Construction of the Normal Form for Elliptic Tori in Planetary Systems. Part I: Numerical Validation of the Semi-Analytic Algorithm*

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Abstract

We adapt the Kolmogorov's normalization algorithm (which is the key element of the original proof scheme of the KAM theorem) to the construction of a suitable normal form related to an invariant elliptic torus. As a byproduct, our procedure can also provide some analytic expansions of the motions on elliptic tori. By extensively using algebraic manipulations on a computer, we explicitly apply our method to a planar four–body model not too different with respect to the real Sun–Jupiter–Saturn–Uranus system. The frequency analysis method allows us to check that our location of the initial conditions on an invariant elliptic torus is really accurate.

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1 Introduction

Since the birth of the KAM theory (see [20], [36] and [1]), the invariant tori are expected to be the key dynamical object which explains the (nearly perfect) quasi-periodicity of the planetary motions of our Solar System.

Among the consequences of the KAM theory, which concern the tori of maximal dimension, the following one looks natural. In a planetary system including n planets, one expects that the persistence under small perturbations should hold also for the n–dimensional invariant tori, which are a slight deformation of the composition of *n coplanar circular* Keplerian orbits. However, a separate proof is needed in order to ensure the existence of these lower dimensional invariant tori which are said to be elliptic, because they correspond to stable equilibrium points of the secular motions. Such a theorem has been recently proved by Biasco, Chierchia and Valdinoci in two different cases: for the spatial three–body planetary problem and for a planar system with a central star and n planets (see [4] and [5], respectively). In our opinion, their approach is deep from a theoretical point of view, but is not suitable for explicit applications, even if one is interested just in finding the locations of the elliptic invariant tori. In order to clarify this point, let us roughly summarize the scheme of their proofs as follows: first, they carry out all the preliminary canonical transformations that are necessary to bring the Hamiltonian in a particular form, to which they can subsequently apply a theorem due to Pöschel (see [40] and [41]), so to ensure the existence of elliptic lower dimensional tori. Moreover, Pöschel's versions of this theorem are based on a careful adaptation of the usual Arnold's proof scheme for non-degenerate systems: the perturbation is removed by a sequence of canonical transformations which are defined on a subset of the phase space excluding the "resonant regions" (see [1] and [2]). Since resonances are everywhere dense (but the width of the regions eliminated around them is suitably decreased, when the order of the resonances increases), therefore the change of coordinates giving the shape of the invariant elliptic tori is defined on a Cantor set which *does not contain any open subset*. The efficiency of an eventual explicit application based of such an approach is highly questionable and, as far as we know, it has never been used to calculate an orbit of a Celestial Mechanics problem.

The original proof scheme of the KAM theorem, introduced by Kolmogorov himself, is in a much better position for what concerns the translation into an explicit algorithm constructing invariant tori (see [20], [3], [13] and [14]). In fact, this approach has been successfully used to calculate the orbits for some interesting problems in Celestial Mechanics (see [32], [33], [34] and [11]). The present work aims to adapt the Kolmogorov's algorithm, in order to construct a suitable normal form related to the elliptic tori. Moreover, this will allow us to explicitly integrate the equations of motion on those invariant surfaces, by using a so called semi-analytic procedure.

When one is interested in showing the long term stability of a planetary system, the construction of a normal form related to some fixed elliptic torus could be a relevant milestone. In fact, it is possible to ensure the effective stability in the neighborhood of such an invariant surface by implementing a partial construction of the Birkhoff's normal form (see, e.g., [18] and [15], where this approach is used in order to study the stability nearby an invariant KAM torus having maximal dimension). For what concerns our Solar System, such an approach might be applied to some asteroids with small orbital eccentricities and inclinations. However, as explained in [43], this same approach can not yet succeed in proving the long-time stability of the major planets of our Solar System.

The location of the elliptic tori can be useful also for practical purposes. In fact, the regions close to them are exceptionally stable, being mainly filled by invariant tori of maximal dimension. Therefore, they can be of interest for spatial missions aiming, for instance, to observe asteroids not far from the elliptic tori. Moreover, our technique should adapt quite easily also to the construction of hyperbolic tori that can be used in the design of spacecraft missions with transfers requiring low energy. Also in view of this kind of applications, lower dimensional tori of elliptic, hyperbolic and mixed type have been studied in the vicinity of the Lagrangian points for both the restricted three–body problem and the bicircular restricted four-body problem (see, e.g., [17], [19], [8] and [10]).

The present paper is organized as follows. The search for elliptic tori is applied just to a model not far from the SJSU *planar* system (let us recall that the real orbits of the planets of our Solar System *are not* lying on lower dimensional tori). Therefore, sect. 2 is devoted to the introduction of our Hamiltonian model and to the description of its expansion in canonical coordinates. This will allow us to write down the form of the Hamiltonian to which our approach can be applied. By the way, we think that with some minor modifications our procedure should adapt also to the more general spatial case, after having performed the reduction of the angular momentum, which is not considered here in order to shorten the description of all the preliminary expansions (for an introduction to some methods performing both the partial and the total reduction, see [9], [35] and [37]).

Our algorithm constructing a normal form for elliptic tori is presented in a purely formal way in sect. 3. Let us recall that our procedure is mainly a reformulation of the classical Kolmogorov's normalization algorithm, that is modified in a suitable way for our purposes. The theoretical background necessary to understand when our algorithm can converge is informally discussed in subsect. 3.4 and it will be fully detailed in a future work (see [16]).

Sect. 4 is devoted to explain an application which is also a test of our procedure. First, in subsects. 4.1–4.2 we describe the way to implement our algorithm, by using an algebraic manipulator on a computer so to produce both the normal form and the semi-analytic integration of the motion on an invariant elliptic torus. The Fourier spectrum of the motions on elliptic tori is strongly characteristic: just the mean-motion frequencies and their linear combinations can show up. This simple remark allows us to check the accuracy of our results by using frequency analysis, as it will be described in sect. 4.3.

2 Classical expansion of the planar planetary Hamiltonian

As claimed in the introduction, in order to fix the ideas, we think it is convenient to focus on a concrete planetary model, to which we will apply our algorithm constructing the elliptic tori in the next sections. Let us consider four point bodies P_0, P_1, P_2, P_3, with masses m_0, m_1, m_2, m_3, mutually interacting according to Newton's gravitational law. For shortness, hereafter we will assume that the indexes 0, 1, 2, 3 correspond[1] to Sun, Jupiter, Saturn and Uranus, respectively.

Let us now recall how the classical Poincaré variables can be introduced so to perform a first expansion of the Hamiltonian around circular orbits, i.e., having zero eccentricity. We basically follow the formalism introduced by Poincaré (see [38] and [39]; for a modern exposition, see, e.g., [27] and [28]). We remove the motion of the center of mass by using heliocentric coordinates $\underline{r}_j = \overrightarrow{P_0 P_j}$, with $j = 1, 2, 3$. Denoting by $\underline{\tilde{r}}_j$ the momenta conjugated to $\underline{r}_j$, the Hamiltonian of the system has 6 degress of freedom, and reads

$$F(\underline{\tilde{r}}, \underline{r}) = T^{(0)}(\underline{\tilde{r}}) + U^{(0)}(\underline{r}) + T^{(1)}(\underline{\tilde{r}}) + U^{(1)}(\underline{r}) \ , \tag{1}$$

where

$$\begin{aligned} T^{(0)}(\underline{\tilde{r}}) &= \tfrac{1}{2}\sum_{j=1}^{3} \tfrac{m_0+m_j}{m_0 m_j} \|\underline{\tilde{r}}_j\|^2 \ , & T^{(1)}(\underline{\tilde{r}}) &= \tfrac{1}{m_0}\Big(\underline{\tilde{r}}_1\cdot\underline{\tilde{r}}_2 + \underline{\tilde{r}}_1\cdot\underline{\tilde{r}}_3 + \underline{\tilde{r}}_2\cdot\underline{\tilde{r}}_3\Big) \ , \\ U^{(0)}(\underline{r}) &= -\mathcal{G}\sum_{j=1}^{3} \tfrac{m_0\, m_j}{\|\underline{r}_j\|} \ , & U^{(1)}(\underline{r}) &= -\mathcal{G}\left(\tfrac{m_1\, m_2}{\|\underline{r}_1-\underline{r}_2\|} + \tfrac{m_1\, m_3}{\|\underline{r}_1-\underline{r}_3\|} + \tfrac{m_2\, m_3}{\|\underline{r}_2-\underline{r}_3\|}\right) \ . \end{aligned}$$

The plane set of Poincaré's canonical variables is introduced as

$$\begin{aligned} \Lambda_j &= \frac{m_0\, m_j}{m_0+m_j}\sqrt{\mathcal{G}(m_0+m_j)a_j} \ , & \lambda_j &= M_j + \omega_j \ , \\ \xi_j &= \sqrt{2\Lambda_j}\sqrt{1-\sqrt{1-e_j^2}}\,\cos\omega_j \ , & \eta_j &= -\sqrt{2\Lambda_j}\sqrt{1-\sqrt{1-e_j^2}}\,\sin\omega_j \ , \end{aligned} \tag{2}$$

for $j = 1\,,2\,,3\,$, where $a_j\,$, $e_j\,$, M_j and ω_j are the semi-major axis, the eccentricity, the mean anomaly and the perihelion argument, respectively, of the j–th planet. One immediately sees that both ξ_j and η_j are of the same order of magnitude as the eccentricity $e_j\,$. Using Poincaré's variables (2), the Hamiltonian F can be rearranged so that one has

$$F(\underline{\Lambda}, \underline{\lambda}, \underline{\xi}, \underline{\eta}) = F^{(0)}(\underline{\Lambda}) + F^{(1)}(\underline{\Lambda}, \underline{\lambda}, \underline{\xi}, \underline{\eta}) \ , \tag{3}$$

where $F^{(0)} = T^{(0)}+U^{(0)}$, $F^{(1)} = T^{(1)}+U^{(1)}$. Let us emphasize that $F^{(0)} = \mathcal{O}(1)$ and $F^{(1)} = \mathcal{O}(\mu)$, where the small dimensionless parameter $\mu = \max\{m_1\,/\,m_0\,,\, m_2\,/\,m_0\,,\, m_3\,/\,m_0\,\}$ highlights the different size of the terms appearing in the Hamiltonian. Therefore, let us remark that the time derivative of each coordinate is $\mathcal{O}(\mu)$ but in the case of the angles $\underline{\lambda}$. Thus, according to the common language in Celestial Mechanics, in the following we will refer to $\underline{\lambda}$ and to their conjugate actions $\underline{\Lambda}$ as the *fast variables*, while $(\underline{\xi}, \underline{\eta})$ will be called *secular variables*.

We proceed now by expanding the Hamiltonian (3) in order to construct the first basic approximation of the normal form for elliptic tori. After having chosen a center value $\underline{\Lambda}^*$

[1]Let us stress that the four considered point bodies have the same masses as Sun, Jupiter, Saturn and Uranus, but the orbits here studied are significantly different with respect to the real ones.

Table 1: Masses m_j and initial conditions for Jupiter, Saturn and Uranus in our planar model. We adopt the UA as unit of length, the year as time unit and set the gravitational constant $\mathcal{G} = 1$. With these units, the solar mass is equal to $(2\pi)^2$. The initial conditions are expressed by the usual heliocentric planar orbital elements: the semi-major axis a_j, the mean anomaly M_j, the eccentricity e_j and the perihelion longitude ω_j. The data are taken by JPL at the Julian Date 2440400.5.

	Jupiter ($j = 1$)	Saturn ($j = 2$)	Uranus ($j = 3$)
m_j	$(2\pi)^2/1047.355$	$(2\pi)^2/3498.5$	$(2\pi)^2/22902.98$
a_j	5.20463727204700266	9.54108529142232165	19.2231635458410572
M_j	3.04525729444853654	5.32199311882584869	0.19431922829271914
e_j	0.04785365972484999	0.05460848595674678	0.04858667407651962
ω_j	0.24927354029554571	1.61225062288036902	2.99374344439246487

for the Taylor expansions with respect to the fast actions (in a way we will explain later), we perform a translation $\mathcal{T}_{\underline{\Lambda}^*}$ defined as

$$L_j = \Lambda_j - \Lambda_j^* \,, \qquad \forall\, j = 1\,,\, 2\,,\, 3\,. \tag{4}$$

This is a canonical transformation that leaves the coordinates $\underline{\lambda}$, $\underline{\xi}$ and $\underline{\eta}$ unchanged. The transformed Hamiltonian $\mathcal{H}^{(\mathcal{T})} = F \circ \mathcal{T}_{\underline{\Lambda}^*}$ can be expanded in power series of $\underline{L}$, $\underline{\xi}$, $\underline{\eta}$ around the origin. Thus, forgetting an unessential constant we rearrange the Hamiltonian of the system as

$$\mathcal{H}^{(\mathcal{T})}(\underline{L}, \underline{\lambda}, \underline{\xi}, \underline{\eta}) = \underline{n}^* \cdot \underline{L} + \sum_{j_1=2}^{\infty} h_{j_1\,,0}^{(\mathrm{Kep})}(\underline{L}) + \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} h_{j_1\,,j_2}^{(\mathcal{T})}(\underline{L}, \underline{\lambda}, \underline{\xi}, \underline{\eta}) \,, \tag{5}$$

where the functions $h_{j_1\,,j_2}^{(\mathcal{T})}$ are homogeneous polynomials of degree j_1 in the actions $\underline{L}$ and of degree j_2 in the secular variables $(\underline{\xi}, \underline{\eta})$. The coefficients of such homogeneous polynomials do depend analytically and periodically on the angles $\underline{\lambda}$. The terms $h_{j_1\,,0}^{(\mathrm{Kep})}$ of the Keplerian part are homogeneous polynomials of degree j_1 in the actions $\underline{L}$, the explicit expression of which can be determined in a straightforward manner. In the latter equation the term which is both linear in the actions and independent of all the other canonical variables (i.e., $\underline{n}^* \cdot \underline{L}$) has been separated in view of its relevance in perturbation theory, as it will be discussed in the next section. We also expand the coefficients of the power series $h_{j_1\,,j_2}^{(T_F)}$ in Fourier series of the angles $\underline{\lambda}$. The expansion of the Hamiltonian is a traditional procedure in Celestial Mechanics. We work out these expansions for the case of the planar SJSU system using a specially devised algebraic manipulator. The calculation is based on the approach described in sect. 2.1 of [32], which in turn uses the scheme sketched in sect. 3.3 of [42].

The reduction to the planar case is performed as follows. We pick from Table IV of [44] the initial conditions of the planets in terms of heliocentric positions and velocities at the

Julian Date 2440400.5 . Next, we calculate the corresponding orbital elements with respect to the invariant plane (that is perpendicular to the total angular momentum). Finally we include the longitudes of the nodes Ω_j (which are meaningless in the planar case) in the corresponding perihelion longitude ω_j and we eliminate the inclinations by setting them equal to zero. The remaining initial values of the orbital elements are reported in Table 1.

Having fixed the initial conditions we come to determining the *average* values $(a_1^*\,,\,a_2^*\,,\,a_3^*)$ of the semi-major axes during the evolution. To this end we perform a long-term numerical integration of Newton's equations starting from the initial conditions related to the data reported in Table 1. After having computed $(a_1^*\,,\,a_2^*\,,\,a_3^*)$, we determine the values $\underline{\Lambda}^*$ via the first equation in (2). This allows us to perform the expansion (5) of the Hamiltonian as a function of the canonical coordinates $(\underline{L},\underline{\lambda},\underline{\xi},\underline{\eta})$. In our calculations we truncate this initial expansion as follows. (a) The keplerian part is expanded up to the quartic terms. The series where the general summand $h_{j_1\,,j_2}^{(\mathcal{T})}$ appears are truncated so to include: (b1) the terms having degree j_1 in the actions $\underline{L}$ with $j_1\le 3$, (b2) all terms having degree j_2 in the secular variables $(\underline{\xi},\underline{\eta})$, with j_2 such that $2j_1+j_2\le 8$, (b3) all terms up to the trigonometric degree 18 with respect to the angles $\underline{\lambda}$. Let us remark that with respect to the analogous initial expansion we performed in [43], here we preferred to considerably reduce the maximal degree in the secular coordinates, in order to increase those related to the fast ones. This choice is motivated by the fact that the orbits on elliptic tori experience smaller values of the eccentricities (let us recall that both $\xi_j=\mathcal{O}(e_j)$ and $\eta_j=\mathcal{O}(e_j)$ $\forall\ j=1\,,\,2\,,\,3$) than those related to the real motions; moreover, larger limits on the fast coordinates are needed, in order to give a sharp enough numerical evidence of the convergence of the algorithm described in the next section.

Let us now focus on the average with respect to the fast angles of the Hamiltonian written in (5), i.e. $\big\langle\mathcal{H}^{(\mathcal{T})}\big\rangle_{\underline{\lambda}}$. The fast actions $\underline{L}$ are obviously invariant with respect to the flow of $\big\langle\mathcal{H}^{(\mathcal{T})}\big\rangle_{\underline{\lambda}}$, thus, they can be neglected while just the secular motions are considered. The remaining most significant term is given by the lowest order approximation of the secular Hamiltonian, namely its quadratic term $\big\langle h_{0,2}^{(\mathcal{T})}\big\rangle_{\underline{\lambda}}$, which is essentially the one considered in the theory first developed by Lagrange (see [21]) and furtherly improved by Laplace (see [24], [25] and [26]) and by Lagrange himself (see [22], [23]). In modern language, we can say that the origin of the secular coordinates phase space (i.e., $(\underline{\xi},\underline{\eta})=(\underline{0},\underline{0})$) is an elliptic equilibrium point for the secular Hamiltonian. In fact, under mild assumptions on the quadratic part of the Hamiltonian which are satisfied in our case (see sect. 3 of [5], where such hypotheses are shown to be generically fulfilled for a planar model of our Solar System), it is well known that one can find a canonical transformation $(\underline{L},\underline{\lambda},\underline{\xi},\underline{\eta})=\mathcal{D}(\underline{p},\underline{q},\underline{x},\underline{y})$ owning the following properties: (i) $\underline{L}=\underline{p}$ and $\underline{\lambda}=\underline{q}$, (ii) the map $(\underline{\xi},\underline{\eta})=\big(\underline{\xi}(\underline{x}),\underline{\eta}(\underline{y})\big)$ is linear, (iii) $\mathcal{D}$ diagonalizes the quadratic part of the Hamiltonian, so that we can write $\big\langle h_{0,2}^{(\mathcal{T})}\big\rangle_{\underline{\lambda}}$ in the new coordinates as $\sum_{j=1}^{3}\nu_j^{(0)}(x_j^2+y_j^2)/2$, where all the entries of the vector $\underline{\nu}^{(0)}$ have the same sign. Our algorithm constructing a suitable normal form for elliptic tori can be started from the Hamiltonian $H^{(0)}=\mathcal{H}^{(\mathcal{T})}\circ\mathcal{D}$, i.e.

$$H^{(0)}(\underline{p},\underline{q},\underline{x},\underline{y})=\mathcal{H}^{(\mathcal{T})}\big(\mathcal{D}(\underline{p},\underline{q},\underline{x},\underline{y})\big)\ . \tag{6}$$

3 Formal algorithm

In the present section, let us more generically assume that the number of degrees of freedom of our system is n_1+n_2, where the canonical coordinates $(\underline{p},\underline{q},\underline{x},\underline{y})$ can naturally be split in two parts, that are $(\underline{p},\underline{q})\in\mathbb{R}^{n_1}\times\mathbb{T}^{n_1}$ and $(\underline{x},\underline{y})\in\mathbb{R}^{n_2}\times\mathbb{R}^{n_2}$.

In order to better understand our whole procedure, we think it is convenient to immediately state our final goal. We want to determine a canonical transformation $(\underline{p},\underline{q},\underline{x},\underline{y})=\mathcal{K}^{(\infty)}(\underline{P},\underline{Q},\underline{X},\underline{Y})$ such that the Hamiltonian $H^{(\infty)}=H^{(0)}\circ\mathcal{K}^{(\infty)}$ is brought to the following normal form[2]:

$$
\begin{aligned}
H^{(\infty)}(\underline{P},\underline{Q},\underline{X},\underline{Y}) = \underline{\omega}^{(\infty)}\cdot\underline{P} + \sum_{j=1}^{n_2}\frac{\Omega_j^{(\infty)}\left(X_j^2+Y_j^2\right)}{2} + \\
\mathcal{O}\big(\|\underline{P}\|^2\big)+\mathcal{O}\big(\|\underline{P}\|\|(\underline{X},\underline{Y})\|\big)+\mathcal{O}\big(\|(\underline{X},\underline{Y})\|^3\big)\ ,
\end{aligned}
\tag{7}
$$

where the notation means that we want to remove all terms which are linear in $\underline{P}$ and independent of $(\underline{X},\underline{Y})$, or at most quadratic in $(\underline{X},\underline{Y})$ and independent of $\underline{P}$.

When initial conditions of the type $(\underline{P},\underline{Q},\underline{X},\underline{Y})=(\underline{0},\underline{Q}_0,\underline{0},\underline{0})$ (with $\underline{Q}_0\in\mathbb{T}^{n_1}$) are considered, the normal form (7) allows us to easily calculate the solution of the Hamilton equations, i.e.

$$
\big(\underline{P}(t),\underline{Q}(t),\underline{X}(t),\underline{Y}(t)\big)=\big(\underline{0},\underline{Q}_0+\underline{\omega}^{(\infty)}t,\underline{0},\underline{0}\big)\ . \tag{8}
$$

This clearly means that the n_1–dimensional (elliptic) torus corresponding to $\underline{P}=\underline{X}=\underline{Y}=\underline{0}$ is invariant and the orbits are quasi-periodic on it.

Let us start the description of the generic r–th step of our algorithm constructing the normal form. We begin with a Hamiltonian of the following type:

$$
H^{(r-1)}(\underline{p},\underline{q},\underline{x},\underline{y})=\underline{\omega}^{(r-1)}\cdot\underline{p}+\underline{\Omega}^{(r-1)}\cdot\underline{J}+\sum_{s=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\substack{2j_1+j_2=l\\ j_1\ge0\,,\,j_2\ge0}} f_{j_1,j_2}^{(r-1,s)}(\underline{p},\underline{q},\underline{x},\underline{y})\ , \tag{9}
$$

where $J_j=(x_j^2+y_j^2)/2$ is the action which is usually related to the j–th pair of secular canonical coordinates (x_j,y_j), $\forall\ j=1\,,\,\ldots\,,\,n_2$. Moreover, there is a fixed integer value $K>0$ such that the terms $f_{j_1,j_2}^{(r-1,s)}$ satisfy the following hypotheses:

(A) $f_{j_1,j_2}^{(r-1,s)}\in\mathcal{P}_{j_1,j_2}^{(sK)}$, where $\mathcal{P}_{j_1,j_2}^{(sK)}$ is the class of functions such that (a1) they are homogeneous polynomials of degree j_1 in the actions $\underline{p}$, (a2) they are homogeneous polynomials of degree j_2 in the secular variables $(\underline{x},\underline{y})$, (a3) their Fourier expansion is finite with maximal trigonometric degree equal to sK;

(B) the terms $f_{j_1,j_2}^{(r-1,s)}$ are "well Fourier-ordered"; this nonstandard definition means that $\forall\ j_1\ge0\,,\,j_2\ge0\,,\,s\ge1$ every Fourier harmonic $\underline{k}$ appearing in the expansion of

[2]Let us here stress a little abuse of notation. Hereafter, the symbol ω will mean the frequencies vector related to the motion on a torus (as it is usual in KAM theory), while in the previous sections it was used to represent the perihelion longitudes (according to the classical notation in Celestial Mechanics). Analogously, hereafter, Ω will denote the oscillation frequencies transverse to an elliptic torus, while before it was used for the longitudes of the nodes.

$f_{j_1,j_2}^{(r-1,s)}$ is such that its corresponding trigonometric degree $|\underline{k}| = |k_1| + \ldots + |k_{j_1}| > (s-1)K$.

By using formula (5) and the properties (i)–(iii) of the canonical transformation $\mathcal{D}$, one easily sees that the Hamiltonian $H^{(0)}$ (that is defined in (6)) can be expanded in the form written in (9), after having suitably reordered its Fourier expansion so to satisfy the above requirements (A) and (B). Therefore, our constructive algorithm can be concretely applied to the Hamiltonian $H^{(0)}$ by starting with $r = 1$.

The comparison of the expansion in (9) with the normal form in (7) clearly shows that we have to eliminate all the terms $f_{j_1,j_2}^{(0,s)}$ where the index $l = 2j_1 + j_2$ is such that $0 \le l \le 2$. Thus, the r–th step of our algorithm can be naturally divided in three stages, each of ones aims to reduce the perturbative terms with $l = 0,\ 1,\ 2$, respectively.

3.1 First stage of the r-th normalization step: removing of the terms depending just on $\underline{q}$

By making use of the classical Lie series algorithm to calculate canonical transformations (see, e.g., [12] for an introduction), we first introduce the new Hamiltonian $H^{(\mathrm{I};r)} = \exp \mathcal{L}_{\chi_0^{(r)}} H^{(r-1)}$, where the generating function $\chi_0^{(r)}(\underline{q}) \in \mathcal{P}_{0,0}^{(rK)}$ is determined as the solution of the equation

$$\left\{ \chi_0^{(r)} \,,\, \underline{\omega}^{(r-1)} \cdot \underline{p} \right\} + \sum_{s=1}^{r} f_{0,0}^{(r-1,s)}(\underline{q}) = 0 \ , \tag{10}$$

where we used the classical symbol $\{\cdot,\cdot\}$ to represent the Poisson brackets. The previous equation (that is usually said to be of homological type) admits a solution provided the frequency vector $\underline{\omega}^{(r-1)}$ is non-resonant up to order rK , i.e.

$$\min_{0<|\underline{k}|\le rK} \left| \underline{k} \cdot \underline{\omega}^{(r-1)} \right| \ge \alpha_r \qquad \text{with} \ \ \alpha_r > 0 \ , \tag{11}$$

where, for the time being, $\{\alpha_r\}_{r>0}$ is nothing but a sequence of real positive numbers and $|\underline{k}|$ denotes the l^1–norm of the integer vector $\underline{k}$, i.e. $|\underline{k}| = |k_1| + \ldots + |k_{n_1}|$. The solution of the homological equation (10) can be easily recovered by looking at the little more complicate case of $X_2^{(r)}$, which is discussed in the third stage of the r–th normalization step (see formulas (21)–(23)).

In order to avoid the proliferation of too many symbols, let us make a common abuse of notation so to still denote with $(\underline{p},\underline{q},\underline{x},\underline{y})$ the new canonical coordinates $\exp \mathcal{L}_{\chi_0^{(r)}}(\underline{p},\underline{q},\underline{x},\underline{y})$. The expansion of the new Hamiltonian can be written as follows:

$$H^{(\mathrm{I};r)}(\underline{p},\underline{q},\underline{x},\underline{y}) = \underline{\omega}^{(r-1)} \cdot \underline{p} + \underline{\Omega}^{(r-1)} \cdot \underline{J} + \sum_{s=0}^{\infty} \sum_{l=0}^{\infty} \sum_{\substack{2j_1+j_2=l \\ j_1\ge 0\,,\, j_2\ge 0}} f_{j_1,j_2}^{(\mathrm{I};r,s)}(\underline{p},\underline{q},\underline{x},\underline{y}) \ . \tag{12}$$

The mathematical recursive definitions of the terms $f_{j_1,j_2}^{(\mathrm{I};r,s)}$ are lenghty, but it is rather easy to understand how to deal with them when they are translated in a programming language. The main remark is concerned with the classes of functions, i.e.

$$\frac{1}{i!}\mathcal{L}^{i}_{\chi_0^{(r)}} f_{j_1\,,\,j_2}^{(r-1,s)} \in \mathcal{P}_{j_1-i\,,\,j_2}^{((s+i)K)} \qquad \forall\ 0\le i\le j_1\,,\ j_2\ge 0\,,\ s\ge 0\ . \tag{13}$$

Therefore, after having calculated all the Poisson brackets needed by the term $\frac{1}{i!}\mathcal{L}^{i}_{\chi_0^{(r)}} f_{j_1\,,\,j_2}^{(r-1,s)}$, it is enough to know that it contributes to the sum $\sum_{j=0}^{s+i} f_{j_1-i\,,\,j_2}^{(\mathrm{I};r,j)}$. A suitable "reordering of the Fourier series" will allow us to ensure that also the expansion (12) satisfies the conditions (A) and (B), which have been stated at the beginning of the present section.

3.2 Second stage of the r-th normalization step: removing of the terms linear in $(\underline{x},\underline{y})$ and independent of $\underline{p}$

Let us now introduce the new Hamiltonian $H^{(\mathrm{II};r)}=\exp\mathcal{L}_{\chi_1^{(r)}}H^{(\mathrm{II};r)}$, where the generating function $\chi_1^{(r)}(\underline{q},\underline{x},\underline{y})\in\mathcal{P}_{0,1}^{(rK)}$ is determined as the solution of the equation

$$\left\{\chi_1^{(r)}\ ,\ \underline{\omega}^{(r-1)}\cdot\underline{p}+\sum_{j=1}^{n_2}\frac{\Omega_j^{(r-1)}}{2}\big(x_j^2+y_j^2\big)\right\}+\sum_{s=0}^{r} f_{0,1}^{(\mathrm{I};r,s)}(\underline{q},\underline{x},\underline{y})=0\ , \tag{14}$$

In order to explicitly write down the solution of the previous equation, it is convenient to temporarily introduce action–angle coordinates so to replace the secular pairs $(\underline{x},\underline{y})$ by putting $x_j=\sqrt{2J_j}\cos\varphi_j$ and $y_j=\sqrt{2J_j}\sin\varphi_j$ $\forall\ j=1,\,\ldots\,,\,n_2$; therefore, let us assume that the expansion of the known terms appearing in equation (14) is the following one:

$$\sum_{s=0}^{r} f_{0,1}^{(\mathrm{I};r,s)}(\underline{q},\underline{J},\underline{\varphi})=\sum_{0\le|\underline{k}|\le rK}\ \sum_{j=1}^{n_2}\sqrt{2J_j}\left[c_{\underline{k},j}^{(\pm)}\cos\big(\underline{k}\cdot\underline{q}\pm\varphi_j\big)+d_{\underline{k},j}^{(\pm)}\sin\big(\underline{k}\cdot\underline{q}\pm\varphi_j\big)\right]\ , \tag{15}$$

with suitable real coefficients $c_{\underline{k},j}^{(\pm)}$ and $d_{\underline{k},j}^{(\pm)}$. Thus, one can easily check that

$$\chi_1^{(r)}(\underline{q},\underline{J},\underline{\varphi})=\sum_{0\le|\underline{k}|\le rK}\ \sum_{j=1}^{n_2}\sqrt{2J_j}\left[-\frac{c_{\underline{k},j}^{(\pm)}\sin\big(\underline{k}\cdot\underline{q}\pm\varphi_j\big)}{\underline{k}\cdot\underline{\omega}^{(r-1)}\pm\Omega_j^{(r-1)}}+\frac{d_{\underline{k},j}^{(\pm)}\cos\big(\underline{k}\cdot\underline{q}\pm\varphi_j\big)}{\underline{k}\cdot\underline{\omega}^{(r-1)}\pm\Omega_j^{(r-1)}}\right] \tag{16}$$

is a solution of the homological equation (14) and it exists provided the frequency vector $\underline{\omega}^{(r-1)}$ satisfies the so-called *first Melnikov non-resonance condition* up to order rK , i.e.

$$\min_{\substack{0<|\underline{k}|\le rK\\ j=1\,,\,\ldots\,,\,n_2}}\big|\underline{k}\cdot\underline{\omega}^{(r-1)}\pm\Omega_j^{(r-1)}\big|\ge\alpha_r\qquad\text{with}\ \ \alpha_r>0\ , \tag{17}$$

and all the entries of the frequency vector $\underline{\Omega}^{(r-1)}$ are far enough from the origin, i.e.

$$\min_{j=1\,,\,\ldots\,,\,n_2}\big|\Omega_j^{(r-1)}\big|\ge\beta\qquad\text{with}\ \ \beta>0\ . \tag{18}$$

For what concerns planetary Hamiltonians where the D'Alembert rules hold true, let us remark that all the coefficients $c_{\underline{k},j}^{(\pm)}$ and $d_{\underline{k},j}^{(\pm)}$ appearing in (15) and having even values of

$|\underline{k}|$ are equal to zero. In order to solve the equation (14), therefore, we could to not need the condition (18), which however is substantially included in another one (i.e., (31)) that we will be forced to introduce later.

Starting from the expansion (16) of $\chi_1^{(r)}(\underline{q}, \underline{J}, \underline{\varphi})$, one can immediately recover the expression of $\chi_1^{(r)}(\underline{q}, \underline{x}, \underline{y})$ as a function of the original polynomial variables. We can then explicitly calculate the expansion of the new Hamiltonian, which can be written as follows:

$$H^{(\mathrm{II};r)}(\underline{p}, \underline{q}, \underline{x}, \underline{y}) = \underline{\omega}^{(r-1)} \cdot \underline{p} + \underline{\Omega}^{(r-1)} \cdot \underline{J} + \sum_{s=0}^{\infty}\sum_{l=0}^{\infty} \sum_{\substack{2j_1+j_2=l \\ j_1\ge 0\,,\, j_2\ge 0}} f_{j_1\,,\,j_2}^{(\mathrm{II};r,s)}(\underline{p}, \underline{q}, \underline{x}, \underline{y}) \ . \tag{19}$$

Also in this case, providing mathematical recursive definitions of the terms $f_{j_1\,,\,j_2}^{(\mathrm{II};r,s)}$ is a quite ennoying task. Thus, we think it is better to just describe how to deal with them when they are translated in a programming language. First, let us remark that the following relations about the classes of functions hold true:

$$\frac{1}{i!}\mathcal{L}^{i}_{\chi_1^{(r)}} \sum_{2j_1+j_2=l} f_{j_1\,,\,j_2}^{(\mathrm{I};r,s)} \in \bigcup_{2j_1+j_2=l-i} \mathcal{P}_{j_1\,,\,j_2}^{((s+i)K)} \qquad \forall\ 0\le i\le l\,,\ s\ge 0\ . \tag{20}$$

Therefore, after having calculated all the Poisson brackets appearing in the expression of the term $\frac{1}{i!}\mathcal{L}^{i}_{\chi_1^{(r)}}\sum_{2j_1+j_2=l} f_{j_1\,,\,j_2}^{(\mathrm{I};r,s)}$, it is enough to know that it contributes to the sum $\sum_{j=0}^{s+i}\sum_{2j_1+j_2=l-i} f_{j_1\,,\,j_2}^{(\mathrm{II};r,j)}$. Again a suitable "reordering of the Taylor–Fourier series" will allow us to ensure that also the expansion (19) satisfies the conditions (A) and (B), which have been stated at the beginning of the present section.

3.3 Third stage of the r-th normalization step: removing *both* the linear terms in $\underline{p}$ that are independent of $(\underline{x}, \underline{y})$ *and* the quadratic ones in $(\underline{x}, \underline{y})$ which are independent of $\underline{p}$

The Hamiltonian produced at the end of the r–th normalization step is provided by the composition of three canonical transformations[3] which can be given in terms of Lie series, i.e. $H^{(r)} = \exp\mathcal{L}_{\mathcal{D}_2^{(r)}} \circ \exp\mathcal{L}_{Y_2^{(r)}} \circ \exp\mathcal{L}_{X_2^{(r)}} H^{(\mathrm{II};r)}$, where the generating functions belong to three different classes: $X_2^{(r)}(\underline{p}, \underline{q}) \in \mathcal{P}_{1,0}^{(rK)}$, $Y_2^{(r)}(\underline{q}, \underline{x}, \underline{y}) \in \mathcal{P}_{0,2}^{(rK)}$ and $\mathcal{D}_2^{(r)}(\underline{x}, \underline{y}) \in \mathcal{P}_{0,2}^{(0)}$. The explicit expressions of these generating functions are given below, in formulas (21), (24) and (28), respectively.

[3]When one focuses on the estimates needed to prove the convergence of the algorithm, it is certainly simpler to introduce the generating function $\chi_2^{(r)}(\underline{p}, \underline{q}, \underline{x}, \underline{y}) = X_2^{(r)}(\underline{p}, \underline{q}) + Y_2^{(r)}(\underline{q}, \underline{x}, \underline{y})$ and to consider the new Hamiltonian $\exp\mathcal{L}_{\mathcal{D}_2^{(r)}} \circ \exp\mathcal{L}_{\chi_2^{(r)}} H^{(\mathrm{II};r)}$ which slightly differs from $H^{(r)}$, because $X_2^{(r)}$, and $Y_2^{(r)}$ do not commute with respect to the Poisson brackets. Since in the present work we do not want to theoretically study the problem of the convergence of our algorithm, we think here is better to distinguish the present third stage of the r–th normalization step in other three parts, so to highlight their different roles. Moreover, this choice looks more natural to us, when one implements the constructive algorithm by algebraic manipulations on a computer.

We start with $X_2^{(r)}(\underline{p},\underline{q}) \in \mathcal{P}_{1,0}^{(rK)}$, which is determined as the solution of the equation

$$\left\{X_2^{(r)}\,,\,\underline{\omega}^{(r-1)}\cdot\underline{p}\right\} + \sum_{s=1}^{r} f_{1,0}^{(\mathrm{II};r,s)}(\underline{p},\underline{q}) = 0\ . \tag{21}$$

This implies that

$$X_2^{(r)}(\underline{p},\underline{q}) = \sum_{0<|\underline{k}|\le rK}\sum_{j=1}^{n_1} p_j\left[-\frac{c_{\underline{k}\,,\,j}\sin\left(\underline{k}\cdot\underline{q}\right)}{\underline{k}\cdot\underline{\omega}^{(r-1)}} + \frac{d_{\underline{k}\,,\,j}\cos\left(\underline{k}\cdot\underline{q}\right)}{\underline{k}\cdot\underline{\omega}^{(r-1)}}\right]\ , \tag{22}$$

where we preliminarly assumed that the expansion of the known terms appearing in equation (21) has the form

$$\sum_{s=1}^{r} f_{1,0}^{(\mathrm{II};r,s)}(\underline{p},\underline{q}) = \sum_{0<|\underline{k}|\le rK}\sum_{j=1}^{n_1} p_j\left[c_{\underline{k}\,,\,j}\cos\left(\underline{k}\cdot\underline{q}\right) + d_{\underline{k}\,,\,j}\sin\left(\underline{k}\cdot\underline{q}\right)\right]\ , \tag{23}$$

with suitable real coefficients $c_{\underline{k}\,,\,j}$ and $d_{\underline{k}\,,\,j}$. Let us here recall that the solution (22) for the equation (21) exists provided the frequency vector $\underline{\omega}^{(r-1)}$ satisfies the non-resonance condition (11).

Let us now consider $Y_2^{(r)}(\underline{q},\underline{x},\underline{y}) \in \mathcal{P}_{0,2}^{(rK)}$, which is a solution of the following homological equation:

$$\left\{Y_2^{(r)}\,,\,\underline{\omega}^{(r-1)}\cdot\underline{p} + \sum_{j=1}^{n_2}\frac{\Omega_j^{(r-1)}}{2}\left(x_j^2+y_j^2\right)\right\} + \sum_{s=1}^{r} f_{0,2}^{(\mathrm{II};r,s)}(\underline{q},\underline{x},\underline{y}) = 0\ , \tag{24}$$

In order to explicitly write down the expansion of $Y_2^{(r)}$, it is convenient to temporarily reintroduce the action–angle coordinates $(\underline{J},\underline{\varphi})$ so to replace the secular pairs $(\underline{x},\underline{y})$; therefore, let us assume that the expansion of the known terms appearing in equation (24) has the form

$$\begin{aligned}\sum_{s=1}^{r} f_{0,2}^{(\mathrm{II};r,s)}(\underline{q},\underline{J},\underline{\varphi}) = \sum_{0<|\underline{k}|\le rK}\sum_{i\,,\,j\,=\,1}^{n_2} 2\sqrt{J_iJ_j}\Big[&c_{\underline{k}\,,\,i\,,\,j}^{(\pm\,,\,\pm)}\cos\left(\underline{k}\cdot\underline{q}\pm\varphi_i\pm\varphi_j\right) +\\ &d_{\underline{k}\,,\,i\,,\,j}^{(\pm\,,\,\pm)}\sin\left(\underline{k}\cdot\underline{q}\pm\varphi_i\pm\varphi_j\right)\Big]\ ,\end{aligned} \tag{25}$$

with suitable real coefficients $c_{\underline{k}\,,\,i\,,\,j}^{(\pm\,,\,\pm)}$ and $d_{\underline{k}\,,\,i\,,\,j}^{(\pm\,,\,\pm)}$. Thus, one can easily check that

$$\begin{aligned}Y_2^{(r)}(\underline{q},\underline{J},\underline{\varphi}) = \sum_{0<|\underline{k}|\le rK}\sum_{i\,,\,j\,=\,1}^{n_2} 2\sqrt{J_iJ_j}\Bigg[&-\frac{c_{\underline{k}\,,\,i\,,\,j}^{(\pm\,,\,\pm)}\sin\left(\underline{k}\cdot\underline{q}\pm\varphi_i\pm\varphi_j\right)}{\underline{k}\cdot\underline{\omega}^{(r-1)}\pm\Omega_i^{(r-1)}\pm\Omega_j^{(r-1)}}\\ &+\frac{d_{\underline{k}\,,\,i\,,\,j}^{(\pm\,,\,\pm)}\cos\left(\underline{k}\cdot\underline{q}\pm\varphi_i\pm\varphi_j\right)}{\underline{k}\cdot\underline{\omega}^{(r-1)}\pm\Omega_i^{(r-1)}\pm\Omega_j^{(r-1)}}\Bigg]\end{aligned} \tag{26}$$

is a solution of equation (24) and it exists provided the frequency vector $\underline{\omega}^{(r-1)}$ satisfies the so-called *second Melnikov non-resonance condition* up to order rK , i.e.

$$\min_{\substack{0<|\underline{k}|\le rK \\ i\,,j=1\,,\ldots,\,n_2}} \left|\underline{k}\cdot\underline{\omega}^{(r-1)} \pm \Omega_i^{(r-1)} \pm \Omega_j^{(r-1)}\right| \ge \alpha_r \qquad \text{with} \;\; \alpha_r > 0\ . \tag{27}$$

Let us here remark that the previous assumption includes also the non-resonance condition (11) as a special case, i.e. when $i=j$ and the signs appearing in the expression $\pm\Omega_i^{(r-1)} \pm \Omega_j^{(r-1)}$ are opposite.

Also for what concerns the generating function $\mathcal{D}_2^{(r)}$, once again it is convenient to replace the secular pairs $(\underline{x},\underline{y})$ with the action–angle coordinates $(\underline{J},\underline{\varphi})$. Let us here remark that $\underline{\Omega}^{(r-1)}\cdot\underline{J}$ and $f_{0\,,\,2}^{(\mathrm{II};r,0)}(\underline{x},\underline{y})$ are the only terms appearing in expansion (19), quadratic in $(\underline{x},\underline{y})$ (so they also are $\mathcal{O}(\underline{J})$) and not depending on $\underline{p}$ and $\underline{q}$. The canonical transformation induced by the Lie series $\exp\mathcal{L}_{\mathcal{D}_2^{(r)}}$ aims to eliminate the part of $f_{0\,,\,2}^{(\mathrm{II};r,0)}$ depending on the secular angles $\underline{\varphi}$. Therefore, the generating function $\mathcal{D}_2^{(r)}$ is defined so to solve the following equation:

$$\left\{\mathcal{D}_2^{(r)}\,,\,\underline{\Omega}^{(r-1)}\cdot\underline{J}\right\} + f_{0\,,\,2}^{(\mathrm{II};r,0)}(\underline{J},\underline{\varphi}) - \left\langle f_{0\,,\,2}^{(\mathrm{II};r,0)}\right\rangle_{\underline{\varphi}} = 0\ , \tag{28}$$

where $\langle\cdot\rangle_{\underline{\varphi}}$ denotes the average with respect to the angles $\underline{\varphi}$. This implies that

$$\begin{aligned}\mathcal{D}_2^{(r)}(\underline{J},\underline{\varphi}) = \sum_{i\,,j=1}^{n_2} \sum_{\substack{s_i\,,\,s_j=\pm1 \\ s_i\cdot i+s_j\cdot j\neq 0}} 2\sqrt{J_iJ_j}\Bigg[&-\frac{c_{i\,,j\,,s_i\,,s_j}\sin\left(s_i\varphi_i+s_j\varphi_j\right)}{s_i\Omega_i^{(r-1)}+s_j\Omega_j^{(r-1)}} \\ &+\frac{d_{i\,,j\,,s_i\,,s_j}\cos\left(s_i\varphi_i+s_j\varphi_j\right)}{s_i\Omega_i^{(r-1)}+s_j\Omega_j^{(r-1)}}\Bigg]\ ,\end{aligned} \tag{29}$$

where we preliminarly assumed that the expansion of the known terms appearing in equation (28) is the following one:

$$\begin{aligned}f_{0,2}^{(\mathrm{II};r,0)}(\underline{J},\underline{\varphi}) = \sum_{i\,,j=1}^{n_2} \sum_{\substack{s_i=\pm1 \\ s_j=\pm1}} 2\sqrt{J_iJ_j}\Big[&c_{i\,,j\,,s_i\,,s_j}\cos\left(s_i\varphi_i+s_j\varphi_j\right)+ \\ &d_{i\,,j\,,s_i\,,s_j}\sin\left(s_i\varphi_i+s_j\varphi_j\right)\Big]\ ,\end{aligned} \tag{30}$$

with suitable real coefficients $c_{i\,,j\,,s_i\,,s_j}$ and $d_{i\,,j\,,s_i\,,s_j}$. Let us remark that the solution (22) for the equation (21) exists provided the frequency vector $\underline{\Omega}^{(r-1)}$ satisfies the following finite non-resonance condition:

$$\min_{|\underline{l}|=2}\left|\underline{l}\cdot\underline{\Omega}^{(r-1)}\right| \ge \beta \qquad \text{with} \;\; \beta>0\ . \tag{31}$$

At this point of the algorithm, it is convenient to slightly modify the frequencies $\underline{\omega}$ and $\underline{\Omega}$, so to include the terms which are linear with respect to the actions, do not depend on the angles and, then, can not be eliminated by our normalization procedure. More definitely, we define $\underline{\omega}^{(r)}$ and $\underline{\Omega}^{(r)}$, so that

$$\underline{\omega}^{(r)} \cdot \underline{p} = \underline{\omega}^{(r-1)} \cdot \underline{p} + f_{1,0}^{(\mathrm{II};r,0)}(\underline{p}) \ , \qquad \underline{\Omega}^{(r)} \cdot \underline{J} = \underline{\Omega}^{(r-1)} \cdot \underline{J} + \left\langle f_{0\,,2}^{(\mathrm{II};r,0)} \right\rangle_{\underline{\varphi}} \ . \tag{32}$$

Standard utilities provided by any computer algebra system should allow everyone to get the expansions of $Y_2^{(r)}(\underline{q},\underline{x},\underline{y})$ and $\mathcal{D}_2^{(r)}(\underline{x},\underline{y})$, starting from those of $Y_2^{(r)}(\underline{q},\underline{J},\underline{\varphi})$ and $\mathcal{D}_2^{(r)}(\underline{J},\underline{\varphi})$, which are written in (26) and (29), respectively. Thus, we are now able to explicitly produce the expansion of the new Hamiltonian, which can be written as follows:

$$H^{(r)}(\underline{p},\underline{q},\underline{x},\underline{y}) = \underline{\omega}^{(r-1)} \cdot \underline{p} + \underline{\Omega}^{(r-1)} \cdot \underline{J} + \sum_{s=0}^{\infty} \sum_{l=0}^{\infty} \sum_{\substack{2j_1+j_2=l \\ j_1\ge 0\,,\, j_2\ge 0}} f_{j_1\,,\,j_2}^{(r,s)}(\underline{p},\underline{q},\underline{x},\underline{y}) \ . \tag{33}$$

Let us remark that this expansion of $H^{(r)}$ has exactly the same form of that written for $H^{(r-1)}$ in (9), but we stress that the algorithm is arranged so to make smaller and smaller the contribution of the terms $f_{j_1\,,\,j_2}^{(r,s)}$, when the value of r is increased, $\forall\ s \ge 0$ and $l = j_1 + j_2 = 0,\, 1,\, 2$.

In this case too, we avoid to write down the lenghty mathematical recursive definitions of the terms $f_{j_1\,,\,j_2}^{(r,s)}$. Instead, we provide some relations about the classes of functions, which are useful to understand how to translate this third stage of the r–th normalization step in a programming language. For what concerns the generating function $X_2^{(r)}$, the following relations about the classes of the functions hold true:

$$\frac{1}{i!}\mathcal{L}^i_{X_2^{(r)}} f_{j_1\,,\,j_2}^{(\mathrm{II};r,s)} \in \mathcal{P}_{j_1\,,\,j_2}^{((s+i)K)} \qquad \forall\ i \ge 0\,,\ j_1 \ge 0\,,\ \ j_2 \ge 0\,,\ s \ge 0 \ . \tag{34}$$

The relations involving the generating function $Y_2^{(r)}$ are a little more complicated:

$$\frac{1}{i!}\mathcal{L}^i_{Y_2^{(r)}} \sum_{2j_1+j_2=l} f_{j_1\,,\,j_2}^{(\mathrm{II};r,s)} \in \bigcup_{2j_1+j_2=l} \mathcal{P}_{j_1\,,\,j_2}^{((s+i)K)} \qquad \forall\ i \ge 0\,,\ l \ge 0\,,\ s \ge 0 \ . \tag{35}$$

Finally, one can easily remark that each class of function is invariant with respect to a Poisson bracket with the generating function $\mathcal{D}_2^{(r)}$, therefore:

$$\frac{1}{i!}\mathcal{L}^i_{\mathcal{D}_2^{(r)}} f_{j_1\,,\,j_2}^{(\mathrm{II};r,s)} \in \mathcal{P}_{j_1\,,\,j_2}^{(sK)} \qquad \forall\ i \ge 0\,,\ j_1 \ge 0\,,\ \ j_2 \ge 0\,,\ s \ge 0 \ . \tag{36}$$

By taking into account the relations (34)–(36) about the classes of functions, the definition (32) of the new frequencies vectors and by suitably "reordering" the Taylor–Fourier series, it is possible to ensure that also the expansion (33) satisfies the conditions (A) and (B), which have been stated at the beginning of the present section about the equation (9). Therefore, the whole normalization procedure, that has been here described for the r–th step can be iteratively repeated.

3.4 Some remarks about the convergence of the normalization algorithm

We devote this section to an informal discussion of the relations between the normalization procedure for an elliptic torus, which is the subject of the present paper, and the Kolmogorov's algorithm for a torus of maximal dimension. Our aim is to bring into evidence, on one hand, the differences that make the case of an elliptic lower dimensional torus definitely more difficult and, on the other hand, the impact that these differences have on the explicit calculation.

The main hypotheses of the classical KAM theorem are (a) that the perturbation should be small enough and (b) that a strong non-resonance condition must be satisfied by the frequencies of the unperturbed torus. Both these conditions appear also in the proof of existence of elliptic tori, but the condition of non-resonance presents some critical peculiarities.

Concerning the smallness of the perturbation, the main problem remains that the analytical estimates are extremely restrictive. Nevertheless, one can obtain realistic results by using algebraic manipulation in order to implement a computer-assisted proof, as it has been made sevelar times about the classical KAM theorem (see, e.g., [32]). A computer-assisted procedure takes advantage of the preliminar application of the algorithm constructing the normal form (which is explicitly performed for a *finite* number of steps R, as large as possible), because a suitable version of the KAM theorem is finally applied to the Hamiltonian $H^{(R)}$ having the perturbing terms strongly reduced with respect to the initial $H^{(0)}$. In the case of lower dimensional elliptic tori, by comparing the Hamiltonian normal form (7) with the expansion (33) of $H^{(r)}$, one easily realizes that the initial expression of the perturbation (making part of the Hamiltonian $H^{(0)}$, written in (6)) is given by

$$\sum_{s=0}^{\infty}\sum_{l=0}^{2}\sum_{\substack{2j_1+j_2=l\\ j_1\geq 0\,,\,j_2\geq 0}} f_{j_1\,,\,j_2}^{(0,s)}\ . \tag{37}$$

Looking at all the preliminary expansions, which have been described in sect. 2 and allowed us to introduce the initial Hamiltonian $H^{(0)}$, one immediately sees that all the perturbing terms appearing in (37) are proportional to μ. Let us also recall that the small parameter μ is equal to the mass ratio between the biggest planet and the central star (according to its definition given in the discussion following formula (3)). In the present context, the explicit application of the normalization algorithm mainly requires to translate in a programming language the method described in the previous sections.

From a conceptual point of view, a much more difficult problem is here concerned with the conditions on non-resonance for the frequencies. In the case of a torus having maximal dimension, one must choose the n frequencies $\omega_1,\ldots,\omega_n$ so as to satisfy a strong non-resonance condition. A typical request is that they obey a Diophantine condition, i.e., that the sequence $\{\alpha_r\}_{r\geq 1}$ appearing in the inequality (11) must be such that $\alpha_r\geq\gamma/(rK)^{\tau}$ with suitable positive values of the constant γ and τ. This choice must be made at the very beginning of the procedure, and the perturbed invariant torus that is found at the end has *the same* frequencies as the unperturbed one. The reason is that at every step a small translation of the actions is introduced in order to keep the frequencies constant.

In the case of the elliptic lower dimensional torus one deals instead with two separate set of frequencies, namely $\underline{\omega}^{(0)} \in \mathbb{R}^{n_1}$ which characterize the orbits on the torus, and the transverse frequencies $\underline{\Omega}^{(0)} \in \mathbb{R}^{n_2}$ that are related to the oscillation of orbits close to but not lying on the torus. By the way, this justifies the adjective "transverse" that is commonly used. Now, the frequencies $\underline{\omega}^{(0)}$ on the torus can be chosen in an arbitrary manner, but the transverse frequencies $\underline{\Omega}^{(0)}$ are functions of $\underline{\omega}^{(0)}$, being given by the Hamiltonian. This is easily understood by considering the case of a periodic orbit, i.e., $n_1 = 1$, since in that case the transverse frequencies are related to the eigenvalues of the monodromy matrix.

The striking fact is that, due precisely to the dependence of the transverse frequencies $\underline{\Omega}^{(0)}$ on $\underline{\omega}^{(0)}$, the algorithm forces us to change these frequencies at every step. That is, one actually deals with infinite sequences $\underline{\omega}^{(r)}$ and $\underline{\Omega}^{(r)}$, all required to satisfy at every order a non-resonance condition of the form (27). Moreover, both sequences should converge to a final set of frequencies $\underline{\omega}^{(\infty)} = \underline{\omega}^{(\infty)}\big(\underline{\omega}^{(0)}\big)$ and $\underline{\Omega}^{(\infty)} = \underline{\Omega}^{(\infty)}\big(\underline{\omega}^{(0)}\big)$ which must be non-resonant (e.g., Diophantine). Thus, we are forced to conclude that, depending on the initial choice of $\underline{\omega}^{(0)}$, it may happen that the algorithm stops at some step because the frequencies fail to satisfy at least one of the non-resonance conditions (11), (17), (18), (27) and (31). This is indeed one of the main difficulties in working out the proof of existence of an elliptic torus.

Let us first consider the analytical aspect, in such a way that we can sketch some of the ideas that will be exploited in detail in a future more theoretical work, dedicated to the same subject studied here. One initially focus on an open ball $\mathcal{B} \subset \mathbb{R}^{n_1}$ such that the Diophantine condition *at finite order* required for the first step is satisfied by every $\underline{\omega}^{(0)} \in \mathcal{B} \subset \mathbb{R}^{n_1}$ and by the corresponding transverse frequencies $\underline{\Omega}^{(0)}$. This can be done, because only a finite number of non-resonance relations are considered. Therefore, one shows that at every step there exists a subset of frequencies in $\mathcal{B}$ which satisfies the non-resonance conditions (still at finite but increasing order) required in order to perform the next step, together with the corresponding transverse frequencies. This is obtained by a procedure which is reminiscent of Arnold's proof scheme of KAM theorem: at every step one removes from $\mathcal{B}$ a finite number of intersections of $\mathcal{B}$ with a small strip around a resonant plane in $\mathbb{R}^{n_1}$, assuring that the width of the strip decreases fast enough so that the remaining set always takes a non-empty interior part. By the way, this is also strongly reminiscent of the process of construction of a Cantor set. The final goal is to prove just that one is left with a Cantor set on non-resonant frequencies which satisfy the required resonance conditions and has positive Lebesgue measure. Moreover, the relative measure with respect to $\mathcal{B}$ tends to 1 when the size of the perturbation is decreased to zero. This is the idea underlying the proof that will be expanded in [16]. We emphasize that the procedure outlined here is strongly inspired by the proof scheme of KAM theorem introduced by Arnold, which is quite different from Kolmogorov's one (compare [20] with [1]).

Let us now come to the numerical aspect. At first sight the formal algorithm seems to require a cumbersome trial and error procedure in order to find the good frequencies: when some of the non-resonance conditions fail to be satisfied at a given step one should change the initial frequencies and restart the whole process. Moreover, since the non-resonance condition must be satisfied by the final frequencies, which obviously can not be calculated, the whole process seems to be unsuitable for a rigorous proof. We explain here in which sense the computer-assisted proofs can help to improve the results also in this context. We make two remarks.

The first remark is connected with the use of interval arithmetic while performing the actual construction. Following the suggestion of the analytic scheme of proof, we look for uniform estimates on a small open ball $\mathcal{B}$, such that $\forall\, \underline{\omega}^{(0)} \in \mathcal{B}$ we explicitly perform R normalization steps, with R as large as possible. Essentially, we may reproduce numerically the process of eliminating step by step the unwanted resonant frequencies by suitably determining the intervals. Once R steps have been explicitly performed, we may apply to the partially normalized Hamiltonian $H^{(R)}$ a suitable formulation of the KAM theorem for elliptic tori. This means that we recover the scheme that we have already applied to the case of tori with maximal dimension. That is, we can take advantage of the fact that the perturbing terms are much smaller than the corresponding ones for the initial Hamiltonian $H^{(0)}$; thus, in principle we could ensure that for realistic values of μ the relative measure of the invariant tori is so large that the set of those $\underline{\omega}^{(0)}$ for which the algorithm can not work (i.e., $\mathcal{B} \setminus \mathcal{S}$) is so small that can be neglected when we are dealing with a practical application.

Taking a more practical attitude, we may rely on the fact that the set of good frequencies, according to the theory, has Lebesgue measure close to one, so that the case of frequencies which are resonant at some finite order occurs with very low probability. Thus, we just make a choice of the initial frequencies and proceed with the construction, checking at every order that the non-resonance conditions that we need at that order are fulfilled. We emphasize that the most extended resonant regions are those of low order, so that it is not very difficult to check initially that the chosen frequencies will likely be good enough. It may happen, of course, that the whole procedure must be restarted with different frequencies, but we expect that this will rarely occur. However, since the size of the perturbation is expected to decrease geometrically, we may confidently expect that the probability of failure will decrease, too. This is confirmed by the actual calculations.

When R steps have been made, in principle we can apply the theorem to a small neighborhood of the calculated frequencies by choosing a suitable initial ball around the frequencies approximated at that step.

4 Elliptic tori for the SJSU system

We come now to the application of the formal algorithm for the construction of an elliptic torus to the planar SJSU system.

The initial Hamiltonian is written in (5), with a suitable rearrangement of terms so that it is given the form (9) with $r = 1$. This requires also a diagonalization of the quadratic part in the secular variables, which is performed as described at the end of sect. 2.

In the present section, we explicitly construct the normal form at a finite order checking

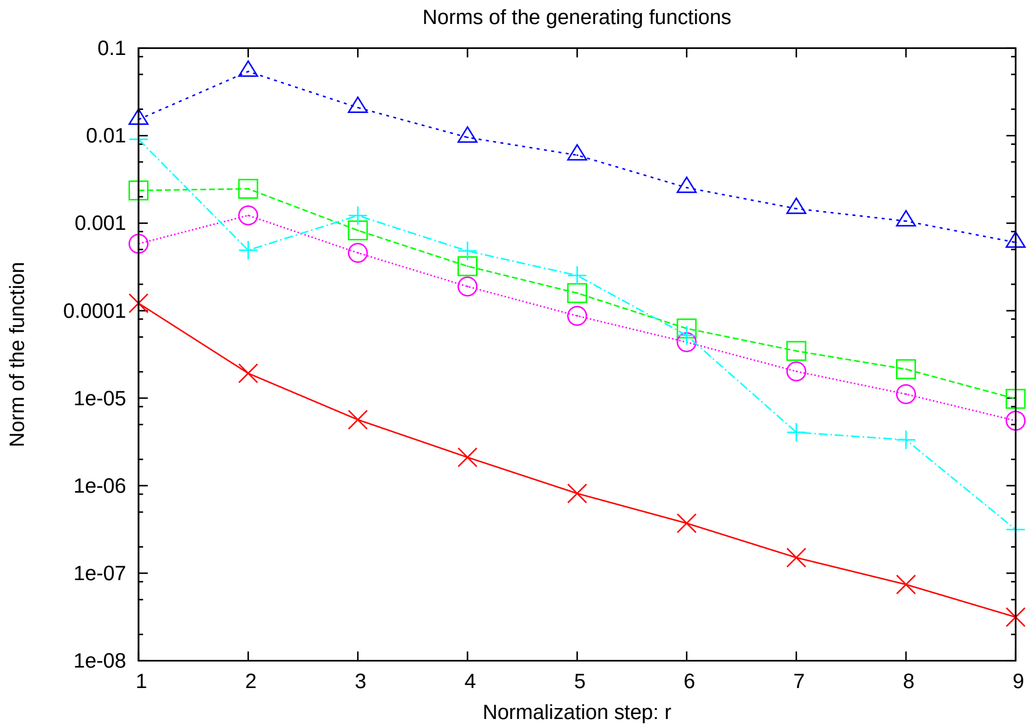

Figure 1: Algorithm constructing the normal form related to an elliptic torus for the planar SJSU system: plot of the norm of the generating functions as a function of the normalization step r ; more precisely, the symbols $\times$, $\square$, $\triangle$, $\bigcirc$ and $+$ refer to the norm of the generating functions $\chi_0^{(r)}$, $\chi_1^{(r)}$, $X_2^{(r)}$, $Y_2^{(r)}$ and $\mathcal{D}_2^{(r)}$, respectively, which are defined during the normalization algorithm, as described in sect. 3. The norm is calculated by simply adding up the absolute values of all the coefficients appearing in the expansion of each generating function.

that the norms of the generating function decrease as predicted by the theory. Then we perform a numerical check by comparing the orbit obtained via the normal form with the numerically integrated one.

4.1 Constructing the elliptic torus by using computer algebra

We applied the algorithm constructing elliptic tori (which has been widely described in sect. 3) to the Hamiltonian $H^{(0)}$ (that is defined in (6) and has been obtained as described in sect. 2). The parameters have been fixed according to the specific values of the planar SJSU system, which are reported in Table 1. Our software package for computer algebra allowed us to explicitly calculate all the expansions (6) of $H^{(r)}$ with index r ranging between 0 and 9 , so to include: (c1) the terms having degree j_1 in the actions $\underline{p}$ with $j_1 \leq 3$, (c2) all terms having degree j_2 in the variables $(\underline{x}, \underline{y})$, with j_2 such that $2j_1 + j_2 \leq 8$, (c3) all terms up to the trigonometric degree 18 with respect to the angles $\underline{q}$. Let us recall that the truncation rules (c1)–(c3) are in agreement with those prescribed about the expansion (5) in sect. 2 at points (b1)–(b3). Let us remark that both the truncation rules (c1) and (c2) are preserved by all the canonical transformations included in our algorithm. Moreover, we have found that fixing $K = 2$ is a suitable choice to have a rather regular decreasing of the size of the generating functions when the normalization step r is increased, as shown in Fig. 1. Since the maximal trigonometric degree of the generating functions $\chi_0^{(r)}$, $\chi_1^{(r)}$, $X_2^{(r)}$ and $Y_2^{(r)}$ is equal to rK , the choice to set $K = 2$ and the rule (c3) explain why we stopped the algorithm after having ended the normalization step with $r = 9$.

The behavior of the norms of the generating functions is reported in Fig. 1. Let us make a few comments. The theoretical estimates predict that the norms should decrease geometrically with the order in order to assure the convergence of the normal form. The figure shows that this is indeed the behavior in our case. We emphasize that the presence of a dangerous resonance would be reflected in a sudden increase of the coefficients; thus, the plot gives a practical confirmation that the frequencies are well chosen.

Let us stress that performing the construction of the normal form up to order $r = 9$ has been very stressing for the computational resources available to us, although the length of the calculation is not reflected in a corresponding length of the present subsection.

4.2 Explicit calculation of the orbits on the elliptic torus

We now perform a check on the approximation of the elliptic torus. To this end, we calculate the orbit on the torus using the analytic expression and we compare it with a numerical integration of Hamilton's equations. In this subsection we explain how the calculation of the orbit via normal form is performed.

According to the theory of Lie series, the canonical transformation $(\underline{p}, \underline{q}, \underline{x}, \underline{y}) =$

$\mathcal{K}^{(r)}\big(\underline{p}^{(r)},\underline{q}^{(r)},\underline{x}^{(r)},\underline{y}^{(r)}\big)$ inducing the normalization up to the step r is given by

$$\begin{aligned}\mathcal{K}^{(r)}\big(\underline{p}^{(r)},\underline{q}^{(r)},\underline{x}^{(r)},\underline{y}^{(r)}\big) = {}& \exp\mathcal{L}_{\mathcal{D}_2^{(r)}}\circ\exp\mathcal{L}_{Y_2^{(r)}}\circ\exp\mathcal{L}_{X_2^{(r)}}\circ \\ & \exp\mathcal{L}_{\chi_1^{(r)}}\circ\exp\mathcal{L}_{\chi_0^{(r)}}\circ\ldots\circ\exp\mathcal{L}_{\mathcal{D}_2^{(1)}}\circ\exp\mathcal{L}_{Y_2^{(1)}}\circ \\ & \exp\mathcal{L}_{X_2^{(1)}}\circ\exp\mathcal{L}_{\chi_1^{(1)}}\circ\exp\mathcal{L}_{\chi_0^{(1)}}\big(\underline{p}^{(r)},\underline{q}^{(r)},\underline{x}^{(r)},\underline{y}^{(r)}\big)\ ,\end{aligned} \tag{38}$$

where $\big(\underline{p}^{(r)},\underline{q}^{(r)},\underline{x}^{(r)},\underline{y}^{(r)}\big)$ are meant to be the new coordinates. Thus, the canonical transformation $(\underline{p},\underline{q},\underline{x},\underline{y}) = \mathcal{K}^{(\infty)}(\underline{P},\underline{Q},\underline{X},\underline{Y})$ brings $H^{(0)}$ in the normal form $H^{(\infty)} = H^{(0)}\circ\mathcal{K}^{(\infty)}$, which is written in (7), with $\mathcal{K}^{(\infty)} = \lim_{r\to\infty}\mathcal{K}^{(r)}$. Let us introduce a new symbol to denote the composition of all the canonical change of coordinates defined in sects. 2 and 3, i.e.

$$\mathcal{C}^{(r)} = \mathcal{E}\circ\mathcal{T}_{\underline{\Lambda}^*}\circ\mathcal{D}\circ\mathcal{K}^{(r)}\ , \tag{39}$$

where $(\underline{\tilde{r}},\underline{r}) = \mathcal{E}(\underline{\Lambda},\underline{\lambda},\underline{\xi},\underline{\eta})$ is the canonical transformation giving the heliocentric positions $\underline{r}$ and their conjugated momenta $\underline{\tilde{r}}$ as a function of the Poincaré variables. If $\big(\underline{\tilde{r}}(0),\underline{r}(0)\big)$ is an initial condition on an invariant elliptic torus, in principle we might use the following calculation scheme to integrate the equation of motion:

$$\begin{array}{ccc} \big(\underline{\tilde{r}}(0),\underline{r}(0)\big) & \overset{\left(\mathcal{C}^{(\infty)}\right)^{-1}}{\longrightarrow} & \big(\underline{P}(0)=\underline{0}\,,\ \underline{Q}(0)\,,\ \underline{X}(0)=\underline{0}\,,\ \underline{Y}(0)=\underline{0}\big) \\ & & \Big\downarrow\ \Phi^t_{\underline{\omega}^{(\infty)}\cdot\underline{P}} \\ \big(\underline{\tilde{r}}(t),\underline{r}(t)\big) & \overset{\mathcal{C}^{(\infty)}}{\longleftarrow} & \big(\underline{P}(t)=\underline{0}\,,\ \underline{Q}(t)=\underline{Q}(0)+\underline{\omega}^{(\infty)}t\,,\ \underline{X}(t)=\underline{0}\,,\ \underline{Y}(t)=\underline{0}\big) \end{array}\ , \tag{40}$$

where $\Phi^t_{\underline{\omega}^{(\infty)}\cdot\underline{P}}$ induces the quasi-periodic flow related to the frequencies vector $\underline{\omega}^{(\infty)}$. Of course, the previous scheme requires an unlimited computing power; from a practical point of view, we can just approximate it, by replacing $\mathcal{C}^{(\infty)}$ with $\mathcal{C}^{(R)}$, where R is as large as possible. Thus, the integration via normal form actually reduces to a transformation of the coordinates of the initial point to the coordinates of the normal form, the calculation of the flow at time t in the latter coordinates, which is a trivial matter since the flow is exactly quasi-periodic with known frequencies, followed by a transformation back to the original coordinates.

Such an approximated semi-analytic calculation scheme can be directly compared with the results provided by a numerical integrator. As it has been shown in [32], [33], [34] and [11], this kind of comparisons provide a very stressing test for the accuracy of the whole algorithm constructing the normal form.

4.3 Validation of the results by using frequency analysis

The ideal calculation scheme (40) highlights that the Fourier spectrum of each component of the motion law $t \mapsto \big(\underline{\tilde{r}}(t), \underline{r}(t)\big)$ is the very peculiar one

$$\sum_{j=0}^{\infty} c_j \exp\big(\mathrm{i}\zeta_j t\big) \ , \quad \text{where, } \forall\ j \ge 0\,,\ c_j \in \mathbb{C} \text{ and } \exists\ \underline{k}_j \in \mathbb{Z}^{n_1} \text{ such that } \zeta_j = \underline{k}_j \cdot \underline{\omega}^{(\infty)}\ . \tag{41}$$

In other words, the Fourier spectrum of the planetary motions on elliptic tori is so characteristic, because all its frequencies are given by linear combinations of the fast frequencies. From a strictly mathematical point of view, let us recall that the previous formula for the Fourier spectrum can be deduced by the scheme (40), because of the analyticity of the so called conjugacy function $\underline{Q} \mapsto \mathcal{C}^{(\infty)}\big(\underline{0}, \underline{Q}, \underline{0}, \underline{0}\big)$ and this will be ensured as a byproduct of the theoretical (future) study of the convergence of the constructive algorithm.

In the present subsection we aim to check the peculiar quasi-periodicity of the motions on our approximation of an elliptic torus, by using the frequency map analysis (see, e.g., [29] and [30] for an introduction). We focus on the following initial conditions:

$$\big(\mathcal{C}^{(9)}\big)^{-1}\,(\underline{0}, \underline{0}, \underline{0}, \underline{0}) \ ; \tag{42}$$

according to the calculation method described in the previous subsect. 4.2, this should be an accurate approximation of a point on an elliptic torus. Therefore, we preliminarly integrated the motion of the planar SJSU system over a time interval of 2^{24} years, by using the symplectic method $\mathcal{SBAB}_3$ (see [31]) with a time-step of 0.04 years.

Here we should add a remark concerning the precision. In order to have a signal clean enough to be analyzed a particular care about the precision is mandatory. After some trials tuning the parameters of the numerical integration, we found that the 80 bits floating point numbers provided by the current AMD and INTEL CPUs fits our needs. Technically this is obtained by using the `long double` types of the GNU **C** compiler under a Linux operating system.

The orbits have been sampled with a time interval of 1 year. The signals related to the secular Poincaré variables, that are $\xi_l(t) + \mathrm{i}\eta_l(t)$ with $l = 1, 2, 3$, have been submitted to the frequency analysis method using the so-called Hanning filter.

In Table 2 we report our numerical results about the first 25 summands of the decomposition (41) for the Uranus secular signal, i.e. $\xi_3(t) + \mathrm{i}\eta_3(t)$. Let us point out that the values of the fast frequencies vector $\underline{\omega}^{(\infty)}$ have been preliminarly calculated by looking at the main components of the Fourier spectrum of the signals $\Lambda_l(t)\exp\big(\mathrm{i}\lambda_l(t)\big)$, with $l = 1, 2, 3$. Moreover, we stress that the vectors $\underline{k}_j \in \mathbb{Z}^{n_1}$ listed in the third column are determined so to minimize the absolute difference $|\zeta_j - \underline{k}_j \cdot \underline{\omega}^{(\infty)}|$ with $|\underline{k}_j| \le 20$; indeed, one has to fix some limits on the absolute value of $\underline{k}_j$, in order to make consistent its calculation, and our choice is motivated by the fact that the Fourier decay of the analytic conjugacy function $\mathcal{C}^{(\infty)}\big(\underline{0}, \underline{Q}, \underline{0}, \underline{0}\big)$ is such that the main contributions to the spectrum are related to low order harmonics.

If the initial conditions (42) were exactly on an elliptic torus, each value $|\zeta_j - \underline{k}_j \cdot \underline{\omega}^{(\infty)}|$ reported in the fourth column of Table 2 should be equal to zero. All of them, except for

Table 2: Decomposition of the Fourier spectrum of the signal $\xi_3(t) + \mathrm{i}\eta_3(t)$, which is related to the Uranus secular motion. The following numerical values have been obtained by applying the frequency analysis method. See the text for more details.

j	ζ_j	$\underline{k}_j$	$\lvert\zeta_j - \underline{k}_j \cdot \underline{\omega}^{(\infty)}\rvert$	$\lvert c_j\rvert$
0	$-7.45980878285529281 \times 10^{-2}$	$(0, 0, -1)$	$0.0 \times 10^{+00}$	2.9778×10^{-4}
1	$3.80570419432301466 \times 10^{-1}$	$(1, 0, -2)$	5.6×10^{-17}	5.5358×10^{-5}
2	$6.35855778812917105 \times 10^{-2}$	$(0, 1, -2)$	2.8×10^{-17}	1.8220×10^{-5}
3	$2.01769243591136460 \times 10^{-1}$	$(0, 2, -3)$	5.6×10^{-17}	1.7402×10^{-5}
4	$3.39952909300980821 \times 10^{-1}$	$(0, 3, -4)$	2.8×10^{-16}	6.0978×10^{-6}
5	$8.35738926693155748 \times 10^{-1}$	$(2, 0, -3)$	$0.0 \times 10^{+00}$	3.7192×10^{-6}
6	$4.78136575010826514 \times 10^{-1}$	$(0, 4, -5)$	7.8×10^{-16}	2.4468×10^{-6}
7	$-2.12781753538397789 \times 10^{-1}$	$(0, -1, 0)$	1.9×10^{-16}	1.4515×10^{-6}
8	$6.16320240720669266 \times 10^{-1}$	$(0, 5, -6)$	1.2×10^{-15}	1.0438×10^{-6}
9	$-3.50965419248241817 \times 10^{-1}$	$(0, -2, 1)$	4.4×10^{-16}	1.0174×10^{-6}
10	$1.29090743395401009 \times 10^{+0}$	$(3, 0, -4)$	$0.0 \times 10^{+00}$	7.7418×10^{-7}
11	$-4.89149084958088287 \times 10^{-1}$	$(0, -3, 2)$	1.3×10^{-15}	7.1150×10^{-7}
12	$7.54503906430515681 \times 10^{-1}$	$(0, 6, -7)$	5.6×10^{-16}	4.6095×10^{-7}
13	$-9.84935102350262381 \times 10^{-1}$	$(-2, 0, 1)$	7.8×10^{-16}	4.1629×10^{-7}
14	$-6.27332750667927597 \times 10^{-1}$	$(0, -4, 3)$	4.0×10^{-15}	3.8136×10^{-7}
15	$-5.29766595089407821 \times 10^{-1}$	$(-1, 0, 0)$	5.6×10^{-16}	2.9632×10^{-7}
16	$-1.10827474865547476 \times 10^{-5}$	$(0, 0, 0)$	1.1×10^{-05}	2.3288×10^{-7}
17	$-7.65516416377781117 \times 10^{-1}$	$(0, -5, 4)$	4.8×10^{-15}	1.9146×10^{-7}
18	$8.92687572140363206 \times 10^{-1}$	$(0, 7, -8)$	3.4×10^{-15}	2.0808×10^{-7}
19	$1.74607594121486587 \times 10^{+0}$	$(4, 0, -5)$	1.4×10^{-15}	1.7591×10^{-7}
20	$-1.44010360961111750 \times 10^{+0}$	$(-3, 0, 2)$	1.6×10^{-15}	1.3340×10^{-7}
21	$2.29680677496932953 \times 10^{-2}$	$(-1, 4, -4)$	2.8×10^{-13}	1.1411×10^{-7}
22	$1.03087123785019941 \times 10^{+0}$	$(0, 8, -9)$	5.0×10^{-15}	9.5369×10^{-8}
23	$-9.03700082087618650 \times 10^{-1}$	$(0, -6, 5)$	2.3×10^{-15}	9.3899×10^{-8}
24	$1.16905490356004615 \times 10^{+0}$	$(0, 9, -10)$	2.9×10^{-15}	4.4199×10^{-8}

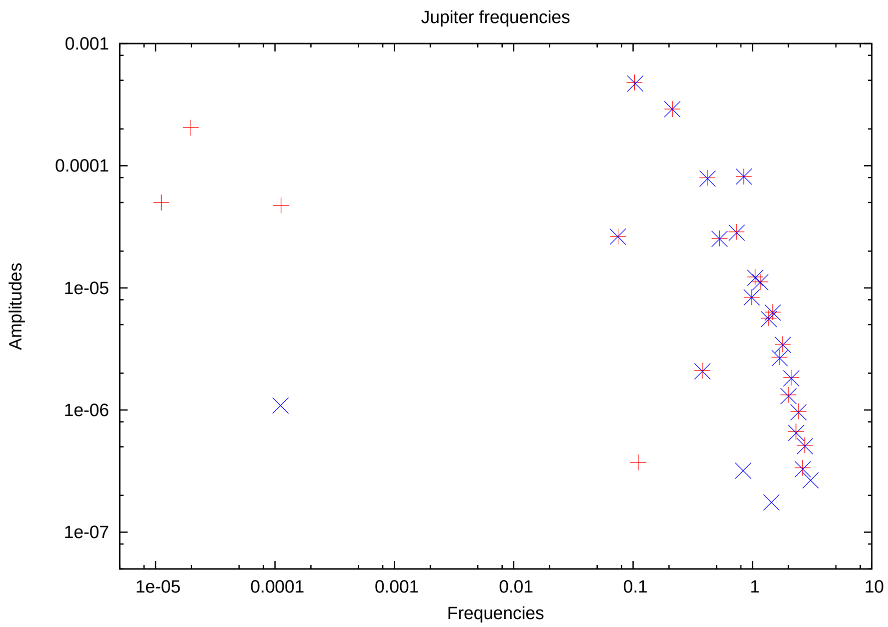

Figure 2: Frequency analysis of the secular signal related to the secular Jupiter motion: $\xi_1(t)+\mathrm{i}\eta_1(t) = \sum_{j=0}^{\infty} c_j \exp\big(\mathrm{i}\zeta_j t\big)$. Plot of the amplitudes $|c_j|$ as a function of the frequencies ζ_j in Log-Log scale. The symbol $\times$ [$+$, resp.] refer to the signal related to the motion starting from the initial conditions (42) [(43), resp.], i.e. the approximation of a point on an elliptic torus after having performed 9 [0, resp.] steps of the algorithm constructing the corresponding normal form. In both cases, the results for just the first 25 components have been reported in the figure above.

the case corresponding to $j = 16$, are actually small enough to be considered as generated by round-off errors. On the other hand, we can definitely say that $\zeta_{16} \simeq -1.1 \times 10^{-5}$ is a "secular frequency", because its value is $\mathcal{O}(\mu)$. Indeed, let us recall that $\mu \simeq 10^{-3}$, but the mass ratio for Uranus, i.e. $m_3/m_0 \simeq 4.4 \times 10^{-5}$, is even smaller.

Let us say that the occurrence of secular frequencies in the Fourier decomposition of the signal should be expected. Indeed, they could be completely avoided only in a very ideal situation, namely: (i) all the calculations described in sects. 2 and 3 should be carried out without performing any truncations on the expansions, (ii) the initial conditions (42) should be replaced with $\left(\mathcal{C}^{(\infty)}\right)^{-1}(\underline{0},\underline{0},\underline{0},\underline{0})$, (iii) no numerical errors should be there. In a practical calculation the orbit can not be exactly placed on an elliptic torus, so the presence of secular frequencies just means that we are just close to it. Nevertheless, it is very remarkable that the amplitude of the first found secular frequency is three orders of magnitude smaller than the main component of the spectrum. In our opinion, this is a first clear indication that our algorithm is properly working.

Other components corresponding to secular frequencies are expected to be even smaller than that found with $j = 16$. In fact, let us recall that the frequency analysis method detects the summands $c_j \exp\left(\mathrm{i}\zeta_j t\right)$ appearing in (41) in a nearly decreasing order with respect to the amplitude $|c_j|$ (for instance, one can easily see that just one exchange is needed in order to rewrite Table 2 in the correct decreasing order); moreover, we calculated that the discrepancy $\left|\xi_3(t)+\mathrm{i}\eta_3(t)-\sum_{j=0}^{24} c_j \exp\left(\mathrm{i}\zeta_j t\right)\right|$ is smaller than about $\simeq 3.2\times 10^{-7}$ for all the time values t for which we sampled the signal. Let us emphasize that such an upper bound on the maximal discrepancy is just a little larger than the amplitude $|c_{16}|$.

A similar decomposition has been calculated for both the signals $\xi_1(t) + \mathrm{i}\eta_1(t)$ and $\xi_2(t)+\mathrm{i}\eta_2(t)$ (which are related to the secular motions of Jupiter and Saturn, respectively). The behavior is very similar to that of Table 2, so we omit the corresponding tables because the interesting results are more evident from the figures that we are going to present.

The most relevant information about such decompositions of the secular motions of the three planets is summarized in the plots done with the $\times$ symbol appearing in Figs. 2–4.

Those figures contain also a comparison with the results provided by a, say, trivial approximation of an orbit on an elliptic torus. In fact, the dots marked with the $+$ symbol appearing in Figs. 2–4 refer to a frequency analysis which is performed exactly in the same way as that corresponding to the $\times$ symbol, except the fact that the numerical integration of the equations of motion is started from the following initial conditions

$$\left(\mathcal{C}^{(0)}\right)^{-1}(\underline{0},\underline{0},\underline{0},\underline{0}) \ , \tag{43}$$

instead of that reported in formula (42). Let us remark that $\left(\mathcal{C}^{(0)}\right)^{-1}(\underline{0},\underline{0},\underline{0},\underline{0}) = \mathcal{E}\circ\mathcal{T}_{\underline{\Lambda}^*}\circ\mathcal{D}\,(\underline{0},\underline{0},\underline{0},\underline{0})$ is a sort of trivial approximation of a point on the elliptic torus as it is provided by simply avoiding to apply the part of our algorithm constructing the normal form, as it is described in sect. 3. In order to discuss in a more definite way, we have already assumed that the secular frequencies are $\mathcal{O}(\mu)$, thus, let us separate them from the fast ones, when they are smaller than 10^{-3}. By looking at the right side of Figs. 2–4, one can immediately remark that the parts of the spectra related to the fast frequencies are nearly indistinguishable when the initial conditions (42) or (43) are

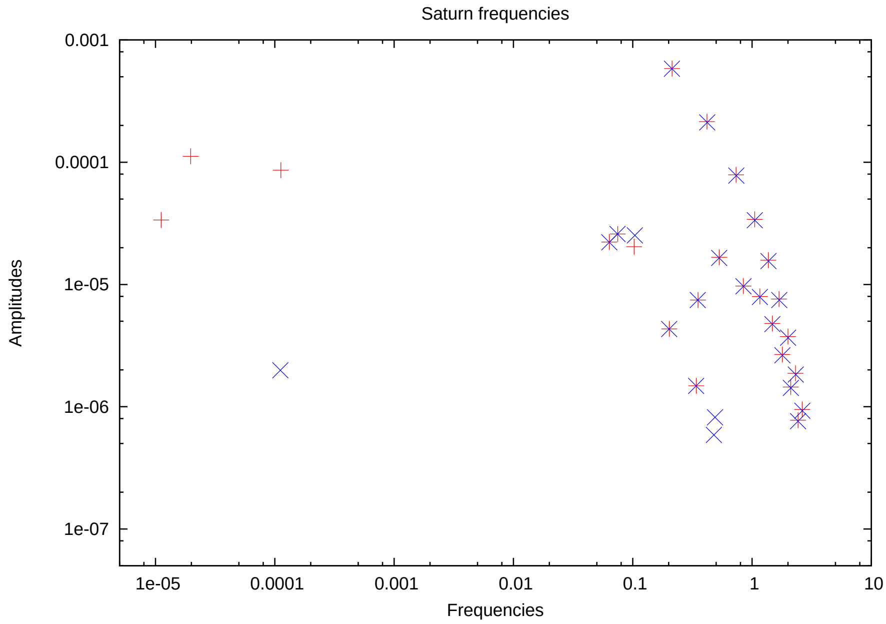

Figure 3: Frequency analysis of the secular signal related to the secular Saturn motion: $\xi_2(t) + \mathrm{i}\eta_2(t) = \sum_{j=0}^{\infty} c_j \exp\big(\mathrm{i}\zeta_j t\big)$. Plot of the amplitudes $|c_j|$ as a function of the frequencies ζ_j in Log-Log scale. The meaning of the symbols $\times$ and $+$ is the same as in Fig. 2.

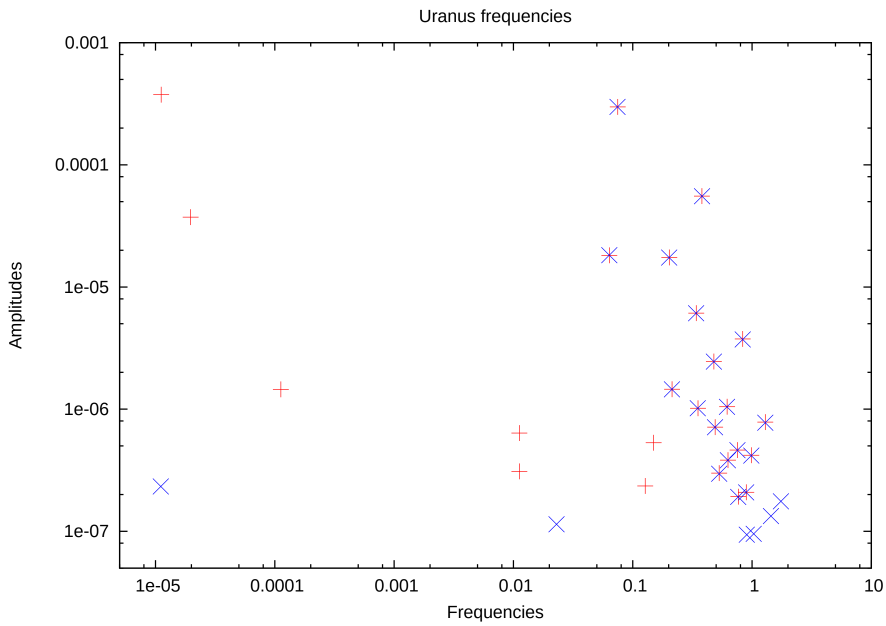

Figure 4: Frequency analysis of the secular signal related to the secular Uranus motion: $\xi_3(t) + \mathrm{i}\eta_3(t) = \sum_{j=0}^{\infty} c_j \exp\big(\mathrm{i}\zeta_j t\big)$. Plot of the amplitudes $|c_j|$ as a function of the frequencies ζ_j in Log-Log scale. The meaning of the symbols $\times$ and $+$ is the same as in Fig. 2.

considered, because the dots marked with the symbols $\times$ and $+$ superpose each other in a nearly exact way for what concerns all the main components. On the other hand, the secular parts of the spectra (that are in the left side of Figs. 2–4) strongly differ. In fact, when the initial conditions (43) (that trivially approximate a point on the elliptic torus) are considered, three secular frequencies are detected; while just one is found in the case of the more accurate initial data (42). Moreover, by comparing the amplitudes, one can see that the unique secular component detected by both the frequency analyses is decreased by at least two orders of magnitude when our algorithm is applied. In our opinion, this comparison makes evident the effectiveness of our procedure constructing the normal form for an elliptic torus.

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