CONSTRUCTING FREE ACTIONS OF P-GROUPS ON PRODUCTS OF SPHERES.

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Abstract. We prove that, for p an odd prime, every finite p-group of rank 3 acts freely on a finite complex X homotopy equivalent to a product of three spheres.

Contents

1. Introduction.

The origin of the study of group actions on spheres goes back to Hopf and the spherical space form problem, which asked for a classification of finite groups that can act freely on some sphere. The first result was due to P.A. Smith [23], who showed that if a finite group G acts freely on a sphere, then it has periodic cohomology. Later Milnor [20] gave a second necessary condition: If a finite group G acts freely on a sphere, then any element in G of order 2 must be central. Finally Thomas, Wall and Madsen [19] were able to prove the these two necessary conditions were in fact sufficient: a finite group G acts freely on a sphere if and only if it has periodic cohomology and all involutions are central.

From the homotopy point of view, Swan [26] proved that a finite group has periodic cohomology if and only if it acts freely on a finite dimensional CW-complex homotopy equivalent to a sphere. It is a classical result that a finite group has periodic cohomology if and only if all of its abelian p-subgroups are cyclic. Based on that and on their own algebraic results, Benson and Carlson [5] suggested the rank conjecture: for any finite group G we have that $rk(G) = hrk(G)$, where $hrk(G) = min\{k \in \mathbb{N}|\ G \textit{ acts freely on a finite dimensional CW-complex } X \simeq S^{n_1} \times ... \times S^{n_k}\}$ is the homotopy rank of G. With this notation, Swan's result says that $rk(G) = 1$ if and only if $hrk(G) = 1$. In the same period Heller [14] showed that $(\mathbb{Z}_p)^3$ cannot act freely on a finite dimensional CW-complex homotopy equivalent to a product of two spheres. More recently Adem and Smith [2] showed that if $rk(G) = 2$ then $hrk(G) = 2$ for G a p-group or G a simple group (different from $PSL_3(\mathbb{F}_p)$). Adem [4] proved also that every odd order rank two group acts freely an a finite CW-complex $X \simeq S^n \times S^m$. This also follows from a more general result of Jackson [17]. Our main theorem is the following:

Theorem 1.1. *For p an odd prime, every finite p-group of rank 3 acts freely on a finite CW-complex homotopy equivalent to the product of three spheres.*

Note that a converse to theorem 1.1 is given by Hanke in [15] in the sense that: if $(\mathbb{Z}/p)^r$ acts freely on $X = S^{n_1} \times ... \times S^{n_k}$ and if $p > 3dim(X)$, then $r \leq k$. We outline now the structure of the paper. Let p be an odd prime, let G be a p-group and $S(V)$ a complex representation G-sphere. In section 2 we first prove that, for all integer $k \geq 0$, there exists a positive integer q such that the group $\pi_k(Aut_G(S(V^{\oplus q})))$ is finite. We then incorporate this result in an outline of a known construction ([2], [10], [29]) that, in favorable conditions, gives a strategy to build group actions on products of spheres with controlled isotropy subgroups.

In section 3 we apply section 2 to prove that for G a rank 3 p-group, p odd, there is a free finite G-CW-complex $X \simeq S^m \times S^d \times S^k$. In section 4 we use section 2 to generalize theorem 3.2 in [2] for a p-group G: if X is a finite dimensional G-CW-complex with abelian isotropy, we show that there is a free finite dimensional G-CW-complex $Y \simeq X \times S^1 \times ... \times S^{n_k}$. As a corollary we will be able to build free finite G-CW-complexes $X \simeq S^{n_1} \times ... \times S^{n_{rk(G)}}$ for G a central extension of abelian p-groups. Our results overlap here with those of Űnlű and Yalçin [30].

In section 5 we discuss the rank conjecture for some infinite groups. The motivation comes from a result of Connolly and Prassidis [10] stating that: a group with finite virtual cohomological dimension, which is countable and with rank 1 finite subgroups, acts freely on a finite dimensional CW-complex $X \simeq S^m$. We show that an effective Γ-sphere does not need to exist but that the algebraic analogue still holds. More in detail: First, we define an amalgamated product Γ such that every finite dimensional Γ-space homotopy equivalent to a sphere has an isotropy subgroup of rank 2. Secondly, we prove that for all groups Γ with finite virtual cohomological dimension, there is a finite dimensional $\mathbb{Z}[\Gamma]$-projective complex $\mathbf{D}$ with $H^*(\mathbf{D}) \cong H^*(S^{n_1} \times ... \times S^{n_{rk(\Gamma)}})$.

2. The general construction.

The main result of this section is the construction of proposition 2.7. A key ingredient of the construction is proposition 2.5, which says that under some conditions $\pi_k(Aut_G(S^n))$ is finite. We begin with a series of lemmas and corollaries that we assemble into a proof of proposition 2.5.

Lemmas 2.1, 2.2 and 2.4 are individual results needed in the proof of proposition 2.5. Lemma 2.3 serves the proof of lemma 2.4.

Lemma 2.1. *Let X be a G-CW-complex and let $Aut_G(X)$ be the monoid of G-equivariant self-homotopy equivalences of X. For $k > 0$, the map of unbased homotopy classes $\varphi : \left[S^k, Aut_G(X)\right] \to \left[S^k \times X, X\right]_G$ is injective and, up to the choice of a connected component, factors through:*

$$\begin{array}{ccc} \left[S^k, Aut_G(X)\right] & \xrightarrow{\varphi} & \left[S^k \times X, X\right]_G \\ \downarrow & \nearrow & \\ \pi_k(Aut_G(X)) & & \end{array}$$

In particular all G-equivariant homotopies $H : I \times S^k \times X \to X$ between maps in $Im(\varphi)$ can be taken to satisfy $H(t, \star, x) = H(t', \star, x)$ for all $t, t' \in I$ and $x \in X$.

Proof. The map $\varphi : \left[S^k, Aut_G(X)\right] \to \left[S^k \times X, X\right]_G$ is clearly well defined. To see that it is injective, consider a G-equivariant homotopy $H : I \times S^k \times X \to X$ from $\varphi(f)$ to $\varphi(g)$. Clearly $H|_{\{0\}\times\{x_0\}\times X} = \varphi(f)(x_0) \in Aut_G(X)$. Which implies that $H|_{\{t\}\times\{x\}\times X} \in Aut_G(X)$ for all $(t, x) \in I \times S^k$ because $H|_{\{t\}\times\{x\}\times X} \simeq H|_{\{0\}\times\{x_0\}\times X}$ via a path in $I \times S^k$ from $(0, x_0)$ to (t, x). As a result, H defines an homotopy from f to g. To prove that φ factors, up to the choice of a connected component, through:

$$\begin{array}{ccc} \left[S^k, Aut_G(X)\right] & \xrightarrow{\varphi} & \left[S^k \times X, X\right]_G \\ \downarrow & \nearrow & \\ \pi_k(Aut_G(X)) & & \end{array}$$

we want to show that the map $\pi_k(Aut_G(X)) \to \left[S^k, Aut_G(X)\right]$ is a bijection. Observe that $Aut_G(X)$ is a monoid, thus an H-space so that $\pi_1(Aut_G(X), Id)$ acts trivially on $\pi_k(Aut_G(X), Id)$. The monoid $Aut_G(X)$ is very nice because all of its connected components are homotopy equivalent through maps of the form: $(Aut_G(X), Id) \to (Aut_G(X), f)$ with $g \mapsto f \circ g$. Consequently $\pi_1(Aut_G(X), f)$ acts trivially on $\pi_k(Aut_G(X), f)$ for all $f \in Aut_G(X)$. We conclude that $\pi_k(Aut_G(X)) \to \left[S^k, Aut_G(X)\right]$ is a bijection. The last claim directly follows from the diagram. □

Lemma 2.2. *Let G be a finite group acting on a space X. Let $H_1 < G$ be an isotropy subgroup maximal among isotropy subgroups. Set $X_1 = \{x \in X \mid G_x \in (H_1)\}$, where (H_1) denotes the conjugacy class of H_1. We then have that $Aut_G(X_1) \cong Aut_{WH_1}(X^{H_1})$ (here $WH_1 = NH_1/H_1$ is the Weil group).*

Proof. Let's begin by studying X_1. Clearly $X_1 \subset \cup_{H\in(H_1)} X^H$. Since $H \in (H_1)$ is supposed to be maximal, we must have that if $x \in X^H$, then $G_x = H$ so that $X_1 = \cup_{H\in(H_1)} X^H$. Similarly, if $x \in X^H \cap X^{H'}$, then $H = G_x = H'$. As a result $X_1 = \coprod_{H\in(H_1)} X^H$.

Observe next that a G-equivariant map $f : X_1 \to X_1$ restricts to a WH_1-equivariant map $f_1 : X^{H_1} \to X^{H_1}$ because $WH_1 = NH_1/H_1$ and H_1 acts trivially on X_1. The same holds for a G-equivariant homotopy $F : I \times X_1 \to X_1$, so that we have a well defined map $res : Aut_G(X_1) \to Aut_{WH_1}(X^{H_1})$.

We want to show now that the map $res : Aut_G(X_1) \to Aut_{WH_1}(X^{H_1})$ has an inverse given by $res^{-1}(f)(x) = gf(g^{-1}x)$, where $g \in G$ is such that $g^{-1}x \in X^{H_1}$. We begin by showing that res^{-1} is well defined. For all $x \in X_1 = \coprod_{H\in(H_1)} X^H$ there is a $g \in G$ such that $x \in X^{gH_1g^{-1}}$, so that $g^{-1}x \in X^{H_1}$. Assume that $\gamma \in G$ is also such that $\gamma^{-1}x \in X^{H_1}$. Clearly $gH_1g^{-1} = H = \gamma H_1\gamma^{-1}$, where $x \in X^H$. Thus $\gamma^{-1}gH_1g^{-1}\gamma = H_1$ so that $\gamma^{-1}g \in NH_1$. For $f \in Aut_{WH_1}(X^{H_1})$ we then have $gf(g^{-1}x) = gg^{-1}\gamma f(\gamma^{-1}gg^{-1}x) = \gamma f(\gamma^{-1}x)$ because f is NH_1-equivariant. Therefore res^{-1} is well defined.

Next, we show that $res^{-1}(f)$ is G-equivariant: For $x \in X_1$, let again $g \in G$ be such that $g^{-1}x \in X^{H_1}$. For all $g_0 \in G$ we have that $(g_0g)^{-1}g_0x \in X^{H_1}$. As a result $res^{-1}(f)(g_0x) = g_0gf((g_0g)^{-1}g_0x) = g_0gf(g^{-1}x) = g_0res^{-1}(f)(x)$. In the same way one can check that if f^{-1} is the homotopy inverse of f via homotopies H and H^{-1}, then $res^{-1}f^{-1}$ is the homotopy inverse of $res^{-1}f$ via homotopies $res^{-1}H$ and $res^{-1}H^{-1}$. Finally we observe that $res \circ res^{-1} = Id$ by choosing $g = 1$, while $res^{-1} \circ res = Id$ because $res^{-1}f = f$ when f is G-equivariant. □

Lemma 2.3. *Let G be a finite group and S^n a linear G-sphere. If $0 < k < n$ then $H^n(S^k \times S^n/G, \{*\} \times S^n/G, \mathbb{Z})$ is finite.*

Proof. Consider the long exact sequence of the pair $(S^k \times S^n/G, \{*\} \times S^n/G)$ with integer coefficients:

$$\begin{array}{ccccc} H^{n-1}(S^k \times S^n/G) & \twoheadrightarrow & H^{n-1}(\{*\} \times S^n/G) & & \\ & & \downarrow & & \\ & & H^n(S^k \times S^n/G, \{*\} \times S^n/G) \longrightarrow & H^n(S^k \times S^n/G) & \xrightarrow{i^*} H^n(\{*\} \times S^n/G) \end{array}$$

Clearly $H^n(S^k \times S^n/G, \{*\} \times S^n/G) \subset ker(i^*)$. But $H^n(S^k \times S^n/G) = H^n(\{*\} \times S^n/G) \oplus H^{n-k}(\{*\} \times S^n/G)$. Thus for $i^* : H^n(S^k \times S^n/G) \to H^n(\{*\} \times S^n/G)$ we have that $Ker(i^*) = H^{n-k}(\{*\} \times S^n/G)$. Finally, the groups $H^{n-k}(S^n/G)$ are finite for $0 < k < n$ because $H^{n-k}(S^n/G, \mathbb{Q}) = 0$ by the Vietoris-Begle theorem. □

Lemma 2.4. *Let G be a finite group and $S(V)$ a linear G-sphere. For $H < G$ write $n_r(H)$ for the integer such that $S(V^{\oplus r})^H = S^{n_r(H)}$. For all $k > 0$ there is an integer $q > 0$ such that the groups:*

$$H^{n_q(H_i)}(S^k \times S^{n_q(H_i)}/WH_i, \cup_{H>H_i} S^k \times S^{n_q(H)}/WH_i \cup \{\star\} \times S^{n_q(H_i)}/WH_i, \mathbb{Z})$$

are finite for all H_i with $n_1(H_i) > 0$.

Proof. Fix a subgroup $H_i < G$ such that $n_1(H_i) > 0$. If there is $H > H_i$ with $n_1(H) = n_1(H_i)$, then the required cohomology group is zero (it is of the form $H^{n(H_i)}(X, X, \mathbb{Z})$). Assume that for all

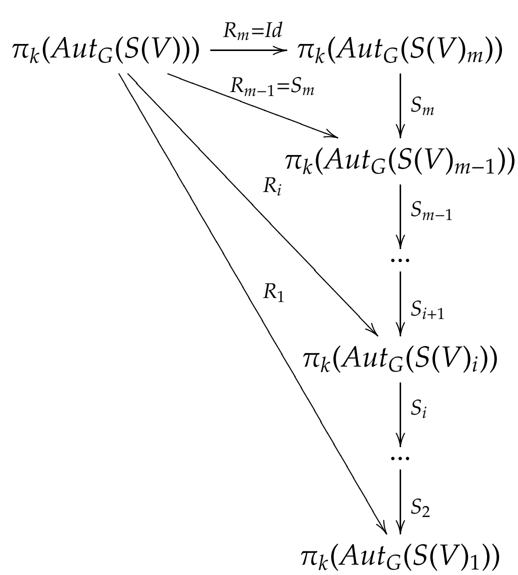
$H > H_i$ we have $n_1(H) < n_1(H_i)$. In this case we want so show that we can take enough direct sums to be in the situation of corollary 2.3.

Let $n_{r,i} = max_{H>H_i}\{n_r(H)\}$ and $m_{r,i} = n_r(H_i) - n_{r,i} > 0$. Observe that $n_r(H) = rn_1(H) + (r-1)$ so that $n_{r,i} = rn_{1,i} + (r-1)$ and $m_{r,i} = n_r(H_i) - n_{r,i} = rn_1(H_i) + (r-1) - (rn_{1,i} + (r-1)) = rm_{1,i}$. Therefore there is a q_i big enough such that $m_{r,i} > k+2$. In other words $n_r(H_i) - k - 2 > n_{r,i}$. We have found an integer $q_i > 0$ such that all the cells τ of the CW-complex $S^{n_{q_i}(H_i)}$ of dimension $dim(\tau) \geq n_{q_i}(H_i) - k - 2$, are also cells of the relative CW-complex $(S^{n_{q_i}(H_i)}, \cup_{H>H_i} S^{n_{q_i}(H)})$.

We turn now our attention to the announced cohomology group. By our condition on the cells of $S^{n_{q_i}(H_i)}$, we have that the cells τ of the CW-complex $S^k \times S^{n_{q_i}(H_i)}/WH_i$ of dimension $dim(\tau) \geq n_{q_i}(H_i) - 2$, are also cells of the relative CW-complex $(S^k \times S^{n_{q_i}(H_i)}/WH_i, \cup_{H>H_i} S^k \times S^{n_{q_i}(H)}/WH_i)$. Henceforth: $H^{n_{q_i}(H_i)}(S^k \times S^{n_{q_i}(H_i)}/WH_i, \cup_{H>H_i} S^k \times S^{n_{q_i}(H)}/WH_i \cup \{\star\} \times S^{n_{q_i}(H_i)}, \mathbb{Z}) = H^{n(H_i)}(S^k \times S^{n_{q_1}(H_i)}/WH_i, \{\star\} \times S^{n_{q_i}(H_i)}/WH_i, \mathbb{Z})$. This last group is finite, by virtue of lemma 2.3. We conclude by observing that we can then set $q = max_{H_i<G}\{q_i\}$. □

Proposition 2.5. *Let G be a finite p-group. Let $S(V)$ be a complex representation G-sphere. For all integer $k \geq 0$ there exists an integer $q > 0$ such that $\pi_k(Aut_G(S(V^{\oplus q})))$ is finite.*

Proof. If $k = 0$, the result has been proven in [12]. Assume that $k > 0$. Before explaining how the proof proceeds, we set up some notation: Choose an ordering of the conjugacy classes of isotropy subgroups $\{(H_1), ..., (H_m)\}$ such that if $(H_j) < (H_i)$ then $i < j$. Consider the filtration $S(V)_1 \subset ... \subset S(V)_m = S(V)$ given by $S(V)_i = \left\{x \in S(V) | \ (G_x) = (H_j); \ j \leq i\right\}$. Observe that we have homomorphisms $R_i : \pi_k(Aut_G(S(V))) \to \pi_k(Aut_G(S(V)_i))$ because $H(I \times S(V)_i) \subset S(V)_i$ for all equivariant $H : I \times S(V) \to S(V)$. Similarly we have homomorphisms $S_i : \pi_k(Aut_G(S(V)_i)) \to \pi_k(Aut_G(S(V)_{i-1}))$. Here is how the proof runs. Look at the commutative diagram:



Clearly to prove that $\pi_k(Aut_G(S(V)))$ is finite is the same as to prove that $Im(R_m)$ is finite. To prove that $Im(R_m)$ is finite, we will show by induction over i that $Im(R_i)$ is finite. Such an induction can be performed by showing that $Im(R_1)$ is finite and that $S_i^{-1}(R_{i-1}(f)) \cap Im(R_i)$ is finite for all i and for all $f \in \pi_k(Aut_G(S(V)))$. This outline can only be carried out up to replacing $S(V)$ with some power $S(V^{\oplus q})$.

We begin by showing that there is $q_1 > 0$ such that $\pi_k(Aut_G(S(V^{\oplus q_1})_1))$ is finite. In particular we will have that $Im(R_1) \subset \pi_k(Aut_G(S(V^{\oplus q_1})_1))$ is finite. For $H < G$ write $n_r(H)$ for the integer such that $S(V^{\oplus r})^H = S^{n_r(H)}$. Observe that $n_r(H) = rn(H) + (r-1)$. By lemma 2.2 we have that $\pi_k(Aut_G(S(V)_1)) = \pi_k(Aut_{WH_1}(S^{n_1(H_1)}))$. The WH_1-action on $S^{n_1(H_1)}$ is free because H_1 is maximal among isotropy subgroups. Therefore proposition 2.4 of [10] says that $\pi_k(Aut_{WH_1}(S^{n_1(H_1)}))$ is finite if $k < n_1(H_1) - 1$. If $k \geq n_1(H_1) - 1$, then there is a $q_1 > 0$ for which $k < q_1 n_1(H_1) + (q_1 - 1) - 1 = n_{q_1}(H_1) - 1$. As a result $\pi_k(Aut_G(S(V^{\oplus q_1})_1)) = \pi_k(Aut_{WH_1}(S^{n_{q_1}(H_1)}))$ is finite (always by proposition 2.4 of [10]).

As explained above, the second and last step is to prove that there is $q \geq q_1$ such that $S_i^{-1}(R_{i-1}(f)) \cap Im(R_i)$ is finite for all i and for all $f \in \pi_k(Aut_G(S(V^{\oplus q})))$. For that purpose we are going to use equivariant obstruction theory a la Tom Dieck (see [27] section 8 and [28] chapter 2). We begin with some preliminaries. As in lemma 2.4, let $q' > 0$ be such that the groups:

$$H^{n_{q'}(H')}(S^k \times S^{n_{q'}(H')}/WH', \cup_{H>H'} S^k \times S^{n_{q'}(H)}/WH' \cup \{\star\} \times S^{n_{q'}(H')}/WH', \mathbb{Z})$$

are finite for all $H' < G$ with $n_1(H') > 0$. Let $q = max\{q_1, q'\}$. To simplify the notation we write $W = V^{\oplus q}$, $X = S^k \times S(W)$ and $\bar{X}^{H_i} = \cup_{H>H_i} X^H \cup \{\star\} \times S(W)^{H_i}$. With this notation we have that the group:

$$H^{n_q(H_i)}(X^{H_i}/WH_i, \bar{X}^{H_i}/WH_i, \pi_{n_q(H_i)}(S^{n_q(H_i)}))$$

is finite by lemma 2.4, while if $r \neq n_q(H_i)$ then the groups:

$$H^r(X^{H_i}/WH_i, \bar{X}^{H_i}/WH_i, \pi_r(S^{n_q(H_i)}))$$

are finite because they are finitely generated torsion abelian groups. (The fixed points of a complex representation spheres are odd-dimensional spheres whose homotopy groups are all but one finite).

To use equivariant obstruction theory we need one last observation: By lemma 2.1 there is an injection $\pi_k(Aut_G(S(W)_i)) \to \left[S^k \times S(W)_i, S(W)_i\right]_G$ yielding a commutative diagram with injective columns:

$$\begin{array}{ccc}
\pi_k(Aut_G(S(W)_i)) & \xrightarrow{S_i} & \pi_k(Aut_G(S(W)_{i-1})) \\
\downarrow{\scriptstyle \varphi_i} & & \downarrow{\scriptstyle \varphi_{i-1}} \\
\left[S^k \times S(W)_i, S(W)_i\right]_G & \xrightarrow[s_i]{} & \left[S^k \times S(W)_{i-1}, S(W)_{i-1}\right]_G
\end{array}$$

As a consequence, to prove that $S_i^{-1}(R_{i-1}(f)) \cap Im(R_i)$ is finite, it is enough to prove that $s_i^{-1}(\varphi_{i-1}(R_{i-1}(f))) \cap \varphi_i(Im(R_i))$ is finite. By abuse of notation we will keep on writing S_i and $R_{i-1}(f)$, but we will think of them as living in the bottom row of the diagram.

We are now in condition of applying equivariant obstruction theory inductively over r to each of the diagrams:

$$\begin{array}{ccc} [Sk_{r+1}(X_i, X_{i-1}), S(W)_i]_G & \xrightarrow{S_{i,r+1}} & [X_{i-1}, S(W)_{i-1}]_G \\ \downarrow{\scriptstyle Sk_r} & \nearrow{\scriptstyle S_{i,r}} & \\ [Sk_r(X_i, X_{i-1}), S(W)_i]_G & & \end{array}$$

If $r = 0$, then $Sk_0(X_i, X_{i-1}) = X_{i-1} \coprod \{x_0, ..., x_l\}$. Consequently, $S_{i,0}^{-1}(R_{i-1}(f)) \cap Sk_0(Im(R_i))$ depends on the connected components of $S(W)_{i-1}$. But $S(W)_{i-1}$ has finitely many connected components because it is a finite CW-complex, therefore $S_{i,0}^{-1}(R_{i-1}(f)) \cap Sk_0(Im(R_i))$ is finite. From now on, to simplify the notation, we are going to write $f_i = R_i(f)$ for all possible i and f. Assume that $S_{i,r}^{-1}(f_{i-1}) \cap Sk_r(Im(R_i)) = \left\{g_{i,r}^1, ..., g_{i,r}^t\right\}$ is finite of order t (i.e. $g_{i,r}^j \neq g_{i,r}^l$ if $j \neq l$). For each $g_{i,r+1} \in S_{i,r+1}^{-1}(f_{i-1}) \cap Sk_{r+1}(Im(R_i))$ there is a unique $g_{i,r}^j$ and a homotopy h from $g_{i,r} = g_i|_{Sk_r(X_i, X_{i-1} \cup \{\star\} \times S(W))}$ to $g_{i,r}^j = g_i^j|_{Sk_r(X_i, X_{i-1} \cup \{\star\} \times S(W))}$.

The crucial observation here is that the homotopy h can be supposed to be constant over the subcomplex $\{\star\} \times S(W)$: if h extends to a homotopy between g_i and g_i^j, then, by lemma 2.1, the homotopy $H : I \times S^k \times S(W)_i \rightarrow S(W)_i$ between g_i and g_i^j can be taken to satisfy $H(t, \star, y) = H(t', \star, y)$ for all $t, t' \in I$ and $y \in S(W)_i$.

We write $d(g_{i,r}, h, g_{i,r}^j) \in H^{r+1}(X^{H_i}/WH_i, \bar{X}^{H_i}/WH_i, \pi_{r+1}((S^n)^{H_i}))$ for the difference cocycle as in [27] section 8 and [28] chapter 2 (see also [24]). Notice that $d(g_{i,r}, h, g_{i,r}^j)$ is a cocycle because $g^j, g \in \pi_k(Aut_G(S(W)))$ (see lemma 3.14 in [28]). The properties of the difference cocycle are given in 3.13 of chapter 2 of [28]. In particular we have that, if $d(g'_{i,r}, h', g_{i,r}^j) = d(g_{i,r}, h, g_{i,r}^j)$, then $d(g'_{i,r}, h' + h^{-1}, g_{i,r}) = d(g'_{i,r}, h', g_{i,r}^j) + d(g_{i,r}^j, h^{-1}, g_{i,r}) = d(g'_{i,r}, h', g_{i,r}^j) - d(g_{i,r}, h, g_{i,r}^j) = 0$, so that $g_{i,r+1} \simeq g'_{i,r+1}$. We can therefore define an injection:

$$d : S_{i,r+1}^{-1}(f_{i-1}) \cap Sk_{r+1}(Im(R_i)) \rightarrow \coprod_{j=1}^{t} \left\{(g_{i,r}^j)\right\} \times H^{r+1}(X^{H_i}/WH_i, \bar{X}^{H_i}/WH_i, \pi_{r+1}(S^{n_q(H_i)}))$$

by setting $g_i \mapsto \left\{g_i^j\right\} \times d(g_{i,r}, h, g_{i,r}^j)$. Since we chose the integer q in order to have all the cohomology groups on the right hand side to be finite, we must have that the left hand side is finite as well.

Summarizing, by induction we have that $S_{i,r+1}^{-1}(f_{i-1}) \cap Sk_{r+1}(Im(R_i))$ is finite for all r. Since X is finite dimensional, this shows that $S_i^{-1}(f_{i-1})$ is finite. We conclude as explained in the outline at the beginning of this proof. $\square$

We introduce next the following notation:

Notation 2.6. *Let G be a finite group and X a G-CW-complex. We write:*

$$rk_X(G) = max\,\{n \in \mathbb{N}|\ there\ exists\ \ G_\sigma\ \ with\ \ rk(G_\sigma) = n\}$$

We can now state the main result of the section:

Proposition 2.7. *Let G be a finite group and let X be a finite dimensional G-CW-complex. Assume that to each isotropy subgroup G_σ we can associate a representation $\rho_\sigma : G_\sigma \to U(n)$ such that $\rho_\sigma|_{G\tau} \cong \rho_\tau$ whenever $G_\tau < G_\sigma$. If ρ_σ is fixed point free for all G_σ with $rk(G_\sigma) = rk_X(G)$, then there exists a finite dimensional G-CW-complex $E \cong X \times S^m$ with $rk_E(G) = rk_X(G) - 1$. Moreover, if X is finite then E is finite as well.*

Proof. The proof follows [10]. We refer the reader to [29] for the details. Write S^{2n-1}_σ for the linear sphere associated to ρ_σ. We want to glue these spheres into a G-equivariant spherical fibration over X. We will proceed by induction over the skeleton of X. For every G-orbit of the 0-skeleton, choose a representant σ and define a map $G \times_{G_\sigma} S^{2n-1}_\sigma \to X^0$ by $(g, x) \mapsto g \cdot \sigma$. This defines a G-equivariant spherical fibration $S^{2n-1} \to E_0 \to Sk_0(X)$ whose total space is a finite dimensional G-CW-complex. Clearly if ρ_σ is fixed point free for all G_σ with $rk(G_\sigma) = rk_X(G)$, then $rk(E_0) = rk_X(G) - 1$.

The inductive step is next. Suppose given a G-equivariant spherical fibration over the $(k-1)$-skeleton $*^{q_{k-1}} S^{2n-1} \to E_{k-1} \to Sk_{k-1}(X)$ whose total space is a finite dimensional G-CW-complex. Assume also that if ρ_σ is fixed point free for all G_σ with $rk(G_\sigma) = rk_X(G)$, then $rk(E_{k-1}) = rk_X(G)-1$. Now, for every G-orbit of a k-cell, choose a representative σ. The G_σ-equivariant fibration $*^{q_{k-1}} S^{2n-1} \to E_{k-1}|_{\partial\sigma} \to \partial\sigma$ is classified by an element $a_\sigma \in \pi_{k-2}(Aut_{G_\sigma}(*^{q_{k-1}} S^{2n-1}))$.

We want to have $a_\sigma = 0$: Observe that, in general, for two complex G-spheres $S(V)$ and $S(W)$, we have that $S(V \oplus W) \cong S(V) * S(W)$ as G-spheres. Therefore, by lemma 2.5, we can take enough Whitney sums of the fibration $*^{q_{k-1}} S^{2n-1} \to E_{k-1} \to Sk_{k-1}(X)$ to guarantee that $a_\sigma = 0$ (see lemma 2.3 and proposition 2.4 in [10]). We can then extend the G_σ-equivariant fibration $*^{q_k} S^{2n-1} \to E_{k-1}|_{\partial\sigma} \to \partial\sigma$ equivariantly across the cell σ. We define a G-equivariant spherical fibration over the orbit of σ by $G \times_{G_\sigma} *^{q_k} S^{2n-1}_\sigma \to G\sigma$ with $(g, x) \mapsto g \cdot \sigma$.

Repeating the procedure for all the representatives of the G-orbits of k-cells, we recover a G-equivariant spherical fibration $*^{q_k} S^{2n-1} \to E_k \to Sk_k(X)$ with total space a finite dimensional G-CW-complex. Clearly if ρ_σ is fixed point free for all G_σ with $rk(G_\sigma) = rk_X(G)$, then $rk(E_k) = rk_X(G) - 1$. We conclude noticing that, by proposition 2.8 in [2], up to taking further fiber joins, we can assume that the total fibration $*^q S^{2n-1} \to E \to X$ is a product one.

For the last statement, one can observe that all the constructions take place in the category of finite CW-complexes, providing that the initial space X is a finite CW-complex. □

3. Rank 3 p-groups (p odd).

The results of this section have also been announced by Jackson in [16]. We give a proof which uses the group theory developed there. For the convenience of the reader we reproduce it here.

Lemma 3.1. *If G is a finite p-group with $rk(G) = 3$ and $rk(Z(G)) = 1$, then there exists a normal abelian subgroup $Q < G$ of type (p, p) with $Q \cap Z(G) \neq 0$.*

Proof. This is proven by Suzuki in [25], section 4. □

Proposition 3.2. *Let G be a finite p-group with $p > 2$, $rk(G) = 3$ and $rk(Z(G)) = 1$. Let Q be an abelian normal subgroup of type (p, p) as above. Suppose that $H < G$ with $H \cap Z(G) = 0$ and $|H| = p^n$. Then either H is cyclic, $H < C_G(Q)$, H is abelian of type (p, p^{n-1}) or $H \cong M(p^n) =< x, y | x^{p^{n-1}} = y^p = 1, y^{-1}xy = x^{1+p^{n-2}} >$.*

Proof. If $rk(H) = 1$ then H is cyclic since $p > 2$. Suppose that $rk(H) = 2$ and $H \cap Q \neq 0$. By assumption $Z(G) \cap Q = \mathbb{Z}/p$, $H \cap Z(G) = 0$ and $Q \cap H = \mathbb{Z}/p$. The map $c : H \to Aut(H \cap Q)$ given by $c_h(x) = hxh^{-1}$ is well defined because Q is normal (by lemma 3.1). Since $|H| = p^n$ and $|Aut(H \cap Q)| = p - 1$, we have that the map c is trivial. As a result we have that $H < C_G(Q)$.

Assume now that $H \cap Q = 0$. In this case $H \cap C_G(Q) \neq H$ since otherwise we would have $rk(G) > 3$. Set $K = H \cap C_G(Q)$ and observe that K is cyclic (else we would have $rk(G) > 3$). Assume for a moment that $[G : C_G(Q)] = p$. In this case $[H : K] = p$, in other words H has a maximal cyclic subgroup. By [25] section 4, H needs then to be abelian of type (p, p^{n-1}) or $M(p^n)$.

We still have to prove that $[G : C_G(Q)] = p$. The group G acts on Q by conjugation and for each element q of Q not in center of G, we have that $G_q = C_G(Q)$. As a result $|C_G(Q)| = |G_q| = |G|/p$ since $Q \cong (Z/p)^2$ with the first coordinate in the center $Z(G)$. □

Proposition 3.3. *Let G be a finite p-group with $p > 2$, $rk(G) = 3$ and $rk(Z(G)) = 1$. There exists a class function $\beta : G \to \mathbb{C}$ such that for any subgroup $H \subset G$, with $H \cap Z(G) = 0$, the restriction $\beta|_H$ is a complex character of H. If in addition H is a rank two elementary abelian subgroup, then the character $\beta|_H$ corresponds to an isomorphism class of fixed-point free representations.*

Proof. Define $\beta : G \to \mathbb{C}$ as follows:

$$x \mapsto \begin{cases} (p^2 - p)|G|, & \text{if } x = 0; \\ 0, & \text{if } x \in Z(G) \setminus 0; \\ -p|G|, & \text{if } x \in Q \setminus Z(G); \\ 0, & \text{if } x \in C_G(Q) \setminus Q; \\ -|G|, & \text{if } x \in G \setminus C_G(Q) \text{ of order p;} \\ 0 & \text{if } x \in G \setminus C_G(Q) \text{ of order greater than p.} \end{cases}$$

The map β is a class function because we have the following sequence of subgroups each normal in G:

$$0 < Z(G) < Q < C_G(Q) < G.$$

Consider first an elementary abelian subgroup H of G of rank 2 and which intersects trivially the center $Z(G)$. If $H \cap Q \neq 0$ then:

$$\beta|_H = |G| \sum_{i=0}^{p-1} \sum_{j=1}^{p-1} \phi_i \phi_j$$

where ϕ_k is the k-th irreducible character of $\mathbb{Z}/p$. If $H \cap Q = 0$ then:

$$\beta|_H = (p-1)|G|/p \left(\sum_{i=0}^{p-1} \sum_{j=1}^{p-1} \phi_i \phi_j \right) + |G| \sum_{i=1}^{p-1} \phi_i \phi_0.$$

Consider now a subgroup H of G with $H \cap Z(G) = 0$. We will proceed case by case using the classification above.

(1) If $H \cap Q \neq 0$ then $H \subset C_G(Q)$ and $|K| = p$ with $K = Q \cap H$. Let ϕ be the character of K which is $p-1$ on the identity and -1 for each other element of K. Then:

$$\beta|_H = \frac{p|G|}{|H:K|} Ind_K^H \phi.$$

(2) If $H \cap Q = 0$ and $H \subset C_G(Q)$ then H is cyclic and $\beta|_H = |G|/|H|(p^2 - p)\phi$ where ϕ is the character of H that is $|H|$ on the identity and 0 elsewhere.

(3) If $H \cap Q = 0$ and H is cyclic with $H \cap C_G(Q) = 0$, then $|H| = p$ and $\beta|_H = |G|/p\phi$ where ϕ is $(p^3 - p^2)$ on the identity and $-p$ elsewhere.

(4) Assume that $H \cap Q = 0$ and that H is abelian of type (p, p^{n-1}). Write $H =< x, y \in H | x^p = y^{p^{n-1}} = 1, [x, y] = 1 >$. Notice that $< y >= H \cap C_G(Q)$. For each $1 \leq i \leq p-1$ set $H_i =< xy^{ip^{n-2}} >$. Clearly $|H_i| = p$, $H_i \cap C_G(Q) = 0$ and $H_i \cap H_j = 0$ if $i \neq j$.

Let ϕ_i be the character of H_i which is $p-1$ on the identity and -1 elsewhere. Set:

$$\phi = \sum_{i=1}^{p-1} Ind_{H_i}^H \phi_i.$$

Since $\phi(1) = |H|(p-1)$, $\phi(z) = -|H|/p$ for $z \in H \backslash < y >$ and $\phi(z) = 0$ for $z \in < y >$; we conclude that $\beta|_H = p|G|/|H|\phi$.

(5) If $H \cap Q = 0$ and $H \cong M(p^n)$, we can write $H =< x, y | x^{p^{n-1}} = y^p = 1, y^{-1}xy = x^{1+p^{n+2}} >$. Let $N =< x^{p^{n-2}}, y >\cong (\mathbb{Z}/p)^2$ which is normal in H. Let ϕ be the character of H which is $p-1$ on the identity and such that $\phi(x^i) = -1$ for all $1 \leq i \leq p-1$ and $\phi(z) = 0$ when $z \in N \backslash < y >$. Then $\beta|_H = Ind_N^H p|G|/|H:N|\phi$.

□

We can now turn our attention to the topological problem:

Proposition 3.4. *For every odd order rank 3 p-group G, there is a finite dimensional G-CW-complex* $X \simeq S^m \times S^n$ *with cyclic isotropy subgroups.*

Proof. If $Z(G)$ is not cyclic, then it is enough to consider the linear spheres of representations of G induced from free representations of some $\mathbb{Z}/p \times \mathbb{Z}/p < Z(G)$.

Assume that $Z(G)$ is cyclic and let S^m be the linear G-sphere obtained by inducing from a free linear action of $Z(G)$. The isotropy subgroups for this action are the one described in proposition 3.2. The conditions of proposition 2.7 are fulfilled by proposition 3.3. The conclusion follows. □

As a direct consequence of theorem 3.2 in [2] we obtain:

Theorem 3.5. *For every odd order rank 3 p-group G, there is a free finite G-CW-complex* $X \cong S^m \times S^n \times S^k$.

Note that a converse to theorem 3.5 is given by Hanke in [15] in the sense that: if $(\mathbb{Z}/p)^r$ acts freely on $X = S^{n_1} \times ... \times S^{n_k}$ and if $p > 3dim(X)$, then $r \leq k$.

Remark. For $p = 2$ the situation is more complicated because of the classification of subgroups. A 2-group of rank 1 can be either cyclic or generalized quaternion. A 2-group with a maximal abelian subgroup can be cyclic, generalized quaternion, dihedral, $M(2^n)$ (see proposition 3.2) or $S_{4m} =< x, y | x^{2m} = y^2 = 1, y^{-1}xy = x^{m-1} >$ (see [25] section 4 chapter 4). For $p = 2$, the class function of proposition 3.3 does not restrict to characters over the subgroups, in general.

4. Abelian isotropy p-groups.

We begin by generalizing theorem 3.2 in [2] for p-groups.

Theorem 4.1. *Let G be a finite p-group and let X be a finite dimensional G-CW-complex with* G_σ *abelian for all cells* $\sigma \subset X$. *Then there is a free finite dimensional G-CW-complex* $Y \simeq X \times S^{n_1} \times ... \times S^{rk_X(G)}$. *Moreover, if X is finite, then Y is finite as well.*

Proof. We prove the theorem by induction over $rk_X(G)$. If $rk_X(G) = 1$, the theorem has been proven by Adem and Smith (3.2 in [2]).

The inductive step follows. By virtue of proposition 2.7, we only need to associate to each isotropy subgroup G_σ a representation $\rho_\sigma : G_\sigma \to U(m)$ such that $\rho_\sigma|_{G\tau} \cong \rho_\tau$ whenever $G_\tau < G_\sigma$ and such that ρ_σ is fixed point free for all G_σ with $rk(G_\sigma) = rk_X(G)$.

Consider the class function $\beta : G \to \mathbb{C}$ given by:

$$x \mapsto \begin{cases} |G|(p^{rk_X(G)} - 1), & \text{if } x = 0; \\ -|G|, & \text{if } o(x) = p; \\ 0, & \text{otherwise.} \end{cases}$$

To simplify the notation write $A = G_\sigma$ for an isotropy subgroup (which is abelian by hypothesis). We need to prove that $\beta|_A$ is a character which is fixed point free whenever $A \cong (\mathbb{Z}/p)^{rk_X(G)}$. Set $A_p = \{0\} \cup \{x \in A | o(x) = p\}$. Since A is abelian we have $A_p \triangleleft A$. Fix an injection $f : A_p \to (\mathbb{Z}/p)^{rk_X(G)}$. Write $\rho_0 : (\mathbb{Z}/p)^{rk_X(G)} \to U(p^{rk_X(G)} - 1)$ for the reduced regular representation and let $\rho = \rho_0 \circ f$ be the representation $A_p \to (\mathbb{Z}/p)^{rk_X(G)} \to U(p^{rk_X(G)} - 1)$. Consider finally the representation of

A given by $\eta = |G||A_p|/|A|Ind_{A_p}^{A}\rho$. Clearly $\eta(0) = |G|(p^{rk_X(G)} - 1)$, $\eta(x) = -|G|$ if $x \in A_p \setminus 0$ while $\eta(x) = 0$ if $x \notin A_p$. As a result $\beta|_A = \eta$. Let now $A \cong (\mathbb{Z}/p)^{rk_X(G)}$. Clearly $\beta|_A$ is a multiple of the reduced regular representation, thus fixed point free. □

Corollary 4.2. *Let G be a p-group. Assume that G is a central extension of abelians, then there is a free finite G-CW-complex $X \simeq S^{n_1} \times ... \times S^{n_{rk(G)}}$. The result in particular holds for extraspecial p-groups.*

Proof. Let $X = S^{n_1} \times ... \times S^{n_{rk(Z(G))}}$ be the product of the G-spheres arising from suitable representations of the center. Clearly $rk_X(G) = rk(G) - rk(Z(G))$ and G_σ is abelian. The conclusion follows. □

5. Infinite groups.

As pointed out in [10], there is a class of infinite groups which is worth considering, when studying the rank conjecture mentioned in the introduction (which is usually stated for finite groups). This is the class of groups Γ of finite virtual cohomological dimension. Recall that, by definition, a group Γ has finite virtual cohomological dimension, if it has a finite index subgroup $\Gamma' < \Gamma$ with finite cohomological dimension (that is to say: $H^n(\Gamma', M) = 0$ for all coefficients M and for all n big enough). Writing *vcd* for virtual cohomological dimension and *cd* for cohomological dimension, one can show that the number $vcd(\Gamma) = cd(\Gamma')$ is well defined. See for example [8] for background on groups with finite virtual cohomological dimension. The crucial property that makes them interesting to us is the following: for any such group Γ there exists a finite dimensional Γ-CW-complex $\mathfrak{E}\Gamma$ with $|\Gamma_x| < \infty$ for all $x \in \mathfrak{E}\Gamma$.

It is already known that a group with finite virtual cohomological dimension, which is countable and with rank 1 finite subgroups, acts freely on a finite dimensional CW-complex $X \simeq S^m$ [10]. The next step would be to prove the analogue result for groups Γ with rank 2 finite subgroups. The easiest examples to consider are amalgamated products $\Gamma = G_1 *_{G_0} G_2$, where G_i is a finite group for $i = 0, 1, 2$ and $G_0 < G_i$ for $i = 1, 2$. In this case, for every finite subgroup $H < \Gamma$, there is $\gamma \in \Gamma$ such that $\gamma H \gamma^{-1} < G_i$ for $i = 1$ or $i = 2$ (see [22]). In particular $rk(\Gamma) = max\{rk(G_1), rk(G_2)\}$. The first attempt would be to find an effective Γ-sphere, i.e. a Γ-sphere with rank 1 isotropy subgroups. In the first subsection we exhibit an amalgamation of two p-groups which doesn't have an effective Γ-sphere.

Recall from [5] that, for a finite group G, we have that $rk(G) = r$ if and only if there are r finite dimensional $\mathbb{Z}[G]$-complexes $\mathbf{C}_1, ..., \mathbf{C}_r$ such that $\mathbf{C} = \mathbf{C}_1 \otimes ... \otimes \mathbf{C}_r$ is a complex of projective $\mathbb{Z}[G]$-modules with $H^*(\mathbf{C}) \cong H^*(S^{n_1} \times ... \times S^{n_r})$. In the second subsection we prove a similar result: for every group Γ with $vcd(\Gamma) < \infty$, there is a finite dimensional, contractible $\mathbb{Z}[\Gamma]$-complex $\mathbf{C}$ and $rk(\Gamma)$ finite dimensional $\mathbb{Z}[\Gamma]$-complexes $\mathbf{C}_1, ..., \mathbf{C}_{rk(\Gamma)}$ such that $\mathbf{D} = \mathbf{C} \otimes \mathbf{C}_1 \otimes ... \otimes \mathbf{C}_{rk(\Gamma)}$ is a complex of projective $\mathbb{Z}[\Gamma]$-modules with $H^*(\mathbf{D}) \cong H^*(S^{n_1} \times ... \times S^{n_{rk(\Gamma)}})$. As a result, the group Γ introduced in the first subsection here below satisfies the algebraic analogue of the rank conjecture but doesn't have an effective Γ-sphere. The geometric problem of knowing whether or not Γ acts freely on a product of two spheres is still open.

5.1. **A group without effective action on a sphere.** Let E and E' be two copies of the extraspecial p-group of order p^3 and exponent p. (Such a group can be identified with the upper triangular 3×3 matrices over $\mathbb{F}_p$ with 1 on the diagonal). Consider the amalgamated product $\Gamma = E' *_{\mathbb{Z}/p} E$ given by $\mathbb{Z}/p = Z(E)$ and an injective map $f : \mathbb{Z}/p \to E$ with $f(\mathbb{Z}/p) \cap Z(E') = 1$. Clearly $rk(\Gamma) = 2$. Let Γ act on a finite dimensional CW-complex $X \simeq S^n$. Consider the restriction of this action to E and E'. It is well known that the dimension function of a p-group action on a sphere is realized by a representation over the real numbers [11]. Therefore, an even multiple of the dimension functions for E and E' must be realized by characters χ_E and $\chi_{E'}$.

Clearly the dimension functions of χ_E and $\chi_{E'}$ must agree over $Z(E)$ and $f(\mathbb{Z}/p)$. Looking at the character table of E, we observe that every irreducible character α, giving rise to an effective sphere, vanishes outside $Z(E)$ while $\alpha(z) = m\zeta_p$ for all $z \in Z(E) \setminus \{0\}$ (here ζ_p is a p-root of the unity). Thus, χ_E and $\chi_{E'}$ cannot be both characters giving rise to effective spheres. We deduce that the original action must have some finite isotropy subgroups of rank 2. This provides an example of an infinite group, with rank 2 finite p-subgroups, not acting with effective Euler class on any sphere.

5.2. **Algebraic spheres.** Let Γ be a group with $vcd(\Gamma) < \infty$ and rank r. As announced in the introduction of section 5, we want to show that there is a finite dimensional, contractible $\mathbb{Z}[\Gamma]$-complex $\mathbf{C}$ and $rk(\Gamma)$ finite dimensional $\mathbb{Z}[\Gamma]$-complexes $\mathbf{C}_1, \dots, \mathbf{C}_r$ such that $\mathbf{C} = \mathbf{C}_1 \otimes \dots \otimes \mathbf{C}_r$ is a complex of projective $\mathbb{Z}[\Gamma]$-modules with $H^*(\mathbf{C}) \cong H^*(S^{n_1} \times \dots \times S^{n_r})$. We begin by recalling some preliminaries concerning the cohomology of finite groups. We follow here [6] and [7]. Let G be a finite group. Consider $\zeta \in H^n(G, R) \cong Ext^n_{RG}(R, R) \cong Hom_{RG}(\hat{\Omega}^n R, R)$, where $\hat{\Omega}^n R$ is the nth kernel in a RG-projective resolution $\mathbf{P}$ of R. We choose a cocycle $\hat{\zeta} : \hat{\Omega}^n R \to R$ representing ζ. By making $\mathbf{P}$ large enough we can assume that $\hat{\zeta}$ is surjective. We denote L_ζ its kernel and form the pushout diagram:

$$\begin{array}{ccccccccccccccc}
 & & L_\zeta & \xrightarrow{=} & L_\zeta & & & & & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & & & & & \\
0 & \longrightarrow & \hat{\Omega}^n R & \longrightarrow & P_{n-1} & \longrightarrow & P_{n-2} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & R & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \hat{\zeta}} & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & R & \longrightarrow & P_{n-1}/L_\zeta & \longrightarrow & P_{n-2} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & & & & & & & \\
 & & 0 & & 0 & & & & & & & & & &
\end{array}$$

We denote by $\mathbf{C}_\zeta$ the chain complex:

$$0 \to P_{n-1}/L_\zeta \to P_{n-2} \to ... \to P_0 \to R \to 0$$

formed by truncating the bottom row of this diagram. Thus we have that $H_0(\mathbf{C}_\zeta) = H_{n-1}(\mathbf{C}_\zeta) = R$ while $H_i(\mathbf{C}_\zeta) = 0$ if $i \neq 0, n-1$. A useful result is given in the proof of theorem 3.1 in [3]:

Proposition 5.1. *Let G be a finite group. For all positive integer r, there exist classes $\xi_1, ..., \xi_r \in H^*(G, \mathbb{Z})$ such that, for all $H < G$ with $rk(H) \leq r$, the complex $\mathbb{Z}[G/H] \otimes L_{\xi_1} \otimes ... \otimes L_{\xi_r}$ is $\mathbb{Z}[G]$-projective.*

Proof. See the proof of theorem 3.1 in [3] □

Corollary 5.2. *Let G be a finite group. For all positive integer r, there exist r finite dimensional $\mathbb{Z}[G]$-complexes $\boldsymbol{C}_1, ..., \boldsymbol{C}_r$ such that $H^*(\boldsymbol{C}_1 \otimes ... \otimes \boldsymbol{C}_r) = H^*(S^{n_1} \times ... \times S^{n_r})$; with $\boldsymbol{C}_1 \otimes ... \otimes \boldsymbol{C}_r$ a complex of $\mathbb{Z}[H]$-projective modules for all $H < G$ with $rk(H) \leq r$.*

Proof. Let $\xi_1, ..., \xi_r \in H^*(G, \mathbb{Z})$ be the classes given in proposition 5.1. Consider the chain complex $\mathbf{C} = \mathbf{C}_{\xi_1} \otimes ... \otimes \mathbf{C}_{\xi_r}$. Clearly $H^*(\mathbf{C}) = H^*(S^{n_1} \times ... \times S^{n_r})$. For the second part of the claim, observe that all the modules in $\mathbf{C}_{\xi_i}$ are $\mathbb{Z}[G]$-projective except the module P_{n_i-1}/L_{ξ_i}. Recall that the tensor product of any module with a projective module is projective, so that it remains to examine the module $P_{n_1-1}/L_{\xi_1} \otimes ... \otimes P_{n_r-1}/L_{\xi_r}$. Let $H < G$ be such that $rk(H) \leq r$. Since $\mathbb{Z}[G/H] \otimes L_{\xi_1} \otimes ... \otimes L_{\xi_r}$ is $\mathbb{Z}[G]$-projective by proposition 5.1, we conclude that $\mathbb{Z}[G/H] \otimes P_{n_1-1}/L_{\xi_1} \otimes ... \otimes P_{n_r-1}/L_{\xi_r}$ is $\mathbb{Z}[G]$-projective as in 5.14.2 of [7]. It then easily follows that $P_{n_1-1}/L_{\xi_1} \otimes ... \otimes P_{n_r-1}/L_{\xi_r}$ is $\mathbb{Z}[H]$-projective. □

We go back now to our group Γ with $vcd(\Gamma) < \infty$ and rank r. Write Γ' for a torsion-free normal subgroup of Γ with $G = \Gamma/\Gamma'$ finite. Write also $\mathfrak{E}\Gamma$ for a contractible finite dimensional proper Γ-CW-complex. Here proper means that the isotropy subgroups are finite. We apply corollary 5.2 to Γ/Γ' with $r = rk(\Gamma)$. We recover a $\mathbb{Z}[\Gamma]$-complex $\mathbf{C} = \mathbf{C}_1 \otimes ... \otimes \mathbf{C}_r$ such that $H^*(\mathbf{C}_1 \otimes ... \otimes \mathbf{C}_r) = H^*(S^{n_1} \times ... \times S^{n_r})$; with $\mathbf{C}_1 \otimes ... \otimes \mathbf{C}_r$ a complex of $\mathbb{Z}[H]$-projective modules for all finite $H < \Gamma$. On the other hand, we have the contractible $\mathbb{Z}\Gamma$-complex $C_*(\mathfrak{E}\Gamma)$. Therefore the complex $\mathbf{D} = C_*(\mathfrak{E}\Gamma) \otimes \mathbf{C}$ is such that $H^*(\mathbf{D}) = H^*(S^{n_1} \times ... \times S^{n_r})$.

Lemma 5.3. *With the notation above, the complex $\boldsymbol{D}$ is $\mathbb{Z}[\Gamma]$-projective.*

Proof. The complex $C_*(\mathfrak{E}\Gamma)$ decomposes as a direct sum of permutation modules: $C_*(\mathfrak{E}\Gamma) = \oplus_\sigma \mathbb{Z}[\Gamma/\Gamma_\sigma] = \oplus_\sigma \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_\sigma]} \mathbb{Z}$. Here σ spans the cells of $\mathfrak{E}\Gamma/\Gamma$. Consequently $\mathbf{D} = \oplus_\sigma(\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_\sigma]} \mathbf{C}_1 \otimes ... \otimes \mathbf{C}_r)$, so that we only need to prove that $\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_\sigma]} \mathbf{C}_1 \otimes ... \otimes \mathbf{C}_r$ is $\mathbb{Z}[\Gamma]$-projective. Let Q_σ be a $\mathbb{Z}[\Gamma_\sigma]$-module such that $(\mathbf{C}_1 \otimes ... \otimes \mathbf{C}_r) \oplus Q_\sigma$ is $\mathbb{Z}[\Gamma_\sigma]$-free. We then have that $(\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_\sigma]} \mathbf{C}_1 \otimes ... \otimes \mathbf{C}_r) \oplus (\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_\sigma]} Q_\sigma) = \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_\sigma]} ((\mathbf{C}_1 \otimes ... \otimes \mathbf{C}_r) \oplus Q_\sigma)$ is $\mathbb{Z}[\Gamma]$-free.

□

We summarize the main result of this subsection in the following:

Corollary 5.4. *For a group Γ with $vcd(\Gamma) < \infty$ and $rk(\Gamma) = r$, there exist a finite dimensional contractible complex $\boldsymbol{C}$ and r finite dimensional $\mathbb{Z}[\Gamma]$-complexes $\boldsymbol{C}_1,...,\boldsymbol{C}_r$ such that $\boldsymbol{D} = \boldsymbol{C} \otimes \boldsymbol{C}_1 \otimes ... \otimes \boldsymbol{C}_r$ is a $\mathbb{Z}[\Gamma]$-projective complex with $H^*(\boldsymbol{D}) \cong H^*(S^{n_1} \times ... \times S^{n_r})$.*

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