A 2-COMPONENT μ-HUNTER-SAXTON EQUATION

DAFENG ZUO

Abstract. In this paper, we propose a two-component generalization of the generalized Hunter-Saxton equation obtained in [22]. We will show that this equation is a bihamiltonian Euler equation, and also can be viewed as a bi-variational equation.

1. Introduction

V.I.Arnold in [1] suggested a general framework for the Euler equations on an arbitrary (possibly infinite-dimensional) Lie algebra $\mathcal{G}$. In many cases, the Euler equations on $\mathcal{G}$ describe geodesic flows with respect to a suitable one-side invariant Riemannian metric on the corresponding group G. Now it is well-known that Arnold's approach to the Euler equation works very well for the Virasoro algebra and its extensions, see [6, 10, 13, 14, 15, 19] and references therein.

Let $\mathcal{D}(\mathbb{S}^1)$ be a group of orientation preserving diffeomorphisms of the circle and $G = \mathcal{D}(\mathbb{S}^1) \oplus \mathbb{R}$ be the Bott-Virasoro group. In [6], Ovsienko and Khesin showed that the KdV equation is an Euler equation, describing a geodesic flow on G with respect to a right invariant L^2 metric. Another interesting example is the Camassa-Holm equation, which was originally derived in [4] as an abstract equation with a bihamiltonian structure, and independently in [9] as a shallow water approximation. In [10], Misiolek showed that the Camassa-Holm equation is also an Euler equation for a geodesic flow on G with respect to a right-invariant Sobolev H^1-metric.

In [13], Khesin and Misiolek extended the Arnold's approach to homogeneous spaces and provided a beautiful geometric setting for the Hunter-Saxton equation, which firstly appeared in [8] as an asymptotic equation for rotators in liquid crystals, and its relatives. They showed that the Hunter-Saxton equation is an Euler equation describing the geodesic flow on the homogeneous spaces of the Bott-Virasoro group G modulo rotations with respect to a right invariant homogeneous $\dot{H}^1$ metric.

Furthermore, by using extended Bott-Virasoro groups, Guha etc. [11, 16, 21] generalized the above results to two-component integrable systems, including several coupled KdV type systems and 2-component peak type systems, especially 2-component Camassa-Holm equation which was introduced by Chen, Liu and Zhang [17] and independently by Falqui [18]. Another interesting topic is to discuss the super or supersymmetric analogue, see [6, 12, 16, 20, 23, 24] and references therein.

Recently Khesin, Lenells and Misiolek in [22] introduced a generalized Hunter-Saxton (μ-HS in brief) equation lying mid-way between the periodic Hunter-Saxton and Camassa-Holm equations,

$$-f_{txx} = -2\mu(f)f_x + 2f_x f_{xx} + f f_{xxx}, \tag{1.1}$$

2000 *Mathematics Subject Classification.* 37K10, 35Q51.

where $f = f(t,x)$ is a time-dependent function on the unit circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and $\mu(f) = \int_{\mathbb{S}^1} f dx$ denotes its mean. This equation describes evolution of rotators in liquid crystals with external magnetic field and self-interaction.

Let $\mathcal{D}^s(\mathbb{S}^1)$ be a group of orientation preserving Sobolev H^s diffeomorphisms of the circle. They proved that the μ-HS equation (1.1) describes a geodesic flow on $\mathcal{D}^s(\mathbb{S}^1)$ with a right-invariant metric given at the identity by the inner product

$$\langle f, g\rangle_\mu = \mu(f)\mu(g) + \int_{\mathbb{S}^1} f'(x)g'(x)dx. \tag{1.2}$$

They also showed that (1.2) is bihamiltonian and admits both cusped as well as smooth traveling-wave solutions which are natural candidates for solitons. In this paper, we want to generalize these to a two-component μ-HS (2-μHS in brief) equation. Our main object is the Lie algebra $\mathcal{G} = \mathrm{Vect}^s(\mathbb{S}^1) \ltimes \mathrm{C}^\infty(\mathbb{S}^1)$ and its three-dimensional central extension $\widehat{\mathcal{G}}$. Firstly, we introduce an inner product on $\widehat{\mathcal{G}}$ given by

$$\langle \hat{f}, \hat{g}\rangle_\mu = \mu(f)\mu(g) + \int_{\mathbb{S}^1} (f'(x)g'(x) + a(x)b(x))dx + \overrightarrow{\alpha} \cdot \overrightarrow{\beta}, \tag{1.3}$$

where $\hat{f} = (f(x)\frac{d}{dx}, a(x), \overrightarrow{\alpha})$, $\hat{g} = (g(x)\frac{d}{dx}, b(x), \overrightarrow{\beta})$ and $\overrightarrow{\alpha}, \overrightarrow{\beta} \in \mathbb{R}^3$. Afterwards, we have

Theorem 1.1. [=*Theorem 2.2*]. *The Euler equation on* $\widehat{\mathcal{G}}^*_{reg}$ *with respect to* (1.3) *is a 2-μHS equation*

$$\begin{cases} -f_{xxt} = 2\mu(f)f_x - 2f_x f_{xx} - f f_{xxx} + v_x v - \gamma_1 f_{xxx} + \gamma_2 v_{xx}, \\ v_t = (vf)_x - \gamma_2 f_{xx} + 2\gamma_3 v_x, \end{cases} \tag{1.4}$$

where $\gamma_j \in \mathbb{R}$, $j = 1, 2, 3$.

Actually from the geometric view, if we extend the inner product (1.3) to a left invariant metric on $\widehat{G} = \mathcal{D}^s(\mathbb{S}^1) \ltimes \mathrm{C}^\infty(\mathbb{S}^1) \oplus \mathbb{R}^3$, we could view the 2-$\mu$HS equation (1.4) as a geodesic flow on $\widehat{G}$ with respect to this left invariant metric. Obviously, if we choose $v = 0$ and $\gamma_j = 0$, $j = 1, 2, 3$ and replace t by $-t$, (1.4) reduces to (1.1). Furthermore, we show that

Theorem 1.2. [=*Theorem 3.1 and 4.1*]. *The 2-μHS equation* (1.4) *can be viewed as a bihamiltonian and bi-variational equation.*

This paper is organized as follows. In section 2, we calculate the Euler equation on $\widehat{\mathcal{G}}^*_{reg}$. In section 3, we study the Hamiltonian nature and the Lax pair of the 2-μHS equation (1.4). Section 4 is devoted to discuss the variational nature of (1.4). In the last section we describe the interrelation between bihamiltonian natures and bi-variational natures.

Acknowledgement. The author would like to thank Prof. Khesin Boris and Prof. Partha Guha for references [6] and [21], respectively, and the anonymous referee for several useful suggestions. This work is partially supported by the Fundamental Research Funds for the Central Universities and NSFC(10971209,10871184).

2. Eulerian nature of the 2-μHS equation

Let $\mathcal{D}^s(\mathbb{S}^1)$ be a group of orientation preserving Sobolev H^s diffeomorphisms of the circle and let $T_{id}\mathcal{D}^s(\mathbb{S}^1)$ be the corresponding Lie algebra of vector fields, denoted by $\mathrm{Vect}^s(\mathbb{S}^1) = \{f(x)\frac{d}{dx} | f(x) \in H^s(\mathbb{S}^1)\}$.

The main objects in our paper will be the group $\mathcal{D}^s(\mathbb{S}^1) \ltimes \mathrm{C}^\infty(\mathbb{S}^1)$, its Lie algebra $\mathcal{G} = \mathrm{Vect}^s(\mathbb{S}^1) \ltimes \mathrm{C}^\infty(\mathbb{S}^1)$ with the Lie bracket given by

$$[(f(x)\frac{d}{dx}, a(x)), (g(x)\frac{d}{dx}, b(x))] = \left((f(x)g'(x) - f'(x)g(x))\frac{d}{dx}, f(x)b'(x) - a'(x)g(x)\right),$$

and their central extensions. It is well known in [3, 7] that the algebra $\mathcal{G}$ has a three dimensional central extension given by the following nontrivial cocycles

$$\begin{aligned}
\omega_1\left((f(x)\frac{d}{dx}, a(x)), (g(x)\frac{d}{dx}, b(x))\right) &= \int_{\mathbb{S}^1} f'(x)g''(x)dx, \\
\omega_2\left((f(x)\frac{d}{dx}, a(x)), (g(x)\frac{d}{dx}, b(x))\right) &= \int_{\mathbb{S}^1} [f''(x)b(x) - g''(x)a(x)]dx, \\
\omega_3\left((f(x)\frac{d}{dx}, a(x)), (g(x)\frac{d}{dx}, b(x))\right) &= 2\int_{\mathbb{S}^1} a(x)b''(x)dx,
\end{aligned} \tag{2.1}$$

where $f(x)$, $g(x) \in H^s(\mathbb{S}^1)$ and $a(x)$, $b(x) \in \mathrm{C}^\infty(\mathbb{S}^1)$. Notice that the first cocycle ω_1 is the well-known Gelfand-Fuchs cocycle [2, 5]. The Virasoro algebra $Vir = \mathrm{Vect}^s(\mathbb{S}^1) \oplus \mathbb{R}$ is the unique non-trivial central extension of $\mathrm{Vect}^s(\mathbb{S}^1)$ via the Gelfand-Fuchs cocycle ω_1. Sometimes we would like to use the modified Gelfand-Fuchs cocycle

$$\tilde{\omega}_1\left((f(x)\frac{d}{dx}, a(x)), (g(x)\frac{d}{dx}, b(x))\right) = \int_{\mathbb{S}^1} (c_1 f'(x)g''(x) + c_2 f'(x)g(x))dx, \tag{2.2}$$

which is cohomologeous to the Gelfand-Fuchs cocycle ω_1, where c_1, $c_2 \in \mathbb{R}$.

Definition 2.1. *The algebra $\widehat{\mathcal{G}}$ is an extension of $\mathcal{G}$ defined by*

$$\widehat{\mathcal{G}} = Vect^s(\mathbb{S}^1) \ltimes C^\infty(\mathbb{S}^1) \oplus \mathbb{R}^3 \tag{2.3}$$

with the commutation relation

$$[\hat{f}, \hat{g}] = \left((fg' - f'g)\frac{d}{dx}, fb' - a'g, \overrightarrow{\omega}\right), \tag{2.4}$$

where $\hat{f} = (f(x)\frac{d}{dx}, a(x), \overrightarrow{\alpha})$, $\hat{g} = (g(x)\frac{d}{dx}, b(x), \overrightarrow{\beta})$ and $\overrightarrow{\alpha}, \overrightarrow{\beta} \in \mathbb{R}^3$ and $\overrightarrow{\omega} = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$.

Let

$$\widehat{\mathcal{G}}^*_{reg} = \mathrm{C}^\infty(\mathbb{S}^1) \oplus \mathrm{C}^\infty(\mathbb{S}^1) \oplus \mathbb{R}^3 \tag{2.5}$$

denote the regular part of the dual space $\widehat{\mathcal{G}}^*$ to the Lie algebra $\widehat{\mathcal{G}}$, under the pairing

$$\langle \hat{u}, \hat{f} \rangle^* = \int_{\mathbb{S}^1} (u(x)f(x) + a(x)v(x))dx + \overrightarrow{\alpha} \cdot \overrightarrow{\gamma}, \tag{2.6}$$

where $\hat{u} = (u(x)(dx)^2, v(x), \overrightarrow{\gamma}) \in \widehat{\mathcal{G}}^*$. Of particular interest are the coadjoint orbits in $\widehat{\mathcal{G}}^*_{reg}$.

On $\widehat{\mathcal{G}}$, let us introduce an inner product

$$\langle \hat{f}, \hat{g} \rangle_\mu = \mu(f)\mu(g) + \int_{\mathbb{S}^1} (f'(x)g'(x) + a(x)b(x))dx + \overrightarrow{\alpha} \cdot \overrightarrow{\beta}. \tag{2.7}$$

A direct computation gives

$$\langle \hat{f}, \hat{g}\rangle_\mu = \langle \hat{f}, (\Lambda(g)(dx)^2, b(x), \overrightarrow{\beta})\rangle^*, \quad \Lambda(g) = \mu(g) - g''(x),$$

which induces an inertia operator $\mathcal{A} : \widehat{\mathcal{G}} \longrightarrow \widehat{\mathcal{G}}^*_{reg}$ given by

$$\mathcal{A}(\hat{g}) = (\Lambda(g)(dx)^2, b(x), \overrightarrow{\beta}). \tag{2.8}$$

Theorem 2.2. *The 2-μHS equation* (1.4) *is an Euler equation on* $\widehat{\mathcal{G}}^*_{reg}$ *with respect to the inner product* (2.7)*.*

Proof. By definition,

$$\begin{aligned}\langle ad^*_{\hat{f}}(\hat{u}), \hat{g}\rangle^* &= -\langle \hat{u}, [\hat{f}, \hat{g}]\rangle^* \qquad \text{by using integration by parts}\\ &= \langle \big((2uf_x + u_x f + a_x v - \alpha_1 f_{xxx} + \alpha_2 a_{xx})(dx)^2, (vf)_x - \alpha_2 f_{xx} + 2\alpha_3 a_x, 0\big), \hat{g}\rangle^*.\end{aligned}$$

This gives

$$ad^*_{\hat{f}}(\hat{u}) = ((2uf_x + u_x f + a_x v - \alpha_1 f_{xxx} + \alpha_2 a_{xx})(dx)^2, (vf)_x - \alpha_2 f_{xx} + 2\alpha_3 a_x, 0).$$

By definition in [13], the Euler equation on $\widehat{\mathcal{G}}^*_{reg}$ is given by

$$\frac{d\hat{u}}{dt} = ad^*_{\mathcal{A}^{-1}\hat{u}}\hat{u} \tag{2.9}$$

as an evolution of a point $\hat{u} \in \widehat{\mathcal{G}}^*_{reg}$. That is to say, the Euler equation on $\widehat{\mathcal{G}}^*_{reg}$ is

$$\begin{aligned}u_t &= 2uf_x + u_x f + v_x v - \gamma_1 f_{xxx} + \gamma_2 v_{xx},\\ v_t &= (vf)_x - \gamma_2 f_{xx} + 2\gamma_3 v_x,\end{aligned}$$

where $u(x,t) = \Lambda(f(x,t)) = \mu(f) - f_{xx}$. By integrating both sides of this equation over the circle and using periodicity, we obtain

$$\mu(f_t) = \mu(f)_t = 0.$$

This yields that

$$\begin{aligned}-f_{xxt} &= 2\mu(f)f_x - 2f_x f_{xx} - ff_{xxx} + v_x v - \gamma_1 f_{xxx} + \gamma_2 v_{xx},\\ v_t &= (vf)_x - \gamma_2 f_{xx} + 2\gamma_3 v_x,\end{aligned}$$

which is the 2-μHS equation (1.4). □

Remark 2.3. *If we replace the Gelfand-Fuchs cocycle* ω_1 *by the modified cocycle* $\tilde{\omega}_1$*, the Euler equation* $\widehat{\mathcal{G}}^*_{reg}$ *is of the form*

$$\begin{aligned}-f_{xxt} &= 2\mu(f)f_x - 2f_x f_{xx} - ff_{xxx} + v_x v - \gamma_1 c_1 f_{xxx} + \gamma_2 v_{xx} + \gamma_1 c_2 f_x,\\ v_t &= (vf)_x - \gamma_2 f_{xx} + 2\gamma_3 v_x.\end{aligned}$$

3. Hamiltonian nature of the 2-μHS equation

In this section, we want to study the Hamiltonian nature of the 2-μHS equation (1.4) and its geometric meaning. We will show that

Theorem 3.1. *The 2-μHS equation* (1.4) *is bihamiltonian.*

Proof. Let us define $u(x,t)=\Lambda(f)=\mu(f)-f_{xx}$ and

$$H_1=\frac{1}{2}\int_{\mathbb{S}^1}(uf+v^2)dx \tag{3.1}$$

and

$$H_2=\int_{\mathbb{S}^1}(\mu(f)f^2+\frac{1}{2}ff_x^2+\frac{1}{2}fv^2-\gamma_2 vf_x+\gamma_3 v^2-\frac{\gamma_1}{2}ff_{xx})dx. \tag{3.2}$$

It is easy to check that the 2-μHS equation can be written as

$$\begin{pmatrix} u \\ v \end{pmatrix}_t=\mathcal{J}_1\begin{pmatrix} \frac{\delta H_2}{\delta u} \\ \frac{\delta H_2}{\delta v} \end{pmatrix}=\mathcal{J}_2\begin{pmatrix} \frac{\delta H_1}{\delta u} \\ \frac{\delta H_1}{\delta v} \end{pmatrix}, \tag{3.3}$$

where the Hamiltonian operators are

$$\mathcal{J}_1=\begin{pmatrix} \partial_x\Lambda & 0 \\ 0 & \partial_x \end{pmatrix}, \quad \mathcal{J}_2=\begin{pmatrix} u\partial_x+\partial_x u-\gamma_1\partial_x^3 & v\partial_x+\gamma_2\partial_x^2 \\ \partial_x v-\gamma_2\partial_x^2 & 2\gamma_3\partial_x \end{pmatrix}. \tag{3.4}$$

By a direct and lengthy calculation we could show that Hamiltonian operators $\mathcal{J}_1$ and $\mathcal{J}_2$ are compatible. □

Next we want to explain the geometric meaning of the bihamiltonian structures of the 2-μHS equation (1.4). Let $F_i:\widehat{\mathcal{G}}^*_{reg}\to\mathbb{R}$, $i=1,2$, be two arbitrary smooth functionals. It is well-known that the dual space $\widehat{\mathcal{G}}^*_{reg}$ carries the canonical Lie-Poisson bracket

$$\{F_1,F_2\}_2(\hat{u})=\langle\hat{u},[\frac{\delta F_1}{\delta\hat{u}},\frac{\delta F_2}{\delta\hat{u}}]\rangle^*, \tag{3.5}$$

where $\hat{u}=(u(x,t)(dx)^2,v(x,t),\vec{\gamma})\in\widehat{\mathcal{G}}^*_{reg}$ and $\frac{\delta F_i}{\delta\hat{u}}=(\frac{\delta F_i}{\delta u},\frac{\delta F_i}{\delta v},\frac{\delta F_i}{\delta\vec{\gamma}})\in\widehat{\mathcal{G}}, i=1,2$. By definition of the Euler equation (2.9), we know that the Lie-Poisson structure (3.5) is exactly the second Poisson bracket, induced by $\mathcal{J}_2$, of the 2-μHS equation (1.4).

To explain the first Hamiltonian structure, in the following we will use the "frozen Lie-Poisson" method introduced in [13]. Let us define a frozen (or constant) Poisson bracket

$$\{F_1,F_2\}_1(\hat{u})=\langle\hat{u}_0,[\frac{\delta F_1}{\delta\hat{u}},\frac{\delta F_2}{\delta\hat{u}}]\rangle^*, \tag{3.6}$$

where $\hat{u}_0=(u_0(dx)^2,v_0,\vec{\gamma}_0)\in\widehat{\mathcal{G}}^*_{reg}$. The corresponding Hamiltonian equation for any functional $F:\widehat{\mathcal{G}}^*_{reg}\to\mathbb{R}$ reads

$$\frac{d\hat{u}}{dt}=ad^*_{\frac{\delta F}{\delta\hat{u}}}\hat{u}_0 \tag{3.7}$$

which gives

$$\begin{aligned} u_t&=2u_0(\frac{\delta F}{\delta u})_x+(\frac{\delta F}{\delta v})_x v_0-\gamma_1^0(\frac{\delta F}{\delta u})_{xxx}+\gamma_2^0(\frac{\delta F}{\delta v})_{xx},\\ v_t&=(v_0\frac{\delta F}{\delta u})_x-\gamma_2^0(\frac{\delta F}{\delta u})_{xx}+2\gamma_3^0(\frac{\delta F}{\delta v})_x,\\ \vec{\gamma}_{0,t}&=0. \end{aligned} \tag{3.8}$$

Let us take the Hamiltonian functional F to be

$$H_2=\int_{\mathbb{S}^1}(\mu(f)f^2+\frac{1}{2}ff_x^2+\frac{1}{2}fv^2-\gamma_2 vf_x+\gamma_3 v^2-\frac{\gamma_1}{2}ff_{xx})dx \tag{3.9}$$

and set $u(x,t) = \Lambda(f(x,t)) = \mu(f) - f_{xx}$. Then we have

$$\frac{\delta F}{\delta u} = \Lambda^{-1}(\mu(f^2) + 2f\mu(f) - \frac{1}{2}f_x^2 - ff_{xx} - \gamma_1 f_{xx} + \gamma_2 v_x),$$

$$\frac{\delta F}{\delta v} = vf - \gamma_2 f_x + 2\gamma_3 v. \tag{3.10}$$

Let us choose a fixed point

$$\hat{u}_0 = (u_0, v_0, \vec{\gamma}_0) = (0, 0, (1, 0, \frac{1}{2})).$$

Observe that $\partial_x^3 \Lambda^{-1} = -\partial_x$. By substituting (3.10) into (3.8), we obtain the 2-μHS equation (1.4). According to the **Proposition 5.3** in [13], $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$ are compatible for every freezing point $\hat{u}_0$. Consequently we have

Theorem 3.2. *The 2-μHS equation* (1.4) *is Hamiltonian with respect to two compatible Poisson structures* (3.5) *and* (3.6) *on* $\widehat{\mathcal{G}}^*_{reg}$, *where the first bracket is frozen at the point* $\hat{u}_0 = (u_0, v_0, \vec{\gamma}_0) = (0, 0, (1, 0, \frac{1}{2}))$.

Let us point out that the constant bracket depends on the choice of the freezing point $\hat{u}_0$, while the Lie-Poisson bracket is only determined by the Lie algebra structure.

To this end we want to derive a Lax pair of 2-μHS equation (1.4) with $\overrightarrow{\gamma} = 0$, i.e.,

$$-f_{xxt} = 2\mu(f)f_x - 2f_x f_{xx} - ff_{xxx} + v_x v, \quad v_t = (vf)_x. \tag{3.11}$$

Motivated by the Lax pair of the two-component Camassa-Holm equation in [17], we could assume that the Lax pair of (3.11) has the following form

$$\Psi_x = U\Psi, \quad \Psi_t = V\Psi \tag{3.12}$$

with

$$U = \begin{pmatrix} 0 & 1 \\ \lambda\Lambda(f) - \lambda^2 v^2 & 0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} p & r \\ q & -p \end{pmatrix},$$

where λ is a spectral parameter. The compatibility condition

$$U_t - V_x + UV - VU = 0$$

in componentwise form reads

$$p = -\frac{r_x}{2}, \quad q = p_x + r(\lambda\Lambda(f) - \lambda^2 v^2),$$
$$2\lambda^2 v v_t + \lambda f_{xxt} + q_x - 2p(\lambda\Lambda(f) - \lambda^2 v^2) = 0.$$

By choosing $r = f - \frac{1}{2\lambda}$, we have

$$p = -\frac{f_x}{2}, \quad q = -\frac{f_{xx}}{2} + (f - \frac{1}{2\lambda})(\lambda\Lambda(f) - \lambda^2 v^2)$$

and

$$f_{xxt} + 2\mu(f)f_x - 2f_x f_{xx} - ff_{xxx} + v_x v + 2\lambda v(v_t - (vf)_x) = 0$$

which yields the system (3.11). Let us write $\Psi = \begin{pmatrix} \psi \\ \psi_x \end{pmatrix}$, we have

Proposition 3.3. *The system* (3.11) *has a Lax pair given by*

$$\psi_{xx} = (\lambda\Lambda(f) - \lambda^2 v^2)\psi, \quad \psi_t = (f - \frac{1}{2\lambda})\psi_x - \frac{1}{2}f_x\psi,$$

where $\lambda \in \mathbb{C} - \{0\}$ *is a spectral parameter.*

4. Variational nature of the 2-μHS equation

In [22], they have shown that the μ-HS equation (1.1) can be obtained from two distinct variational principles. In this section we will show that the 2-μHS equation (1.4) also arises as the equation

$$\delta\mathcal{S} = 0$$

for the action functional

$$\mathcal{S} = \int (\int \mathcal{L} dx) dt$$

with two different densities $\mathcal{L}$. That is to say,

Theorem 4.1. *The 2-μHS equation* (1.4) *satisfies two different variational principles.*

Proof. Motivated by the Lagrangian densities for the μ-HS equation (1.1) in [22], by some conjectural computations we find two generalized Lagrangian densities for the 2-μHS equation (1.4). More precisely,

Case I. Let us consider the first Lagrangian density

$$\mathcal{L}_1 = \frac{1}{2} f_x^2 + \frac{1}{2}\mu(f) f + \frac{1}{2} v^2 - v z_x + w(f z_x - z_t + \tilde{\gamma}_3 v) + \gamma_2 w_x f - 2\gamma_1 f, \tag{4.1}$$

where $\tilde{\gamma}_3 = \gamma_3 - \frac{1}{2}\gamma_1$. Varying the corresponding action with respect to f, v, w and z respectively, we get

$$\begin{aligned} &f_{xx} = \mu(f) + w z_x + \gamma_2 w_x - 2\gamma_1, \\ &z_x = v + \tilde{\gamma}_3 w, \\ &z_t = f z_x + \tilde{\gamma}_3 v - \gamma_2 f_x, \\ &w_t = (wf)_x - v_x. \end{aligned} \tag{4.2}$$

By using (4.2), we have

$$\begin{aligned} v_t &= z_{xt} - \gamma_3 w_t = [f(v + \tilde{\gamma}_3 w) + \tilde{\gamma}_3 v - \gamma_2 f_x]_x - \tilde{\gamma}_3((wf)_x - v_x), \\ &= (vf)_x - \gamma_2 f_{xx} + (2\gamma_3 - \gamma_1) v_x, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} -f_{xxt} &+ f_x f_{xx} + f f_{xxx} \\ &= -(\mu(f) + w z_x + \gamma_2 w_x)_t + f_x(\mu(f) + w z_x + \gamma_2 w_x - 2\gamma_1) + f(\mu(f) + w z_x + \gamma_2 w_x)_x \\ &= -w_t z_x - w z_{xt} + \gamma w_{xt} + f_x w z_x + f w_x z_x + f w z_{xx} + \gamma_2 f w_{xx} + 2\gamma_1 f_x \\ &= v v_x + 2\mu(f) f_x + \gamma_2 v_{xx} - 2\gamma_1 f_x. \end{aligned} \tag{4.4}$$

Notice that if we replace f by $f + \gamma_1$ in the system (4.3) and (4.4), this gives the 2-μHS equation (1.4).

Case II. The second variational representation can be obtained from the Lagrangian density

$$\mathcal{L}_2 = -f_x f_t + 2\mu(f) f^2 + f f_x^2 + f\phi_x^2 - \gamma_1 f f_{xx} - 2\gamma_2 \phi_x f_x + 2\gamma_3 \phi_x^2 - \phi_x \phi_t. \tag{4.5}$$

The variational principle $\delta\mathcal{S} = 0$ gives the Euler-Lagrange equation

$$\begin{aligned} &-f_{xt} = 2\mu(f) f + \mu(f^2) - \tfrac{1}{2} f_x^2 - f f_{xx} + \tfrac{1}{2}\phi_x^2 - \gamma_1 f_x + \gamma_2 \phi_{xx}, \\ &\phi_{xt} = (f\phi_x)_x - \gamma_2 f_{xx} + 2\gamma_3 \phi_{xx}. \end{aligned} \tag{4.6}$$

If we set $\phi_x = v$ and take the x-derivative of the first term in (4.6), this yields the 2-μHS equation (1.4). □

5. Relation between Hamiltonian nature and Variational nature

Recall that we have shown that the 2-μHS equation (1.4) is bihamiltonian and has two different variational principles. In the last section we want to study the relation between Hamiltonian natures and bi-variational principles and prove that

Theorem 5.1. *The two variational formulations for the 2-μHS equation* (1.4) *formally correspond to the two Hamiltonian formulations of this equation with Hamiltonian functionals H_1 and H_2.*

Proof. The action is related to the Lagrangian by $\mathcal{S} = \int(\int \mathcal{L}dx)dt$. The first variational principle has the Lagrangian density,

$$\mathcal{L}_1 = \frac{1}{2}f_x^2 + \frac{1}{2}\mu(f)f + \frac{1}{2}v^2 - vz_x + w(fz_x - z_t + \tilde{\gamma}_3 v) + \gamma_2 w_x f - 2\gamma_1 f.$$

The momenta conjugate to the velocities f_t, v_t, z_t and w_t, respectively, are

$$\frac{\partial \mathcal{L}_1}{\partial f_t} = 0, \quad \frac{\partial \mathcal{L}_1}{\partial w_t} = 0, \quad \frac{\partial \mathcal{L}_1}{\partial z_t} = -w, \quad \frac{\partial \mathcal{L}_1}{\partial w_t} = 0.$$

Consequently, the Hamiltonian density is

$$\begin{aligned}\mathcal{H} &= -z_t w - \mathcal{L}_1\\ &= -\frac{1}{2}f_x^2 - \frac{1}{2}\mu(f)f - \frac{1}{2}v^2 + vz_x - w(fz_x + \tilde{\gamma}_3 v) - \gamma_2 w_x f + 2\gamma_1\\ &= \frac{1}{2}\mu(f)f - \frac{1}{2}f_x^2 + \frac{1}{2}v^2 - ff_{xx}, \quad \text{by using (4.2).}\end{aligned}$$

Therefore, the Hamiltonian is

$$\begin{aligned}H &= \int \mathcal{H}dx = \int (\frac{1}{2}\mu(f)f - \frac{1}{2}f_x^2 + \frac{1}{2}v^2 - ff_{xx})dx\\ &= \frac{1}{2}\int (\mu(f)f - ff_{xx} + v^2)dx,\end{aligned}$$

which is exactly H_1 defined in (3.1).

In the second principle the Lagrangian density is

$$\mathcal{L}_2 = -f_x f_t + 2\mu(f)f^2 + ff_x^2 + f\phi_x^2 - \gamma_1 ff_{xx} - 2\gamma_2\phi_x f_x + 2\gamma_3\phi_x^2 - \phi_x\phi_t.$$

The momenta conjugate to the velocities f_t and ϕ_t, respectively, are

$$\frac{\partial \mathcal{L}_2}{\partial f_t} = -f_x, \quad \frac{\partial \mathcal{L}_2}{\partial \phi_t} = -\phi_x.$$

Consequently, the Hamiltonian density is

$$\begin{aligned}\mathcal{H} &= -f_x f_t - \phi_x\phi_t - \mathcal{L}_2\\ &= -2\mu(f)f^2 - ff_x^2 - f\phi_x^2 + \gamma_1 ff_{xx} + 2\gamma_2\phi_x f_x - 2\gamma_3\phi_x^2\end{aligned}$$

Now let us set $\phi_x = v$ and so

$$H = \int(-2\mu(f)f^2 - ff_x^2 - fv^2 + -\gamma_1 ff_{xx} + 2\gamma_2 vf_x - 2\gamma_3 v^2)dx = -\frac{H_2}{2},$$

where H_2 is defined in (3.2). □

References

[1] V. I. Arnold, Sur la géométrie différentielle des groupes de Lie de dimenson infinie et ses applications à l'hydrodynamique des fluids parfaits, Ann. Inst. Fourier Grenoble 16 (1966) 319–361.

[2] I.M.Gelfand and D.B.Fuchs,The cohomology of the Lie algebra of vector field in a circle, Functional Anal.Appl., 2(1968) 342-343.

[3] Ebin, David G. and Marsden, Jerrold, Groups of diffeomorphisms and the notion of an incompressible fluid, Ann. of Math., 92(1970) 102–163.

[4] A. S. Fokas and B. Fuchssteiner, Symplectic structures, their Backlund transformation and hereditary symmetries, Physica D 4 (1981) 47–66.

[5] D.B.Fuchs, Cohomology of infinite dimensional Lie algegras, Plenum Press, New York, 1986.

[6] V. Yu.Ovsienko and B. Khesin, KdV super equation as an Euler equation, Funct. Anal. Appl., 21 (1987) 329–331.

[7] E. Arbarello, C. De Concini, V. G. Kac, and C. Procesi, Moduli spaces of curves and representation theory, Comm. Math. Phys. Volume 117, Number 1 (1988)1–36.

[8] J. K. Hunter and R. Saxton, Dynamics of director fields, SIAM J. Appl. Math., 51 (1991) 1498–1521.

[9] R. Camassa and D. D. Holm, An integrable shallow water equation with peaked solitons, Phys. Rev. Lett., 71 (1993) 1661–1664.

[10] G. Misiolek, A shallow water equation as a geodesic flow on the Bott-Virasoro group, J. Geom. Phys., 24 (1998)203–208.

[11] P. Guha, Integrable geodesic flows on the (super)extension of the Bott-Virasoro group. Lett. Math. Phys., 4 (2000) 311–328.

[12] C. Devchand and J. Schiff, The supersymmetric Camassa-Holm equation and geodesic flow on the superconformal group. J. Math. Phys., 1(2001)260–273.

[13] B.Khesin and G.Misiolek, Euler equations on homogeneous spaces and Virasoro orbits, Adv. Math., 176(2003)116–144..

[14] A. Constantin, and B. Kolev, Geodesic flow on the diffeomorphism group of the circle, Comment. Math. Helv. 78(2003)787–804.

[15] A. Constantin, and B. Kolev, Integrability of invariant metrics on the diffeomorphism group of the circle, J. Nonlinear Sci., 16(2006)109–122.

[16] P.Guha and P. J. Olver, Geodesic flow and two (super) component analog of the Camassa-Holm equation, SIGMA Symmetry Integrability Geom. Methods Appl., 2 (2006) Paper 054, 9 pp.

[17] Chen, Ming; Liu, Si-Qi; Zhang, Youjin, A two-component generalization of the Camassa-Holm equation and its solutions, Lett. Math. Phys., 75 (2006)1–15.

[18] G. Falqui, On a Camassa-Holm type equation with two dependent variables, J. Phys. A 39(2006) 327–342.

[19] A. Constantin, T. Kappeler, B. Kolev, and P. Topalov, On geodesic exponential maps of the Virasoro group, Ann. Global Anal. Geom., 31(2007)155–180.

[20] J.Lenells, A bi-Hamiltonian supersymmetric geodesic equation, Lett. Math. Phys., 85 (2008)55–63.

[21] P. Guha, Geodesic flow on extended Bott-Virasoro group and generalized two-component peakon type dual systems, Rev. Math. Phys., 20 (2008) 1191–1208.

[22] B.Khesin, J.Lenells, and G. Misiolek, Generalized Hunter-Saxton equation and the geometry of the group of circle diffeomorphisms, Math. Ann., 342 (2008)617–656.

[23] J. Lenells and O. Lechtenfeld, On the $N = 2$ supersymmetric Camassa-Holm and Hunter-Saxton equations, J. Math. Phys., 50 (2009)no. 1, 012704, 17 pp.

[24] C. Sachse, P.Guha and C.Devchand, Superconformal algebras and supersymmetric integrable flows, arXiv:0912.4458v1.

Department of Mathematics,University of Science and Technology of China, Hefei 230026, P.R.China

E-mail address: `dfzuo@ustc.edu.cn`