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Majorana meets Coxeter: Non-Abelian Majorana Fermions and Novel Non-Abelian Statistics

Shigehiro Yasui,[1] Kazunori Itakura,[1,2] and Muneto Nitta[3]

[1]*KEK Theory Center, Institute of Particle and Nuclear Studies, High Energy Accelerator Research Organization (KEK), 1-1 Oho, Tsukuba, Ibaraki 305-0801, Japan*

[2]*Department of Particle and Nuclear Studies, Graduate University for Advanced Studies (SOKENDAI), 1-1 Oho, Tsukuba, Ibaraki 305-0801, Japan*

[3]*Department of Physics, and Research and Education Center for Natural Sciences, Keio University, 4-1-1 Hiyoshi, Yokohama, Kanagawa 223-8521, Japan*

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Abstract

Statistics of particles is one of the most fundamental aspects of quantum systems. Majorana fermions in vortices on a two-dimensional plane obey novel statistics called non-Abelian statistics. Non-Abelian statistics leads to the mixing of particle states, which will enable quantum computation. It has been known that non-Abelian statistics exists in a system of 'Abelian' vortices, in which zero-energy Abelian Majorana fermions are trapped. In this article, we discuss vortices possessing a 'non-Abelian' internal symmetry; these vortices are called non-Abelian vortices and have the zero-energy non-Abelian Majorana fermions inside them. Considering the system of multiple non-Abelian vortices, we derive a novel non-Abelian statistics that is different from the previously derived non-Abelian statistics. This new non-Abelian statistics is given by a product of two different groups, namely the non-Abelian statistics obeyed by the Abelian Majorana fermions and the Coxeter group. The Coxeter group describes reflections and induces the geometry of polytopes in a high-dimensional space. As the simplest non-Abelian symmetry, we consider the $SO(3)$ group with its vector representation, and we concretely present the Coxeter group and its geometry in the high-dimensional Hilbert space spanned by non-Abelian Majorana fermions.

I. INTRODUCTION

Majorana fermions proposed by Ettore Majorana in the 20th century have a peculiar property in that an antiparticle is equivalent to a particle [1]. His original conjecture that neutrinos might be such particles has been rejected; however, Majorana fermions are now attracting much attention in condensed matter physics [2]. It has been recently recognized that Majorana fermions with exact zero energy are trapped inside the core of half-quantized vortices in chiral p-wave superconductors or p-wave superfluids and that they appear on the edge of topological superconductors/insulators [3]. A system of multiple vortices provides an exchange statistics that is different from that of bosons, fermions and anyons. Such statistics, called 'non-Abelian statistics' is mathematically described by the braid group [4, 5]. More precisely, one Dirac fermion is defined by a set of two Majorana fermions trapped in two different vortices, and the Hilbert space is constructed from these Dirac fermions [5]. The dimension (2^m) of the Hilbert space increases exponentially with an increase in the even number ($2m$) of vortices. Consequently, non-Abelian statistics provides one possible candidate for quantum computation [6], and hence, p-wave superconductors or superfluids may be used as a device for quantum computers. This is why Majorana fermions and non-Abelian statistics have attracted the attention of many researchers in recent years [2, 6]. Recently, it has been proposed that even in three dimensions, non-Abelian statistics is realized by Majorana fermions trapped on monopole-like objects [7].

Though the exchange statistics discussed above is non-Abelian, Majorana fermions on vortices or other defects studied thus far are all 'Abelian' in the sense that only a single Majorana fermion is trapped in each defect. On the other hand, if multiple Majorana fermions are trapped inside a defect and can continuously mix with each other, one may be able to interpret them as a single 'non-Abelian Majorana fermion', namely a Majorana fermion having a non-Abelian internal symmetry. Then, these fermions must belong to representations of the underlying Lie group.

In this article, we show that non-Abelian Majorana fermions obey a novel non-Abelian exchange statistics. Since Majorana fermions are real (*i.e.* not complex) fields, their group representation must also be real. The simplest of such representation is the vector representation, triplet, of the $SO(3)$ Lie group. We explicitly construct the non-Abelian exchange statistics of non-Abelian Majorana fermions belonging to the vector representation of $SO(3)$. We not only show that the Hilbert space for $2m$ vortices has dimension 2^{3m} much larger than 2^m of Abelian fermions, but also show that it contains a new component. In addition to the non-Abelian statistics already derived by Ivanov [5], we find another structure, *i.e.* the Coxeter group [8, 9]: the entire non-Abelian statistics is a direct product of these two. The Coxeter group is a symmetry group of higher-dimensional generalization of polytopes such as a triangle or a tetrahedron, which was found by Harold Scott MacDonald 'Donald' Coxeter, one of the great Mathematicians in the 20th century. The large Hilbert space spanned by non-Abelian Majorana fermions contains high-dimensional internal spaces of various representations of $SO(3)$, not only singlets and triplets but also quintets and higher representations in general, where the Coxeter group acts on them to exchange multiple states in the same representations.

One question arising immediately may be whether there exist physical systems realizing such non-Abelian Majorana fermions in reality. The answer is yes, such systems probably exist in the universe, *i.e.* in quark matter at extremely high density in neutron stars or quark stars, which are expected to exhibit the so-called 'colour superconductivity' [10, 11]. These stars rotate rapidly, and consequently, stable vortices are created; these vortices are non-Abelian vortices with colour magnetic fluxes confined inside them [12]. We have shown in our previous paper [13] that non-Abelian Majorana fermions of an $SO(3)$ triplet indeed exist in the core of a non-Abelian vortex. The origin of this $SO(3)$ group is the 'colour-flavour locked' symmetry $SU(3)_{\rm C+F}$ in the ground state of a colour superconductor [10, 11], which is spontaneously broken down to its subgroup $SU(2)\times U(1)$ in the core of a vortex [14]. Since there remains an unbroken symmetry $SU(2)\sim SO(3)$ inside the core, the zero modes trapped in it must belong to the representations of $SO(3)$. We have found triplet and singlet Majorana fermions [13]; however, only the triplet is a new object to be considered in this article. Theoretically, fermionic modes in vortices are treated by the Bogoliubov-de Gennes equation, which is an equation for fermions coupled to the vortex profile. For example, the particle (φ) and hole (η) components of the triplet zero modes (chirally right-handed) are approximately given, in two-component Weyl spinor representation, as

$$\varphi(r,\theta) = \mathrm{e}^{-|\Delta_{\rm CFL}|r} \begin{pmatrix} J_0(\mu r) \\ iJ_1(\mu r)\,\mathrm{e}^{i\theta} \end{pmatrix}, \quad \eta(r,\theta) = \mathrm{e}^{-|\Delta_{\rm CFL}|r} \begin{pmatrix} -J_1(\mu r)\,\mathrm{e}^{-i\theta} \\ iJ_0(\mu r) \end{pmatrix}, \tag{1}$$

where $\Delta_{\rm CFL}$ is the gap value of the bulk colour superconductor (in the colour-flavour locked phase), μ is the chemical potential, and r,θ are the polar coordinates perpendicular to the z direction. The triplet zero modes ψ^a $(a=1,2,3)$ are then compactly expressed as $\psi^a \propto (\varphi,(-1)^{a+1}\eta)^t$, and they satisfy the Majorana condition $\psi^a=(\psi^a)^C$, where $(\psi^a)^C \propto (\eta^C,(-1)^{a+1}\varphi^C)^t$ with C being the charge conjugation. It should be noted that these zero-mode solutions (non-Abelian Majorana fermions) are well localized around the centre of the vortex and do not depend on the z coordinate, indicating that they are essentially local objects on a two-dimensional plane. Hence,

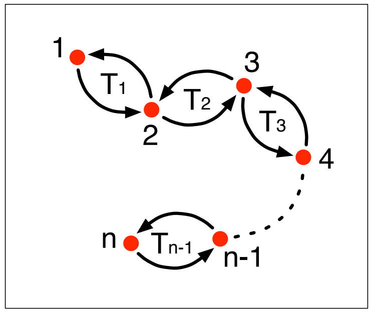

FIG. 1: Schematic of n particles on a two-dimensional plane and exchange of the kth and $(k+1)$th particles denoted by T_k $(k = 1, \cdots, n-1)$.

the non-Abelian vortices appearing in high-density matter provide an example of realization of the non-Abelian Majorana fermions.

However, it should be emphasized that our conclusion in this article does not rely on any specific model; all that we need is non-Abelian Majorana fermions of an $SO(3)$ triplet. Thus, our analysis raises a possibility to realize such a statistics in table top samples, for instance in cold atomic gasses that can be well controlled through experiments. It should also be noted that the $SO(3)$ group and its vector representation are chosen only for illustration as the simplest example and that our method works for arbitrary Lie groups and arbitrary (real) representations, opening up a new possibility of Majorana fermions, non-Abelian statistics, and quantum computations.

II. STATISTICS ON TWO-DIMENSIONAL PLANE

The exchange of particles on a two-dimensional plane is described by the braid group. Let us suppose n particles (braids) and label them as shown in Fig. 1. The braid group is defined as a set of operations T_k $(k = 1, \cdots, n-1)$ that involve the exchange of the positions of the neighbouring kth and $(k+1)$th braids in a way such that the kth braid always goes around the $(k+1)$th braid in an anti-clockwise direction. The operations T_k satisfy the following braid relations: i) $T_k T_l = T_l T_k$ for $|k-l| > 1$ and ii) $T_k T_l T_k = T_l T_k T_l$ for $|k-l| = 1$, which are schematically shown in Fig. 2. It should be noted that $T_k^{-1} \neq T_k$ because the operation is directed. This simple definition of the braid group allows for various representations with rich non-trivial structures, as we will see below.

The representation of the braid group is expressed by a linear group $\{\tau_k | k = 1, \cdots, n-1\}$ acting on a vector space. Here, τ_k's also satisfy the following braid relations: 1) $\tau_k \tau_l = \tau_l \tau_k$ for $|k-l| > 1$ and 2) $\tau_k \tau_l \tau_k = \tau_l \tau_k \tau_l$ for $|k-l| = 1$. The representations contain information about the

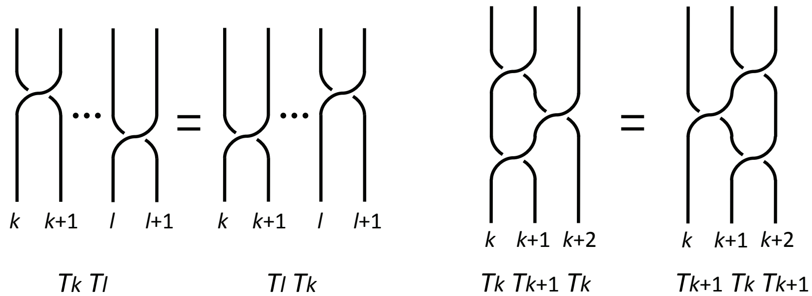

FIG. 2: Schematic of the braid relations i) $T_k T_l = T_l T_k$ with $|k-l| > 1$ and ii) $T_k T_{k+1} T_k = T_{k+1} T_k T_{k+1}$.

statistics on the exchange of particles. For example, the one-dimensional representation allows for the anyon statistics, which gives a wave function a complex factor under the exchange of particles; $\tau_1 = \tau_2 = \cdots = \tau_{n-1} = \mathrm{e}^{i\theta}$ with $0 \leq \theta < 2\pi$ being a real number. Although the anyon statistics is a characteristic statistics in two dimensions, it is still Abelian. More generally, the braid group allows for non-Abelian statistics in which neighbouring τ_k's are non-commutative: $[\tau_k, \tau_l] \neq 0$ for $|k-l| = 1$.

III. NON-ABELIAN STATISTICS OF MULTIPLE NON-ABELIAN VORTICES

Let us consider n non-Abelian vortices with the $SO(3)$ symmetry. There are correspondingly n non-Abelian Majorana fermions belonging to the triplet of $SO(3)$ at each vortex site. Let us introduce operators γ_k^a ($a = 1,\ 2,\ 3$) for creation operators of the triplet non-Abelian Majorana fermions in the kth non-Abelian vortex. These operators satisfy the anti-commutation relation $\{\gamma_k^a, \gamma_l^b\} = 2\delta_{kl}\delta^{ab}$ and the self-conjugation relation $\gamma_k^{a\dagger} = \gamma_k^a$. This formulation is an extension of the Abelian Majorana fermions discussed in [5] to the non-Abelian case. An exchange operation of neighbouring kth and $(k+1)$th non-Abelian vortices, denoted by T_k as shown in Fig. 1, induces an exchange of the non-Abelian Majorana fermions. Because the Majorana fermion turning around a vortex changes the sign of the wave function owing to the odd winding number and the Majorana condition, a cut should be introduced to keep the phase single valued [5]. Consequently, the operation T_k induces the following transformation (see Fig. 3):

$$T_k : \begin{cases} \gamma_k^a & \to \gamma_{k+1}^a \\ \gamma_{k+1}^a & \to -\gamma_k^a \end{cases}, \quad \text{for all } a \tag{2}$$

with the rest γ_l^a ($l \neq k,\ k+1$) unchanged. It should be noted that the presence of this minus sign is crucial to satisfy the braid relations i) and ii).

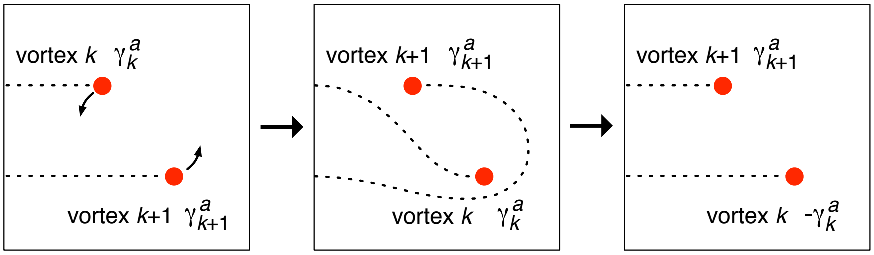

FIG. 3: Exchange of two non-Abelian Majorana fermions γ_k^a and γ_{k+1}^a on the two-dimensional plane. The dotted lines denote the cuts attached to each non-Abelian Majorana fermion. From the middle figure to the rightmost figure, it can be seen that the non-Abelian Majorana fermion γ_k^a jumps the cut of the $(k+1)$th vortex, to obtain an extra minus sign. Therefore, T_k induces the transformation: $\gamma_k^a \to \gamma_{k+1}^a$ and $\gamma_{k+1}^a \to -\gamma_k^a$.

Now, we discuss the representation τ_k for the exchange of non-Abelian Majorana fermions. First, it should be noted that transformation (2) can be realized by $SO(3)$ invariant unitary operators $\hat{\tau}_k = \hat{\tau}_k^1\hat{\tau}_k^2\hat{\tau}_k^3$, where $\hat{\tau}_k^a$ is made of non-Abelian Majorana operators γ_k^a and γ_{k+1}^a as $\hat{\tau}_k^a = (1+\gamma_{k+1}^a\gamma_k^a)/\sqrt{2}$ (not summed over a). It is easily verified that $\hat{\tau}_k\gamma_l^a\hat{\tau}_k^{-1}$ indeed generates the above transformation. Once the Hilbert space is defined, one obtains the representations of $\hat{\tau}_k$ as matrices. In order to define the bases of the Hilbert space, we first introduce an operator of non-Abelian Dirac fermions $\hat{\Psi}_k^a = (\gamma_{2k-1}^a + i\gamma_{2k}^a)/2$ $(k = 1, \cdots, n/2)$ using an even number (n) of non-Abelian vortices. The non-Abelian Dirac fermions satisfy the anti-commutation relations $\{\hat{\Psi}_k^a, \hat{\Psi}_l^{b\dagger}\} = \delta_{kl}\delta^{ab}$, $\{\hat{\Psi}_k^a, \hat{\Psi}_l^b\} = \{\hat{\Psi}_k^{a\dagger}, \hat{\Psi}_l^{b\dagger}\} = 0$, and the operators $\hat{\Psi}_k^a$ and $\hat{\Psi}_l^{a\dagger} (\neq \hat{\Psi}_k^a)$ correspond to annihilation and creation operators, respectively. Then, we can construct the Hilbert space by acting successively creation operators $\hat{\Psi}_k^{a\dagger}$'s on the 'vacuum state' $|0\rangle$ defined by $\hat{\Psi}_k^a|0\rangle = 0$. In what follows, we concretely show representations τ_k in two cases with two $(n = 2)$ and four $(n = 4)$ non-Abelian vortices.

First, we discuss the case with two non-Abelian vortices $(n = 2)$, where we can define only one operation T_1 and only one non-Abelian Dirac fermion $\hat{\Psi}_1^a$. The Hilbert space is spanned by the following four basis states: singlet-even $|\mathbf{1}_0\rangle = |0\rangle$ (vacuum), singlet-odd $|\mathbf{1}_3\rangle = \frac{1}{3!}\epsilon^{abc}\hat{\Psi}_1^{a\dagger}\hat{\Psi}_1^{b\dagger}\hat{\Psi}_1^{c\dagger}|0\rangle$ (filled by three Dirac fermions), triplet-even $|\mathbf{3}_2\rangle = \frac{1}{2!}\epsilon^{abc}\hat{\Psi}_1^{b\dagger}\hat{\Psi}_1^{c\dagger}|0\rangle$ (occupied by two Dirac fermions), and triplet-odd $|\mathbf{3}_1\rangle = \hat{\Psi}_1^{a\dagger}|0\rangle$ (occupied by one Dirac fermion), where the subscript denotes the number of non-Abelian Dirac fermions. With these bases $\{|\mathbf{1}_0\rangle, |\mathbf{1}_3\rangle, |\mathbf{3}_2\rangle, |\mathbf{3}_1\rangle\}$, we find the representation τ_1 to be an 8×8 matrix; $\tau_1 = \mathrm{diag}(\mathrm{e}^{-i3\pi/4}, \mathrm{e}^{i3\pi/4}, \mathrm{e}^{i\pi/4}I_{3\times3}, \mathrm{e}^{-i\pi/4}I_{3\times3})$. The 3×3 unit matrix $I_{3\times3}$ means the three components in each triplet state. The obtained matrix τ_1 is diagonal, and thus, the system of two non-Abelian Majorana fermions follows the (anyon-like)

Abelian statistics.

Second, we discuss the representation τ_k ($k = 1, 2, 3$) for four non-Abelian vortices ($n = 4$). We can construct the Hilbert space in a manner similar to that described above, and this is expressed by singlet ($\mathbf{1}$), triplet ($\mathbf{3}$), and quintet ($\mathbf{5}$) states, which are further specified by even ($\mathcal{E}$) and odd ($\mathcal{O}$) numbers of Dirac fermions (see Appendix A). Then, we obtain the representations τ_k ($k = 1, 2, 3$) as 64×64 matrices; $\tau_k = \mathrm{diag}(\tau_k^{\mathbf{1},\mathcal{E}}, \tau_k^{\mathbf{1},\mathcal{O}}, \tau_k^{\mathbf{3},\mathcal{E}} I_{3\times3}, \tau_k^{\mathbf{3},\mathcal{O}} I_{3\times3}, \tau_k^{\mathbf{5},\mathcal{E}} I_{5\times5}, \tau_k^{\mathbf{5},\mathcal{O}} I_{5\times5})$. The submatrices $\tau_k^{\mathcal{M},\mathcal{P}}$ ($\mathcal{M} = \mathbf{1}, \mathbf{3}, \mathbf{5}$, $\mathcal{P} = \mathcal{E}, \mathcal{O}$) are found to have the following structure:

$$\tau_k^{\mathcal{M},\mathcal{P}} = \sigma_k^{\mathcal{M}} \otimes h_k^{\mathcal{P}} \,. \tag{3}$$

Here, the first matrices $\sigma_k^{\mathcal{M}}$ have different expressions depending on the multiplets:

$$\sigma_1^{\mathbf{1}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2^{\mathbf{1}} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \quad \sigma_3^{\mathbf{1}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4}$$

for the singlet states,

$$\sigma_1^{\mathbf{3}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_2^{\mathbf{3}} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}, \quad \sigma_3^{\mathbf{3}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{5}$$

for the triplet states, and

$$\sigma_1^{\mathbf{5}} = \sigma_2^{\mathbf{5}} = \sigma_3^{\mathbf{5}} = 1, \tag{6}$$

for the quintet states. On the other hand, the second matrices $h_k^{\mathcal{P}}$ are common for the multiplets:

$$h_1^{\mathcal{E}} = h_1^{\mathcal{O}} = \begin{pmatrix} e^{i\frac{\pi}{4}} & 0 \\ 0 & e^{-i\frac{\pi}{4}} \end{pmatrix}, \quad h_2^{\mathcal{E}} = h_2^{\mathcal{O}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad h_3^{\mathcal{E}} = h_3^{\mathcal{O}\dagger} = h_1^{\mathcal{E}} \tag{7}$$

for even and odd numbers of Dirac fermions. These are our main results. We have obtained matrix representations for the exchange statistics in the system of four non-Abelian vortices. It should be noted that both the matrices $\sigma_2^{\mathcal{M}}$ ($\mathcal{M} = \mathbf{1}, \mathbf{3}$) and $h_2^{\mathcal{P}}$ in the submatrices $\tau_2^{\mathcal{M},\mathcal{P}}$, and consequently the matrix τ_2, are non-diagonal. Therefore, *the system of four non-Abelian Majorana fermions follows the non-Abelian statistics.* It should be emphasized that the non-Abelian matrices we have derived are essentially new and are considered as the generalizations of the corresponding matrices obtained by Ivanov. Indeed, while the matrices $h_k^{\mathcal{P}}$ (common for the multiplets) are the same as those that Ivanov obtained for 'Abelian' vortices (thus we call them the Ivanov matrices), the matrices $\sigma_k^{\mathcal{M}}$ are new matrices that only appear in 'non-Abelian' vortices. Hence, the representation matrices we have found are the direct products of the new matrices $\sigma_k^{\mathcal{M}}$ and Ivanov matrices $h_k^{\mathcal{P}}$.

IV. COXETER GROUP

Unexpectedly, the new matrices $\sigma_k^{\mathcal{M}}$ are identified with the elements in the Coxeter group, which is related to the operations of reflections shown below. The Coxeter group S is defined as a group with generators $s_i \in S$ $(i = 1, 2, 3, \cdots)$ satisfying the following two conditions: a) $s_i^2 = 1$ and b) $(s_i\, s_j)^{m_{i,j}} = 1$ with a positive integer $m_{i,j} \geq 2$ for $i \neq j$. It should be noted that condition a) gives $m_{i,i} = 1$. It is easy to check that the matrices $\sigma_k^{\mathcal{M}}$ indeed satisfy conditions a) and b). Elements $m_{i,j}$ can be summarized as the Coxeter matrix $(\mathsf{M})_{ij} = m_{i,j}$. In the present case, from the matrices $\sigma_k^{\mathcal{M}}$, we obtain the Coxeter matrix

$$\mathsf{M}_3 = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \tag{8}$$

where the top-left submatrices correspond to each multiplet; the 1×1 submatrix (1) corresponds to the quintet, the 2×2 submatrix $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ corresponds to the singlet, and the 3×3 whole matrix corresponds to the triplet.

The Coxeter group is closely related to the geometry in a high-dimensional space, because conditions a) and b) can be interpreted as geometrical operations in the Hilbert space: the condition a) corresponds to a reflection and the condition b) corresponds to a rotation by an angle $2\pi/m_{s,t}$. Therefore, the Coxeter group leads to the existence of several polytopes that are invariant under reflections and rotations, such as a 2-simplex (triangle) under the reflections $\sigma_1^{\mathbf{1}}$ and $\sigma_2^{\mathbf{1}}$ for the singlet and a 3-simplex (tetrahedron) under the reflections $\sigma_1^{\mathbf{3}}$, $\sigma_2^{\mathbf{3}}$, and $\sigma_3^{\mathbf{3}}$ for the triplet, as shown in Fig. 4. The fact that the Coxeter group appears in the exchange statistics of the system of the non-Abelian vortices provides a new insight that the non-Abelian statistics of non-Abelian Majorana fermions can be intuitively understood with the help of the geometry of polytopes.

We may extend our discussion to the system of any even number $(n = 2m)$ of non-Abelian vortices. From the analysis given above, we expect $m_{i,j} = 3$ for $|i - j| = 1$ and $m_{i,j} = 2$ for $|i - j| > 1$ for the representation matrices σ_i $(i = 1, \cdots, 2m - 1)$, which would be obtained from the decomposition of τ_k. Therefore, the Coxeter matrix will be given by the $(2m - 1) \times (2m - 1)$

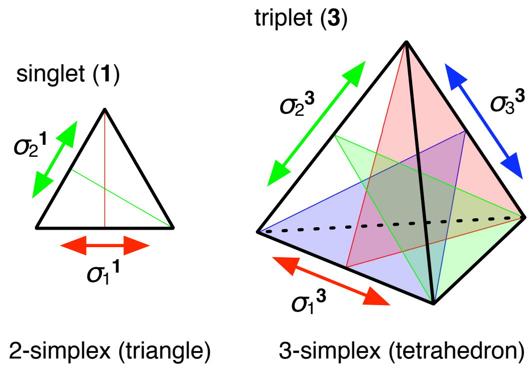

FIG. 4: 2-simplex (triangle) for the singlet and 3-simplex (tetrahedron) for the triplet induced from the reflections $\sigma_k^{\mathcal{M}}$ ($k = 1$ and 2 for $\mathcal{M} = \mathbf{1}$, and $k = 1$, 2 and 3 for $\mathcal{M} = \mathbf{3}$). These simplexes are invariant under reflections by $\sigma_k^{\mathcal{M}}$.

matrix as follows:

$$\mathsf{M}_{2m-1} = \begin{pmatrix} 1 & 3 & 2 & 2 & \cdots \\ 3 & 1 & 3 & 2 & \\ 2 & 3 & 1 & 3 & \\ 2 & 2 & 3 & 1 & \\ \vdots & & & & \ddots \end{pmatrix}. \tag{9}$$

In other words, a product $\sigma_i\sigma_j$ of reflections σ_i and σ_j makes a rotation by an angle $2\pi/3$ for $|i-j| = 1$ and $2\pi/2 = \pi$ for $|i-j| > 1$. Therefore, the system of an even number ($2m$) of non-Abelian vortices would lead to the existence of a $(2m-1)$-simplex as the highest-dimensional object (see also Appendix B).

It is known that the Coxeter group is classified into several types [9]. The Coxeter group summarized by the matrix (9) is called A_{2m-1}. Then, it would be interesting to ask whether the system of non-Abelian Majorana fermions can lead to other types of Coxeter groups, such as B_l ($l \geq 2$), D_l ($l \geq 4$), E_l ($l = 6, 7, 8$), F_4, G_2, H_l ($l = 3, 4$), $I_2(l)$ ($l = 5$, $l \geq 7$), or which new symmetry of the non-Abelian Majorana fermions would be connected to them. These questions would open an extensive view about the novel relationship between the non-Abelian Majorana fermions and the Coxeter groups.

Acknowledgments

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Appendix A: Bases of the Hilbert space for the system of four non-Abelian vortices

We explicitly present the bases of the Hilbert space of four non-Abelian vortices. There are four bases,

$$
\begin{aligned}
|\mathbf{1}_{00}\rangle &= |0\rangle, \\
|\mathbf{1}_{33}\rangle &= i\frac{1}{3!}\epsilon^{abc}\frac{1}{3!}\epsilon^{def}\hat{\Psi}_1^{a\dagger}\hat{\Psi}_1^{b\dagger}\hat{\Psi}_1^{c\dagger}\hat{\Psi}_2^{d\dagger}\hat{\Psi}_2^{e\dagger}\hat{\Psi}_2^{f\dagger}|0\rangle, \\
|\mathbf{1}_{11}\rangle &= i\frac{1}{\sqrt{3}}\hat{\Psi}_1^{a\dagger}\hat{\Psi}_2^{a\dagger}|0\rangle, \\
|\mathbf{1}_{22}\rangle &= \frac{1}{\sqrt{3}}\frac{1}{2!}\epsilon^{abc}\frac{1}{2!}\epsilon^{ade}\hat{\Psi}_1^{b\dagger}\hat{\Psi}_1^{c\dagger}\hat{\Psi}_2^{d\dagger}\hat{\Psi}_2^{e\dagger}|0\rangle,
\end{aligned}
\tag{A1}
$$

for the singlet-even $(\mathbf{1}, \mathcal{E})$ states, and

$$
\begin{aligned}
|\mathbf{1}_{03}\rangle &= \frac{1}{3!}\epsilon^{abc}\hat{\Psi}_2^{a\dagger}\hat{\Psi}_2^{b\dagger}\hat{\Psi}_2^{c\dagger}|0\rangle, \\
|\mathbf{1}_{30}\rangle &= -i\frac{1}{3!}\epsilon^{abc}\hat{\Psi}_1^{a\dagger}\hat{\Psi}_1^{b\dagger}\hat{\Psi}_1^{c\dagger}|0\rangle, \\
|\mathbf{1}_{21}\rangle &= -\frac{1}{\sqrt{3}}\frac{1}{2!}\epsilon^{abc}\hat{\Psi}_1^{a\dagger}\hat{\Psi}_1^{b\dagger}\hat{\Psi}_2^{c\dagger}|0\rangle, \\
|\mathbf{1}_{12}\rangle &= i\frac{1}{\sqrt{3}}\frac{1}{2!}\epsilon^{abc}\hat{\Psi}_1^{a\dagger}\hat{\Psi}_2^{b\dagger}\hat{\Psi}_2^{c\dagger}|0\rangle,
\end{aligned}
\tag{A2}
$$

for the singlet-odd $(\mathbf{1}, \mathcal{O})$ states. There are six bases,

$$
\begin{aligned}
|\mathbf{3}_{02}\rangle &= \frac{1}{2!}\epsilon^{abc}\hat{\Psi}_2^{b\dagger}\hat{\Psi}_2^{c\dagger}|0\rangle, \\
|\mathbf{3}_{31}\rangle &= -i\frac{1}{3!}\epsilon^{bcd}\hat{\Psi}_1^{b\dagger}\hat{\Psi}_1^{c\dagger}\hat{\Psi}_1^{d\dagger}\hat{\Psi}_2^{a\dagger}|0\rangle, \\
|\mathbf{3}_{22}\rangle &= \frac{1}{\sqrt{2}}\epsilon^{abc}\frac{1}{2!}\epsilon^{bde}\frac{1}{2!}\epsilon^{cfg}\hat{\Psi}_1^{d\dagger}\hat{\Psi}_1^{e\dagger}\hat{\Psi}_2^{f\dagger}\hat{\Psi}_2^{g\dagger}|0\rangle, \\
|\mathbf{3}_{11}\rangle &= i\frac{1}{\sqrt{2}}\epsilon^{abc}\hat{\Psi}_1^{b\dagger}\hat{\Psi}_2^{c\dagger}|0\rangle, \\
|\mathbf{3}_{20}\rangle &= -\frac{1}{2!}\epsilon^{abc}\hat{\Psi}_1^{b\dagger}\hat{\Psi}_1^{c\dagger}|0\rangle, \\
|\mathbf{3}_{13}\rangle &= i\frac{1}{3!}\epsilon^{bcd}\hat{\Psi}_1^{a\dagger}\hat{\Psi}_2^{b\dagger}\hat{\Psi}_2^{c\dagger}\hat{\Psi}_2^{d\dagger}|0\rangle,
\end{aligned}
\tag{A3}
$$

for the triplet-even ($\mathbf{3}$, $\mathcal{E}$) states, and

$$
\begin{aligned}
|\mathbf{3}_{01}\rangle &= \hat{\Psi}_2^{a\dagger}|0\rangle, \\
|\mathbf{3}_{32}\rangle &= i\frac{1}{3!}\epsilon^{bcd}\hat{\Psi}_1^{b\dagger}\hat{\Psi}_1^{c\dagger}\hat{\Psi}_1^{d\dagger}\frac{1}{2!}\epsilon^{aef}\hat{\Psi}_2^{e\dagger}\hat{\Psi}_2^{f\dagger}|0\rangle, \\
|\mathbf{3}_{21}\rangle &= \frac{1}{\sqrt{2}}\epsilon^{abc}\frac{1}{2!}\epsilon^{bde}\hat{\Psi}_1^{d\dagger}\hat{\Psi}_1^{e\dagger}\hat{\Psi}_2^{c\dagger}|0\rangle, \\
|\mathbf{3}_{12}\rangle &= -i\frac{1}{\sqrt{3}}\epsilon^{abc}\frac{1}{2!}\epsilon^{cde}\hat{\Psi}_1^{b\dagger}\hat{\Psi}_2^{d\dagger}\hat{\Psi}_2^{e\dagger}|0\rangle, \\
|\mathbf{3}_{23}\rangle &= \frac{1}{2!}\epsilon^{abc}\hat{\Psi}_1^{b\dagger}\hat{\Psi}_1^{c\dagger}\frac{1}{3!}\epsilon^{def}\hat{\Psi}_2^{d\dagger}\hat{\Psi}_2^{e\dagger}\hat{\Psi}_2^{f\dagger}|0\rangle, \\
|\mathbf{3}_{10}\rangle &= i\hat{\Psi}_1^{a\dagger}|0\rangle,
\end{aligned} \tag{A4}
$$

for the triplet-odd ($\mathbf{3}$, $\mathcal{O}$) states. There are two bases,

$$
\begin{aligned}
|\mathbf{5}_{22}\rangle &= i\mathcal{N}\left[\frac{1}{2}\left\{\frac{1}{2!}\epsilon^{acd}\hat{\Psi}_1^{c\dagger}\hat{\Psi}_1^{d\dagger}\frac{1}{2!}\epsilon^{bef}\hat{\Psi}_2^{e\dagger}\hat{\Psi}_2^{f\dagger} + \frac{1}{2!}\epsilon^{bcd}\hat{\Psi}_1^{c\dagger}\hat{\Psi}_1^{d\dagger}\frac{1}{2!}\epsilon^{aef}\hat{\Psi}_2^{e\dagger}\hat{\Psi}_2^{f\dagger}\right\}\right. \\
&\qquad \left. -\frac{\delta^{ab}}{3}\frac{1}{2!}\epsilon^{cde}\hat{\Psi}_1^{d\dagger}\hat{\Psi}_1^{e\dagger}\frac{1}{2!}\epsilon^{cfg}\hat{\Psi}_2^{f\dagger}\hat{\Psi}_2^{g\dagger}\right]|0\rangle, \\
|\mathbf{5}_{11}\rangle &= -\mathcal{N}\left[\frac{1}{2}\left\{\hat{\Psi}_1^{a\dagger}\hat{\Psi}_2^{b\dagger} + \hat{\Psi}_1^{b\dagger}\hat{\Psi}_2^{a\dagger}\right\} - \frac{\delta^{ab}}{3}\hat{\Psi}_1^{c\dagger}\hat{\Psi}_2^{c\dagger}\right]|0\rangle,
\end{aligned} \tag{A5}
$$

for the quintet-even ($\mathbf{5}$, $\mathcal{E}$) states, and

$$
\begin{aligned}
|\mathbf{5}_{21}\rangle &= -i\mathcal{N}\left[\frac{1}{2}\left\{\frac{1}{2!}\epsilon^{acd}\hat{\Psi}_1^{c\dagger}\hat{\Psi}_1^{d\dagger}\hat{\Psi}_2^{b\dagger} + \frac{1}{2!}\epsilon^{bcd}\hat{\Psi}_1^{c\dagger}\hat{\Psi}_1^{d\dagger}\hat{\Psi}_2^{a\dagger}\right\} - \frac{\delta^{ab}}{3}\frac{1}{2!}\epsilon^{cde}\hat{\Psi}_1^{c\dagger}\hat{\Psi}_1^{d\dagger}\hat{\Psi}_2^{e\dagger}\right]|0\rangle, \\
|\mathbf{5}_{12}\rangle &= -\mathcal{N}\left[\frac{1}{2}\left\{\hat{\Psi}_1^{a\dagger}\frac{1}{2!}\epsilon^{bcd}\hat{\Psi}_2^{c\dagger}\hat{\Psi}_2^{d\dagger} + \hat{\Psi}_1^{b\dagger}\frac{1}{2!}\epsilon^{acd}\hat{\Psi}_2^{c\dagger}\hat{\Psi}_2^{d\dagger}\right\} - \frac{\delta^{ab}}{3}\frac{1}{2!}\epsilon^{cde}\hat{\Psi}_1^{c\dagger}\hat{\Psi}_2^{d\dagger}\hat{\Psi}_2^{e\dagger}\right]|0\rangle,
\end{aligned} \tag{A6}
$$

for the quintet-odd ($\mathbf{5}$, $\mathcal{O}$) states with $\mathcal{N} = \sqrt{3/2}$ for $a = b$ and $\mathcal{N} = \sqrt{2}$ for $a \neq b$. With these bases we obtain the matrices τ_k ($k = 1, 2, 3$) as presented in Eqs. (3)–(7).

Appendix B: Coxeter matrix for arbitrary number of non-Abelian vortices

In the text, we show the direct-product structure of the matrices $\tau_k = \sigma_k^{\mathcal{M}} \otimes h_k^{\mathcal{P}}$ ($\mathcal{M} = \mathbf{1}$, $\mathbf{3}$, $\mathbf{5}$, and $\mathcal{P} = \mathcal{E}$, $\mathcal{O}$) obtained in the Hilbert space with bases presented in Subsec. A. In fact, such a product structure holds even at the operator level. The operator $\hat{\tau}_k = \hat{\tau}_k^1\hat{\tau}_k^2\hat{\tau}_k^3$ with $\hat{\tau}_k^a = (1 + \gamma_{k+1}^a\gamma_k^a)/\sqrt{2}$ is expressed as a product of two $SO(3)$ invariant unitary operators, *i.e.*

$$
\hat{\tau}_k = \hat{\sigma}_k\hat{h}_k\,, \tag{B1}
$$

where

$$
\hat{\sigma}_k = \frac{1}{2}\Big(1 - \gamma_{k+1}^1\gamma_{k+1}^2\gamma_k^1\gamma_k^2 - \gamma_{k+1}^2\gamma_{k+1}^3\gamma_k^2\gamma_k^3 - \gamma_{k+1}^3\gamma_{k+1}^1\gamma_k^3\gamma_k^1\Big) \tag{B2}
$$

and

$$\hat{h}_k = \frac{1}{\sqrt{2}}\Big(1 - \gamma^1_{k+1}\gamma^2_{k+1}\gamma^3_{k+1}\gamma^1_k\gamma^2_k\gamma^3_k\Big). \tag{B3}$$

It is easily verified that operators $\hat{\sigma}_k$ satisfy relations a) and b) of the Coxeter group, such that

$$\begin{aligned} \hat{\sigma}_k^2 &= 1, \\ (\hat{\sigma}_k\hat{\sigma}_l)^3 &= 1 \quad \text{for } |k-l| = 1, \\ (\hat{\sigma}_k\hat{\sigma}_l)^2 &= 1 \quad \text{for } |k-l| > 1. \end{aligned} \tag{B4}$$

Therefore, we confirm the Coxeter matrix (9) for an arbitrary number of non-Abelian vortices.

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