OVERLOAD BEHAVIOR OF CONE SCHEDULES FOR PROCESSING SYSTEMS

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Abstract

This note complements the analysis of [1] and [2] addressing the case where the traffic load is *not* within the stability region, that is, the system operates in *overload*. For the case where the cone schedule matrix [1] is diagonal, it is shown that the job backlog explodes on a particular ray, as opposed to various subsequences exploding on diverse ones. The context and model used here are those described in [1]. The analysis technique draws on and parallels closely those in [1] and [2].

1 Introduction

Consider the model of [1], where the PCS matrix $\mathbf{B}$ is now diagonal $\mathbf{\Delta}$ (and positive-definite, hence, all its diagonal elements are positive). The system operates in overload, in the sense that $\rho \notin \mathcal{P}$, where

$$\mathcal{P} = \left\{ \rho \in \mathbb{R}_+^Q : \langle \rho, \Delta v \rangle \leq \max_{S \in \mathcal{S}} \langle S, \Delta v \rangle \text{ for every } v \in \mathbb{R}^Q \right\}, \tag{1.1}$$

as defined in [1]. We consider the limit defined below:

$$H = \limsup_{t \to \infty} \left\langle \frac{X(t)}{t}, \mathbf{\Delta} \frac{X(t)}{t} \right\rangle \tag{1.2}$$

and select a convergent increasing unbounded subsequence $\{t_c\}$ on which the 'limsup' is attained[4] – hence,

$$\lim_{c \to \infty} \frac{X(t_c)}{t_c} = \eta \tag{1.3}$$

and

$$\limsup_{c \to \infty} \left\langle \frac{X(t_c)}{t_c}, \mathbf{\Delta} \frac{X(t_c)}{t_c} \right\rangle = \langle \eta, \mathbf{\Delta} \eta \rangle = H. \tag{1.4}$$

Lemma 1.1 *We have*

$$\rho \notin \mathcal{P} \implies \eta \neq 0 \tag{1.5}$$

Proof: See [2], Proposition 2.1. We have that

$$\rho \notin \mathcal{P} \implies \limsup_{t \to \infty} \frac{X_q(t)}{t} > 0 \text{ for some } q \in \mathcal{Q} \implies \eta \neq 0 \text{ and } \limsup_{t \to \infty} \left\langle \frac{X(t)}{t}, \mathbf{\Delta} \frac{X(t)}{t} \right\rangle = \langle \eta, \Delta \eta \rangle = H > 0. \tag{1.6}$$

■

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[4]See footnote 12 of [1] concerning why such a subsequence exists

2 Overload Regime: $\rho \notin \mathcal{P} \implies \lim_{\mathbf{t}\to\infty} \frac{\mathbf{X(t)}}{\mathbf{t}} = \eta \neq \mathbf{0}$.

Theorem 2.1 *When $\rho \notin \mathcal{P}$, we have*

$$\lim_{t\to\infty} \frac{X(t)}{t} = \eta \neq 0. \tag{2.1}$$

That is, the workload explodes on the same non-zero ray η on any subsequence.

Proof:

From Section V of [1] on PCS cone geometry, recall that $\mathcal{C}_S = \{x \in \mathbb{R}^Q : \langle S, \mathbf{\Delta} x\rangle = \max_{S'\in\mathcal{S}} \langle S', \mathbf{\Delta} x\rangle\}$ is a cone, and when $X(t) \in \mathcal{C}_S^o$ (the interior of $\mathcal{C}_S$) the PCS will choose $S(t) = S$. Moreover, the *surrounding cone* of any non-zero vector η is the cone

$$\mathcal{C}(\eta) = \bigcup_{S\in\mathcal{S}^*(\eta)-\mathcal{S}^\dagger} C_S \tag{2.2}$$

where $S^*(\eta) = \text{argmax}_{S\in\mathcal{S}} \langle S, \mathbf{\Delta}\eta\rangle$ is the set of service vectors of that PCS would select for backlog η and $\mathcal{S}\dagger$ is the set of *non-essential* ones (see [1], end of Section IV). We have

$$X(t) \in \mathcal{C}^o(\eta) \implies \langle S(t), \mathbf{\Delta}\eta\rangle = \max_{S\in\mathcal{S}} \langle S, \mathbf{\Delta}\eta\rangle \tag{2.3}$$

where $\mathcal{C}^o(\eta)$ is the interior of $\mathcal{C}(\eta)$. Define now

$$\mathcal{K}(\eta) = \{x \in \mathbb{R}^Q_{0+} : x_q > \max_{S\in\mathcal{S}}\{S_q\} \text{ for each } q \text{ with } \eta_q > 0\}, \tag{2.4}$$

which is upward-scalable; indeed, $x \in \mathcal{K}(\eta)$ implies $\alpha x \in \mathcal{K}(\eta)$ for any scalar $\alpha > 1$. Note that when $X(t) \in \mathcal{K}(\eta)$ we have $X_q(t) > \max_{S\in\mathcal{S}}\{S_q\}$ from all $q \in \mathcal{Q}$ with $\eta_q > 0$, so $D_q(t) = \min\{X_q(t), S_q(t)\} = S_q(t)$. Therefore,

$$X(t) \in \mathcal{K}(\eta) \implies D_q(t) = S_q(t) \text{ for all } q \in \mathcal{Q} \text{ with } \eta_q > 0. \tag{2.5}$$

that is, all service capacity allocated at slot t to queue q with $\eta_q > 0$ is used; there is no idling in that time slot. Consider now the set

$$\mathcal{V}(\eta) = \mathcal{K}(\eta) \bigcap \mathcal{C}^o(\eta) \tag{2.6}$$

and note that it is upward-scalable, that is, $x \in \mathcal{V}(\eta)$ implies $\alpha x \in \mathcal{V}(\eta)$ for any scalar $\alpha > 1$. Thus, the set $\mathcal{V}(\eta)$ is 'cone-like' for large backlog vectors.

2.1 Structural Properties

Lemma 2.1 *For every sequence $\{t'_c\}$ such that $t'_c < t_c$ and*

$$X(t) \in \mathcal{V}(\eta) \quad \textit{for every} \quad X(t) \in (t'_c, t_c] \tag{2.7}$$

for every c, we have

$$\left\langle \frac{X(t_c) - X(t'_c)}{t_c - t'_c}, \mathbf{\Delta}\eta \right\rangle = \left\langle \frac{\sum_{t=t'_c}^{t_c-1} A(t)}{t_c - t'_c}, \mathbf{\Delta}\eta \right\rangle - \max_{S\in\mathcal{S}} \langle S(t), \mathbf{\Delta}\eta\rangle \,. \tag{2.8}$$

Proof: We write (using similar arguments like in equations of A.22 to A.27 of [1]),

$$
\begin{aligned}
\left\langle X(t_c) - X(t'_c), \mathbf{\Delta}\eta \right\rangle &= \left\langle \sum_{t=t'_c}^{t_c-1} A(t), \mathbf{\Delta}\eta \right\rangle - \left\langle \sum_{t=t'_c}^{t_c-1} D(t), \mathbf{\Delta}\eta \right\rangle \\
&= \left\langle \sum_{t=t'_c}^{t_c-1} A(t), \mathbf{\Delta}\eta \right\rangle - \sum_{t=t'_c}^{t_c-1} \left\langle D(t), \mathbf{\Delta}\eta \right\rangle \\
&= \left\langle \sum_{t=t'_c}^{t_c-1} A(t), \mathbf{\Delta}\eta \right\rangle - \sum_{t=t'_c}^{t_c-1} \left[\sum_{q:\eta_q>0} D_q(t) \mathbf{\Delta}_{qq}\eta_q + \sum_{q:\eta_q=0} D_q(t) \mathbf{\Delta}_{qq}\eta_q \right] \\
&= \left\langle \sum_{t=t'_c}^{t_c-1} A(t), \mathbf{\Delta}\eta \right\rangle - \sum_{t=t'_c}^{t_c-1} \left[\sum_{q:\eta_q>0} S_q(t) \mathbf{\Delta}_{qq}\eta_q + \sum_{q:\eta_q=0} S_q(t)\, 0 \right] \\
&= \left\langle \sum_{t=t'_c}^{t_c-1} A(t), \mathbf{\Delta}\eta \right\rangle - \sum_{t=t'_c}^{t_c-1} \left\langle S(t), \mathbf{\Delta}\eta \right\rangle \\
&= \left\langle \sum_{t=t'_c}^{t_c-1} A(t), \mathbf{\Delta}\eta \right\rangle - \max_{S\in\mathcal{S}} \left\langle S(t), \mathbf{\Delta}\eta \right\rangle (t_c - t'_c),
\end{aligned}
\tag{2.9}
$$

To see the above steps, recall the following. First, $X(t) \in \mathcal{V}(\eta)$ for every $t \in (t'_c, t_c]$ and any c, by assumption. Therefore, $X(t) \in \mathcal{K}(\eta)$, hence, from (2.5) we get $D_q(t) = S_q(t)$ for $q \in \mathcal{Q}$ with $\eta_q > 0$, for every $t \in (t'_c, t_c]$ and any c. Moreover, $X(t) \in \mathcal{C}(\eta)$, hence, from (2.3) we get $\langle S(t), \mathbf{\Delta}\eta \rangle = \max_{S\in\mathcal{S}} \langle S, \mathbf{\Delta}\eta \rangle$, for all $t \in (t'_c, t_c]$ and any c. ■

Lemma 2.2 *For any increasing unbounded time sequences $\{t_n\}$ and $\{t'_n\}$, we have*

$$
\lim_{n\to\infty} \frac{t_n - t'_n}{t_n} = \chi \in (0, 1] \implies \lim_{n\to\infty} \frac{\sum_{t=t'_n}^{t_n-1} A(t)}{t_n - t'_n} = \rho \tag{2.10}
$$

Proof: Note that $t'_n < t_n$ eventually (for any large n), expand the terms as follows:

$$
\frac{\sum_{t=t'_n}^{t_n-1} A(t)}{t_n - t'_n} = \frac{\sum_{t=0}^{t_n-1} A(t)}{t_n - t'_n} - \frac{\sum_{t=0}^{t'_n-1} A(t)}{t_n - t'_n} = \frac{\sum_{t=0}^{t_n-1} A(t)}{t_n} \frac{t_n}{t_n - t'_n} - \frac{\sum_{t=0}^{t_n-1} A(t)}{t'_n} \frac{t'_n}{t_n - t'_n}, \tag{2.11}
$$

and observe that letting $n\to\infty$ we get

$$
\lim_{n\to\infty} \frac{\sum_{t=t'_n}^{t_n-1} A(t)}{t_n - t'_n} = \rho\, \frac{1}{\chi} - \rho\, (\frac{1}{\chi} - 1) = \rho, \tag{2.12}
$$

since $\lim_{n\to\infty} \frac{\sum_{t=0}^{T} A(t)}{T} = \rho$. This completes the proof of the lemma. ■

Lemma 2.3 *For any increasing unbounded subsequence $\{t_m\}$ with $\lim_{m\to\infty} \frac{X(t_m)}{t_m} = \mu$, we have*

$$
\langle \mu, \Delta\eta \rangle \geq \langle \rho, \mathbf{\Delta}\eta \rangle - \max_{S\in\mathcal{S}} \langle S, \mathbf{\Delta}\eta \rangle \tag{2.13}
$$

Proof: We write

$$X(t_m) = \sum_{t=0}^{t_m-1} A(t) - \sum_{t=0}^{t_m-1} D(t) \tag{2.14}$$

and observe that $D_q(t) = \min\{S_q(t), X_q(t)\} \le S_q(t)$ for every $q \in \mathcal{Q}$, hence, $-D_q(t) \ge -S_q(t)$. Therefore, since $\mathbf{\Delta}$ is diagonal (with positive elements), we have $-\langle D(t), \mathbf{\Delta}\eta\rangle \ge \langle S(t), \mathbf{\Delta}\eta\rangle \ge -\max_{S\in\mathcal{S}} \langle S, \mathbf{\Delta}\eta\rangle$. Projecting on $\mathbf{\Delta}\eta$ we get

$$\langle X(t_m), \mathbf{\Delta}\eta\rangle \ge \left\langle \sum_{t=0}^{t_m-1} A(t), \mathbf{\Delta}\eta \right\rangle - \max_{S\in\mathcal{S}} \langle S, \mathbf{\Delta}\eta\rangle (t_m) \tag{2.15}$$

Dividing by t_m and letting $m\to\infty$, we get

$$\langle \mu, \mathbf{\Delta}\eta\rangle \ge \langle \rho, \mathbf{\Delta}\eta\rangle - \max_{S\in\mathcal{S}} \langle S, \mathbf{\Delta}\eta\rangle > 0 \tag{2.16}$$

This completes the proof of the lemma. ■

Lemma 2.4 *For any increasing unbounded subsequence* $\{t_m\}$ *with* $\lim_{m\to\infty} \frac{X(t_m)}{t_m} = \mu$*, we have*

$$\langle \mu, \mathbf{\Delta}\eta\rangle \ge \langle \eta, \mathbf{\Delta}\eta\rangle \implies \mu = \eta \tag{2.17}$$

Proof: Indeed (recalling that $\mathbf{\Delta}$ is positive-definite), we have

$$\begin{aligned} 0 \le \langle \mu - \eta, \mathbf{\Delta}(\mu - \eta)\rangle &= \langle \mu, \mathbf{\Delta}\mu\rangle - 2\langle \mu, \mathbf{\Delta}\eta\rangle + \langle \eta, \mathbf{\Delta}\eta\rangle \\ &\le \langle \mu, \mathbf{\Delta}\mu\rangle - 2\langle \eta, \mathbf{\Delta}\eta\rangle + \langle \eta, \mathbf{\Delta}\eta\rangle = \langle \mu, \mathbf{\Delta}\mu\rangle - \langle \eta, \mathbf{\Delta}\eta\rangle, \end{aligned} \tag{2.18}$$

so $\langle \mu, \mathbf{\Delta}\mu\rangle \ge \langle \eta, \mathbf{\Delta}\eta\rangle$. But since $\langle \eta, \mathbf{\Delta}\eta\rangle = \limsup t\to\infty \left\langle \frac{X(t)}{t}, \mathbf{\Delta}\frac{X(t)}{t}\right\rangle$, we must have $\langle \mu, \mathbf{\Delta}\mu\rangle = \langle \eta, \mathbf{\Delta}\eta\rangle$, therefore, $\langle \mu - \eta, \mathbf{\Delta}(\mu - \eta)\rangle = 0$, which implies $\mu = \eta$. This completes the proof of the lemma. ■

Lemma 2.5 *For every* $\epsilon \in (0, 1)$ *we have*

$$-\left[\langle \rho, \mathbf{\Delta}\eta\rangle - \max_{S\in\mathcal{S}} \langle S, \mathbf{\Delta}\eta\rangle\right] \frac{\epsilon}{1-\epsilon} + \langle \eta, \mathbf{\Delta}\eta\rangle \frac{1}{1-\epsilon} \ge \langle \eta, \mathbf{\Delta}\eta\rangle \tag{2.19}$$

Proof: Rewrite the inequality as $-[\langle \rho, \mathbf{\Delta}\eta\rangle - \max_{S\in\mathcal{S}} \langle S, \mathbf{\Delta}\eta\rangle]\,\epsilon + \langle \eta, \mathbf{\Delta}\eta\rangle \ge (1-\epsilon)\langle \eta, \mathbf{\Delta}\eta\rangle$, since $1 - \epsilon > 0$. This is equivalent (since $\epsilon > 0$) to

$$\langle \eta, \mathbf{\Delta}\eta\rangle \ge \langle \rho, \mathbf{\Delta}\eta\rangle - \max_{S\in\mathcal{S}} \langle S, \mathbf{\Delta}\eta\rangle \tag{2.20}$$

But this is true by Lemma 2.3 applied to the sequence $\{t_c\}$ with $\lim_{c\to\infty} \frac{X(t_c)}{t_c} = \eta$. This complete the proof of the lemma. ■

2.2 Uniqueness of limit $\lim_{\mathbf{t}\to\infty} \frac{\mathbf{X(t)}}{\mathbf{t}}$ on an individual arrival trace

Proposition 2.1 *There is no subsequence* $\{\mathbf{t_a}\}$ *with* $\lim_{\mathbf{a}\to\infty} \frac{\mathbf{X(t_a)}}{\mathbf{t_a}} = \psi \ne \eta$.

Arguing by contradiction, assume that there is some other convergent subsequence $\{X(t_a)\}$ such that $\lim_{a\to\infty}\frac{X(t_a)}{t_a} = \psi \neq \eta$. We shall show that this is impossible. Note that $\psi_q < \infty$ for all q. This is easy to see since $\psi_q = \lim_{a\to\infty}\frac{X_q(t_a)}{t_a} \leq \lim_{a\to\infty}\frac{A_q(t_a)}{t_a} = \rho_q < \infty$. Define first

$$s_c = \max\{t_a : t_a < t_c\} < t_c \tag{2.21}$$

Lemma 2.6 *We have that*

$$\liminf_{c\to\infty}\frac{t_c - s_c}{t_c} = \epsilon \in (0,1) \tag{2.22}$$

Proof: A) We first show that $\epsilon > 0$. We start by showing that there is no increasing unbounded subsequence $\{t_b\}$ of $\{t_c\}$ such that $\lim_{b\to\infty}\frac{t_b - s_b}{t_b} = 0$, where $s_b = \max\{t_a < t_b\}$. Note that this also implies that $\lim_{b\to\infty}\frac{s_b}{t_b} = 1$. Arguing by contradiction, suppose it exists. Observe that for every $q \in \mathcal{Q}$ we have

$$-\bar{S}_q(t_b - s_b) \leq X_q(t_b) - X_q(s_b) \leq \bar{A}_q(t_b - s_b), \tag{2.23}$$

where $\bar{A}_q < \infty$ is the maximum workload that can arrive in queue q in any time slot (see model in [1] for assumption of boundedness) and $\bar{S}_q = \max_{S\in\mathcal{S}}\{S_q\} < \infty$ is the maximum workload that can be removed from queue q in any time slot. Dividing by t_b, letting $b\to\infty$, we get

$$\lim_{b\to\infty}\frac{X(t_b) - X(s_b)}{t_b} = 0 = \lim_{b\to\infty}\left[\frac{X(t_b)}{t_b} - \frac{X(s_b)}{s_b}\frac{s_b}{t_b}\right] = \eta - \psi \tag{2.24}$$

which implies that $\eta = \psi$ and establishes the desired contradiction.

B) We still need to show that $\epsilon \neq 1$ (note that $\frac{t_c - s_c}{t_c} \leq 1$). Arguing by contradiction, suppose there exists a subsequence $\{t_i\}$ of $\{t_c\}$ (and corresponding subsequence $\{s_i\}$ of $\{s_c\}$) such that $\lim_{i\to\infty}\frac{t_i - s_i}{t_i} = 1$, hence, $\lim_{i\to\infty}\frac{s_i}{t_i} = 0$. Applying Lemmas 2.1 and 2.2 with $\{t'_i\} = \{s_i\}$

$$\lim_{i\to\infty}\left\langle\frac{X(t_i) - X(s_i)}{t_i - s_i}, \mathbf{\Delta}\eta\right\rangle = \langle\rho, \mathbf{\Delta}\eta\rangle - \max_{S\in\mathcal{S}}\langle S(t), \mathbf{\Delta}\eta\rangle \tag{2.25}$$

It follows that

$$\begin{aligned}
\langle\eta, \mathbf{\Delta}\eta\rangle &= \lim_{i\to\infty}\left\langle\frac{X(t_i)}{t_i}, \mathbf{\Delta}\eta\right\rangle \\
&= \lim_{i\to\infty}\left\langle\frac{X(t_i) - X(s_i)}{t_i - s_i}\ \frac{t_i - s_i}{t_i} + \frac{X(s_i)}{s_i}\ \frac{s_i}{t_i}, \mathbf{\Delta}\eta\right\rangle \\
&= \lim_{i\to\infty}\left\langle\frac{X(t_i) - X(s_i)}{t_i - s_i}, \mathbf{\Delta}\eta\right\rangle\ \frac{t_i - s_i}{t_i} + \left\langle\frac{X(s_i)}{s_i}, \mathbf{\Delta}\eta\right\rangle\ \frac{s_i}{t_i} \\
&= \left[\langle\rho, \mathbf{\Delta}\eta\rangle - \max_{S\in\mathcal{S}}\langle S(t), \mathbf{\Delta}\eta\rangle\right]\cdot 1 + \langle\psi, \mathbf{\Delta}\eta\rangle\cdot 0 \\
&= \langle\rho, \mathbf{\Delta}\eta\rangle - \max_{S\in\mathcal{S}}\langle S(t), \mathbf{\Delta}\eta\rangle
\end{aligned} \tag{2.26}$$

Now applying Lemma 2.3 on the subsequence $\{s_i\}$ with $\lim_{s_i\to\infty}\frac{X(s_i)}{s_i} = \psi$ we get

$$\langle\psi, \Delta\eta\rangle \geq \langle\rho, \mathbf{\Delta}\eta\rangle - \max_{S\in\mathcal{S}}\langle S, \mathbf{\Delta}\eta\rangle = \langle\eta, \mathbf{\Delta}\eta\rangle\,, \tag{2.27}$$

using (2.26). Hence, $\langle \psi, \Delta\eta \rangle \geq \langle \eta, \mathbf{\Delta}\eta \rangle$, which implies $\psi = \eta$ by Lemma 2.4. But this is impossible since by definition of subsequence $\{s_c\}$, $\psi \neq \eta$, which completes the proof of the lemma. ■

Select now a subsequence of $\{t_c\}$ on which this 'liminf' is attained, but keep the same indexing c of the original one for notational simplicity, hence,

$$\lim_{c\to\infty} \frac{t_c - s_c}{t_c} = \epsilon \in (0, 1). \tag{2.28}$$

Therefore, $\lim_{c\to\infty} \frac{t_c}{s_c} = \frac{1}{1-\epsilon}$ and $\lim_{c\to\infty} \frac{t_c - s_c}{s_c} = \frac{\epsilon}{1-\epsilon}$.

Again, applying Lemmas 2.1 and 2.2 with $\{t'_c\} = \{s_c\}$, dividing by $t_c - s_c$ and letting $c\to\infty$, we get

$$\lim_{c\to\infty} \left\langle \frac{X(t_c) - X(s_c)}{t_c - s_c}, \mathbf{\Delta}\eta \right\rangle = \langle \rho, \mathbf{\Delta}\eta \rangle - \max_{S\in\mathcal{S}} \langle S, \mathbf{\Delta}\eta \rangle . \tag{2.29}$$

Then, we can write

$$\begin{aligned} \langle \psi, \mathbf{\Delta}\eta \rangle &= \lim_{c\to\infty} \left\langle \frac{X(s_c)}{s_c}, \mathbf{\Delta}\eta \right\rangle \\ &= \lim_{c\to\infty} \left\langle -\frac{X(t_c) - X(s_c)}{t_c - s_c} \, \frac{t_c - s_c}{s_c} + \frac{X(t_c)}{t_c} \, \frac{t_c}{s_c}, \mathbf{\Delta}\eta \right\rangle \\ &= -\left[\langle \rho, \mathbf{\Delta}\eta \rangle - \max_{S\in\mathcal{S}} \langle S, \mathbf{\Delta}\eta \rangle \right] \frac{\epsilon}{1-\epsilon} + \langle \eta, \mathbf{\Delta}\eta \rangle \frac{1}{1-\epsilon} \\ &\geq \langle \eta, \mathbf{\Delta}\eta \rangle \end{aligned} \tag{2.30}$$

The last equality is due to Lemma 2.5. Therefore, $\langle \psi, \mathbf{\Delta}\eta \rangle \geq \langle \eta, \mathbf{\Delta}\eta \rangle$, which implies $\psi = \eta$ from Lemma 2.4. But this contradicts the assumption that $\psi \neq \eta$. This establishes the sought after contradiction. So for each individual arrival trace, there exists a unique limit $\lim_{t\to\infty} \frac{X(t)}{t} = \eta$, which concludes the proof of the proposition. Moreover, since $\lim_{t\to\infty} \frac{X(t)}{t} = \eta$ this implies that there exists $t_o < \infty$ such that $X(t)$ is in $\mathcal{V}(\eta)$ for all $t > t_o$. It remains to show that the limit, η, is independent of the particular arrival trace.

2.3 Characterizing the limit η

The purpose of this section is to characterize η in terms of ρ and service vectors $\mathcal{S}$ to establish the independence of η on the individual arrival trace. Knowing that $\lim_{t\to\infty} \frac{X(t)}{t} = \eta$, we now turn to identifying a couple of the characteristic properties of η.

Lemma 2.7 *Every limit is a fixed point. That is,*

$$\eta = \lim_{t\to\infty} \frac{X(t)}{t} = \left[\rho - \sum_{m=1}^{N} \alpha_m S_m \right]^+ \tag{2.31}$$

for some $\alpha_m \geq 0, \sum_m \alpha_m = 1$. *Furthermore,* $\alpha_m > 0$ *implies that* $\eta \in C_{S_m}$

Proof: Consider a subsequence $\{t_n\}$ such that for each m:

$$\alpha_m = \lim_{n\to\infty} \frac{\sum_{t=0}^{t_n - 1} \mathbf{1}_{\{S(t)=S_m\}}}{t_n} \tag{2.32}$$

Note that by definition: $\alpha_m \in [0, 1]$ and $\sum_m \alpha_m \leq 1$. Further, because $\rho \notin \mathcal{P}$, there exist q and $T < \infty$, such that $X_q(t) > 0$ for all $t > T$; hence, PCS will never idle for $t > T$ and $\sum_m \alpha_m = 1$.

We have for q such that $\eta_q > 0$:

$$\begin{aligned}
\eta_q &= \lim_{n\to\infty} \frac{X_q(t_n)}{t_n} \\
&= \lim_{n\to\infty} \frac{\sum_{t=0}^{t_n-1}\left[A_q(t) - D_q(t)\right]}{t_n} \\
&= \lim_{n\to\infty} \frac{X_q(t_o) + \sum_{t=t_o}^{t_n-1}\left[A_q(t) - \sum_m S_{m,q}\mathbf{1}_{\{S(t)=S_m\}}\right]}{t_n} \\
&= \rho_q - \sum_m \alpha_m S_{m,q}
\end{aligned} \tag{2.33}$$

Where $t_o < \infty$ such that for all $t > t_o$, $X(t) \in \mathcal{V}(\eta)$. It's existence is given by Proposition 2.1. For q such that $\eta_q = 0$, we have:

$$\begin{aligned}
0 = \eta_q &= \lim_{n\to\infty} \frac{X_q(t_n)}{t_n} \\
&= \lim_{n\to\infty} \frac{\sum_{t=0}^{t_n-1}\left[A_q(t) - D_q(t)\right]}{t_n} \\
&= \lim_{n\to\infty} \frac{X_q(t_o) + \sum_{t=t_o}^{t_n-1}\left[A_q(t) - \sum_m \min\{X_q(t), S_{m,q}\}\mathbf{1}_{\{S(t)=S_m\}}\right]}{t_n} \\
&\geq \lim_{n\to\infty} \frac{X_q(t_o) + \sum_{t=t_o}^{t_n-1}\left[A_q(t) - \sum_m S_{m,q}\mathbf{1}_{\{S(t)=S_m\}}\right]}{t_n} \\
&= \rho - \sum_m \alpha_m S_{m,q}
\end{aligned} \tag{2.34}$$

Which means that $\rho_q - \sum_m \alpha_m S_{m,q} \leq 0$ and

$$0 = \eta_q = \left[\rho_q - \sum_m \alpha_m S_{m,q}\right]^+, \tag{2.35}$$

which gives us that $\eta = \left[\rho - \sum_m \alpha_m S_m\right]^+$.

Finally, we have to show that if $\alpha_m > 0$, then $\eta \in C_{S_m}$. We have seen that α_m is the proportion of time that service vector S_m is used under PCS once $X(t) \in \mathcal{V}(\eta)$ for all $t > t_o$. By Proposition 2.1, we know that t_o exists. By contradiction, suppose that $\eta \notin C_{S_m}$. This implies that there exists $m' \neq m$ such that $\langle \eta, \mathbf{\Delta} S_{m'} \rangle > \langle \eta, \mathbf{\Delta} S_m \rangle$. Since $\alpha_m > 0$, we must use S_m for some $t > t_o$. This contradicts the definition of $\mathcal{V}(\eta)$, which by (2.3) says that PCS would use $S_{m'}$ rather than S_m which would imply that $\alpha_m = 0$. Hence, if $\alpha_m > 0$, $\eta \in C_{S_m}$. ■

Lemma 2.8 *We have*

$$\langle \eta, \mathbf{\Delta}\eta \rangle = \langle \rho, \mathbf{\Delta}\eta \rangle - \max_{S\in\mathcal{S}} \langle S, \mathbf{\Delta}\eta \rangle \tag{2.36}$$

Proof: This follows from Lemma 2.7. Replacing $\eta = [\rho - \sum_{m=1}^{N} \alpha_m S_m]^+$ we have

$$\begin{aligned}
\langle \eta, \mathbf{\Delta}\eta \rangle &= \left\langle \left[\rho - \sum_{m=1}^{N} \alpha_m S_m\right]^+, \mathbf{\Delta}\eta \right\rangle \\
&= \sum_{q:\eta_q>0} \left[\rho - \sum_{m=1}^{N} \alpha_m S_m\right]_q \mathbf{\Delta}_{qq}\eta_q + \sum_{q:\eta_q=0} \left[\rho - \sum_{m=1}^{N} \alpha_m S_m\right]_q^+ \mathbf{\Delta}_{qq}\eta_q \\
&= \sum_{q:\eta_q>0} \left[\rho - \sum_{m=1}^{N} \alpha_m S_m\right]_q \mathbf{\Delta}_{qq}\eta_q + \sum_{q:\eta_q=0} \left[\rho - \sum_{m=1}^{N} \alpha_m S_m\right]_q \mathbf{\Delta}_{qq}0 \\
&= \left\langle \rho - \sum_{m=1}^{N} \alpha_m S_m, \mathbf{\Delta}\eta \right\rangle \\
&= \langle \rho, \mathbf{\Delta}\eta \rangle - \sum_{m=1}^{N} \langle \alpha_m S_m, \mathbf{\Delta}\eta \rangle \\
&= \langle \rho, \mathbf{\Delta}\eta \rangle - \max_{S \in \mathcal{S}} \langle S, \mathbf{\Delta}\eta \rangle
\end{aligned} \tag{2.37}$$

where the last equality follows from the fact that η is a fixed point. ■

Lemma 2.9 *The vector $\eta = \lim_{t\to\infty} \frac{X(t)}{t}$ is the unique minimizer of*

$$\langle \eta, \mathbf{\Delta}\eta \rangle = \min_{\eta' \in \Psi(\rho,\mathcal{S})} \langle \eta', \mathbf{\Delta}\eta' \rangle \tag{2.38}$$

where

$$\Psi(\rho, \mathcal{S}) = \{\eta' : \eta' = (\rho - r)^+ \text{ with } r \in \mathcal{P}\} \tag{2.39}$$

and $\mathcal{P}$ is the stability region given by $\mathcal{S}$. Therefore, $r = \sum_{S\in\mathcal{S}} \alpha_S S$ with $\sum_{S\in\mathcal{S}} \alpha_S \le 1$ and $\alpha_S \ge 0$ for each $S \in \mathcal{S}$, where $\mathcal{S}$ is the set of service vectors.

Proof: From Lemma 2.7, we have that $\eta \in \Psi(\rho, \mathcal{S})$. Arbitrarily choose any vector

$$\bar{\eta} = \Big(\rho - \sum_{S\in\mathcal{S}} \alpha_S S\Big)^+, \text{ with } \sum_{S\in\mathcal{S}} \alpha_S \le 1, \text{ and } \alpha_S \ge 0, S \in \mathcal{S}. \tag{2.40}$$

Projecting on $\mathbf{\Delta}\eta$ we get

$$\begin{aligned}
\langle \bar{\eta}, \mathbf{\Delta}\eta \rangle &= \left\langle \left[\rho - \sum_{S\in\mathcal{S}} \alpha_S S\right]^+, \mathbf{\Delta}\eta \right\rangle \\
&\ge \left\langle \rho - \sum_{S\in\mathcal{S}} \alpha_S S, \mathbf{\Delta}\eta \right\rangle \\
&= \langle \rho, \mathbf{\Delta}\eta \rangle - \sum_{S\in\mathcal{S}} \alpha_S \langle S, \mathbf{\Delta}\eta \rangle \\
&\ge \langle \rho, \mathbf{\Delta}\eta \rangle - \max_{S\in\mathcal{S}} \langle S, \mathbf{\Delta}\eta \rangle = \langle \eta, \mathbf{\Delta}\eta \rangle
\end{aligned} \tag{2.41}$$

The first inequality comes from the fact that $\mathbf{\Delta}_{qq} > 0$ and $\eta_q \geq 0$. The last equality comes from Lemma 2.8. Therefore, $\langle \bar{\eta}, \mathbf{\Delta}\eta \rangle \geq \langle \eta, \mathbf{\Delta}\eta \rangle$. This implies (recalling that $\mathbf{\Delta}$ is positive-definite) that

$$\begin{aligned} 0 &\leq \langle \bar{\eta} - \eta, \mathbf{\Delta}(\bar{\eta} - \eta) \rangle \\ &= \langle \bar{\eta}, \mathbf{\Delta}\bar{\eta} \rangle - 2\langle \bar{\eta}, \mathbf{\Delta}\eta \rangle + \langle \eta, \mathbf{\Delta}\eta \rangle \\ &\leq \langle \bar{\eta}, \mathbf{\Delta}\bar{\eta} \rangle - 2\langle \eta, \mathbf{\Delta}\eta \rangle + \langle \eta, \mathbf{\Delta}\eta \rangle \\ &= \langle \bar{\eta}, \mathbf{\Delta}\bar{\eta} \rangle - \langle \eta, \mathbf{\Delta}\eta \rangle , \end{aligned} \tag{2.42}$$

so $\langle \bar{\eta}, \mathbf{\Delta}\bar{\eta} \rangle \geq \langle \eta, \mathbf{\Delta}\eta \rangle$ and η is the minimizer of $\langle \eta', \mathbf{\Delta}\eta' \rangle$.

We still need to prove that the minimizer η is unique. This is done by showing that 1) $\langle \eta, \mathbf{\Delta}\eta \rangle$ is strictly convex in η and 2) the set $\Psi(\rho, \mathcal{S})$ is convex–uniqueness will follow from convex programming theory. 1) It is trivial to show that $\langle \eta, \mathbf{\Delta}\eta \rangle$ is strictly convex in η since $\mathbf{\Delta} > 0$ is a positive definite matrix. 2) We now show that the set $\Psi \equiv \Psi(\rho, \mathcal{S})$ is convex. First, we see that for any $r \in \mathcal{P}$ and corresponding $x = (\rho - r)^+ \in \Psi$, there exists $\bar{x} = \rho - \bar{r} = (\rho - r)^+ = x$ with $\bar{r} \in \mathcal{P}$. Let $\bar{r}_k = \min(\rho_k, r_k) \leq r_k$. Since $\bar{r} \leq r \in \mathcal{P}$, then $\bar{x} \in \Psi$. Now, consider two vectors $x, x' \in \Psi$ with corresponding $r, r' \in \mathcal{P}$ such that $x = (\rho - r)^+$ and $x' = (\rho - r')^+$. What remains to be shown is that for any $a \in [0, 1]$, $ax + (1-a)x' \in \Psi$. Indeed, we have:

$$\begin{aligned} ax + (1-a)x' &= a\bar{x} + (1-a)\bar{x}' \\ &= a(\rho - \bar{r}) + (1-a)(\rho - \bar{r}') \\ &= \rho - (a\bar{r} + (1-a)\bar{r}') \end{aligned} \tag{2.43}$$

By the convexity of $\mathcal{P}$, we know that $a\bar{r} + (1-a)\bar{r}' \in \mathcal{P}$ and subsequently, $\rho - (a\bar{r} + (1-a)\bar{r}') \in \Psi$. This concludes the proof. ■

We have just shown that on all arrival traces with system load ρ, $\lim_{t\to\infty} \frac{X(t)}{t} = \eta$, is unique. Furthermore, the limit, η, is identical across all such traces. This concludes the proof of Theorem 2.1.

References

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