Notes On a Continued Fraction of Ramanujan

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Abstract

We study the properties of a general continued fraction of Ramanujan. In some certain cases we evaluate it completely.

keywords Continued Fractions; Ramanujan;

1 Introduction

Let
$$(a;q)_k = \prod_{n=0}^{k-1}(1-aq^n) \tag{1}$$
Then we define
$$f(-q) = (q;q)_\infty \tag{2}$$
and
$$\Phi(-q) = (-q;q)_\infty \tag{3}$$
Also let
$$K(x) = \int_0^{\pi/2} \frac{1}{\sqrt{1-x^2\sin^2(t)}}dt \tag{4}$$
be the elliptic integral of the first kind
The function k_r is defined from the equation
$$\frac{K(k_r')}{K(k_r)} = \sqrt{r} \tag{5}$$
where r is positive , $q = e^{-\pi\sqrt{r}}$ and $k' = \sqrt{1-k^2}$. Note also that whenever r is positive rational, the k are algebraic numbers.

In Berndt's book: Ramanujan's Notebook Part III, ([B3] pg.21), one can find the following expansion

Theorem.
Suppose that either q, a and b are complex numbers with $|q| < 1$, or q, a, and b are complex numbers with $a = bq^m$ for some integer m. Then
$$U = U(a,b;q) = \frac{(-a;q)_\infty(b;q)_\infty - (a;q)_\infty(-b;q)_\infty}{(-a;q)_\infty(b;q)_\infty + (a;q)_\infty(-b;q)_\infty} =$$

$$\frac{a-b}{1-q+}\frac{(a-bq)(aq-b)}{1-q^3+}\frac{q(a-bq^2)(aq^2-b)}{1-q^5+}\frac{q^2(a-bq^3)(aq^3-b)}{1-q^7+}\cdots \tag{6}$$

Suppose now

$$X=\frac{(-a;q)_\infty(b;q)_\infty}{(a;q)_\infty(-b;q)_\infty} \tag{7}$$

Then holds

$$\frac{X-1}{X+1}=U \tag{8}$$

2 Propositions

Proposition 1.
Set

$$\phi(q)=\sum_{n=-\infty}^{\infty}q^{n^2} \tag{9}$$

then

$$\frac{\phi(q)-1}{\phi(q)+1}=\frac{q}{1+q+}\frac{-q^3}{1+q^3+}\frac{-q^5}{1+q^5+}\frac{-q^7}{1+q^7+}\cdots \tag{10}$$

Proof.
Take $q\to q^2$ in (6) and then set $a\to q$ and $b\to q^2$.

Proposition 2.

$$\frac{\Phi(-q)-f(-q)}{\Phi(-q)+f(-q)}=\frac{q}{1-q+}\frac{q^3}{1-q^3+}\frac{q^5}{1-q^5+}\frac{q^7}{1-q^7+}\cdots \tag{11}$$

Proof.
Set $b=0$ in (6) and then $a=q$.

Proposition 3.

$$\frac{\Phi(-q)-f(-q)}{\Phi(-q)+f(-q)}=-\frac{\phi(-q)-1}{\phi(-q)+1} \tag{12}$$

Proof.
It follows from Propositions 1, 2

Proposition 4.

$$\sum_{n=0}^{\infty}\frac{q^n}{1-a^2q^{2n}}=\frac{1}{1-q+}\frac{-a^2(1-q)^2}{1-q^3+}\frac{-qa^2(1-q^2)^2}{1-q^5+}\frac{-q^2a^2(1-q^3)^2}{1-q^7+}\cdots \tag{13}$$

Proof.
Divide relation (6) by $a-b$ and then take the limit $b\to a$.

Proposition 5.

$$\frac{K(k_r)}{2\pi}+\frac{1}{4}=\frac{1}{1-q+}\frac{(1-q)^2}{1-q^3+}\frac{q(1-q^2)^2}{1-q^5+}\frac{q^2(1-q^3)^2}{1-q^7+}\ldots \tag{14}$$

Proof.
Set in (13) $a=i$, and $q=e^{-\pi\sqrt{r}}$.

Now set

$$u(a,q)=\frac{2a}{1-q+}\frac{a^2(1+q)^2}{1-q^3+}\frac{a^2q(1+q^2)^2}{1-q^5+}\frac{a^2q^2(1+q^3)^2}{1-q^7+}\ldots \tag{15}$$

and

$$P=\left(\frac{(-a;q)_\infty}{(a;q)_\infty}\right)^2 \tag{16}$$

Then

$$\frac{P-1}{P+1}=u(a,q) \tag{17}$$

or

Proposition 6.

$$\left(\frac{(-a;q)_\infty}{(a;q)_\infty}\right)^2=-1+\frac{2}{1-}\frac{2a}{1-q+}\frac{a^2(1+q)^2}{1-q^3+}\frac{a^2q(1+q^2)^2}{1-q^5+}\frac{a^2q^2(1+q^3)^2}{1-q^7+}\ldots \tag{18}$$

Proposition 7.

$$4\sum_{n=0}^{\infty}\frac{a^{2n+1}}{(2n+1)(1-q^{2n+1})}=\log\left(-1+\frac{2}{1-u(a,q)}\right) \tag{19}$$

Proof.
Take the logarithms in both sides of (18) and expand in Taylor series. Then rearange the double sum to get easily the desired result.

Here we must mention that holds the more general formula

$$2\sum_{n=0}^{\infty}\frac{a^{2n+1}-b^{2n+1}}{(2n+1)(1-q^{2n+1})}=\log\left(-1+\frac{2}{1-U(a,b;q)}\right) \tag{20}$$

Thus

$$\left(-1+\frac{2}{1-U(a,b;q)}\right)=\frac{\left(-1+\frac{2}{1-u(a,q)}\right)}{\left(-1+\frac{2}{1-u(b,q)}\right)} \tag{21}$$

and for to study U we have to study only u. In some cases the u fraction can calculated in terms of elliptic functions. For example:

$$-1+\frac{2}{1-u(q,q)}=\frac{\pi}{2k'_r K(k_r)} \tag{22}$$

In general holds

$$4\sum_{n=0}^{\infty}\frac{q^{\nu(2n+1)}}{(2n+1)(1-q^{2n+1})}=-4\sum_{j=1}^{\nu-1}\operatorname{arctanh}(q^j)-\log\left(\frac{2k'_r K(k_r)}{\pi}\right)$$

from which we lead to the following:

Proposition 8.
Let ν be positive integer, then

$$-1+\frac{2}{1-u(q^\nu,q)}=-1+\frac{2}{1-}\frac{2q^\nu}{1-q+}\frac{q^{2\nu}(1+q)^2}{1-q^3+}\frac{q^{2\nu+1}(1+q^2)^2}{1-q^5+}\frac{q^{2\nu+2}(1+q^3)^2}{1-q^7+}\ldots=$$

$$=\frac{\pi}{2k'_r K(k_r)}\exp\left(-4\sum_{j=1}^{\nu-1}\operatorname{arctanh}(q^j)\right) \tag{23}$$

Proposition 9.
Let ν_1, ν_2 be positive integers, then

$$-1+\frac{2}{1-U(q^{\nu_1},q^{\nu_2},q)}=\exp\left(-4\left(\sum_{j_1=1}^{\nu_1-1}\operatorname{arctanh}(q^{j_1})-\sum_{j_2=1}^{\nu_2-1}\operatorname{arctanh}(q^{j_2})\right)\right) \tag{24}$$

Proof.
The proof follows easily from (21) and (24).

Note. One can find many useful results in pages stored on the Web one is:

http://pi.physik.uni-bonn.de/ dieckman/InfProd/InfProd.html

Another formula related with u continued fraction is when $q = e^{-\pi\sqrt{r}}$

$$-1+\frac{2}{1-u(q^{\nu+1/2},q)} = \exp\left(-4\sum_{n=0}^{\infty}\frac{q^{(2n+1)(\nu+1/2)}}{(2n+1)(1-q^{2n+1})}\right)$$

$$= \exp\left(-4\sum_{j=0}^{\nu-1}\operatorname{arctanh}(q^{j+1/2})+\operatorname{arctanh}(k_r)\right) \tag{25}$$

Hence also

$$k_r = \tanh\left(4\sum_{j=0}^{\nu-1}\operatorname{arctanh}(q^{j+1/2})+\log\left(-1+\frac{2}{1-u(q^{\nu+1/2},q)}\right)\right)$$

For every ν positive integer.
Hence we obtain a continued fraction for k_r

$$\frac{k'_r}{1-k_r} = -1+\frac{2}{1-u(q^{1/2},q)} \tag{26}$$

Inspired from the above relations and Propositions we have

Theorem(Unproved)
If c is positive real and ν_1, ν_2 positive integers then:

$$-1+\frac{2}{1-U(q^{\nu_1+c},q^{\nu_2+c},q)} \stackrel{?}{=}$$

$$= \exp\left(-4\left(\sum_{j_1=1}^{\nu_1-1}\operatorname{arctanh}(q^{j_1+c})-\sum_{j_2=1}^{\nu_2-1}\operatorname{arctanh}(q^{j_2+c})\right)\right) \tag{27}$$

or better
If $U = U(a,b,q)$, where $q = e^{-\pi\sqrt{r}}$ and

$$c = \left\{\frac{-\log(a)}{\pi\sqrt{r}}\right\} = \left\{\frac{-\log(b)}{\pi\sqrt{r}}\right\} \tag{28}$$

then

$$-1+\frac{2}{1-U(a,b,q)} =$$

$$= \exp\left(-4\left(\sum_{j_1=1}^{\left[\frac{-\log(a)}{\pi\sqrt{r}}\right]-1}\operatorname{arctanh}(q^{j_1+c})-\sum_{j_2=1}^{\left[\frac{-\log(b)}{\pi\sqrt{r}}\right]-1}\operatorname{arctanh}(q^{j_2+c})\right)\right)$$

where $\{x\}$ is the fractional part of x and $[x]$ is the largest integer that not exiding x.
ii) Observe that for $\nu_1 = \nu_2 = \nu$

$$-1 + \frac{2}{1 - U(q^{\nu+c}, q^{\nu+c}, q)} = 1$$

iii) Also we observe that holds and

$$\left(-1 + \frac{2}{1 - U(a, -b; q)}\right) = \left(-1 + \frac{2}{1 - u(a, q)}\right)\left(-1 + \frac{2}{1 - u(b, q)}\right) \quad (29)$$

This relation is similarly to (21). We also get the following unproved

Proposition 10.(Unproved)
Let $w \in \mathrm{Im}(\mathbf{C})$, then

$$\left|-1 + \frac{2}{1 - U(q^{\nu_1+c}, -wq^{\nu_2+c}, q)}\right| \overset{?}{=} \left(-1 + \frac{2}{1 - u(q^{\nu_1+c}, q)}\right) \quad (30)$$

Seting $c = 0$ in (30) and using Proposition 9 we get

Proposition 11.
When $w, z \in \mathrm{Im}(\mathbf{C})$ and $q = e^{-\pi\sqrt{r}}$, $r > 0$, then
(i)

$$\left|-1 + \frac{2}{1 - U(q^{\nu_1}, -wq^{\nu_2}, q)}\right| = \frac{\pi}{2k_r' K(k_r)} \exp\left(-4\sum_{j=1}^{\nu_1-1} \operatorname{arctanh}(q^j)\right)$$

(ii)

$$\left|-1 + \frac{2}{1 - U(-zq^{\nu_1+c}, -wq^{\nu_2+c}, q)}\right| = 1$$

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