# Decay of Resonance Structure and Trapping Effect in Potential Scattering Problem of Self-Focusing Wave Packet

Hironobu Fujishima*, Makoto Mine[1] †, Masahiko Okumura[2,3] ‡ and Tetsu Yajima[4] §

*Optics R&D Center, CANON INC., 23-10, Kiyohara-Kogyodanchi, Utsunomiya 323-3298, Japan*
[1]*Waseda University Honjo Senior High School, 1136 Nishitomida Honjo Saitama 367-0035 Japan*
[2]*Computational Condensed Matter Physics Laboratory, RIKEN, Wako, Saitama 351-0198, Japan*
[3]*CREST(JST), Kawaguchi, Saitama 332-0012, Japan*
[4]*Department of Information Systems Science, Graduate School of Engineering, Utsunomiya University, Yoto 7-1-2, Utsunomiya 321-8585, Japan*

Potential scattering problems governed by the time dependent Gross-Pitaevskii equation are investigated numerically for various values of coupling constants. The initial condition is assumed to have the Gaussian type envelope, which differs from the soliton solution. The potential is chosen to a box or well type. We estimate dependencies of reflectance and transmittance on width of the potential and compare these results with those given by the stationary Schrödinger equation. We attribute the behaviors of these quantities to limitation on width of nonlinear wave packet. The coupling constant and the width of the potential play an important role in distribution of the waves appearing in the final state of the scattering.

KEYWORDS: Gross-Pitaevskii equation, nonlinear Schrödinger equation, potential scattering, Gaussian initial condition, numerical analysis, Bose-Einstein condensation

## 1. Introduction

The nonlinear Schrödinger equation (NLSE), which has a cubic nonlinear term,

$$\mathrm{i}\phi_t + \phi_{xx} + 2|\phi|^2\phi = 0, \tag{1}$$

appears in various fields of physics.[1)] The NLSE can be derived as an amplitude equation of the system whose dispersion relation depends dominantly on the square of the amplitude. Among them is envelope motion of coupled nonlinear oscillators with cubic interaction.[2)] In nonlinear optics, both self-focusing effect in two-dimensional (2D) systems and optical soliton propagation in one-dimensional (1D) systems are governed by the NLSE.[3,4)] Another important example is the Bose-Einstein condensed (BEC) system[6,7)] where macroscopic wave function of condensate atoms appears as the order parameter accompanied with the spontaneous breakdown of the $U(1)$ gauge symmetry.[5)] In this case,

*E-mail address: fujishima.hironobu@canon.co.jp
†E-mail address: mine@waseda.jp
‡E-mail address: okumura@riken.jp
§E-mail address: yajimat@is.utsunomiya-u.ac.jp

the NLSE is regarded as mean-field approximation of the Heisenberg equation for field operators and describes the time-evolution of this macroscopic wave function in good accuracy.[6,7]

One of the striking feature of 1D NLSE is its integrability. In particular, exact solutions under a given initial condition can be uniquely solved by the inverse scattering transformation (IST) method.[8,9] It is based on an auxiliary linear eigenvalue problem,

$$\begin{pmatrix} \psi_{1x} \\ \psi_{2x} \end{pmatrix} = \begin{pmatrix} -\mathrm{i}\zeta & \mathrm{i}\phi^* \\ \mathrm{i}\phi & \mathrm{i}\zeta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{2}$$

where $\zeta$ is the eigenvalue which is independent of time, $\phi$ is the solution of the NLSE and asterisk means its complex conjugate. This formulation takes a form of a potential scattering problem for auxiliary field $\psi$ where $\phi$ works as a potential. The eigenvalue spectrum consists of discrete and continuous parts, where the former generates soliton solutions and the latter corresponds to small ripples. For the sech-type initial condition with suitable amplitude, the eigenvalue consists only of discrete part. In this case, whole initial value problem is solved analytically and results in famous $N$-soliton solution.[10]

Time evolution of the auxiliary field is defined by another linear equation

$$\begin{pmatrix} \psi_{1t} \\ \psi_{2t} \end{pmatrix} = \begin{pmatrix} 2\mathrm{i}\zeta^2 - \mathrm{i}|\phi|^2 & \phi_x^* - 2\mathrm{i}\zeta\phi^* \\ -\phi_x - 2\mathrm{i}\zeta\phi & -2\mathrm{i}\zeta^2 + \mathrm{i}|\phi|^2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{3}$$

However, the time evolution of the wave packet from non-soliton initial condition is relatively unclear since analytic expressions are hardly feasible.

On the other hand, in the BEC system, it is natural to assume the existence of external field to express the effect of gravity or quadratic traps for atoms.[11] Thus, the term of external field is added to conventional NLSE (1), and the equation is called the time dependent Gross-Pitaevskii equation (TDGPE). So far, most of analytic studies have assumed linear or quadratic potentials only, in which case the integrability of the systems are not spoiled and one can obtain analytical results by systematic application of the IST method.[12,13] In such cases, soliton initial conditions are also assumed.

However, we can also consider spatially-localized potentials where the integrability is manifestly violated. It is important to evaluate the role of nonlinearity on such potential scattering problems like the tunneling effect. In particular, time dependent analysis of moving wave packets is intriguing since *in situ* observation of condensed atoms is possible in the BEC system,[6,7] although most of studies deal with these kinds of problems as stationary ones.[14]

When we analyze the stationary potential scattering problems, we entirely adopt the wave-like nature and the resonant phenomena brought about by the interference effect. On the the other hand, spatially-localized pulse is expected to exhibit the so-called wave packet effect in the scattering process. In addition, nonlinear effects are also interesting in the potential scattering problem. To investigate the influence of this wave packet effect under nonlinearity on the potential scattering problem, it is necessary to trace the dynamics of the system. Some authors have reported on this kind of problem assuming soliton initial conditions.[15,16] However, examples which take non-soliton solutions as initial conditions have been rare because of extra complexities. In this paper, we numerically trace and examine the dynamics of the wave packets governed by 1D TDGPE with the box or well type potential under the Gaussian type initial conditions, different from soliton solutions.

This paper is organized as follows. In the next section, the non-soliton dynamics of the wave packets without external field are analyzed. In section 3, we evaluate and characterize the nonlinear wave packet on the reflectance or transmittance, changing the magnitude of the nonlinearity, position of the initial wave packet, and the width of the potential. The section 4 is devoted to discussion, and we interpret the results on the basis of squeezed width of wave packets. Extra complexities intrinsic to non-soliton initial conditions are also argued in detail. The summary is given in the section 5.

## 2. Time Dependent Gross-Pitaevskii Equation and Scattering Problem

In this section, we briefly summarize mathematical descriptions of the system to be considered. We restrict ourselves to 1D case throughout this paper. By virtue of scale transformation, we can put both of the coefficients of $\phi_t$ and $\phi_{xx}$ of the TDGPE to be unity and we shall consider

$$\mathrm{i}\phi_t + \phi_{xx} + V(x)\phi + g|\phi|^2\phi = 0, \tag{4}$$

where $V(x)$ is an external potential applied to the system, and $g$ is the coupling constant. To investigate the nonlinear and wave packet effects, the sign of $g$ is important and we discard the possibility of $g < 0$ throughout this paper. If we take $g$ to be negative, which means the repulsively interacting field, the wave packet immediately expands and this rapid diffusing makes the amplitude of the wave packet very small. Therefore, excitation of higher harmonic waves is extremely suppressed, and manifestation of nonlinear effect is less expected. Moreover, these widespread wave packets share most of scattering features with stationary plane waves in the linear limit, because the broaden wave packets have narrow spectra in the Fourier space. Therefore, we focus on the $g > 0$ case, which means the attractively interacting field, in order to investigate the nonlinear and wave packet effects on the potential scattering problem. According to the theory of partial differential equation, finite and unique solution of eq. (4) exists for arbitrary initial conditions for 1D case and instability or explosion of the solution observed in multi dimensions never occurs even if we take $g$ to be positive.

Hereafter, we shall normalize the wave function $\phi$ as $\int_{\mathbb{R}} |\phi|^2 \mathrm{d}x = 1$. The initial condition of the wave packet is fixed to be the Gaussian type,

$$\phi(x, 0) = \frac{1}{\sqrt[4]{\pi}} \mathrm{e}^{-\frac{1}{2}(x+x_0)^2 + \mathrm{i}\nu(x+x_0)}, \tag{5}$$

where $-x_0$ is the position of the center of the initial wave packet and $v$ gives the half of its velocity. We assume right-forward propagation of the wave packet, i.e., $x_0 > 0$ and $v > 0$. The energy functional $E$, the Hamiltonian of the system, is defined as

$$E = \int \left( |\phi_x|^2 - V(x)|\phi|^2 - \frac{1}{2} g|\phi|^4 \right) \mathrm{d}x. \tag{6}$$

Equation (4) can be derived straightforwardly from the Hamiltonian through standard canonical procedure. Since we have assumed $g$ to be positive, $E$ might take negative value. In fact, for the initial wave packet located sufficiently far from the potential, the initial value of $E$ becomes negative under the condition

$$\frac{1}{2} + v^2 - \frac{g}{\sqrt{8\pi}} < 0. \tag{7}$$

We consider the box and well type external potential

$$V_{\mathrm{box}} = -\theta(x) + \theta(x-a), \tag{8}$$

$$V_{\mathrm{well}} = V_0(\theta(x) - \theta(x-a)), \tag{9}$$

where $a$ is the width of the potential and $\theta(x)$ denotes the step function. We define the reflectance and transmittance from these potentials as

$$R_{\mathrm{box}} = \lim_{t\to\infty} \int_{-\infty}^{0} |\phi|^2 \mathrm{d}x, \tag{10}$$

$$R_{\mathrm{well}} = \lim_{t\to\infty} \int_{-\infty}^{-b} |\phi|^2 \mathrm{d}x, \tag{11}$$

$$T_{\mathrm{well}} = \lim_{t\to\infty} \int_{a+b}^{\infty} |\phi|^2 \mathrm{d}x. \tag{12}$$

The reason why we introduce $b$ for the definition of $R_{\mathrm{well}}$ (11) and $T_{\mathrm{well}}$ (12) is as follow. For well-type potential, a part of the wave packet is trapped by the potential well and oscillates around the potential area. The distance of $b$ is provided as a margin to distinguish trapped portion and reflected or transmitted ones. The trapped portion never completely separates from the other parts of the wave packet, and continues to exchange very small amount of their norms, and the limits in eqs. (11) and (12) do not exist in strict meaning. However, for an evaluating measure, we use the values at $t = 80$ as if they were limiting ones. In the next sections, we variate $g$, $x_0$ and $a$ and investigate their influence on $R_{\mathrm{box}}$ and $T_{\mathrm{well}}$. For numerical integration, we employ the symplectic Fourier method[18] throughout this paper.

## 3. Results

In this section, effects of nonlinearity on free propagation and potential scattering problems under the wave packet initial condition are considered.

### 3.1 Free propagation

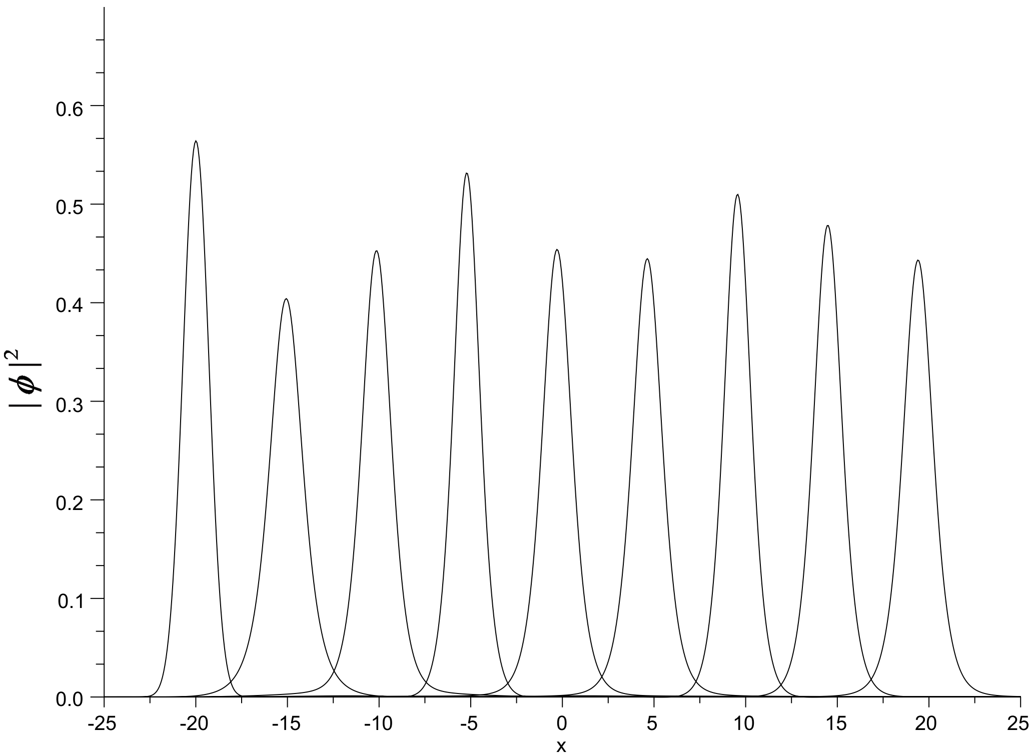


Fig. 1. Free propagating breather-like motion of a wave packet starting from the initial wave packet (5) with $x_0 = 20$, $v = \sqrt{1.5}$ and $g = 4$. The nine wave packets show $|\phi|^2$ at $t$=0, 2, . . . , 14 and 16 from the left to the right.

In this subsection, we discuss free propagation of a wave packet where no external potential exists. In this case, 1-soliton solution of eq. (4) with $V(x) = 0$ can be written as

$$\phi(x,t) = \sqrt{\frac{2}{g}}\eta\,\mathrm{sech}(x\eta + 2t\eta\xi)\mathrm{e}^{\mathrm{i}\{x\xi - t(\eta^2 - \xi^2)\}}, \tag{13}$$

where $\eta$ and $\xi$ are independent parameters and responsible for amplitude and velocity of the soliton, respectively. Once this form of solution is taken to be the initial condition, it never diffuses and keeps its own shape during time evolution.

However, the Gaussian initial condition (5) leads to quasi-breather-like solutions. We show time evolution of the wave profile $|\phi|^2$ in Fig. 1. We have also calculated the wave function in wave-number

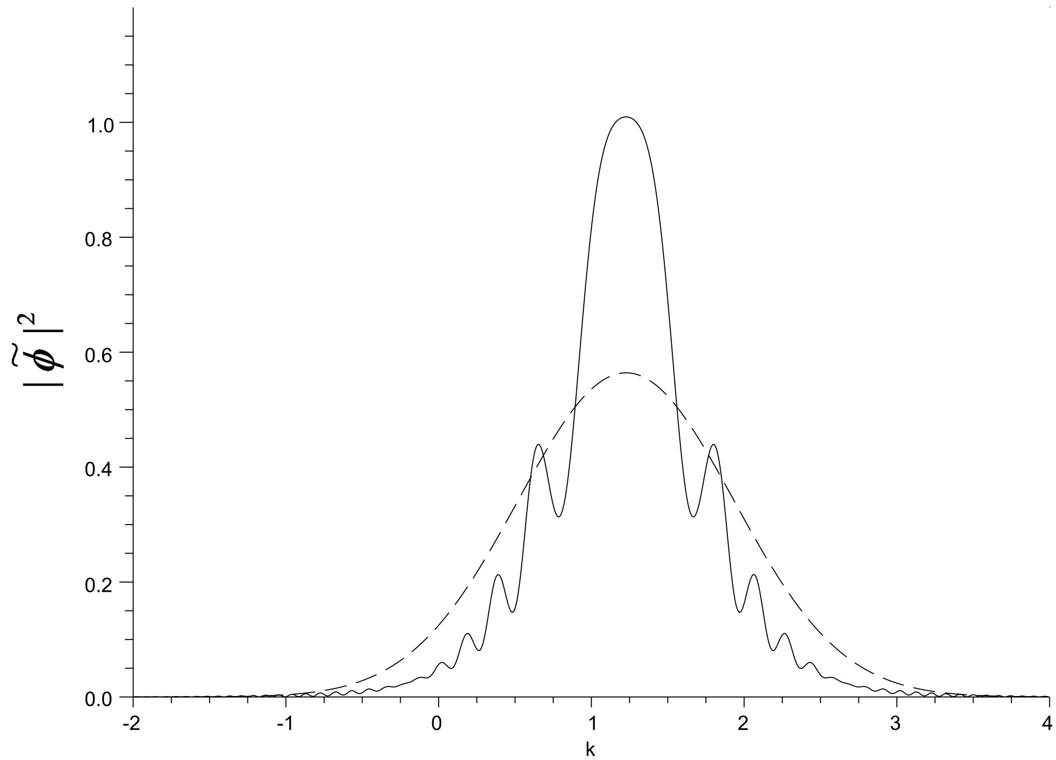


Fig. 2. Solid line shows $|\tilde{\phi}|^2$, freely propagating breather-like wave packet observed in wave number space at $t = 16$. The parameters are the same as the ones used in Fig. 1. The dashed line for $t = 0$.

space $\tilde{\phi}$ as

$$\tilde{\phi}(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x,t) e^{\mathrm{i}kx} \mathrm{d}x. \tag{14}$$

We show a snapshot of $|\tilde{\phi}|^2$ taken at $t = 16$ in Fig. 2. In wave number space, the breathing motion is also observed and a notched structure grows on the surface of the wave packets. This structure is the result of repeated expansion and contraction in the wave number space, i.e. expansion by the higher harmonic excitation and contraction by the dispersion effect (suppression of higher harmonic excitation).

It is known that any solitary waves governed by eq. (4) with $V(x) = 0$ finally split into a complete soliton part and a small oscillating tail (radiation) which rapidly leaves the soliton part in the long run.[19] Therefore, this quasi-breather-like behavior is considered to have finite life time and the decay process is rather transient phenomenon. Fortunately, this life time is sufficiently long to observe the breather-like motion.

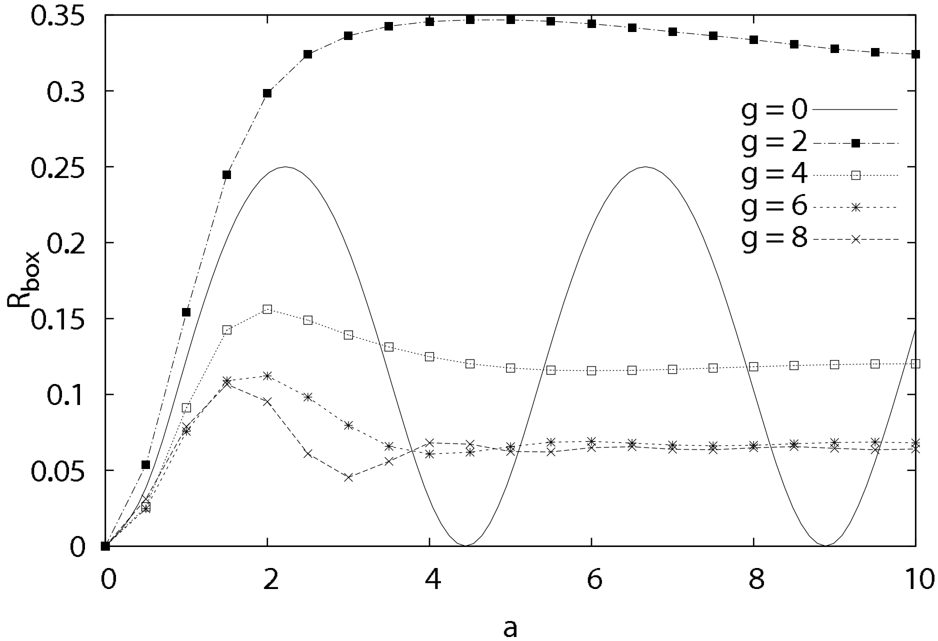


Fig. 3. Reflectance $R_{\text{box}}$ from the box type potential (8) for various values of $g$. The initial condition is the Gaussian type wave packet (5) with $x_0 = 5$ and $v = \sqrt{1.5}$ except for $g = 0$. The curve for $g = 0$ corresponds to linear case given by eq. (15).

### *3.2 Box type potential*

We go on to the main issue, the influence of the nonlinear wave packet effect on the potential scattering problems. In this subsection, we consider a box type (repulsive) potential (8).

When we consider a stationary problem with $g = 0$, this is a text book example of quantum mechanics where analytic expression for the reflectance is obtained. The expression reads

$$R_s = \left[1 + \frac{4v^2(v^2-1)}{\sin^2(a\sqrt{v^2-1})}\right]^{-1}, \tag{15}$$

where $v$ is the twice of the wave number of the incident plain wave. When $\sin(a\sqrt{v^2-1}) = 0$, $R_s$ becomes 0 and the perfect transmission is realized. This is a kind of resonance scattering. Hereafter, we set $x_0$ in eq. (5) to be 5 or 100 throughout this paper. The parameter $v$ is also fixed to be $\sqrt{1.5}$.

The dependencies of $R_{\text{box}}$ on $g$ and $a$ are shown in Figs. 3 and 4. The former is for $x_0 = 5$ and the latter $x_0 = 100$, respectively. The curve for $g = 0$ corresponds to linear case given by eq. (15), the reflectance calculated from the stationery Schödinger equation. As mentioned above, the quantity $R_s$ experiences 0 values twice as $a$ increases. This is due to resonance and also expected to occur periodically as the values of $a$ grows larger.

The behavior of $R_{\text{box}}$ given by TDGPE (4) is drastically different. Firstly, maximum values of $R_{\text{box}}$ for each $g$ are totally suppressed for $g > 2$, although $R_{\text{box}}$ is enhanced for the case of $g = 2$. Secondly,

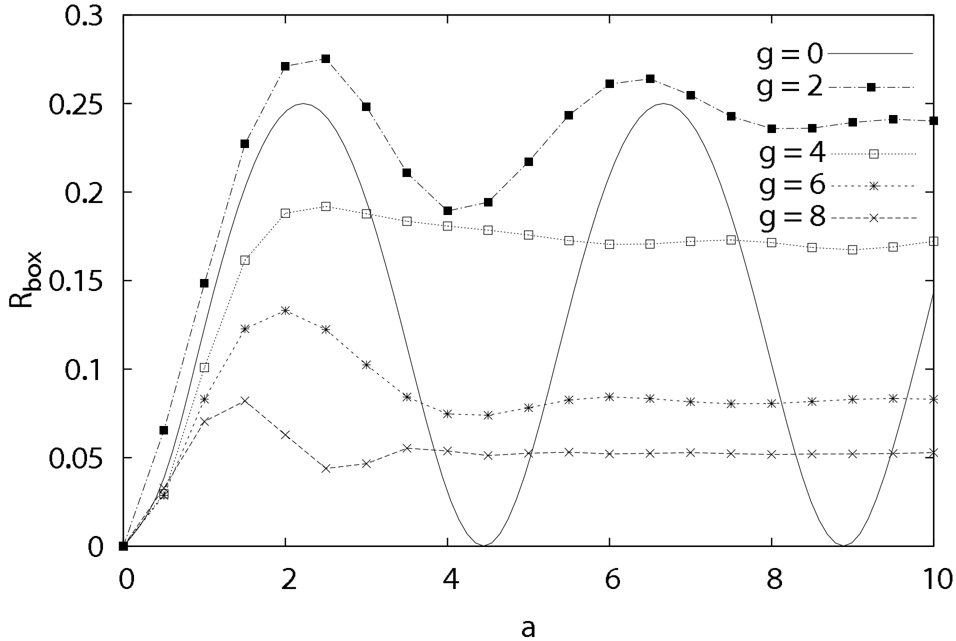


Fig. 4. Reflectance $R_{\mathrm{box}}$ from the box type potential (8) for various values of $g$. The initial condition is the Gaussian type wave packet (5) with $x_0 = 100$ and $v = \sqrt{1.5}$ except for $g = 0$. The curve for $g = 0$ corresponds to linear case given by eq. (15).

they never experience the perfect transmission as the result of the wave packet effect, and they seem to be approaching to their own constant values asymptotically accompanied with small oscillation as $a$ increases, i.e., the periodic resonance structure is destroyed for self-focusing wave packet. Thirdly, wavy resonance structure seems to recover for $g = 2$ case after long free propagation as shown in Fig. 4. This can be reasoned as follows: Since relatively week nonlinearity of $g = 2$ cannot prevent the wave packet from diffusing, it spreads and gains sufficient width for the plain wave approximation to be applied after long propagation. Therefore, the result approaches the linear case.

### *3.3 Well type potential*

Next, we move onto the well type (attractive) potential case. The potential is shown in eq. (9). The parameters are the same as the previous case but the potential depth $V_0$ is taken to be 10. For stationary and linear case, the analytical expression for the transmittance is given as

$$T_{\mathrm{s}} = \left[1 + \frac{100 \sin^2(a\sqrt{v^2+10})}{4v^2(v^2+10)}\right]^{-1}. \tag{16}$$

The perfect transmission is realized when $\sin(a\sqrt{v^2+10}) = 0$. However, in the nonlinear wave packet dynamics, we require more careful definition of the reflectance and the transmittance, since certain amount of wave packet is trapped by attractive potentials. For example, snapshots of the trapped wave profiles are shown in Figs. 5 and 6. We can observe standing wave-like structure in the latter, and it

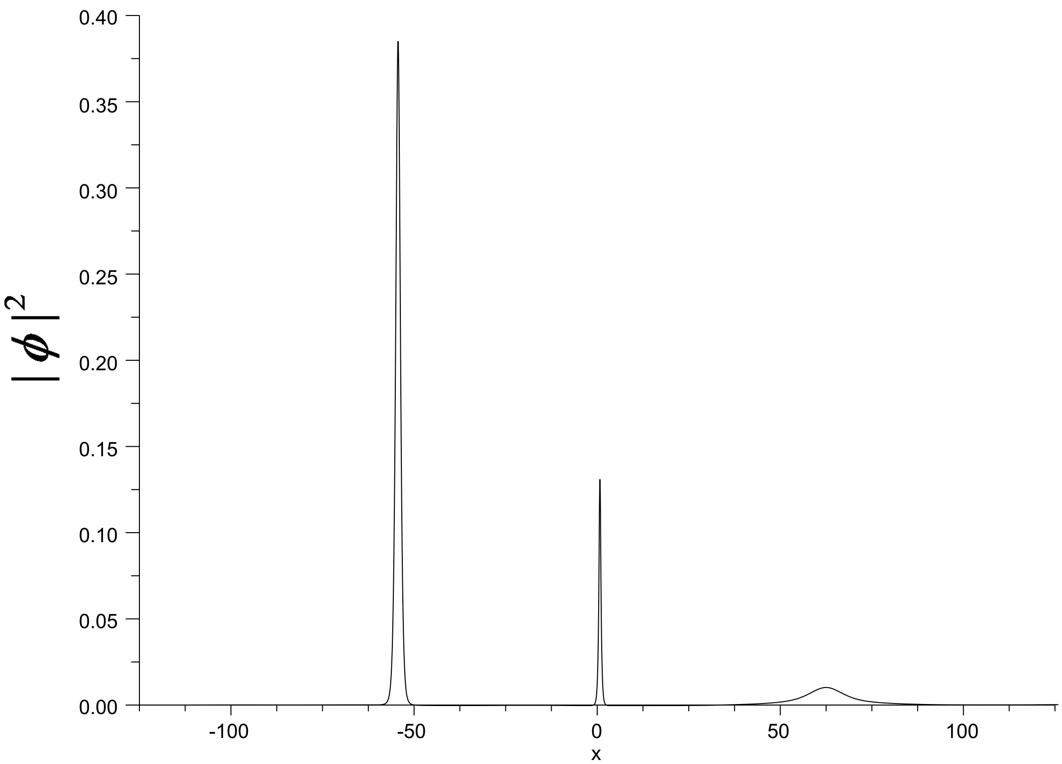


Fig. 5. Typical wave shape including the trapped portion by attractive well type potential (9) with $a = 0.5$. The wave packet located near the origin is the trapped portion. The initial condition is the Gaussian type wave packet (5) with $x_0 = 5$, $v = \sqrt{1.5}$ and $g = 8$. This figure shows the snapshot taken at $t = 30$.

swings back and forth in the potential area. Moreover, these trapping phenomena seem to be intermediate state and the trapped parts continue to gradually emit a part of themselves mainly toward the left. Therefore, the reflectance defined by eq. (10) never converges even after very long time. Here, we employ the expression (11) or (12) instead of (10) to evaluate the nonlinear wave packet effect. The value of the margin $b$ is chosen to be 30 in this paper.

The dependencies of $T_{\mathrm{well}}$ on $g$ and $a$ are shown in Figs. 7 and 8. The former is for $x_0 = 5$ and the latter $x_0 = 100$, respectively. The curve for $g = 0$ corresponds to linear case given by eq. (16), the transmittance calculated from the stationery Schödinger equation. The quantity $T_{\mathrm{s}}$ takes unity several times as $a$ increases. This is also due to resonance and also expected to occur periodically as the values of $a$ grows larger.

Main features resemble the box type potential case. Firstly, the maximum values of $T_{\mathrm{well}}$ for each $g$ are totally suppressed for $g > 2$, although $T_{\mathrm{well}}$ is enhanced for the case of $g = 2$. Secondly, they

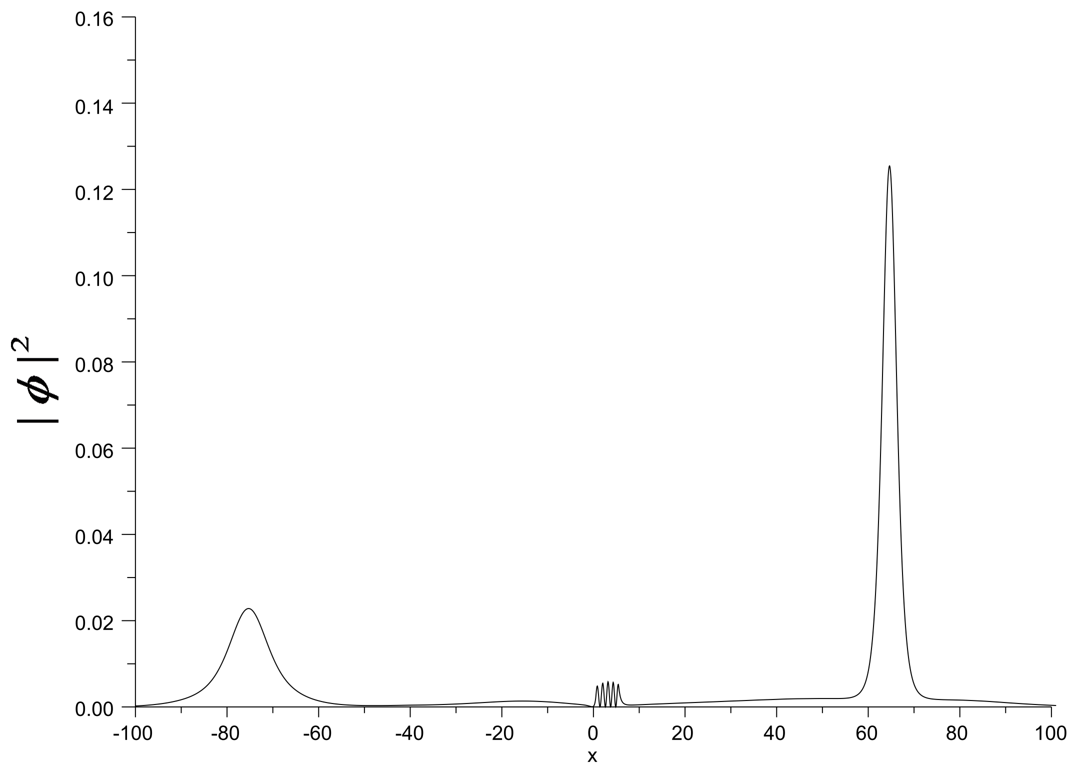


Fig. 6. Typical wave shape including the trapped portion by attractive well type potential (9) with $a = 5$. The trapped portion forms standing wave like structure in the potential well. The initial condition is the Gaussian type wave packet (5) with $x_0 = 5$, $v = \sqrt{1.5}$ and $g = 4$. This figure shows the snapshot taken at $t = 30$.

never experience the perfect transmission as the result of wave packet effect, and periodic resonance structure is destroyed for self-focusing wave packet. Thirdly, the wavy resonance structure seems to recover for $g = 2$ case after long free propagation (Fig. 8). The reason for this restoration is seemed to be the same as the box type potential case.

Here, we evaluate the amount of trapped portion by subtracting the sum of the reflectance (11) and the transmittance (12) from unity, i.e.,

$$N_{\mathrm{trapped}} = \lim_{t\to\infty} \int_{-30}^{a+30} |\phi|^2 \mathrm{d}x. \tag{17}$$

As mentioned before, these values are nothing more than estimates from the values at $t = 80$. Figures 9 and 10 show the dependencies of $N_{\mathrm{trapped}}$ on $g$ and $a$. They basically show that the amount of the trapped portion rises as $g$ and $a$ increase except for the relation between $g = 2$ and 4 cases in the small

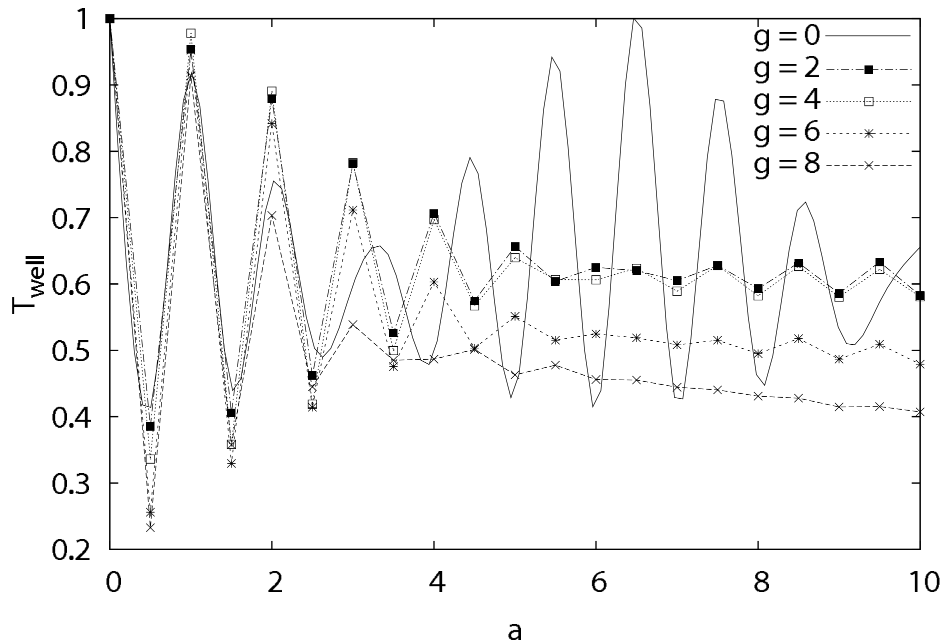


Fig. 7. Transmittance $T_{\rm well}$ over the well type potential (9) for various values of $g$. The initial condition is the Gaussian type wave packet (5) with $x_0 = 5$ and $v = \sqrt{1.5}$. The curve for $g = 0$ corresponds to linear case given by eq. (16).

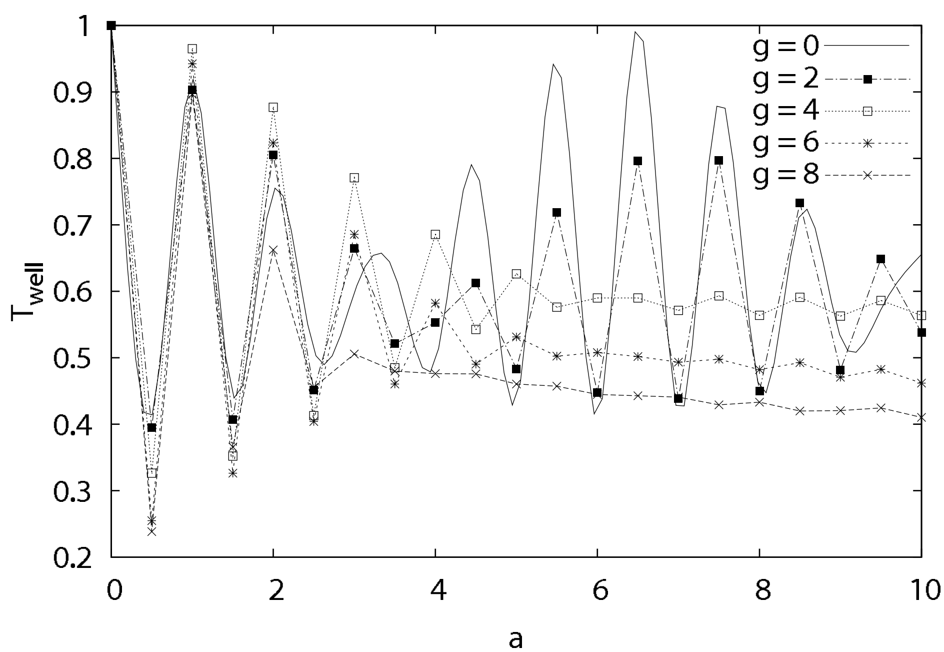


Fig. 8. Transmittance $T_{\rm well}$ over the well type potential (9) for various values of $g$. The initial condition is the Gaussian type wave packet (5) with $x_0 = 100$ and $v = \sqrt{1.5}$. The curve for $g = 0$ corresponds to linear case given by eq. (16).

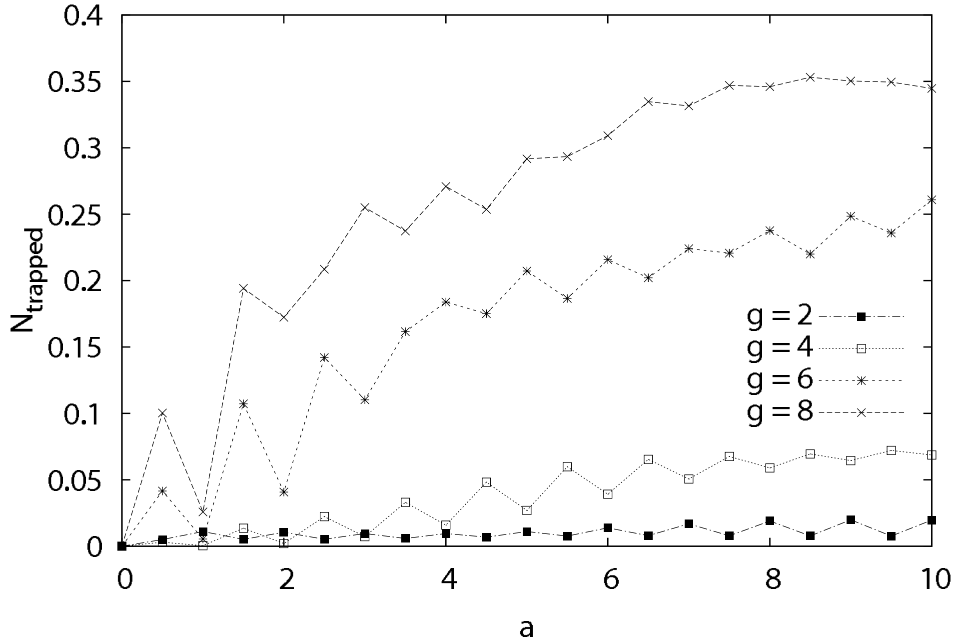


Fig. 9. Trapped portion $N_{\text{trapped}}$ by the well type potential (9) for various values of $g$. The initial condition is the Gaussian type wave packet (5) with $x_0 = 5$ and $v = \sqrt{1.5}$.

$a$ region. Finally, we mention that the wavy structures seem to be accompanied with the resonances.

## 4. Discussions

We have studied the scattering problems of nonlinear wave packets in the previous section. One of the remarkable properties of these problems is that the final reflectance or transmittance is not a function of only $v$ but also the initial position of the wave packet. In section 2, we showed strong modulation of the Fourier spectrum of a wave packet due to the nonlinearity which is shown in Fig. 2. The shape of the Fourier spectrum deforms and oscillates moment by moment during propagation. Therefore, the Fourier spectrum when the wave packet arrives at the potential area, which evidently affects on the reflectance or the transmittance, depends on the parameter $x_0$, i.e., the distance between the starting position of the wave packet and the potential. This is the reason why the final results depend on $x_0$. On the contrary, for linear case, the initial Fourier spectrum is conserved under free propagation, and the role of $x_0$ is not important.

The initial position of the wave packet also affects the result of scattering problem through alteration of the incident kinetic energy. The kinetic energy $K$ is defined as

$$K = \int |\phi_x|^2 \mathrm{d}x, \tag{18}$$

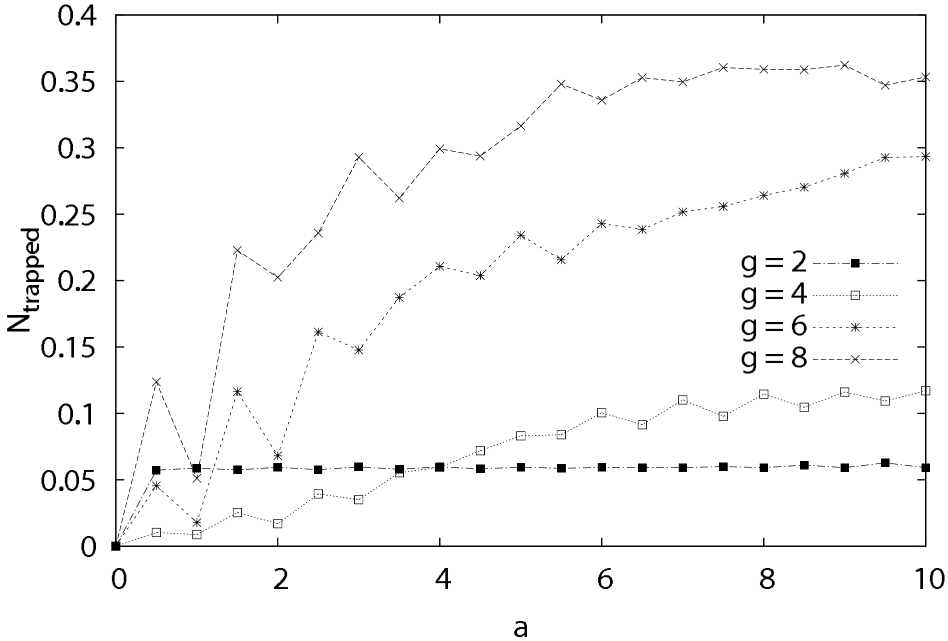


Fig. 10. Trapped portion $N_{\mathrm{trapped}}$ by the well type potential (9) for various values of $g$. The initial condition is the Gaussian type wave packet (5) with $x_0 = 100$ and $v = \sqrt{1.5}$.

and the self-interaction energy

$$I = -\frac{1}{2}g\int |\phi|^4 \mathrm{d}x. \tag{19}$$

The breathing wave packet is always exchanging its kinetic and self-interaction energy even during free propagation. As we can see from eq. (18), when the wave packet gets steeper, the kinetic energy increases. Since total energy $E = K + I$ is a conserved quantity, the negative self-interaction energy decreases to compensate the increase of kinetic energy. Therefore, the incident kinetic energy is also a function of the initial position of the wave packet $x_0$. This incident kinetic energy directly fixes the wave number at the incident and becomes one of the most significant factors of the scattering problem.

From above considerations, any argument on potential scattering problems of nonlinear wave packet requires considerations on the initial position of the wave packet $x_0$, except the soliton initial condition. In this paper, we fixed $x_0$ to be 5 or 100. The reflectance and transmittance might be altered for the different choice of $x_0$, while our main arguments are kept, i.e., the decay of resonance structure and the existence of trapped portion for well type potentials might be observed.

In the previous section, we showed the trapping effect for self-focusing wave packets by an attractive potential. This phenomenon is interpreted as manifestation of the squeezed width of the wave packets. For linear quantum mechanics, this kind of phenomenon never occurs due to prohibition of energy level crossing between scattering and bound states. Because time evolution of a wave packet from an initial state to a final one is fully described by a superposition of elements in the complete set of the scattering state eigenfunctions, i.e.,

$$|\phi(0)\rangle = \int c_\lambda |E_\lambda\rangle \mathrm{d}\lambda \Rightarrow |\phi(t)\rangle = \int c_\lambda \mathrm{e}^{-\mathrm{i}E_\lambda t} |E_\lambda\rangle \mathrm{d}\lambda, \tag{20}$$

where $\lambda$ labels continuous energy eigenvalues and the integral is taken over all the scattering eigenstates. For nonlinear wave packets, however, the distribution of the conserved energy is always changing as argued above and there appears the contribution from potential energy

$$V = -\int_0^a V(x)|\phi|^2 \mathrm{d}x \tag{21}$$

in the potential area. Therefore, there are plenty of chances for attractive potentials that

$$K + I + V < 0 \tag{22}$$

holds temporarily though our choices of parameters do not make inequality (7) holds. This trapped state can be considered "dynamical bound state". Since a part of the wave function is trapped and continues to oscillate dynamically staying around a potential.

It is worth mentioning that for the case of the box type potentials, the reflectance of strongly self-focusing wave packets is observed to approach constant values for larger potential width $a$, and its dependency on parameter $a$ disappears (Figs. 3 and 4). Because the norms of the self-focusing wave packets have finite values only in the narrow limited areas between potential ends and do not have sufficient extent to cover the whole potential area, the wave packets become insensitive to the opposite far end of the potential. In general, the resonance is the result of interference between the forward propagating wave and the backward propagating one. However, the squeezed wave packets merely have backward portion since they hardly interact the other far end of the potential. Such wave packets undergo the effective potential for large $a$,

$$V_{\mathrm{box-eff}} = -\theta(x). \tag{23}$$

For well type potentials, the squeezed wave packets firstly fall off the cliff of the potential,

$$V_{\mathrm{well-eff-1}} = V_0\theta(x). \tag{24}$$

Then, they encounter the other side of the potential wall, and they effectively face

$$V_{\mathrm{well-eff-2}} = -V_0\theta(x-a). \tag{25}$$

The major part of the transmittance shown in Figs. 7 and 8 can be considered as the remainders when we subtract the reflectance shown in Figs. 3 and 4 from unity. The reflected portion by the potential wall repeats reflection in the valley of the potential and is considered to constitute the dynamical bound states.

## 5. Summary

In this paper, we have numerically studied free propagation of wave packets governed by the TDGPE for various values of coupling constants $g$. The initial condition is taken to be the Gaussian form, which is different from the soliton solution. For the strongly self-interacting wave packets, diffusion in real space is suppressed and they exhibited breather-like behaviors. In wave number space, the breathing motion is also observed and a notched structure grows on the surface of the wave packet.

We have also numerically investigated the potential scattering problems under the same developing equation and initial conditions. The potential forms are chosen to be the box or the well type. We have obtained the reflectance $R_{\text{box}}$ and the transmittance $T_{\text{well}}$ for the different values of the coupling $g$ and the width of the potential $a$, and we compared them with the predictions by stationary Schödinger equations. The role of nonlinearity is rather complicated, i.e., it sometimes enhances $R_{\text{box}}$ or $T_{\text{well}}$ but sometimes the opposite. However, there is a tendency that large $g$ decreases both $R_{\text{box}}$ and $T_{\text{well}}$. For larger values of $g$ and $a$, $R_{\text{box}}$ and $T_{\text{well}}$ approach constant values and do not depend on $a$.

We have also observed the dynamically trapped portion of the wave packet. We estimated the amount of it $N_{\text{trapped}}$ changing $g$ and $a$ and found that $N_{\text{trapped}}$ is an increasing function of $g$ and $a$ except for small $g$ and $a$ region. We interpreted these phenomena by squeezing of nonlinear wave packet's width. Whether this trapping effect is a perpetual or just transitional one is not obvious and would be subject of future works.

Finally, we make small remarks on the possibility of real experiments. The control of external environments is relatively easy in the BEC systems where we can confine condensate particles along quasi rectilinear line by tightening laser beam trap. In addition, we can freely change the coupling constants by application of the Feshbach resonance technique.[17)] Soliton-like pulses of BEC have already been created.[20)] If controllable local potential are realized, the possibility to observe and confirm our results by real experiment is promising.

## 6. Acknowledgment

The authors would like to express their sincere gratitude for Professor Miki Wadati for his valuable comments and continuous encouragements. They are also grateful for Professor Yoshiya Yamanaka and Professor Ichiro Ohba of Waseda University for their warm advise. One of the authors H.F. thanks to the hospitality of Utsunomiya University for offering wonderful working spaces and opportunities of fruitful discussion.

## References


1) V.I. Karpman: *Nonlinear Waves in Dispersive Media* (Pergamon Press, Oxford, 1975).
2) A.C. Newell: *Solitons in Mathematical Physics* (SIAM, 1985).
3) A. Hasegawa: *Optical Solitons in Fibers* (Springer-Verlag, Berlin, 1990).
4) G.P. Agrawal: *Nonlinear Fiber Optics* (Academic Press, San Diego, 1989).
5) M. Okumura and Y. Yamanaka: Phys. Rev. A **68** (2003) 013609.
6) F. Dalfovo, S. Giorgini, L.P. Pitaevskii and S. Stringari: Rev. Mod. Phys. **71** (1999) 463.
7) L.P. Pitaevskii and S. Stringari: *Bose-Einstein Condensation* (Oxford University Press, Oxford, 2003).
8) C.S. Gardiner, J.M. Greene, M.D. Kruskal and R.M. Miura: Phys. Rev. Lett. **19** (1967) 1095.
9) V.E. Zakharov and A.B. Shabat: Sov.Phys. JETP **34** (1972) 62.
10) J. Satsuma and N. Yajima: Prog. Theor. Phys. Suppl. **55** (1974) 284.
11) L. Salasnich, A. Parola and L. Reatto: Phys. Rev. A **64** (2001) 023601.
12) H.H. Chen and C.S. Liu: Phys. Rev. Lett. **37** (1976) 693.
13) R. Balakrishnan: Phys. Rev. A **32** (1985) 1145.
14) T. Hyouguchi, S. Adachi and M. Ueda: Phys. Rev. Lett. **88** (2002) 170404.
15) H. Frauenkron and P. Grassberger: Phys. Rev. E **53** (1996) 2823.
16) H. Sakaguchi and M. Tamura: J. Phys. Soc. Jpn **73** (2004) 504.
17) Ph. Courteille, R.S. Freeland, D.J. Heinzen, F.A. van Abeelen and B.J. Verhaar: Phys. Rev. Lett. **81** (1998) 69.
18) N. Sasa and H. Yoshida:Transactions of the Japan Society for Industrial and Applied Mathematics **10** (2000) 119 [in Japanese].
19) M.J. Ablowitz and H. Segur: *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981).
20) L. Khaykovich, F. Schreck, G. Ferrari, T. Bourdel, J. Cubizolles, L. D. Carr, Y. Castin, C. Salomon: Science **296** (2002) 1290.