Rotational Symmetry of Classical Orbits, Arbitrary Quantization of Angular Momentum and the Role of Gauge Field in Two-Dimensional Space

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We study the quantum-classical correspondence in terms of coherent wave functions of a charged particle in two-dimensional central-scalar-potentials as well as the gauge field of a magnetic flux in the sense that the probability clouds of wave functions are well localized on classical orbits. For both closed and open classical orbits, the non-integer angular-momentum quantization with the level-space of angular momentum being greater or less than $\hbar$ is determined uniquely by the same rotational symmetry of classical orbits and probability clouds of coherent wave functions, which is not necessarily 2π-periodic. The gauge potential of a magnetic flux impenetrable to the particle cannot change the quantization rule but is able to shift the spectrum of canonical angular momentum by a flux-dependent value, which results in a common topological phase for all wave functions in the given model. The quantum mechanical model of anyon proposed by Wilczek (Phys. Rev. Lette. 48, 1144) becomes a special case of the arbitrary-quantization.

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I. INTRODUCTION

Recently, renewal of interest has been evoked to the fractional angular momentum (FAM) in two-dimensional (2D) space in relation with the correspondence between quantum mechanical wave-functions and classical periodic-orbits[1, 2]. The FAM is only possible in 2D multiply-connected-space, since in three- or higher-dimensional space, the angular momentum being integer or half-integer is completely determined by the commutation relation of angular-momentum operators. In 2D space, the angular momentum operator has only one component, which does not give rise to a unique determination of the angular-momentum eigenvalue, and the common belief of integer-quantization is based on the 2π-periodic boundary-condition which, however, is not justified.

Wilczek in his pioneering work proposed for the first time a quantum mechanical model consisting of a charged particle and magnetic flux in 2D space to demonstrated the fractional eigenvalues of angular momentum known as anyon [3, 4]. In the existence of gauge field, however, we have both the kinetic angular momentum (KAM) and the canonical angular momentum (CAM), which are different, because of the velocity-dependent forces [5–8]. It is true that "the generator of rotation should be the CAM and is prescribed by Noether's theorem as a conserved quantity"[9], which is gauge dependent and integer-quantized [7–9]. While the KAM is gauge invariant dynamic quantity, generally fractional because of the gauge field [7–9]. Although the consistence of fractional CAM with the Aharonov-Bohm (AB) phase-interference was shown long ago [7] it remains a long standing open question whether or not the fractional CAM plays a role in quantum physics. It is worthwhile to remark that the gauge potential of an AB flux can only shift the angular momentum eigenvalues by a common fractional number but cannot change the integer-quantization rule, namely the eigenvalue-space of angular momentum is still $\hbar$ [7, 11].

The existence of FAM in a wide class of 2D central potentials without gauge field has been discussed more recently [1, 2] by the localization of coherent wave functions on classical orbits, which imposes a special boundary condition leading to the unusual angular phase of wave functions. Following the interesting studies of Ref. [1, 2], which provide an exactly solvable model both quantum mechanically and classically, we consider a charged particle in the gauge field of magnetic flux-string perpendicular to the 2D plane with the central scalar-potentials in addition and study the non-integer quantization of angular momentum (NIQAM) in terms of quantum-classical correspondence. Since the coherent state is constructed by the superposition of angular momentum eigenstates and thus the probability clouds have to possess the same rotational symmetry as that of classical orbits, which is not necessarily 2π-periodic. As a consequence the NIQAM with the level-space being greater or less than $\hbar$ appears naturally along with the correspondence principle. By the explicit calculation it is shown that only the shift of CAM eigenvalues by the gauge field, which gives rise to a common topological phase for all wave functions, is in agreement with the correspondence. The validity of wave functions with a topological phase is further confirmed by the expectation values of KAM operator, which coincide with the classical values of KAM $\pounds^k$. The long standing problem whether or not the FAM is related to the CAM is resolved and the quantum mechanical model of anyon proposed by Wilczek [3, 4] emerges as a special case of $\mu = 1$ with 2π-rotational-symmetry of classical orbits.

More specifically, we refer here to the particular zero-energy states, which can be obtained analytically for both classical- and quantum-solutions [10–12]. On the other hand the zero-energy states are of importance in various fields such as the cold-atom collisions[13, 14], the construction of vortex lattices [15], and quantum cosmology [16].

II. CLASSICAL ORBITS AND ROTATIONAL SYMMETRY

We consider a charged particle of charge e and mass m in a gauge field of infinitely long magnetic flux-line of total flux Φ located at the origin of 2D space and central scalar-potential of form [1, 2]

$$A_0(r) = -\frac{\gamma_\upsilon}{r^{2\mu+2}}, \quad \gamma_\upsilon > 0, \quad -\infty < \upsilon < \infty \tag{1}$$

Where $r = \sqrt{x^2+y^2}$ and $\upsilon = 2\mu + 2$.

Outside the flux-line ($r > 0$), the Lorentz force on the charged particle is always zero because of the vanishing magnetic field $\vec{B} = 0$, while the vector potential in the polar coordinate is seen to be

$$\vec{A} = \frac{\Phi}{2\pi r}\overrightarrow{e}_\varphi$$

with $\overrightarrow{e}_\varphi$ being the unit-vector of angular direction. In order to establish the quantum-classical correspondence, we ought to evoke the canonical variables. In the polar coordinates (r, φ), the Lagrangian of the system is seen to be

$$L = \frac{1}{2}m[\dot{r}^2 + (r\dot{\varphi})^2] - eA_0(r) + L_{WZ} \tag{2}$$

where $L_{WZ} = \alpha\hbar\dot{\varphi}$ is so-called the Wess-Zumino topological interaction-term with the parameter $\alpha = \Phi/\Phi_0$ being the dimensionless magnetic flux in the quantum-unit $\Phi_0 = ch/e$. The Wess-Zumino term does not affect the equation of motion but the initial value of angular momentum. Canonical momentums corresponding to the coordinate variables r, φ are defined by

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \tag{3}$$

$$\pounds^c = \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi} + \alpha\hbar \tag{4}$$

Here $\pounds^c$ is CAM, while $\pounds^k = mr^2\dot{\varphi}$ is the KAM. Then the Hamiltonian is

$$H = \frac{p_r^2}{2m} + \frac{(\pounds^c - \alpha\hbar)^2}{2mr^2} + eA_0(r), \tag{5}$$

From canonical equations we find that both the CAM and KAM are conserved quantity

$$\frac{d\pounds^c}{dt} = -\frac{\partial H}{\partial \varphi} = 0, \quad \pounds^k = \pounds^c - \alpha\hbar$$

For the case of zero-energy and nonvanishing initial KAM $\pounds^k = \pounds^c - \alpha\hbar \neq 0$ we have

$$\frac{1}{2}m(\frac{\pounds^c - \alpha\hbar}{mr^2})^2[(\frac{dr}{d\varphi})^2 + r^2] - e\frac{\gamma_\upsilon}{r^\upsilon} = 0.$$

We assume an initial-value that

$$\pounds^k = \xi_k\hbar, \tag{6}$$

where ξ_k is an arbitrary dimensionless-quantity. Thus in the considered case only the CAM is shifted by the flux. Introducing a dimensionless variable

$$u = r/\tilde{a}_c, \quad \tilde{a}_c = \frac{(2me\gamma_\upsilon)^{1/2\mu}}{[\xi_k\hbar]^{1/\mu}} \tag{7}$$

we obtain the equation of particle trajectories such as [10, 18]

$$(\frac{du}{d\varphi})^2 + u^2 = u^{4-\upsilon} = u^{2-2\mu} \tag{8}$$

The general solution of Eq.(8) is given in Ref.[10, 18]

$$r^\mu = \tilde{a}_c^\mu \cos[\mu(\varphi - \varphi_0)] \tag{9}$$

Using atomic unit $\frac{me_s^2}{\hbar^2}$ with $e_s = e^2/(4\pi\varepsilon_0)$, which is the dimension of length, we set $\sqrt{\frac{2m\gamma_\upsilon e}{\hbar^2}} = 1$ in the numerical evaluation. The classical orbits with the initial angle setting to zero $\varphi_0 = 0$ are shown in Figs. 1-5 (solid green curves), which depend on the initial angular momentum ξ_k only for given scalar potential. In general, we have closed orbits (Figs. 1-4) for $\mu > 0$ ($\upsilon > 2$), while open trajectories for $\mu < -2$ seen from Fig.5. From Eq.(9) it is obvious that the rotation symmetry of classical orbits depends on the power-index μ such that the orbits are invariant under a rotation-angle $\frac{2\pi}{|\mu|}$, and 2π-symmetry holds only for $|\mu| = 1$. We demonstrate in this paper that the angular momentum quantization can be determined only by the rotation symmetry of classical orbits based on the requirement of quantum-classical correspondence. The rotation angles of symmetry for closed orbits in Figs.1-4 are 2π, 6π, $\frac{4\pi}{5}$, and $\frac{2\pi}{5}$ respectively.

III. ANGULAR MOMENTUM QUANTIZATION

In the polar coordinates (r, φ), the zero-energy Schrödinger equation

$$\frac{-\hbar^2}{2m}[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}(\frac{\partial}{\partial \varphi} - i\alpha)^2]\psi + eA_0\psi = 0 \tag{10}$$

becomes

$$(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{\lambda^2}{r^2})R(r) + \frac{2me}{\hbar^2}\frac{\gamma_\upsilon}{r^\upsilon}R(r) = 0,$$

$$(\frac{\partial}{\partial\varphi}-i\alpha)^2\Theta(\varphi)=-\lambda^2\Theta(\varphi).$$

by the separation of variables $\psi=R(r)\Theta(\varphi)$. The eigenvalue solution of angular part Eq. (10) is

$$\Theta_{l^c}(\varphi)=N_\varphi e^{il_c\varphi}. \quad (11)$$

with N_φ being the normalization constant and the CAM eigenvalue l_c to be determined. For the usual requirement of 2π-periodic boundary-condition, $\Theta(\varphi)=\Theta(\varphi+2\pi)$, one can obtain the integer angular momentum quantization and the normalization constant $N_\varphi=\frac{1}{\sqrt{2\pi}}$. However the 2π-periodic boundary-condition is not justified in the $2D$-space, although the potential $A_0(r)=-\frac{\gamma_\upsilon}{r^{2\mu+2}}$ is symmetric under rotation. It has been demonstrated that a macroscopic quantum state, here SU(2) coherent state [1] can be constructed with probability density of wave functions well localized on the classical orbits for the quantum-classical correspondence, which results in a special boundary condition of the angular momentum eigenstates such that rotational period of wave function is not necessarily 2π but should be the same as that of classical orbits $\Theta(\varphi)=\Theta(\varphi+\frac{2\pi}{|\mu|})$[1]. Thus the CAM eigenvalue is no longer integer but should be set as

$$l_n^c=n\left|\mu\right|, \quad (12)$$

where n is a integer. The angular momentum now is quantized with an eigenvalue-space

$$\Delta l=|\mu| \quad (13)$$

The integer-quantization is only a special case of $|\mu|=1$. The normalization constant becomes

$$N_\varphi=\sqrt{\frac{|\mu|}{2\pi}} \quad (14)$$

The radial equation is

$$(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{(l_n^c-\alpha)^2}{r^2})R(r)+\frac{2me}{\hbar^2}\frac{\gamma_\upsilon}{r^\upsilon}R(r)=0, \quad (15)$$

In the choice of Eq. (12) for l_c, the n-th KAM eigenvalue is shifted by the flux number that

$$l_n^k=l_n^c-\alpha=n\left|\mu\right|-\alpha \quad (16)$$

indicating the dynamic effect of the gauge potential in contradiction with the classical solution in which gauge field does not apply a torque on the particle. We do have the other choice of l_c such that

$$l_n^c=n\left|\mu\right|+\alpha \quad (17)$$

while the KAM eigenvalue

$$l_n^k=n\left|\mu\right| \quad (18)$$

does not depend on the flux in consistence with classical solution. The common-phase $e^{i\alpha\varphi}$ factor called the topological phase [7, 11] does not change the angular momentum quantization Eq.(13) nor the normalization constant Eq.(14) since whole eigenfunctions have the same additional angular phase. It is the main goal of the present paper that only the choice of CAM Eq.(17) gives rise to the exact quantum-classical correspondence.

A. Topological phase of the gauge field and exact quantum-classical correspondence

We demonstrate in this section that only the CAM eigenvalues Eq.(17) consist with the quantum-classical correspondence, such that the probability densities of coherent wave functions are well localized on classical orbits. Introducing the dimensionless radius $\chi=r/\tilde{a}_q$ and $y=1/\chi$, where $\tilde{a}_q$ is a quantity with dimension of length, we obtain

$$(y^2\frac{d^2}{dy^2}+y\frac{d}{dy}-(l_n^k)^2+B^2y^{\upsilon-2})R_\tau(y)=0 \quad (19)$$

where the parameter B is defined by

$$B^2\equiv\frac{2me\gamma_\upsilon}{\hbar^2\tilde{a}_q^{\upsilon-2}}$$

and the corresponding KAM eigenvalues l_n^k given by Eq.(18) do not depend on the flux number. Square-integrable solutions of Eq. (19) are found in terms of Bessel functions of the first kind [17, 19–21]

$$R_{l_n^k}(y)=N_{l_n^k}J_{\frac{l_n^k}{|\mu|}}(\frac{1}{|\mu|\,r^\mu}) \quad (20)$$

with the normalization constant given by

$$N_{l_n^k}=\sqrt{2\sqrt{\pi}\left|\mu\right|^{2/\mu}\frac{\Gamma(1+1/\mu)\Gamma[1+l_n^k/\left|\mu\right|+1/\mu]}{\Gamma(1/2+1/\mu)\Gamma[l_n^k/\left|\mu\right|-1/\mu]}} \quad (21)$$

where we have set the parameter $\sqrt{\frac{2m\gamma_\upsilon e}{\hbar^2}}=1$, and thus $B\tilde{a}_q^\mu=1$, $\quad\tilde{a}_q^2B^{2/\mu}=1$ according to the definition of B. If the following conditions are satisfied:

$$\mathrm{Re}(\frac{2l_n^k}{|\mu|}+1)>\mathrm{Re}(\frac{2}{\mu}+1)>0, \quad (22)$$

we have bound states corresponding to the classical closed-orbits as demonstrated in Ref [18]. For the zero-energy $E=0$ there exist classical solutions of closed-orbits with any non-zero angular momentum when $\upsilon>2$, while the condition of bound states in quantum mechanics is $l_n^k>1$. It has been shown that [20] the normalizable quantum solutions can be classified to two classes: (1) bound states ($l_n^k>1$) for $\mu>0$ ($\upsilon>2$) corresponding to the classical closed-orbits (Figs. 1-4, solid green curves) (2) scattering states ($l_k\geqslant 0$) for $\mu<-2$ ($\upsilon<-2$),which are normalizable, corresponding to classical open-orbits (Fig. 5, solid green curves). At the region $-2\leq\upsilon\leq 2$ the solutions of wave functions are not square-integrable [19, 20], which we do not discuss in the present paper. The complete eigenfunctions of the Schrödinger equation can be written in the explicit form as [17, 19–21]

$$\psi_{\mu,l_n^k}(r,\varphi)=N_\varphi e^{i(l_n^k+\alpha)\varphi}N_{l_n^k}J_{\frac{l_n^k}{|\mu|}}(\frac{1}{|\mu|\,r^\mu}). \quad (23)$$

Following Refs.[1, 2] we construct the stationary SU(2) coherent-state for the central-scalar-potentials in the standard way as [22]

$$\Psi_{\mu,N}(r,\varphi) = \frac{1}{(2)^{N/2}} \sum_{n=0}^{N} \binom{N}{n}^{1/2} N_{\varphi} e^{i(l_n^k + c + \alpha)\varphi} N_{l_n^k + c} J_{\frac{l_n^k + c}{|\mu|}}\left(\frac{1}{|\mu| \, r^{\mu}}\right) \tag{24}$$

with l_n^c and l_n^k given in Eqs.(17), (18) respectively. An additional angular momentum number

$$c = n_c \, |\mu| \tag{25}$$

with n_c being a integer-parameter is introduced to adjust the probability density, which does not change the rotational symmetry of wave-function density but the spatial expansion of it. In the following section we will see that the parameter n_c is not arbitrary but is related to the classical KAM $\pounds^k$ such that the expectation value of KAM operator coincides with the classical value. The wave functions are normalized to unity in the angular range of φ being from $-\pi$ to π. From Eq. (24) we can see that the common topological phase-factor resulted by the gauge potential does not affect the probability density of wave functions in agreement with the classical assumption of no torque applied on the charged particle. In Figs. 1-5 the practical values of parameters are given by $\mu = 1, \alpha = 6\frac{1}{2}, n_c = 8$, $\mu = \frac{1}{3}, \alpha = \frac{5}{4}$, $n_c = 9$, $\mu = \frac{5}{2}, \alpha = 31\frac{1}{3}$, $n_c = 14$, $\mu = 5, \alpha = 55\frac{3}{5}, n_c = 12$, and $\mu = -\frac{7}{3}, \alpha = 42\frac{7}{10}$, $n_c = 19$ respectively. In all figures the total number of eigenfunctions is $N = 30$. The density $|\Psi_{\mu,N}(r,\varphi)|^2$ of coherent-state wave function possesses the same rotation symmetry of the classical orbits. Figs. 1-5 (a) show that the probability densities of coherent wave functions are well localized on the classical orbits in each case indicating the exact quantum-classical correspondence. We thus obtain the angular-momentum quantizations with the level-spaces $1, \frac{1}{3}, \frac{5}{2}, 5$, and $\frac{7}{3}$ (of $\hbar$) respectively. The CAM spectrum is shifted by the corresponding flux number α.

Besides the AB interference [7] the fractional spins have many important physical applications, for example, two-vortex system in a superfluid film [23], necklace-ring [24, 25], chiral-wave superconductor[26] or quantum billiard[27] inside a boundary defined by the wedge-shaped section of a circle.

Shift of the KAM eigenvalues by the gauge potential and break down of the correspondence: We, of course, can have the other choice of CAM eigenvalues i.e. Eq.(12), which does not depend on the flux and as a consequence the spectrum of KAM is shifted by the gauge potential as shown in Eq.(16). In this case we do not have the topological phase of the wave functions but have the flux-dependent KAM eigenvalues instead. The quantization

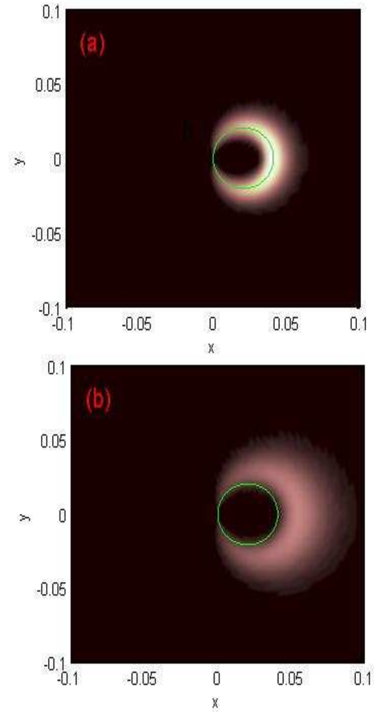

FIG. 1: (Colour online) Probability density image $|\Psi_{\mu,N}(r,\varphi)|^2$ and the closed classical-orbit (solid green curve) for $\mu = 1(\upsilon = 4), \alpha = 6\frac{1}{2}, \xi_k = 23$. (a) shift of CAM; (b) shift of KAM.

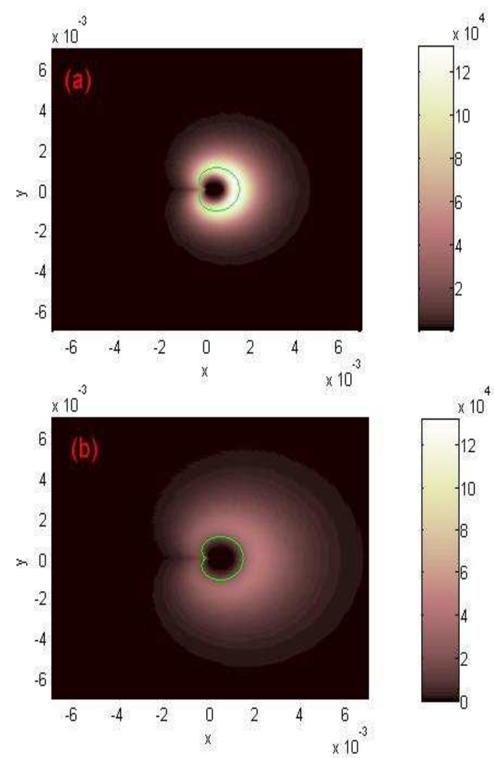

FIG. 2: (Colour online) $\mu = \frac{1}{3}(\upsilon = \frac{8}{3}), \alpha = \frac{5}{4}, \xi_k = 8$.

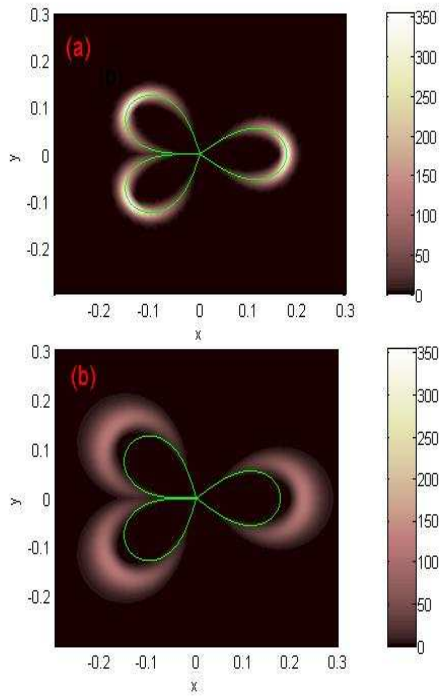

FIG. 3: (Colour online) $\mu = \frac{5}{2}(\upsilon = 7), \alpha = 31\frac{1}{3}, \xi_k = 72.5$.

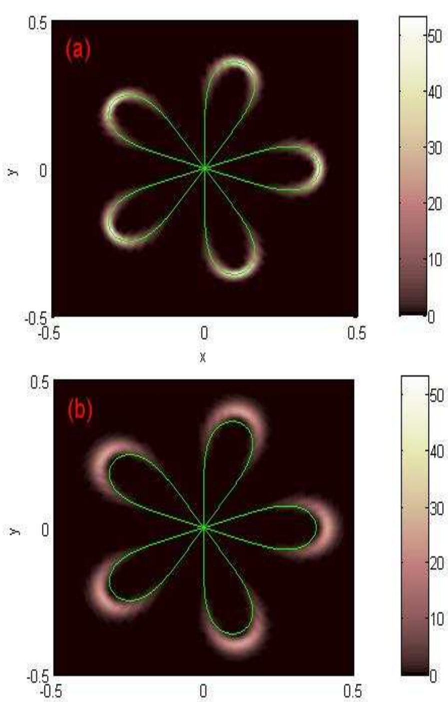

FIG. 4: (Colour online) $\mu = 5(\upsilon = 12), \alpha = 55\frac{3}{5}, \xi_k = 135$.

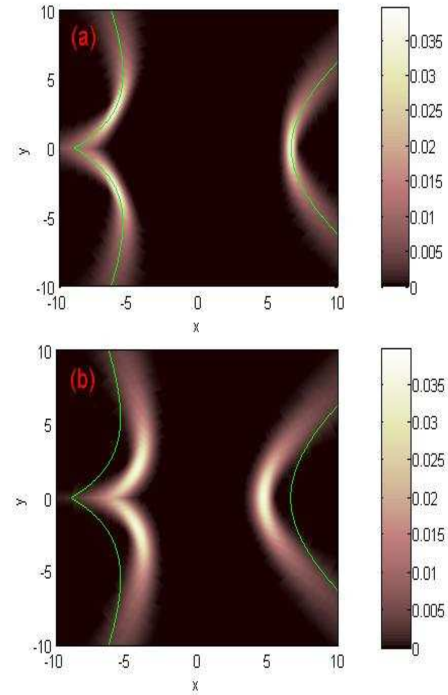

FIG. 5: (Colour online) Probability density image $|\Psi_{\mu,N}(r,\varphi)|^2$ and the open classical-orbit (solid green curve) for $\mu = -\frac{7}{3}(\upsilon = -\frac{8}{3}), \alpha = 42\frac{7}{10}, \xi_k = \frac{238}{3}$.(a) shift of CAM (b) shift of KAM.

of angular momentum, namely the level-space of eigenvalues, is not changed by the gauge potential. Replacing the KAM eigenvalues l_n^k in Eqs.(23,24) by Eq.(16) the corresponding probability densities of coherent states are depicted in Figs.1-5 (b), from which we see that probability densities are no longer localized no the classical orbits indicating additional torques applied on the particle in contradiction with the classical assumption.

Expectation values of angular momentum: We now calculate the expectation values of KAM operator

$$\hat{\pounds}^k = -i\hbar[\frac{\partial}{\partial\varphi} - i\alpha]$$

in the SU(2) coherent-state to find the explicit relation between the adjusting-parameter c and the initial classical-KAM $\pounds^k = \xi_k\hbar$ in order to confirm further the quantum-classical correspondence. The average of KAM operator in the SU(2) coherent state Eq.(24) with the topological phase is evaluated as

$$\left\langle \hat{\pounds}^k \right\rangle = \left\langle \Psi_{\mu,N} \left| \hat{\pounds}^k \right| \Psi_{\mu,N} \right\rangle = \hbar\left|\mu\right|[n_c + \frac{N}{2}] \quad (26)$$

Substituting the corresponding parameter values of μ, n_c, and N into the above formula one can see the exact agreement with the initial-values of classical KAM such that

$$\left\langle \hat{\pounds}^k \right\rangle = \xi_k \quad (27)$$

for $\xi_k = 23, 8, 72.5, 135, \frac{238}{3}$respectively. However, for the CAM eigenvalue-choice of Eq.(12), namely the shift of KAM eigenvalue by the gauge potential**,** the average of KAM operator becomes

$$\left\langle \hat{\mathcal{L}}^k \right\rangle = \hbar[(n_c + \frac{N}{2})\,|\mu| - \alpha] \tag{28}$$

which, of course, disagrees with classical values.

IV. CONCLUSION

In summary, the NIQAM in two-dimensional space is demonstrated in terms of quantum-classical correspondence with an exactly solvable model, such that the probability clouds of macroscopic quantum states, here the SU(2) coherent wave functions, are well localized on the classical orbits. The eigenfunctions of angular momentum have to possess the same rotational symmetry of classical orbits, in which the rotation-period can be greater or less than 2π depending on the power-index μ of the central potential only. As a consequence the level-space of angular momentum spectrum is less or greater than $\hbar$ and the integer-quantization (level-space $\hbar$) is possible only if $\mu = 1$. The gauge potential of AB-flux does not affect the angular momentum quantization but can shift the spectrum of angular momentum by a common value. By explicit calculations it is shown that the quantum-classical correspondence results in the unambiguous determination of CAM eigenvalues with a common topological-phase in the wave functions, the probability densities of which coincide with the classical orbits for any power-index of potentials. The quantum mechanical model of anyon proposed by Wilczek [3, 4] and latter clarified as the fractional CAM in Ref.[7] becomes a special case of the present model with $\mu = 1$.

V. ACKNOWLEDGMENT

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