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Black hole entropy divergence and the uncertainty principle

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Abstract: Black hole entropy has been shown by 't Hooft to diverge at the horizon. The region near the horizon is in a thermal state, so entropy is linear to energy which consequently also diverges. We find a similar divergence for the energy of the reduced density matrix of relativistic and non-relativistic field theories, extending previous results in quantum mechanics. This divergence is due to an infinitely sharp boundary, and it stems from the position/momentum uncertainty relation in the same way that the momentum fluctuations of a precisely localized quantum particle diverge. We show that when the boundary is smoothed the divergence is tamed. We argue that the divergence of black hole entropy can also be interpreted as a consequence of position/momentum uncertainty, and that 't Hooft's brick wall tames the divergence in the same way, by smoothing the boundary.

Contents

1. Introduction

In quantum mechanics some questions can, but should not be asked. If we insist on asking them, the theory itself lets us know in a clear way by giving us a senseless answer. For example, if we ask "what is the typical momentum of a perfectly localized particle?" the formal answer will be infinite because of the position/momentum uncertainty relation. Of course, this just means that the momentum fluctuations will become larger as the particle is localized in a sharper way. Here the observer needs to change the question to "what is the typical momentum of a particle whose wave function has a small finite width in space?" and treat the concept of a sharply localized particle as a limit.

In quantum field theory we can ask another class of questions whose answer is infinity. For example, if we ask "what is the charge of the electron?" the answer comes out infinite. In this case the infinite answer does not mean that we should not have asked the question. Rather, it means that we have misidentified a microscopic parameter in the theory and that this parameter should be "renormalized". After a redefinition of the "bare" (correct) microscopic theory we can ask the question and get a finite answer.

The distinction between the two classes of divergences is the distinction between an ultraviolet (UV) divergence and an ill posed question. The former can be removed by modifying the UV properties of the theory, that is, modifications to the microscopic Hamiltonian of the theory. For the latter, it is necessary to modify the operator whose expectation value is being measured or the state in which it is being measured and not the Hamiltonian.

When we take a non-relativistic particle in a finite box and ask "what is a typical energy in the right half of the box" or "what is a typical momentum in the right half of the box" we get an infinite answer because the fluctuations of momentum and energy are infinite. How should we interpret the answer? In [1] it was shown that the reason for the senseless answer is that the question is inappropriate. The insistence on an infinitely sharp division between left and right is the cause of the divergence. In this case the sensible question should involved a smoothed division of the box. If we limit the resolution with which we divide the box into halves we get a finite answer, inversely proportional to the smoothing width, exactly as in the case of the localized particle.

In black hole physics a similar dichotomy of divergences seems to exist. Two of the possible characterizations of black hole entropy are both proportional to area but they both diverge and apparently for a different reason so they do not appear to coincide. One is entanglement entropy, arising from correlations between that within the black hole and that without [2, 3, 4], and the other is the statistical mechanics definition related to the number of states of a system in the vicinity of a black hole [5, 6]. Both types of entropy can be expressed in terms of a density matrix ρ: $S = -\mathrm{Tr}(\rho \ln \rho)$ of fields exterior to the black hole.

In the statistical mechanics calculation first carried out by 't Hooft, the divergence occurs at the black hole horizon. We will show that the problem of divergence at the boundary can be resolved by asking the right question. It is not a unique black hole characteristic but rather a result of quantum uncertainty, and the correct expression must involve smearing out the boundary. In fact Bekenstein noted in 1994 that if the boundary of the region being traced out were absolutely sharp, the energy would be very large due to the uncertainty principle, and so the boundary must be thought of as "slightly fuzzy" [7], and we will show in detail that this is the case.

Entanglement entropy is divergent for an apparently different reason. There the divergence is seemingly a UV divergence which does not occur at a particular location. We believe that the divergence of the entanglement entropy is of the same character as that of the statistical entropy. However, the detailed investigation of this will be postponed to a subsequent investigation.

This paper is organized as follows. First we review these two characterizations of black hole entropy, entanglement and statistical, focusing on behavior at the horizon. Then we show the relationship between entanglement entropy and energy near the horizon of a black hole. Next, in order to discuss modification of a boundary, we define a smoothing operator. This is then used to examine behavior of energy at a boundary between two subsystems, first for the non-relativistic and then for the relativistic case, and to show that in both cases energy diverges as the boundary becomes sharp. We extend this to Rindler space, as a partitioning of Minkowsky space. Finally, we examine 't Hooft's calculation of black hole entropy, and find that his relocation of the boundary to avoid divergence is equivalent to smearing out the boundary. Therefore here too the divergence is related to sharpness of the boundary, and thus it reflects behavior of any thermal system at a boundary, and is not unique to a black hole. This conclusion lends weight to the possibility that black hole entropy is in fact entanglement entropy, because the peculiar divergence which seemed to single it out from other entangled systems is just a result of quantum position/momentum uncertainty.

2. Review of entanglement and black hole entropies

A quantum system is entangled if it cannot be expressed as a tensor product of its subsystems. In this case although the total state is pure its subsystems are in a mixed state.

Entanglement entropy quantifies the extent to which a state is mixed: $S(\rho) = -Tr(\rho \ln \rho)$, where ρ is the reduced density matrix of either of the subsystems [8]. If the universe is described as a pure state, the black hole horizon divides it into that within and that without the hole, each of which is a mixed state. Therefore black holes have entanglement entropy by definition, and the question is whether black hole entropy is anything more, or whether entanglement entropy saturates the definition.

’t Hooft calculated thermodynamic characteristics of a black hole, among them entropy, and in doing so found a divergence of the integrand at the horizon [6]. He overcame the problem by adjusting the limits of integration to a brick wall a finite infinitesimal distance from the horizon. Entanglement entropy, on the other hand, has ultraviolet divergence, and a UV cutoff must be employed, but it does not diverge at any particular location. Srednicki in his landmark paper [2] calculated entropy by tracing over the degrees of freedom of part of the system, and found it proportional to area. He numerically obtained an expression for entropy $S = 0.30 M^2 R^2$,where $R = (n + \frac{1}{2})a$, a is lattice spacing and n the number of discrete oscillators. He defined M as the inverse lattice spacing a^{-1} so that the actual expression is $S = 0.30(n + \frac{1}{2})^2$. This diverges for an infinite number of oscillators, but not at a particular location. Other treatments of entanglement entropy in general [9, 10] also find UV divergence but not at any particular location.

For a BH in equilibrium, the space just outside the hole near the horizon can be treated as a thermal state in Rindler space [11, 12]. In this case entanglement entropy coincides with thermal entropy, as follows. To find entanglement entropy we take the trace of part of the system, If that part of the system is a thermal state, the partial trace is a thermal density matrix,

$$\rho_{part} = \frac{1}{Z} \sum_i e^{-\beta E_i} |E_i\rangle\langle E_i|. \tag{2.1}$$

Entanglement entropy is given by

$$S = -Tr(\rho_{part} \ln \rho_{part}) \tag{2.2}$$

and the energy is given by

$$\langle E \rangle = \frac{1}{Z} \sum_i E_i e^{-\beta E_i} \tag{2.3}$$

It follows that

$$
\begin{aligned}
S &= -\frac{1}{Z}\sum_i e^{-\beta E_i} \times \left(-\beta \sum_i E_i - \ln Z\right) \\
&= \beta\langle E\rangle + \ln Z.
\end{aligned} \tag{2.4}
$$

For a scalar field at a finite temperature $\ln Z$ is a constant, so the entropy is linear to the expectation value of the energy. Therefore in the case of a black hole the entanglement entropy behaves as does the energy. Thus instead of examining entropy at a barrier dividing the two subsystems, which is a complicated non-local quantity, we can calculate the reduced density matrix of a subsystem and look at the behavior of its energy which is a simpler local quantity.

3. Momentum fluctuations, energy and entropy for smooth partitions

3.1 Smooth restricted operators

We will now define a smoothing function which, when applied to an operator that is restricted to a sub-volume, will soften the sharp partition and serve as a momentum cutoff. Let us discuss a quantum system in a volume Ω which is initially prepared in a pure state $|\psi\rangle$ defined in Ω. We divide the total volume into some sub-volume V, and its complement $\widehat{V}$ so that $\Omega = V \oplus \widehat{V}$. The Hilbert space inherits a natural product structure $\mathcal{H}_\Omega = \mathcal{H}_V \otimes \mathcal{H}_{\widehat{V}}$. We are interested in states $|\psi\rangle$ that are entangled with respect to the Hilbert spaces of V and $\widehat{V}$ so that they can not be brought into a product form $|\psi\rangle = |\psi\rangle_V \otimes |\psi\rangle_{\widehat{V}}$ in terms of a pure state $|\psi\rangle_V$ that belongs to the Hilbert space of V, and another pure state $|\psi\rangle_{\widehat{V}}$ that belongs to the Hilbert space of $\widehat{V}$.

The total density matrix is defined in terms of the total state $|\psi\rangle$

$$
\rho = |\psi\rangle\langle\psi|. \tag{3.1}
$$

The partition of the total volume of the system into two parts

$$
\Omega = V \oplus \widehat{V} \tag{3.2}
$$

induces a product structure on the Hilbert space and allows defining the reduced density matrix by performing a trace over part of the Hilbert space

$$
\rho^V = \mathrm{Tr}_{\widehat{V}}\rho. \tag{3.3}
$$

Operators that act on part of the Hilbert space are defined as integrals over densities in a part of space

$$O^V = \int\limits_V d^3r \mathcal{O}(\vec{r}) \tag{3.4}$$

or alternatively in terms of a theta function

$$\Theta^V(\vec{r}) = \begin{cases} 1 & \vec{r} \in V \\ 0 & \vec{r} \in \widehat{V} \end{cases}, \tag{3.5}$$

$$O^V = \int\limits_\Omega d^3r \mathcal{O}(\vec{r}) \Theta^V(\vec{r}). \tag{3.6}$$

A striking equation relates quantum expectation values of operators that act on part of the Hilbert space to the statistical averages with a reduced density matrix:

$$\langle\psi|O^V|\psi\rangle = \mathrm{Tr}\left(\rho^V O^V\right). \tag{3.7}$$

We can also define a smoothed operator

$$O^V_{\text{smooth}} = \int\limits_\Omega d^3r \mathcal{O}(\vec{r}) \Theta^V_{\text{smooth}}(\vec{r}, w) \tag{3.8}$$

where $\Theta^V_{\text{smooth}}(\vec{r}, w)$ represents a smoothed step function that rather than changing in a discontinuous way from zero to unity on the boundary of V changes in a smooth way over a region of width w near the boundary of V. Expressing Θ^V_{smooth} as the product of a step function and an auxiliary smoothing function $(f(\vec{r}, w))^2$ (the reason for the square will become clear in what follows):

$$\Theta^V_{\text{smooth}}(\vec{r}, w) = (f(\vec{r}, w))^2 \, \Theta^V(\vec{r}) = \begin{cases} \to 1 & \vec{r} \in V \\ 0 \to 1 & \vec{r} \in \partial V \text{ with width } w \\ \to 0 & \vec{r} \in \widehat{V} \end{cases} \tag{3.9}$$

The smooth theta function defined in this way can be made continuous to any fixed desired order in derivatives. So if a class of operators has at most a given order of derivatives it is possible to define a smooth theta function that will be effectively analytic for this class. For example, the one dimensional function

$$\Theta^V_{\text{smooth}}(x, w) = \begin{cases} \frac{x^n}{x^n + w^n} & x \geq 0 \\ 0 & x \leq 0 \end{cases} \tag{3.10}$$

has $n-1$ continuous derivatives at $x=0$.

Rather than using the smoothed step function to modify the operators O^V, we can view the smoothing function $f(\vec{r},w)$ as modifying the wave function (or state) in which the operator is being evaluated

$$\langle\psi|O^V_{\rm smooth}|\psi\rangle = \langle\psi|\left(f(\vec{r},w)\right)^2 O^V|\psi\rangle = \langle f(\vec{r},w)\psi|O^V|f(\vec{r},w)\psi\rangle. \tag{3.11}$$

Defining

$$|\psi_{\rm smooth}\rangle = f(\vec{r},w)|\psi\rangle \tag{3.12}$$

we may express the expectation value of the smoothed operator in the original state $|\psi\rangle$ in terms of an expectation value of the original operator in a smoothed state

$${\rm Tr}\left(\rho^V O^V_{\rm smooth}\right) = {\rm Tr}\left(\rho^V_{\rm smooth} O^V\right) \tag{3.13}$$

where

$$\rho^V_{\rm smooth} = |\psi_{\rm smooth}\rangle\langle\psi_{\rm smooth}|. \tag{3.14}$$

In momentum space

$$|\psi_{\rm smooth}\rangle = \int d^3p f(\vec{p},w)\psi(\vec{p})e^{-i\vec{p}\cdot\vec{r}}. \tag{3.15}$$

Here the smoothing function $f(\vec{p},w)$ looks as if it is a UV cutoff suppressing the the high momentum components of the wave function.

The function $f(\vec{r},w)$ behaves as a window enclosing part of space, and thus it mimics the horizon by "truncating" part of space for the field. We provide it with a varying width, and examine energy as a function of its width. Our aim is to see how sharp localization affects the energy divergence.

3.2 Energy and momentum fluctuations in a restricted non relativistic system

We first write the reduced density matrix for nonrelativistic bosons restricted to one part of space, and we calculate the restricted energy. We take free spinless bosons and consider states that are created by the field operator Ψ acting on the vacuum:

$$|\psi\rangle \;=\; \Psi^\dagger(\vec{r})\,|0\rangle = \sum_{\vec{p}} \frac{e^{-i\vec{p}\vec{r}}}{\sqrt{\Omega}} g(\vec{p})\, a^\dagger_{\vec{p}}\,|0\rangle\,. \tag{3.16}$$

The function $g(\vec{p})$ is the wave function of the state in momentum space[1]

[1] The function g will not be particularly relevant for us and in most cases we will ignore it by setting $g(\vec{p})=1$. All our results can be easily generalized for the case $g(\vec{p})\neq 1$.

The Hamiltonian is given by

$$H = \sum_{\vec{p}} \frac{p^2}{2m} a^\dagger_{\vec{p}} a_{\vec{p}}. \tag{3.17}$$

The energy of a state $|\psi\rangle$ is given by

$$E = \langle\psi|H|\psi\rangle = \left\langle 0|\Psi H \Psi^\dagger|0\right\rangle . \tag{3.18}$$

In configuration space the energy is given by

$$E = \int_{-\infty}^{\infty} d^3r \frac{1}{2m} \left\langle 0|\nabla_r \Psi\left(\vec{r}\right) \nabla_r \Psi^\dagger\left(\vec{r}\right)|0\right\rangle . \tag{3.19}$$

We wish to calculate the energy corresponding to the restricted Hamiltonian $E^V_\psi = \langle\psi|H^V|\psi\rangle = \mathrm{Tr}(\rho^V H^V)$. We replace the restricted Hamiltonian H^V by its smoothed counterpart with the help of a window function $f(\vec{r}, w)$, as discussed above. Alternatively, we can use a restricted smoothed field operator (here we set $g = 1$)

$$\Psi^V_{\text{smoothed}} = \int d^3r\, f\left(\vec{r}\right) \Psi^\dagger\left(\vec{r}\right) = \int d^3r\, f\left(\vec{r}, w\right) \sum_{\vec{p}} \frac{e^{-i\vec{p}\vec{r}}}{\sqrt{V}}\, a^\dagger_{\vec{p}} = \sum_{\vec{p}} f\left(\vec{p}, w\right) a^\dagger_{\vec{p}}, \tag{3.20}$$

with $f\left(\vec{p}, w\right)$ being the Fourier transform of $f\left(\vec{r}, w\right)$. Because $f\left(\vec{r}, w\right)$ is a smooth function its Fourier transform suppresses large momenta and acts effectively as a high momentum cutoff. The result of Eq. (3.20) is substituted into Eq. (3.18). The creation operators on the vacuum give delta functions, resulting in

$$E^V_{\text{smoothed}} = \frac{1}{2m} \int_{-\infty}^{\infty} d^3r \vec{\nabla} f\left(\vec{r}, w\right) \cdot \vec{\nabla} f(\vec{r}, w) \tag{3.21}$$

In appendix A we evaluate explicitly a related case, the restricted smoothed momentum squared $\langle\psi|(P^2_{\text{smooth}})^V|\psi\rangle$.

For specific window functions the smoothed restricted energy can be evaluated explicitly. Consider, for example, a one dimensional case with

$$f(x, w) = \frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{x}{w}\right) . \tag{3.22}$$

The function is depicted by the dashed line in Fig 1. In momentum space (ignoring the singularity at $p = 0$)

$$f(p, w) = \frac{1}{\sqrt{2\pi}} \frac{1}{p} e^{-|p|w}. \tag{3.23}$$

So $1/w$ acts as a high momentum cutoff suppressing any momentum components of the smoothed wavefunction with $|p| > \frac{1}{w}$

The value of the restricted energy, the restriction being the positive half of the x-axis, can be calculated analytically in this case

$$E^V_{\text{smoothed}} = \frac{1}{2m}\frac{1}{2\pi w}. \tag{3.24}$$

This is also shown in Fig 1, taking $m = 1/2$.

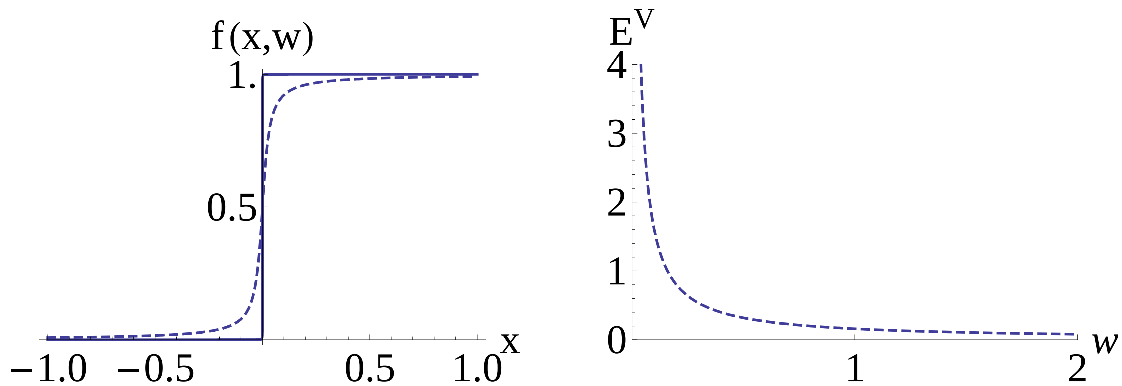

Figure 1: Shown is a one dimensional example of a smooth window function (left) and the corresponding restricted energy as a function of barrier width (right).

Other smoothing functions yield very similar results. The restricted energy is inversely proportional to the smoothing width w and diverges in the limit $w \to 0$.

For the nonrelativistic case, $E^V_{\text{smoothed}} = \frac{1}{2m}\left\langle\psi|(P^2_{\text{smooth}})^V|\psi\right\rangle$. Since $\left\langle\psi|(\vec{P}_{\text{smooth}})^V|\psi\right\rangle = 0$ it follows that $\left\langle\psi|(P^2_{\text{smooth}})^V|\psi\right\rangle = (\Delta P^V_{\text{smooth}})^2$ so the divergence of the energy is equal to the divergence of the momentum fluctuations. The divergence should not be confused with a UV divergence; the two are unrelated. The boundary behaves as if it is a localized particle. Given a function describing barrier slope, energy increases as the barrier grows sharper. That is, the more sharply the position of the dividing barrier is specified, the larger the energy. In the limit that the width tends to zero $w \to 0$ the energy diverges. This is the same phenomenon found in quantum mechanical uncertainty, where the more sharply we specify the position of a particle, the greater the uncertainty of its momentum. The energy in this case is a simple function of momentum and linearly related to momentum uncertainty, so that as the momentum fluctuations diverge so will the energy. Thus the energy divergence here is an indication of position/momentum uncertainty.

3.3 Relativistic smoothed restricted energy

In a relativistic system the energy operator is taken from the energy momentum tensor: $H = T_{00} = \int \frac{d^3k}{(2\pi)^3} k_0 a_k^\dagger a_k$ taking $c, \hbar = 1$.

In order to look for the various expectation values we recall the relativistic scalar product:

$$\langle \varphi | \phi \rangle = -i \int d^3x \left[\varphi \partial_t \phi^* - \left(\partial_t \, \varphi \right) \phi^* \right] \tag{3.25}$$

and the expression for the expectation value of the Hamiltonian in a state $|\varphi\rangle$

$$\langle \varphi \left| H \right| \varphi \rangle = -i \int d^3x \left[\varphi \partial_t \left(H\varphi \right)^* - \left(\partial_t \, \varphi \right) \left(H\varphi \right)^* \right]. \tag{3.26}$$

A smoothed state with window function, as before, can be defined as before $\int d^3r \, f\left(\vec{r}\right) \Psi^\dagger \left(\vec{r}\right) |0\rangle$ where the field operator here is the relativistic one. The resulting smoothed restricted energy is

$$\begin{aligned} E^V_{\text{smooth}} = \left\langle \psi \left| \left(H_{\text{smooth}} \right)^V \right| \psi \right\rangle &= \int d^3p \; f(\vec{p}, w) \; p \; f(-\vec{p}, w) \\ &= \int d^3r \; f(\vec{r}, w) \; \sqrt{\vec{\nabla}^2} \; f(\vec{r}, w) \end{aligned} \tag{3.27}$$

The details of the derivation are given in appendix B. The result clearly has the same behavior as in the non relativistic case. Alternately, since $E^2 \sim P^2$ we may calculate $\left\langle P^2 \right\rangle$ and obtain

$$\frac{1}{2} \int\limits_{-\infty}^{\infty} d^3r \vec{\nabla} f\left(\vec{r}, w\right) \cdot \vec{\nabla} f(\vec{r}, w) \tag{3.28}$$

This is identical to the non relativistic result, and equals $(\Delta P^V_{\text{smooth}})^2$.

We saw that in the nonrelativistic treatment energy tends to diverge the more sharply the boundary between the different parts of space is specified. The relativistic case shows the same phenomenon. Here too, the smoothing function $f(\vec{r}, w)$ acts as a momentum cutoff. In both cases the energy increases as the barrier width becomes narrower, and diverges for a completely sharp barrier with zero width. In the relativistic case $E^2 \sim P^2$ rather than $E \sim P^2$ but we still obtain $E \sim \Delta p$. As before, the energy is proportional to the momentum uncertainty, and just as in the previous section, it diverges when the barrier is made sharp. This can be seen as an example of position/momentum uncertainty.

4. Restricted energy and statistical entropy of the black hole

So far we have discussed restricted operators in flat spacetime. The restriction was implemented in an ad-hoc way by a choice of a (smoothed) theta function. In the case of the BH, spacetime is restricted in a different way. For example, in the Schwarzschild geometry $ds^2 = -(1-\frac{r_s}{r})dt^2 + \frac{1}{1-\frac{r_s}{r}}dr^2 + r^2 d\Omega^2$, the region of space inside the horizon $r < r_s$ is simply absent. So all the operators in Schwarzschild geometry are restricted operators. One can view the redshift factor $\frac{1}{1-\frac{r_s}{r}}$ as implementing the restriction by becoming infinite at the horizon $r = r_s$.

Our goal will be to explain how the redshift, acting as a restriction, creates an infinitely sharp boundary that results in divergence of the energy and entropy. We begin with the simpler case of Rindler spacetime, that is the spacetime of an accelerated observer in Minkowski space. Rindler space has the advantage that it is equivalent to a restriction to half of Minkowski space so this example allows us to explicitly compare the two restriction mechanisms. We will explain how we can implement the ideas of smoothing the boundary by restricting the maximal value of the redshift, and show that when smoothing is implemented all quantities are rendered finite with magnitude inversely proportional to the smoothing parameter, exactly as in the cases that we have encountered before. This will allow us to show that a similar phenomenon occurs for BH's.

4.1 The uncertainty principle in Rindler spacetime

We use the Minkowski space metric:

$$ds^2 = -dt^2 + dz^2 + d\vec{x}_\perp{}^2, \tag{4.1}$$

where z is the coordinate that will be used to separate space into the left and right halves $z < 0$ and $z > 0$ and $\vec{x}_\perp$ stands for the transverse coordinates. An accelerated observer whose acceleration is $a/2\pi$ lives in Rindler space whose metric is

$$ds^2 = -e^{2a\xi}d\eta^2 + e^{2a\xi}d\xi^2 + d\vec{x}_\perp{}^2. \tag{4.2}$$

The Minkowski coordinates and Rindler coordinates are related by:

$$t(\xi,\eta) = \frac{1}{a} e^{a\xi} \sinh a\eta \tag{4.3}$$

$$z(\xi,\eta) = \frac{1}{a} e^{a\xi} \cosh a\eta \tag{4.4}$$

$$\vec{x}_\perp = \vec{x}_\perp . \tag{4.5}$$

Choosing a fixed Rindler time, for example, $\eta = 0$, we see that the ξ coordinate only covers the $z > 0$ half of space. The restriction is implemented by the redshift factor $e^{-a\xi}$ which diverges for $\xi \to -\infty$, corresponding to $z = 0$.

As it stands, the restriction implemented by the redshift is infinitely sharp. The region $z < 0$ simply does not exist in Rindler space. We wish to understand how to implement a smoothed restriction rather than an infinitely sharp one. So we analyze just how the redshift leads to divergence of $(\Delta p)^2$ and vanishing of $(\Delta z)^2$, in order to consider how the divergence may be tamed. We consider a non-relativistic particle whose wave function has some spread Δz in Minkowski space. For example,

$$\psi(z) = \frac{1}{\sqrt{2\pi(\Delta z)^2}} e^{-\frac{1}{2}\frac{z^2}{(\Delta z)^2}} . \tag{4.6}$$

In momentum space the spread of the wave function is inversely proportional to Δz, $(\Delta p)^2 \sim 1/(\Delta z)^2$. Viewed by an accelerated observer, the wave function at the origin $z = 0$ corresponding to $\xi \to -\infty$ would be squeezed in the ξ direction: $\Delta\xi = e^{a\xi}\Delta z$. As required by the uncertainty principle the spread in momentum would increase, $\Delta p_\xi = e^{-a\xi}\Delta p_z$. Thus finite Δz and Δp in Minkowsky space are adjusted by the Rindler metric, so that to the Rindler observer the position fluctuations at the origin will vanish and momentum fluctuations will diverge.

By our choice the particle is localized at the origin (any other choice would simply require a shift in the Rindler time η), so in the limit $\xi \to -\infty$ the momentum fluctuations diverge because the the wave function has been squeezed in space. This divergence obviously does not signal a breakdown of physics. It just means that considering the classical Rindler geometry when viewing a quantum particle requires closer thought. Rindler geometry imposes a restriction on Minkowski space. When the restriction is sharp, equivalent to localizing a particle at the origin, the momentum fluctuations diverge. Limiting the Rindler redshift factor tames the divergence and increases position fluctuations, thus softening the localization, and smoothing the restriction.

4.2 Momentum fluctuations and redshift in Rindler spacetime

In view of the previous discussion, and in preparation for the reinterpration of the 't Hooft calculation which we reviewed in Sect. 1, let us consider a (massless) scalar field ϕ that satisfies the Klein-Gordon equation

$$\frac{1}{\sqrt{-g}}\left(\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\right)\phi=0. \tag{4.7}$$

In Minkowski spacetime there is an exact solution to the Klein-Gordon equation. The z dependent part of the solution is given by

$$\phi(z)=e^{\pm ipz}. \tag{4.8}$$

However, for the purpose of making the calculation more similar to the 't Hooft calculation we can rewrite the solution in a WKB form, where the WKB solution is

$$\phi_{WKB}(z)=e^{\pm i\int^{z}p(z)dz}. \tag{4.9}$$

Obviously, in Minkowski space $p(z)$ is a constant and the WKB solution reduces to the exact solution. The WKB momentum can be expressed as

$$p^2(z)=E^2-p_\perp^2. \tag{4.10}$$

In Rindler spacetime the WKB wave function is

$$\phi_{WKB}(\xi)=e^{\pm i\int^{\xi}d\xi\sqrt{g_{\xi\xi}}p(\xi)} \tag{4.11}$$

with

$$p^2(\xi)=g^{\eta\eta}E^2-p_\perp^2 \tag{4.12}$$

which is space varying. So the WKB wave function is

$$\phi_{WKB}(\xi)=e^{\pm i\int^{\xi}d\xi\sqrt{g_{\xi\xi}}\sqrt{g^{\eta\eta}E^2-p_\perp^2}}. \tag{4.13}$$

Near the horizon $p(\xi)$ diverges as $\sqrt{g^{\eta\eta}E^2}=e^{-a\xi}E$ and the proper length $\widetilde{d\xi}=d\xi\sqrt{g_{\xi\xi}}=d\xi e^{a\xi}$ vanishes. This is a manifestation of the position/momentum uncertainty relation.

Rindler space implements a sharp division of Minkowski space. That is, the Rindler observer sees a sharp cutoff at the horizon $\xi\to-\infty$. Smoothing this cutoff in momentum space means restricting the momentum $p(\xi)$ near the horizon. We saw in the previous section that restricting the redshift widens Δx and shrinks Δp. Therefore restricting the redshift $g^{\eta\eta}$, $g^{\xi\xi}$ will smooth the cutoff.

In 't Hooft's black hole calculation the energy and entropy diverge due to a diverging density of states. In Rindler space too the density of states diverges, and we will see that that this divergence is due to the uncertainty principle. We define the density of states near energy E in Rindler space and evaluate it by counting the number of WKB solutions

$$\begin{aligned}\pi n &= \int d\xi e^{a\xi} \int \frac{d^2 p_\perp}{(2a)^2} p(\xi, E, p_\perp) \\ &= 2\pi \int d\xi e^{a\xi} \int \frac{dp_\perp}{(2a)^2}\, p_\perp \sqrt{e^{-2a\xi} E^2 - p_\perp^2} \\ &= -\frac{2}{3} \frac{\pi}{(2a)^2} E^3 \int d\xi e^{-2a\xi} \qquad (4.14)\end{aligned}$$

where we have performed first the angular integral of $p_\perp$ and then the radial part. This integral diverges because of the diverging redshift factor at the horizon. So the density of states, the entropy and energy are divergent for the same reason and if the redshift factor is restricted, they all become finite.

Rather than making a smooth restriction by limiting the redshift factor we may count the number of smoothed states (states with smoothed wavefunctions) and see that it is finite. The smoothed functions that we need to count are obtained by multiplying the originl unsmoothed function by the smoothing function, $\psi(\xi) \to \psi(\xi) f(\xi, w)$, or in Fourier space $\phi(p) \to \phi(p) f(p, w)$. Recall that in momentum space the function $f(p, w)$ acted as a high momentum cutoff for $p > 1/w$. Then for wavefunctions with energy E we need to effectively restrict the Rindler momentum $p(\xi) = e^{-a\xi}\sqrt{E}$ to be $p(\xi) < 1/w$. In this context it simply means that the redshift factor is limited to some maximal value which can always be expressed as $e^{-a\xi_{min}}$. The "brick wall" model of 't Hooft in this context amounts to a sharp cutoff on the momentum $p(\xi)$. However, clearly, any other cutoff schemes will do the same job. The density of states of smoothed wavefunctions is of course finite,

$$\begin{aligned}\pi n &= \int_{\xi_{min}} d\xi e^{a\xi} \int \frac{d^2 p_\perp}{(2a)^2} p(\xi, E, p_\perp) \\ &= 2\pi \int_{\xi_{min}} d\xi e^{a\xi} \int \frac{dp_\perp}{(2a)^2}\, p_\perp \sqrt{e^{-2a\xi} E^2 - p_\perp^2} \\ &= \frac{2}{3} \frac{\pi}{(2a)^3} E^3 e^{-2a\xi_{min}}. \qquad (4.15)\end{aligned}$$

This makes the energy and entropy finite and inversely proportional to the maximal redshift which determines the smoothing width of the division in Rindler space.

4.3 Momentum fluctuations and entanglement entropy in Schwarzschild spacetime

't Hooft solves the wave equation in the Schwartzschild metric, identifies p, the wave number, and using a WKB approximation he obtains the density of states. However the redshift leads this to diverge at the horizon. The region near the black hole horizon is a thermal state in Rindler space, and indeed just as in Rindler space, limiting the redshift will prevent the divergence.

We recall the calculation in Schwarzschild coordinates. For simplicity we have chosen the scalar field to be massless. The Klein-Gordon equation in these coordinates is

$$\left(1-\frac{2M}{r}\right)^{-1}E^2\phi+\frac{1}{r^2}\partial_r\left(r\left(r-2M\right)\partial_r\right)\phi-\left(\frac{l\left(l+1\right)}{r^2}\right)\phi=0. \tag{4.16}$$

The wave number can be defined as[2]

$$p^2=g^{tt}E^2-\left(\frac{l\left(l+1\right)}{r^2}\right) \tag{4.17}$$

Using a WKB approximation the density of states for a massless scalar field is given by

$$\begin{aligned}\pi n &= \sum_{l,m}\int_{2M} dr\sqrt{g_{rr}}\,p\left(r,l,m\right) \qquad (4.18)\\ &= \int_{2M} dr\sqrt{g_{rr}}\int(2l+1)dl\sqrt{g^{tt}E^2-\frac{l\left(l+1\right)}{r^2}}\end{aligned}$$

where l, m are the angular parameters. Evaluating the integral over l we find

$$\begin{aligned}\pi n &= -\frac{2}{3}\int_{2M} dr\sqrt{g_{rr}}r^2\left(g^{tt}E^2\right)^{3/2}\\ &= -\frac{2}{3}E^3\int_{2M} dr\frac{r^2}{\left(1-\frac{2M}{r}\right)^2} \qquad (4.19)\end{aligned}$$

This integral diverges at the horizon. If we were to limit the redshift, as we did with Rindler space, there would be no divergence. Apparently 't Hooft does otherwise: he takes the lower limit a slight distance away from the horizon, his well known "brick wall," so

[2] This differs by a a factor g_{rr} from 't Hooft's original defintion.

that the lower limit becomes $2M + h$. From this expression he obtains the energy and entropy, which diverge as $h \to 0$.

In fact 't Hooft's adjustment of the lower limit of the integral from $2M$ to $2M + h$ is equivalent to a change of variable which leaves the lower limit at $2M$ but changes the redshift:

$$\int\limits_{2M+h} dr \left(1 - \frac{2M}{r}\right)^{-2} = \int\limits_{2M} d\widetilde{r} \left(1 - \frac{2M}{\widetilde{r} + h}\right)^{-2} \tag{4.20}$$

This clearly does not diverge at the horizon. The new expression is always finite and is limited by $\left(1 - \frac{2M}{2M+h}\right)^{-2} \lesssim (2M/h)^2$ for $h \ll M$.

The altered redshift is equivalent to multiplication of the original redshift in the $\widetilde{r}$ system by a smoothing function:

$$\left(1 - \frac{2M}{\widetilde{r} + h}\right)^{-1} = \left(1 - \frac{2M}{\widetilde{r}}\right)^{-1} f(\widetilde{r}, h) \tag{4.21}$$

with

$$f\left(\widetilde{r}, h\right) = \frac{\left(\widetilde{r} + h\right)\left(\widetilde{r} - 2M\right)}{\widetilde{r}\left(\widetilde{r} - 2M + h\right)}. \tag{4.22}$$

Thus the change of variable implemented by the brick wall has the effect of multiplying the redshift by a smoothing function.

The original divergent integral in eq. (4.20) can be expressed in terms of a sharp step function $\int_{2M} dr \left(1 - \frac{2M}{r}\right)^{-2} = \int_0 dr\ \Theta(r - 2M) \left(1 - \frac{2M}{r}\right)^{-2}$. The altered integral can be expressed in terms of a smoothed step function

$$\int\limits_0 dr \left(1 - \frac{2M}{r}\right)^{-2} f^2(r, h)\ \Theta(r - 2M) = \int\limits_0 dr \left(1 - \frac{2M}{r}\right)^{-2} \widetilde{\Theta}(r - 2M, h) \tag{4.23}$$

Thus we see that 't Hooft's changed lower limit is exactly equivalent to smoothing the step function to a new one $\widetilde{\Theta}(r-2M, h) = f^2(r, h)\ \Theta(r-2M)$ with width h. Formally the brick wall can be seen as either changing the redshift or smoothing the step function. Obviously, any other limiting procedure of the maximal redshift will render the integral finite and make the energy and entropy finite.

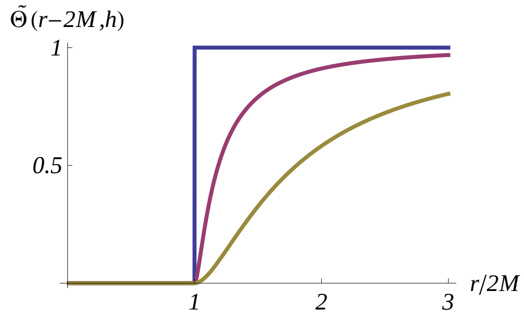

Figure 2: Smoothed step function as function of $r/2M$. Curves have $h = 0$ (sharp step), $h = 0.1$ and $h = 0.9$ (lowest)

5. Summary and conclusions

Energy has been shown to diverge as the boundary between two quantum subsystems becomes sharp. The divergence is due to the fact that the energy is a simple function of the momentum fluctuations. These diverge in the presence of a sharp boundary because of the uncertainty principle, much in the same way that they diverge for a sharply localized particle. For the nonrelativistic case $\langle E \rangle = \frac{1}{2m}(\Delta P)^2$. In the relativistic case $\langle E \rangle = \Delta P$ so in both cases energy divergence at an infinitely sharp boundary is clearly a consequence of position/momentum uncertainty.

In a coordinate system which implements a sharply localized boundary, the density of states and thus energy and entropy diverge at the boundary. Limiting the redshift tames this divergence. We have shown that limiting the redshift smoothes the boundary by widening Δx and limiting Δp. Therefore smoothing the cutoff prevents the energy from diverging. This implies that the divergence of the energy and entropy was a result of the sharp localization of the boundary, and was due to the uncertainty principle.

The region near the boundary of a black hole is a thermal state, where the entropy is linear to energy. Therefore black hole entropy will diverge at the boundary as well. We have shown that regardless of any other cause, there would be divergence at the infinitely sharp boundary as a result of the uncertainty principle. We have also shown that 't Hooft's divergence at the black hole is an example of momentum/position uncertainty, as seen by the fact that the "brick wall" which corrects it in fact smoothes the sharp boundary.

Our result raises the question whether the entanglement and statistical mechanics definitions of black hole entropy might refer to the same quantity. Both are proportional to area. The UV divergence may be tamed with a UV cutoff, and the boundary divergence by smearing out the boundary (both procedures might turn out to be equivalent). So the two expressions could be expressing the same quantity. If this is the case, then the microscopic counting of the number of states becomes tantamount to counting the correlations between the observed and unobserved regions of spacetime. Black hole entropy has also been shown, from thermodynamic considerations as well as explicit calculations in string theory, to equal one fourth of the horizon area. An open problem is to obtain the factor of 1/4 in either definitions of black hole entropy.

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A. Details of nonrelativistic smoothed momentum fluctuations

We wish to calculate the expectation value of the smoothed operators $(P^2)^V$ which can be used to evaluate H^V and other smooth operators. The partial volume V is defined by a window function as described in the text.

The operator P^2 is given by

$$\begin{aligned} P^2 &= \sum_{\vec{p}} \vec{p} a^\dagger_{\vec{p}} a_{\vec{p}} \cdot \sum_{\vec{k}} \vec{k} a^\dagger_{\vec{k}} a_{\vec{k}} \\ &= \sum_{\vec{p},\vec{k}} \vec{p}\cdot\vec{k}\ a^\dagger_{\vec{p}} \left(a^\dagger_{\vec{k}} a_{\vec{p}} + \left[a_{\vec{p}}, a^\dagger_{\vec{k}} \right] \right) a_{\vec{k}} \\ &= \sum_{\vec{p},\vec{k}} \vec{p}\cdot\vec{k} \left(a^\dagger_{\vec{p}} a^\dagger_{\vec{k}} a_{\vec{p}} a_{\vec{k}} + \delta_{\vec{p}\vec{k}} a^\dagger_{\vec{p}} a_{\vec{k}} \right). \\ &= \sum_{\vec{p},\vec{k}} \vec{p}\cdot\vec{k} \left(a^\dagger_{\vec{p}} a^\dagger_{\vec{k}} a_{\vec{p}} a_{\vec{k}} \right) + \sum_{\vec{p}} p^2 a^\dagger_{\vec{p}} a_{\vec{p}} \end{aligned} \tag{A.1}$$

Evaluating the expectation value:

$$\langle\psi|(P^2_{\text{smooth}})^V|\psi\rangle = \int d^3r_1\, d^3r_2\ \langle 0|\, \Psi\left(\vec{r}_1\right) f\left(\vec{r}_1, w\right) \sum_{\vec{p},\vec{k}} \vec{p}\cdot\vec{k} \left(a^\dagger_{\vec{p}} a^\dagger_{\vec{k}} a_{\vec{p}} a_{\vec{k}} + \delta_{\vec{p}\vec{k}} a^\dagger_{\vec{p}} a_{\vec{k}} \right) f\left(\vec{r}_2, w\right) \Psi^\dagger\left(\vec{r}_2\right) |0\rangle$$

$$= \int d^3r\, d^3r_2\, f\left(\vec{r}_1, w\right) f\left(\vec{r}_2, w\right) \sum_{\vec{q},\vec{s}} \frac{e^{i\vec{q}\vec{r}_1}}{\sqrt{\Omega}} \frac{e^{-i\vec{s}\vec{r}_2}}{\sqrt{\Omega}} \langle 0| \sum_{\vec{p},\vec{k}} \vec{p}\cdot\vec{k}\ a_{\vec{q}} a^\dagger_{\vec{p}} a^\dagger_{\vec{k}} a_{\vec{p}} a_{\vec{k}} a^\dagger_{\vec{s}} + \sum_{\vec{p}} p^2 a_{\vec{q}} a^\dagger_{\vec{p}} a_{\vec{p}} a^\dagger_{\vec{s}} |0\rangle\,. \tag{A.2}$$

Since

$$\left\langle 0 \left| a_{\vec{q}} a_{\vec{p}}^{\dagger} a_{\vec{p}} a_{\vec{s}}^{\dagger} \right| 0 \right\rangle = \delta_{\vec{p}\vec{q}} \delta_{\vec{p}\vec{s}} \tag{A.3}$$

and

$$\langle 0 | \, a_{\vec{q}} a_{\vec{p}}^{\dagger} a_{\vec{k}}^{\dagger} a_{\vec{p}} a_{\vec{k}} a_{\vec{s}}^{\dagger} \; | 0 \rangle = 0, \tag{A.4}$$

the expectation value of the smooth operator is then

$$\begin{aligned} \langle \psi | (P^2_{\text{smooth}})^V | \psi \rangle &= \int d^3r \, d^3r_2 \, f\left(\vec{r}_1, w\right) f\left(\vec{r}_2, w\right) \sum_{\vec{q},\vec{s}} \frac{e^{i\vec{q}\vec{r}_1}}{\sqrt{\Omega}} \frac{e^{-i\vec{s}\vec{r}_2}}{\sqrt{\Omega}} p^2 \delta_{\vec{p}\vec{q}} \delta_{\vec{p}\vec{s}} \\ &= \int d^3r \, \vec{\nabla} f\left(\vec{r}, w\right) \cdot \vec{\nabla} f\left(\vec{r}, w\right) \\ &= \sum_{\vec{p}} p^2 f(\vec{p}, w) f(-\vec{p}, w) \; . \end{aligned} \tag{A.5}$$

B. Details of relativistic smoothed energy

In a relativistic theory the hamiltonian is given by $\widehat{H} = \int \frac{d^3k}{(2\pi)^3} k_0 a_k^{\dagger} a_k$ in momentum space. In configuration space, the expectation value of the smoothed restricted hamiltonian is given by the relativistic scalar product,

$$\begin{aligned} \left\langle \psi \left| (H_{\text{smooth}})^V \right| \psi \right\rangle = -i \int d^3r_1 \, d^3r_2 \Big[\Big\langle 0 \Big| \Psi\left(\vec{r}_1, t_1\right) f\left(\vec{r}_1, w\right) \partial_{t_2} \Big(H \, f\left(\vec{r}_2, w\right) \Psi^{\dagger}\left(\vec{r}_2, t_2\right) \Big) - \\ -\partial_{t_1} \Big(\Psi\left(\vec{r}_1, t_1\right) f\left(\vec{r}_1\right) \Big) H \, f\left(\vec{r}_2\right) \Psi^{\dagger}\left(\vec{r}_2\right) \Big| 0 \Big\rangle \Big] \Big|_{t_1 = t_2} \equiv A - B \end{aligned} \tag{B.1}$$

The first term A is given by

$$\begin{aligned} A = \int d^3r_1 \, d^3r_2 \, f\left(\vec{r}_1, w\right) f\left(\vec{r}_2, w\right) \Big\langle 0 \Big| \int \frac{d^3p}{\sqrt{(2\pi)^3 \, 2p_0}} \left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{r}_1 - ip_0 t_1} + a_{\vec{p}}^{\dagger} e^{-i\vec{p}\cdot\vec{r}_1 + ip_0 t_1} \right) \times \\ \int \frac{d^3q}{(2\pi)^3} q_0 a_q^{\dagger} a_q \; \times -i\partial_{t_2} \int \frac{d^3k}{\sqrt{(2\pi)^3 \, 2k_0}} \left(a_{\vec{k}} e^{i\vec{k}\cdot\vec{r}_2 - ik_0 t_2} + a_{\vec{k}}^{\dagger} e^{-i\vec{k}\cdot\vec{r}_2 + ik_0 t_2} \right) \Big| 0 \Big\rangle \Big|_{t_1, t_2 = 0} \end{aligned} \tag{B.2}$$

where $p_0^2 = \vec{p}^2$, $k_0^2 = \vec{k}^2$. The second term B can be expressed in a similar straightforward manner.

We first perform the momentum integrals and evaluate the expectation value. This integral includes the following sets of operators:

$$a_{\vec{p}} a_q^{\dagger} a_q a_{\vec{k}} \, , \; a_{\vec{p}} a_q^{\dagger} a_q a_{\vec{k}}^{\dagger} \, , \; a_{\vec{p}}^{\dagger} a_q^{\dagger} a_q a_{\vec{k}} \, , \; a_{\vec{p}}^{\dagger} a_q^{\dagger} a_q a_{\vec{k}}^{\dagger},$$

but only the second term yields a non-vanishing contribution,

$$\begin{aligned}
&\int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\; d^3q\frac{q_0k_0}{\sqrt{2k_0}\sqrt{2p_0}}\left\langle 0\right|\left(a_{\vec{p}}e^{i\vec{p}\cdot\vec{r}_1}+a^{\dagger}_{\vec{p}}e^{-i\vec{p}\cdot\vec{r}_1}\right)\,a^{\dagger}_{q}a_{q}\left(a_{\vec{k}}e^{i\vec{k}\cdot\vec{r}_2}+a^{\dagger}_{\vec{k}}e^{-i\vec{k}\cdot\vec{r}_2}\right)\left|0\right\rangle \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\; d^3q\frac{q_0k_0}{\sqrt{2k_0}\sqrt{2p_0}}e^{i\vec{p}\cdot\vec{r}_1-i\vec{k}\cdot\vec{r}_2}\left\langle 0\middle|a_{\vec{p}}\;a^{\dagger}_{q}\;a_{q}\;a^{\dagger}_{\vec{k}}\middle|0\right\rangle \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\; d^3q\frac{q_0k_0}{\sqrt{2k_0}\sqrt{2p_0}}e^{i\vec{p}\cdot\vec{r}_1-i\vec{k}\cdot\vec{r}_2}\delta(\vec{p}-\vec{q})\;\delta(\vec{k}-\vec{q}).
\end{aligned} \tag{B.3}$$

Substituting the result of eq. B.3 into eq. (B.2) we find

$$\begin{aligned}
A &= \int d^3r_1\,d^3r_2\,f\left(\vec{r}_1,w\right)f\left(\vec{r}_2,w\right)\int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\; d^3q\frac{q_0k_0}{\sqrt{2k_0}\sqrt{2p_0}}e^{i\vec{p}\cdot\vec{r}_1-i\vec{k}\cdot\vec{r}_2}\delta(\vec{p}-\vec{q})\;\delta(\vec{k}-\vec{q}) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\; d^3q\frac{q_0k_0}{\sqrt{2k_0}\sqrt{2p_0}}f(\vec{p},w)\;f(-\vec{k},w)\;\delta(\vec{p}-\vec{q})\;\delta(\vec{k}-\vec{q}) \\
&= \frac{1}{2}\int d^3p\;p\;f(\vec{p},w)\;f(-\vec{p},w),
\end{aligned} \tag{B.4}$$

where $p^2=\vec{p}^2$, and $f(\vec{p},w)$ is the Fourier transform of $f(\vec{r},w)$.

Repeating the same steps for B we find $B=-A$ so the

$$\begin{aligned}
\left\langle\psi\left|(H_{\text{smooth}})^V\right|\psi\right\rangle &= \int d^3p\;p\;f(\vec{p},w)\;f(-\vec{p},w) \\
&= \int d^3r\;f(\vec{r},w)\;\sqrt{\vec{\nabla}^2}\;f(\vec{r},w)
\end{aligned} \tag{B.5}$$

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