{-1}(t),$$

where $Q = diag(q_1(t), q_2(t), \cdots, q_N(t)$ and

$$R(t) = \begin{pmatrix} P_0(q_1,t) & P_0(q_2,t) & P_0(q_3,t) & \cdots & P_0(q_N,t) \\ P_1(q_1,t) & P_1(q_2,t) & P_1(q_3,t) & \cdots & P_1(q_N,t) \\ P_2(q_1,t) & P_2(q_2,t) & P_2(q_3,t) & \cdots & P_2(q_N,t) \\ \vdots & & & & \\ P_N(q_1,t) & P_N(q_2,t) & P_N(q_3,t) & \cdots & P_N(q_N,t) \end{pmatrix}.$$

For instance, when $N = 3$,

$$R(t) = \begin{pmatrix} 1 & 1 & 1 \\ q_1 & q_2 & q_3 \\ q_1^2 & q_2^2 & q_3^2 \end{pmatrix};$$

N=4,

$$R(t) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ q_1 & q_2 & q_3 & q_4 \\ q_1^2 & q_2^2 & q_3^2 & q_4^2 \\ q_1^3+6t & q_2^3+6t & q_3^3+6t & q_4^3+6t \end{pmatrix};$$

N=5,

$$R(t)=\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ q_1 & q_2 & q_3 & q_4 & q_5 \\ q_1^2 & q_2^2 & q_3^2 & q_4^2 & q_5^2 \\ q_1^3+6t & q_2^3+6t & q_3^3+6t & q_4^3+6t & q_5^3+6t \\ q_1^4+24t & q_2^4+24t & q_3^4+24t & q_4^4+24t & q_5^4+24t \end{array}\right).$$

From the initial value problem, we notice that the polynomials t^n can be replaced by the elementary symmetric polynomials of the roots $q_1, q_2, \cdots, q_N$. Hence one has $R(\vec{q})$. It can be seen that

$$\dot{X}(t) = RLR^{-1},$$

where

$$L = \dot{Q} + [M, Q], \quad M = R^{-1}\dot{R}.$$

For example, when N=3,

$$L(t)=\left(\begin{array}{ccc} \dot{q}_1 & \dot{q}_2\frac{q_2-q_3}{q_3-q_1} & \dot{q}_3\frac{q_3-q_2}{q_2-q_1} \\ \dot{q}_1\frac{q_1-q_3}{q_3-q_2} & \dot{q}_2 & \dot{q}_3\frac{q_3-q_1}{q_1-q_2} \\ \dot{q}_1\frac{q_1-q_2}{q_2-q_3} & \dot{q}_2\frac{q_2-q_1}{q_1-q_3} & \dot{q}_3 \end{array}\right);$$

N=4,

$$L(t)=\left(\begin{array}{cccc} \dot{q}_1 & -\frac{\dot{q}_2(q_2-q_3)(q_2-q_4)+6}{(q_1-q_3)(q_1-q_4)} & -\frac{\dot{q}_3(q_3-q_2)(q_3-q_4)+6}{(q_1-q_2)(q_1-q_4)} & -\frac{\dot{q}_4(q_4-q_2)(q_4-q_3)+6}{(q_1-q_2)(q_1-q_3)} \\ -\frac{\dot{q}_1(q_1-q_3)(q_1-q_4)+6}{(q_2-q_3)(q_2-q_4)} & \dot{q}_2 & -\frac{\dot{q}_3(q_3-q_1)(q_3-q_4)+6}{(q_2-q_4)(q_2-q_1)} & -\frac{\dot{q}_4(q_4-q_1)(q_4-q_3)+6}{(q_2-q_3)(q_2-q_1)} \\ -\frac{\dot{q}_1(q_1-q_2)(q_1-q_4)+6}{(q_2-q_3)(q_4-q_3)} & -\frac{\dot{q}_2(q_2-q_4)(q_2-q_1)+6}{(q_4-q_3)(q_1-q_3)} & \dot{q}_3 & -\frac{\dot{q}_4(q_4-q_1)(q_4-q_2)+6}{(q_2-q_3)(q_1-q_3)} \\ -\frac{\dot{q}_1(q_1-q_2)(q_1-q_3)+6}{(q_4-q_3)(q_4-q_2)} & -\frac{\dot{q}_2(q_2-q_3)(q_2-q_1)+6}{(q_4-q_3)(q_4-q_1)} & -\frac{\dot{q}_3(q_3-q_1)(q_3-q_2)+6}{(q_4-q_1)(q_4-q_2)} & \dot{q}_4 \end{array}\right).$$

Since

$$\frac{dX(t)}{dt} = \begin{cases} a_{i,i-2} = -3(i-1)(i-2), & \text{if } i = 3, 4\cdots, N-1; \\ 0, & \text{otherwise,} \end{cases}$$

we know $\frac{dX(t)}{dt}$ is a nilpotent matrix and hence L is a nilpotent one, too. So

$$tr(L^r) = tr[\frac{dX(t)}{dt}]^r = 0, \quad r = 1, 2, 3, \ldots, ..$$

Actually, a simple calculation yields

$$L^{[\frac{N}{2}]+1} = 0, \quad N \geq 3.$$

Now,

$$\frac{d^2X(t)}{dt^2} = 0$$

will imply the Lax equation

$$\frac{dL(t)}{dt} = [L, M]. \tag{16}$$

For $N = 3, 4, 5$, we see that, by Maple software, q_i satisfies the following Goldfish model [5], a limiting case of the Ruijesenaars-Schneider system:

$$\ddot{q}_i = 2 \sum_{j \neq i} \frac{q_i q_j}{q_i - q_j}. \tag{17}$$

The reason is that $P_i, i = 3, 4, 5$ are linear in t-variable(see the appendix). For $N = 6$, we have from the diagonal terms of the Lax equation (16)

$$\ddot{q}_i = 2 \sum_{j \neq i}^{6} \frac{q_i q_j}{q_i - q_j} + \frac{\sum_{j=1}^{6} (\text{some quadratic terms of } \vec{q}) q_j + 720}{\prod_{i \neq j}^{6} (q_i - q_j)}.$$

Secondly, we consider the general case. Let's define the 2D Appell polynomials $\mathbb{R}_n(z, t)$ by means of the generating function [4]:

$$\mathbb{G}_A(z, t, \lambda) = A(\lambda) e^{\lambda z + \lambda^3 t} = \sum_{n=0}^{\infty} \mathbb{R}_n(z, t) \frac{\lambda^n}{n!},$$

where

$$A(\lambda) = \sum_{0}^{N} \frac{\mathbf{R}_k}{k!} \lambda^k,$$

$\mathbf{R}_k' s$ being constants and $\mathbf{R}_0 = 1$. Then one has the following representation formulas

$$\begin{aligned} \mathbb{R}_N &= \sum_{h=0}^{N} \binom{N}{h} \mathbf{R}_{N-h} P_h(z, t) \\ &= N! \sum_{h=0}^{N} \frac{\mathbf{R}_{N-h}}{(N-h)!} \sum_{r=0}^{[\frac{h}{3}]} \frac{z^{h-3r} t^r}{(h-3r)! r!}. \end{aligned} \tag{18}$$

It's easy to see that the polynomials $\mathbb{R}_n(z, t)$ also satisfy the linear equation (4). When comparing (14) with (18), we have

$$\mathbf{R}_{N-h} = \frac{C_{N-h}}{\binom{N}{h}}.$$

Now, it's suitable to introduce the coefficients of the Taylor expansion

$$\frac{A'(\lambda)}{A(\lambda)} = \sum_{n=0}^{\infty} \alpha_n \frac{\lambda^n}{n!}.$$

It can be seen that the coefficients α_n can be expressed by $\mathbf{R}_0, \mathbf{R}_1, \cdots, \mathbf{R}_{n+1}$ (or the initial datum). The recurrence relation for the 2D Appell polynomial $\mathbb{R}_N(z,t)$ can be written as follows [4]:

$$\begin{aligned}\mathbb{R}_0(z,t) &= 1 \\ \mathbb{R}_N(z,t) &= (z+\alpha_0)\mathbb{R}_{N-1}(z,t) + 3t(N-1)(N-2)\mathbb{R}_{N-3}(z,t) \\ &+ \sum_{k=0}^{N-2} \binom{N-1}{k} \alpha_{N-k-1}\mathbb{R}_k(z,t). \end{aligned} \tag{19}$$

When $A(\lambda)=1$, this recursive relation becomes (11). Hence the relation (19) is a generalization of (11) for arbitrary initial data. From the recurrence relation (19), one can also similarly construct the matrix corresponding to (15). Then we follow the previous procedures and finally can get the Lax equation (16) for general case. Therefore the root dynamics (13) is Lax-integrable. But the computations are more involved and one doesn't pursuit them here.

We notice here that for $N=3,4,5$ the root dynamics of $\mathbb{R}_N$ also satisfies the Gold-fish model (17).

- Asymptotic behavior

It is known that the Gould-Hopper polynomial $P_N(t.z)$ has the scaling property:

$$P_N(t,z) = t^{\frac{N}{3}}\hat{P}_N(\frac{z}{t^{1/3}}), \tag{20}$$

where $\hat{P}_N(\eta)$ is the so-called Appell polynomials [4] (and references therein) in $\eta = \frac{z}{t^{1/3}}$. For example,

$$\begin{aligned} P_8(t,z) &= z^8 + 336tz^5 + 10080t^2z^2 = t^{\frac{8}{3}}[\eta^8 + 336\eta^5 + 10080\eta^2] \\ &= t^{\frac{8}{3}}\hat{P}_8(\eta). \end{aligned}$$

Then the k-th zero $\lambda_N^{(k)}$ of $\hat{P}_N(\eta)$ determines the dynamics of the root q_k, i.e.,

$$q_k(t) = \lambda_N^{(k)} t^{1/3}.$$

Since $\hat{P}_N(\xi\lambda_N^{(k)}) = 0, \xi^3 = 1$, one knows that the roots q_k are located on the circles in the plane with time dependent radius. Finally, from the Initial value Problem (14) and (20), we know that when $t \to \infty$ and $z \to \infty$ such that $|z|^3/t \to$ constant, $P_N(t,z)$ plays the dominant role; hence one yields

$$q_k(t) \to \lambda_N^{(k)} t^{1/3}.$$

Consequently, the roots asymptotically will follow diagonal lines.

4 Singular Rational Solutions of Novikov-Vaselov equation

In this section, we construct singular rational solutions using the Pfaffian and the extended Moutard transformation via the Gould -Hopper polynomials.

In the extended Moutard transformation (7), we see that $\mathcal{P}_1$ and $\mathcal{P}_2$ can be replaced by linear combinations of Gould-Hopper polynomials to construct rational solutions of Novikov-Vaselov equation. In particular, let

$$\mathcal{P}_1(t,z)=\sum_{m=0}^{N}a_mP_m(t,z),\quad \mathcal{P}_2=1,$$

where a_m are constants. Then a simple calculation can get, using (4) and (7),

$$\begin{aligned} W &= 3(\mathcal{P}_1-\bar{\mathcal{P}}_1)-(\sum_{m=0}^{N}a_mz^m-\sum_{m=0}^{N}\bar{a}_m\bar{z}^m)+iC, \\ U(z,\bar{z},t) &= 2\partial\bar{\partial}\ln iW = 2\frac{\partial(3\mathcal{P}_1-\sum_{m=0}^{N}a_mz^m)\bar{\partial}(3\bar{\mathcal{P}}_1-\sum_{m=0}^{N}\bar{a}_m\bar{z}^m)}{[3(\mathcal{P}_1-\bar{\mathcal{P}}_1)-(\sum_{m=0}^{N}a_mz^m-\sum_{m=0}^{N}\bar{a}_m\bar{z}^m)+iC]^2}, \\ V(z,\bar{z},t) &= 2\partial\partial iW = 2\frac{W\partial\partial W-(\partial W)^2}{W^2} \end{aligned} \tag{21}$$

$$=2\frac{[3(\mathcal{P}_1-\bar{\mathcal{P}}_1)-(\sum_{m=0}^{N}a_mz^m-\sum_{m=0}^{N}\bar{a}_m\bar{z}^m)+iC]\partial\partial[3\mathcal{P}_1-\sum_{m=0}^{N}a_mz^m]-[\partial(3\mathcal{P}_1-\sum_{m=0}^{N}a_mz^m)]^2}{[3(\mathcal{P}_1-\bar{\mathcal{P}}_1)-(\sum_{m=0}^{N}a_mz^m-\sum_{m=0}^{N}\bar{a}_m\bar{z}^m)+iC]^2},$$

where C is real constant. From (21), we see that the solution is singular at the imaginary part of W when it is zero. Also, there are infinite wave functions corresponding to them (see the equation (27) below).

As before, let $t\to\infty$ and $z\to\infty$ such that $|z|^3/t\to$ constant. Then from (20), noting that $\eta=\frac{z}{t^{1/3}}$,

$$\mathcal{P}_1(t,z)\to a_NP_N(t,z)=a_Nt^{\frac{N}{3}}\hat{P}_N(\frac{z}{t^{1/3}}).$$

Hence

$$U(z,\bar{z},t)\to 2t^{-2/3}N^2\frac{a_N\bar{a}_N(3\hat{P}_{N-1}(\eta)-\eta^{N-1})(3\bar{\hat{P}}_{N-1}(\bar{\eta})-\bar{\eta}^{N-1})}{[a_N(3\hat{\mathcal{P}}_N(\eta)-\eta^N)-\bar{a}_N(3\bar{\hat{\mathcal{P}}}_N(\bar{\eta})-\bar{\eta}^N)]^2}. \tag{22}$$

Eventually, the solution is singular when the imaginary part of $a_N(3\hat{\mathcal{P}}_N(\eta)-\eta^N)$ is equal to zero. If we let $|z|^3/t=|\eta|\to\infty$, then one can get from (22)

$$|U(z,\bar{z},t)|\in\mathcal{O}(\frac{1}{|z|^2}).$$

In particular, letting

$$\mathcal{P}_1(t,z) = aP_N(t,z) + b, \quad \text{a, b constants,}$$

we have

$$W = a(3P_N(t,z) - z^N) - \bar{a}(3\bar{P}_N(t,\bar{z}) - \bar{z}^N) + iC$$

and using (10), one can reduce the equation (21) to

$$\begin{aligned} U(z,\bar{z},t) &= 2\frac{a\bar{a}(3NP_{N-1} - z^{N-1})(3N\bar{P}_{N-1} - \bar{z}^{N-1})}{[a(3P_N(t,z) - z^N) - \bar{a}(3\bar{P}_N(t,\bar{z}) - \bar{z}^N) + iC]^2} \\ V &= \end{aligned}$$

$$2\frac{aN(N-1)[a(3P_N - z^N) - \bar{a}(3\bar{P}_N - \bar{z}^N) + iC](P_{N-2} - z^{N-2}) - a^2N^2(3P_{N-1} - z^{N-1})^2}{[a(3P_N(t,z) - z^N) - \bar{a}(3\bar{P}_N(t,\bar{z}) - \bar{z}^N) + iC]^2}.$$

For instance, if we choose $\mathcal{P}_1(t,z) = aP_3(t,z) + b = a(z^3 + 6t) + b$, then we obtain by (21)

$$\begin{aligned} U &= \frac{32a\bar{a}z^2\bar{z}^2}{[az^3 + 9at - \bar{a}\bar{z}^3 - 9\bar{a}t + iC/2]^2} \\ &= -8\frac{(\alpha^2 + \beta^2)(x^2 + y^2)^2}{(3\alpha x^2 y - \alpha y^3 + \beta x^3 - 3\beta x y^2 + 9\beta t + C/4)^2} \\ V &= -18\frac{a^2 z^4}{[az^3 + 9at - \bar{a}\bar{z}^3 - 9\bar{a}t + iC/2]^2}, \end{aligned}$$

where $a = \alpha + \beta i$. Also, by (22), we have as $t \to \infty$

$$U \to \frac{9}{2}t^{-2/3}\frac{a\bar{a}\eta^2\bar{\eta}^2}{[a(\eta^3 + 9) - \bar{a}(\bar{\eta}^3 + 9)]^2},$$

which is singular when the imaginary part of $a\eta^3 + 9a$ is zero.

One sees that if $\mathcal{P}_1(t,z)$ is any holomorphic solution of (4), then the formula (21) is still correct.

Next, we use the Gould-Hopper polynomials to construct Pfaffian-type solutions. Given any N Gould-Hopper polynomials $\psi_1, \psi_2, \psi_3, \cdots, \psi_N$, (or their linear combinations), the N-step extended Moutard transformation can be obtained in the Pfaffian [1, 23] (also see [16, 28])

$$P(\psi_1, \psi_2, \psi_3, \cdots, \psi_N) = \begin{cases} Pf(\psi_1, \psi_2, \psi_3, \cdots, \psi_N), & N \quad \text{even}, \\ \widetilde{Pf}(\psi_1, \psi_2, \psi_3, \cdots, \psi_N), & N \quad \text{odd}, \end{cases}$$

$$Pf(\psi_1, \psi_2, \psi_3, \cdots, \psi_N) = \sum_\sigma \epsilon(\sigma) W_{\sigma_1\sigma_2} W_{\sigma_3\sigma_4} \cdots W_{\sigma_{N-1}\sigma_N} \tag{23}$$

$$\widetilde{Pf}(\psi_1, \psi_2, \psi_3, \cdots, \psi_N) = \sum_\sigma \epsilon(\sigma) W_{\sigma_1\sigma_2} W_{\sigma_3\sigma_4} \cdots W_{\sigma_{N-2}\sigma_{N-1}} \psi_{\sigma_N}, \tag{24}$$

where $W_{\sigma_i\sigma_j} = W(\psi_{\sigma(i)}, \psi_{\sigma(j)})$ is defined by the skew product (7), i.e.,

$$\begin{aligned}
W(\psi_{\sigma(i)}, \psi_{\sigma(j)}) &= \sqrt{-1}c_{ij} + \psi_{\sigma(i)}\bar{\psi}_{\sigma(j)} - \psi_{\sigma(j)}\bar{\psi}_{\sigma(i)} + \int_0^z [\psi'_{\sigma(i)}\psi_{\sigma(j)} - \psi_{\sigma(i)}\psi'_{\sigma(j)}]dz \\
&+ \int_0^{\bar{z}} [\bar{\psi}_{\sigma(i)}\bar{\psi}'_{\sigma(j)} - \bar{\psi}'_{\sigma(i)}\bar{\psi}_{\sigma(j)}]d\bar{z} \\
&+ \int_0^t [\psi'''_{\sigma(i)}\psi_{\sigma(j)} - \psi_{\sigma(i)}\psi'''_{\sigma(j)} + 2(\psi'_{\sigma(i)}\psi''_{\sigma(j)} - \psi''_{\sigma(i)}\psi'_{\sigma(j)}) + \bar{\psi}_{\sigma(i)}\bar{\psi}'''_{\sigma(j)} \\
&- \bar{\psi}'''_{\sigma(i)}\bar{\psi}_{\sigma(j)} + 2(\bar{\psi}''_{\sigma(i)}\bar{\psi}'_{\sigma(j)} - \bar{\psi}'_{\sigma(i)}\bar{\psi}''_{\sigma(i)})]dt, \qquad (25)
\end{aligned}$$

where c_{ij} is real constant and $c_{ij} = -c_{ij}$. The summations σ in (23) and (24) run from over the permutations of $\{1, 2, 3, \cdots, N\}$ such that $\sigma_1 < \sigma_2, \sigma_3 < \sigma_4, \sigma_5 < \sigma_6, \cdots$ and

$$\sigma_1 < \sigma_3 < \sigma_5 < \sigma_7 \cdots,$$

with $\epsilon(\sigma) = 1$ for the even permutations and $\epsilon(\sigma) = -1$ for the odd permutations. Then the solution U and V can be expressed as [1]

$$\begin{aligned}
U &= 2\partial\bar{\partial}[\ln P(\psi_1, \psi_2, \psi_3, \cdots, \psi_N)] \\
V &= 2\partial\partial[\ln P(\psi_1, \psi_2, \psi_3, \cdots, \psi_N)], \qquad (26)
\end{aligned}$$

and the corresponding wave function is

$$\varphi = \frac{P(\psi_1, \psi_2, \psi_3, \cdots, \psi_N, \vartheta)}{P(\psi_1, \psi_2, \psi_3, \cdots, \psi_N)}, \qquad (27)$$

where ϑ is an arbitrary Gould-Hopper polynomial different from $\psi_1, \psi_2, \psi_3, \cdots, \psi_N$.

We notice here that since the skew product $W(\psi_i, \psi_j)$ in (25) is purely imaginary number, we sometimes use $iP(\psi_1, \psi_2, \psi_3, \cdots, \psi_N)$ in (26) to replace $P(\psi_1, \psi_2, \psi_3, \cdots, \psi_N)$, depending on $\frac{N}{2}$ (N even) or $\frac{N-1}{2}$ (N odd). For example, when $N = 3$, one has

$$\begin{aligned}
U &= 2\partial\bar{\partial}[\ln iP(\psi_1, \psi_2, \psi_3)] = 2\partial\bar{\partial}\{\ln i[\psi_1 W(\psi_2, \psi_3) - \psi_2 W(\psi_1, \psi_3) + \psi_3 W(\psi_1, \psi_2)]\} \\
V &= 2\partial\partial[\ln iP(\psi_1, \psi_2, \psi_3)] = 2\partial\partial\{\ln i[\psi_1 W(\psi_2, \psi_3) - \psi_2 W(\psi_1, \psi_3) + \psi_3 W(\psi_1, \psi_2)]\},
\end{aligned}$$

and the corresponding wave function is

$$\varphi = \frac{P(\psi_1, \psi_2, \psi_3, \vartheta)}{iP(\psi_1, \psi_2, \psi_3)} = \frac{W(\psi_1, \psi_2)W(\psi_3, \vartheta) - W(\psi_1, \psi_3)W(\psi_2, \vartheta) + W(\psi_1, \vartheta)W(\psi_2, \psi_3)}{i[\psi_1 W(\psi_2, \psi_3) - \psi_2 W(\psi_1, \psi_3) + \psi_3 W(\psi_1, \psi_2)]}.$$

We remark here that in [7, 8], the rational solutions are also obtained using the d-bar dressing method; however, the corresponding wave functions are of different types, i.e., there is a product with the exponential function. A comparison between these rational solutions could be interesting.

5 Concluding Remarks

In this paper we have studied the Novikov-Veselov equation using the Gould-Hopper polynomials. Firstly, one investigates the root dynamics of the so-called σ-flows and gets the Lax pair; moreover, one finds that when $N = 3, 4, 5$, the root dynamics satisfies the Gold-Fish model. Although the Lax pair is established, only two conserved densities are found. The reason is that the Lax operator is nilpotent. Also, the asymptotic behavior is studied. Secondly, we construct singular rational solutions using the Gould-Hopper polynomials and the skew product (25); besides, the Pfaffian-type solutions and the corresponding wave functions are obtained via these polynomials.

Acknowledgments

The author is grateful to Prof. Iskander Taimanov and Prof. Sergy Tsarev for valuable discussions when they visited Taiwan. He also thanks Prof. Wen-Xiu Ma for his suggestions. This work is supported in part by the National Science Council of Taiwan under Grant No. NSC 98-2115-M-606-001-MY2.

Appendix

Gold-Fish Model

For the Goldfish Model

$$\ddot{q}_i = 2\sum_{j\neq i}\frac{\dot{q}_i\dot{q}_j}{\dot{q}_i - q_j},$$

its initial value problem can be solved by the statement: $z = q_i(t), i = 1, 2, \cdots, N$ are the N roots of the equation [5]

$$\sum_{i=1}^{N}\frac{\dot{q}_i(0)}{z - q_i(0)} = \frac{1}{t}.$$

It can be seen that it is a polynomial in z with coefficients linear in t. Then the special choices of initial datum can get the solutions of the root dynamics (13) for the cases $N = 3, 4, 5$. To illustrate it, we take $N = 3$ as an example. When $N = 3$, we have

$$\frac{\dot{q}_1(0)}{z - q_1(0)} + \frac{\dot{q}_2(0)}{z - q_2(0)} + \frac{\dot{q}_3(0)}{z - q_3(0)} = \frac{1}{t}.$$

After some calculations, one yields

$$\begin{aligned} & z^3 - z^2[(q_1(0) + q_2(0) + q_3(0)] + z[q_1(0)q_2(0) + q_2(0)q_3(0) + q_1(0)q_3(0)] - q_1(0)q_2(0)q_3(0) \\ & = tz^2[\dot{q}_1(0) + \dot{q}_2(0) + \dot{q}_3(0)] - tz[\dot{q}_1(0)(q_2(0) + q_3(0)) + \dot{q}_2(0)(q_1(0) + q_3(0)) \\ & +\dot{q}_3(0)(q_1(0) + q_2(0))] + t[\dot{q}_1(0)q_2(0)q_3(0) + \dot{q}_2(0)q_1(0)q_3(0) + \dot{q}_3(0)q_1(0)q_2(0)]. \end{aligned} \tag{A.1}$$

On the other hand, from (14), one knows $q_1(t), q_2(t), q_3(t)$ are the roots of the polynomial

$$\mathbb{P}_3(z, t) = z^3 + 6t + C_1z^2 + C_2z + C_3$$

or

$$z^3 + C_1z^2 + C_2z + C_3 = -6t. \tag{A.2}$$

Comparing (A.1) with (A.2), we are able to get the following linear equations for $\dot{q}_1(0), \dot{q}_2(0), \dot{q}_3(0)$:

$$\begin{aligned} & \dot{q}_1(0) + \dot{q}_2(0) + \dot{q}_3(0) = 0 \\ & \dot{q}_1(0)(q_2(0) + q_3(0)) + \dot{q}_2(0)(q_1(0) + q_3(0)) + \dot{q}_3(0)(q_1(0) + q_2(0)) = 0 \\ & \dot{q}_1(0)q_2(0)q_3(0) + \dot{q}_2(0)q_1(0)q_3(0) + \dot{q}_3(0)q_1(0)q_2(0) = -6. \end{aligned} \tag{A.3}$$

So if the determinant of the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ q_2(0) + q_3(0) & q_1(0) + q_3(0) & q_1(0) + q_2(0) \\ q_2(0)q_3(0) & q_1(0)q_3(0) & q_1(0)q_2(0) \end{pmatrix} \tag{A.4}$$

is not equal to zero, then the initial velocities $\dot{q}_1(0), \dot{q}_2(0), \dot{q}_3(0)$ can be uniquely expressed by the initial positions $q_1(0), q_2(0), q_3(0)$. For the cases $N = 4, 5$, the linear equations (A.3) and the matrix (A.4) can be obtained similarly.

References

[1] C.Athorne and J.J.C.Nimmo: On the Moutard transformation for integrable partial differential equations. Inverse problems 7 (1991), 809-826.

[2] M.Yu. Basalaev, V.G. Dubrovsky and A.V. Topovsky, New exact multi line soliton and periodic solutions with constant asymptotic values at infinity of the NVN integrable nonlinear evolution equation via dibar-dressing method , arXiv: 0912.2155 , 2009, 43 pages,

[3] L.V. Bagdanov, The Veselov-Novikov equation as a natural two-dimensional Generalization of the KdV equation, Ther. Math. Fiz, 70, 309(1987)

[4] Gabriella Bretti and Paolo E. Ricci, Multiextensions of the Bernoulli and Appell polynomials, Taiwan Jour. of Math. , Vol. 8, No. 3 (2004), 415-428,

[5] F. Calogero, The neatest many-body problem amenable to exact treatments (a "goldfish"?), Physica D, 152-153 (2001), pp. 78-84

[6] B.A. Dubrovin, I.M. Krichever,, and S.P. Novikov,: The Schordinger equation in a periodic field and Riemann surfaces. Soviet Math. Dokl. 17 (1976), 947-952.

[7] V.G. Dubrovsky and I.B. Formusatik, The construction of exact rational solutions with constant asymptotic values at infinity of two-dimensional NVN integrable nonlinear evolution equations via dbar-dressing method, J.Phys.A.: Math. and Gen., v.34A, 1837-1851 (2001).

[8] V.G. Dubrovsky and I.B. Formusatik, New lumps of Veselov-Novikov equation and new exact rational potentials of two-dimensional Schrodinger equation via dbar-dressing method, Phys. Lett., 2003, V.313, 1-2, 68-76.

[9] H.W. Gould and A.T. Hopper, Operational formulas connected with two generalizations of Hermite polynomials, Duke Math. J. , 29(1962), 51-63

[10] Oktay K. Pashaev and Zeynep Nilhan Gurkan, Abelian Chern-Simons Vortices and Holomorphic Burgers' Hierarchy, Theor.Math.Phys., 152(2007), 1017-1029

[11] P.G. Grinevich, and S.V. Manakov,: Inverse scattering problem for the two-dimensional Schrodinger operator, the dbar-method and nonlinear equations. Funct. Anal. Appl. 20:2 (1986), 94-103.

[12] P.G. Grinevich, Rational solitons of the Veselov-Novikov equation - reflectionless at fixed energy two-dimensional potentials, Teor. Mat. Fiz., Vol. 69, 307 (1986). 1170-1172

[13] P. Grinevich , A. Mironov and S. Novikov, New Reductions and Nonlinear Systems for 2D Schrodinger Operators , arXiv:1001.4300, 2010

[14] A.E.Mironov, The Novikov-Veselov hierarchy of equations and integrable deformations of minimal Lagrangian tori in CP^2, Siberian Electronic Mathematical Reports, V.1(2004), pp. 38-46

[15] Heng-Chun Hu, Sen-Yue Lou, and Qing-Ping Liu: Darboux transformation and variable separation approach: the Nizhnik-Novikov-Veselov equation. Chinese Phys. Lett. 20 (2003), 1413-1415.

[16] Heng-Chun Hu and Sen-Yue Lou,Construction of the Darboux Transformaiton and Solutions to the Modified Nizhnik-Novikov-Veselov Equation, Chinese Phys. Lett. 21 , No.11 (2004), 2073-2076

[17] I.Krichever, A characterization of Prym varieties, Int. Math. Res. Not. 2006, Art. ID 81476, 36 pp.

[18] B.G. Konopelchenko, Introduction to multidimensional integrable equations: the inverse spectral transform in 2+1-dimensions, New York - London, Plenum Press, 1992.

[19] B. G. Konopelchenko and G. Landolfi, Induced Surfaces and Their Integrable Dynamics II: Generalized Weierstrass Representations in 4-d Spaces and Deformations via DS Hierarchy, Studies in Applied Mathematics, Vol.104(2000), 129-169

[20] Wen-Xiu Ma and Yuncheng You, Solving the Korteweg-de Vries Equation by Its Bilinear Form: Wronskian Solutions, Trans. Amer. Math. Soc., 357 (2005), No.5, 1753-1778

[21] S.V. Manakov, USP. Matem. Nauk, Vol.31, No.5, p.245, 1976

[22] C. R. Moutard, Acad. Sci. Poris 80 729, 1875 ; C. R. Moutard, J. de L'Ecole Polpehnique Cahier 45 I, 1878

[23] Nimmo J.J.C. Darboux transformations in (2 + 1) dimensions, in: Proc. NATO ARW Applications of Analytic and Geometric Methods to Nonlinear Differential Equations (ed. P. Clarkson), NATO ASI Series, Kluwer, 1992, p. 183V192.

[24] S.P. Novikov, Two-dimensional Schrodinger operators in periodic fields, J. Sov. Math., 28(1), 1-20 ,1985

[25] S.P. Novikov and A.P. Veselov, Two-dimensional Schordinger operator: Inverse scattering transform and evolutional equations, Physica D, 18 (1-3), 267-273 (1986).

[26] Si-Qi Liu, Chao-Zhong Wu and Youjin Zhang, On the Drinfeld-Sokolov Hierarchies of D type , arXiv:0912.5273, 2009

[27] V.B. Matveev, M.A. Salle, Darboux Transformations and solitons, in Springer series in Nonlinear Dynamics, Springer, Berlin, Heidelberg, 1991.

[28] Y.Ohta, Pfaffian solution for the Veselov-Novikov equation, Jour. of the Phys. Soc. of Japan, Vol. 61, No. 11(1992), 3928-3933

[29] T.Shiota, Prym varieties and soliton equations. Infinite-dimensional Lie algebras and groups (Luminy-Marseille, 1988), 407-448, Adv. Ser. Math. Phys., 7, World Sci. Publ., Teaneck, NJ, 1989.

[30] I.A. Taimanov and S.P. Tsarev, Two-dimensional rational solitons and their blow-up via the Moutard transformation , Theoret. and Math. Phys. 157 (2007),

[31] Kanehisa Takasaki, Dispersionless Hirota equations of two-component BKP hierarchy, SIGMA, Vol. 2(2006), Paper 057, 22 pages, arXiv:nlin.SI/0604003

[32] A.P. Veselov and S.P. Novikov, Finite-zone, two-dimensional, potential Schordinger operators. Explicit formulas and evolution equations, Sov. Math., Dokl. 30(1984), 588-591