DAMTP-2010-79

Scalar–flat Kähler metrics with conformal Bianchi V symmetry

Maciej Dunajski*
Department of Applied Mathematics and Theoretical Physics,
University of Cambridge,
Wilberforce Road, Cambridge CB3 0WA, UK.
and
Institute of Mathematics of the Polish Accademy of Sciences,
Śniadeckich 8, 00-956 Warsaw, Poland.

Prim Plansangkate†
Centre de Recherches Mathématiques (CRM),
Université de Montréal, C.P. 6128,
Montréal (Québec) H3C 3J7, Canada.

Dedicated to Maciej Przanowski on the occasion of his 65th birthday.

Abstract

We provide an affirmative answer to a question posed by Tod [19], and construct all four–dimensional Kähler metrics with vanishing scalar curvature which are invariant under the conformal action of Bianchi V group. The construction is based on the combination of twistor theory and the isomonodromic problem with two double poles. The resulting metrics are non–diagonal in the left–invariant basis and are explicitly given in terms of Bessel functions and their integrals. We also make a connection with the LeBrun ansatz, and characterise the associated solutions of the $SU(\infty)$ Toda equation by the existence a non–abelian two–dimensional group of point symmetries.

*Email: M.Dunajski@damtp.cam.ac.uk
†Email: plansang@CRM.UMontreal.CA

1 Introduction

Let (M, g) be a Riemannian four–manifold with anti–self–dual (ASD) Weyl curvature. The metric g is said to have cohomogeneity–one if (M, g) admits an isometry group acting transitively on codimension-one surfaces in M. We shall say that a conformal structure $[g]$ on M has cohomogeneity–one, if there exists a cohomogeneity–one metric $g \in [g]$.

The four-dimensional cohomogeneity-one metrics can be classified according to the Bianchi type of the three-dimensional real Lie algebra [1] of the isometry group G. Locally $M = \mathbb{R} \times G$, and the problem of finding ASD cohomogeneity–one metrics reduces to solving a system of ODEs, as the group coordinates do not appear in the Weyl tensor. Moreover, the reduction–integrability dogma suggests that the resulting ODEs will be in some sense solvable as the underlying anti–self–duality equations are integrable by twistor transform [15, 11, 4]. This is indeed the case. If $G = SU(2)$ then the ODEs reduce to Painleve VI, or (if additional assumptions are made about the metric) Painleve III ODEs [17, 8, 18, 19, 13, 3]. In [19] Tod has analysed the general case of conformal ASD cohomogeneity–one metrics with arbitrary group, diagonal in the basis of left–invariant one forms on the group. He has shown that cohomogeneity–one diagonal Bianchi V metrics are conformally flat. The diagonalisabilty assumption can only be made without loss of generality if the underlying metric is Einstein, so the existence of non–diagonal Bianchi V conformal structures has not been ruled out in Tod's work. In particular he has raised a question whether such structures (if they exist) can admit a Kähler metric in a conformal class. In this paper we shall use the method of isomonodromic deformations to construct all such conformal structures.

Let $\mathfrak{g}$ be a Lie algebra of Bianchi type V, and let G be a corresponding simply connected Lie group (see Appendix). The left–invariant vector fields on G satisfy

$$[L_0, L_1] = L_1, \quad [L_0, L_2] = L_2, \quad [L_1, L_2] = 0. \tag{1.1}$$

A general cohomogeneity–one metric on $M = \mathbb{R} \times G$ will be have the form

$$g = dt^2 + h_{jk}(t)\lambda^j \odot \lambda^k, \qquad j, k = 0, 1, 2$$

where λ^j are the Maurer–Cartan one–forms on G such that $L_j \lrcorner \lambda^k = \delta_j{}^k$.

We will say that a group action on a complex manifold is holomorphic if Lie derivatives of the complex stucture along any of the generators vanish.

Theorem 1.1 *Any cohomogeneity–one Bianchi V ASD conformal structure which admits a Kähler metric such that the group acts holomorphically can be locally represented by a cohomogeneity–one metric*

$$g = e^1 \odot \overline{e}^1 + e^2 \odot \overline{e}^2, \tag{1.2}$$

where the complex one–forms e^1, e^2 *are dual to the vector fields*

$$E_1 = \partial_t - \frac{i}{t}(r_0 L_0 + r_1 L_1 + r_2 L_2), \quad E_2 = p_0 L_0 + p_1 L_1 + p_2 L_2$$

and the real functions $\mathbf{r}(t) = (r_0(t), r_1(t), r_2(t))$ *and complex functions* $\mathbf{p}(t) = (p_0(t), p_1(t), p_2(t))$ *satisfy the linear system of ODEs*

$$\frac{d\mathbf{r}}{dt} = tIm\ (\overline{p}_0 \mathbf{p}), \quad t\frac{d\mathbf{p}}{dt} = i(r_0 \mathbf{p} - p_0 \mathbf{r}), \quad \frac{dp_0}{dt} = 0, \quad \frac{dr_0}{dt} = 0 \tag{1.3}$$

such that det $[\mathbf{r}, \mathbf{p}, \overline{\mathbf{p}}] \neq 0$*.*

We shall prove this theorem in Section 2, where we shall also show that equations (1.3) reduce to the Bessel equation and quadratures.

All ASD Kähler metrics have vanishing scalar curvature - we shall follow LeBrun [10] in calling them scalar–flat Kähler - and conversely all scalar–flat Kähler metrics have anti–self–dual Weyl tensor (we choose the natural orientation, where the Kähler two–form is self–dual). In Section 3 we shall construct the Kähler structure in the conformal class of Theorem 1.1. We shall demonstrate that although a G–invariant conformal factor turning the metric (1.2) into a Kähler metric does not exist, there is a simple function on G which does the job. We shall prove

Proposition 1.2 *Any conformally Bianchi–V scalar–flat Kähler metric can locally be put into a form*

$$g_K = \Omega^2\, g, \tag{1.4}$$

where g *is given by* (1.2) *and* $\Omega : G \to \mathbb{R}$ *is a function on the group such that*

$$L_0(\Omega) = \Omega, \quad L_1(\Omega) = 0, \quad L_2(\Omega) = 0. \tag{1.5}$$

The holomorphic (1,0) vector fields are given by E_1 *and* E_2*. Moreover* g_K *is Ricci flat if and only if it is flat.*

The right–invariant vector fields R_j on G generate the conformal transformations of the Kähler metric (1.4)

$$\mathcal{L}_{R_0} g_K = 2g_K, \quad \mathcal{L}_{R_1} g_K = 0, \quad \mathcal{L}_{R_2} g_K = 0,$$

where $\mathcal{L}$ denotes the Lie derivative. If we choose coordinates (ρ, x^1, x^2) on G (see Appendix) such that

$$L_0 = \rho\frac{\partial}{\partial\rho}, \quad L_1 = \rho\frac{\partial}{\partial x^1}, \quad L_2 = \rho\frac{\partial}{\partial x^2}, \qquad \rho \in \mathbb{R}^+, (x^1, x^2) \in \mathbb{R}^2$$

then $\Omega = \rho$.

Equations (1.3) imply that r_0 and p_0 are constants. The remaining equations reduce to the Bessel equation possibly with imaginary order. A particularly simple class of solutions characterised by $r_0 = 0$ is

$$\mathbf{r} = (0, tJ_1(t), tY_1(t)), \quad \mathbf{p} = (1, iJ_0(t), iY_0(t)),$$

where $J_\alpha(t)$ and $Y_\alpha(t)$ are Bessel functions of first and second type of order α. This leads to the following example of a Bianchi-V scalar flat Kähler metric

$$g_K = \frac{1}{4}(d\rho^2 + \rho^2 dt^2) + G_{AB}(t)dx^A dx^B, \qquad A, B = 1, 2 \tag{1.6}$$
$$\text{where} \quad G(t) = \frac{\pi^2 t^2}{16}\begin{pmatrix} Y_0^2 + Y_1^2 & -J_0Y_0 - J_1Y_1 \\ -J_0Y_0 - J_1Y_1 & J_0^2 + J_1^2 \end{pmatrix}.$$

All scalar–flat Kähler metrics admitting a Killing vector which also preserves the Kähler form arise from the LeBrun's ansatz [10], and are determined by a solution to the $SU(\infty)$-Toda field equation

$$u_{xx} + u_{yy} + (e^u)_{zz} = 0, \qquad \text{where} \quad u = u(x, y, z), \tag{1.7}$$

together with a solution to its linearisation. Kähler metrics arising from Proposition 1.2 admit a two–dimensional group of symmetries (generated by the right translations R_1 and R_2) preserving the Kähler from. Thus any linear combination of R_1 and R_2 should lead to a solution of (1.7). In Section 4 we shall characterise the solutions corresponding to the metric (1.6). They admit a two–dimensional non–abelian group of Lie point symmetries and can be found by making an ansatz $u = u(y/z)$.

Acknowledgements. We are grateful to Philip Boalch, Andrew Dancer and Paul Tod for helpful discussions.

2 ASD structures and isomonodromy

A Kähler structure on a four–dimensional real manifold M consists of a pair (g, I) where g is a Riemannian metric and $I : TM \to TM$ is a complex structure such that, for any vector fields X, Y, $g(X, Y) = g(IX, IY)$ and the two–form ω defined by

$$\omega(X, Y) = g(IX, Y)$$

is closed. Given an orientation on M the Hodge operator $* : \Lambda^2 \to \Lambda^2$ satisfies $*^2 = \mathrm{Id}$ and gives a decomposition $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ of two—forms into self–dual (SD) and anti–self–dual (ASD) components. The two–form ω induces a natural orientation on M given by the volume form $\omega \wedge \omega$. With respect to this orientation ω is self–dual. It is well known [16, 4] that if the scalar curvature of a Kähler metric vanishes, then the Weyl tensor of the underlying conformal structure is ASD. Conversely ASD Kähler metrics are scalar–flat.

A convenient way to express the ASD condition on a conformal structure is summarised in the following theorem. The theorem below is originally due to Penrose [15], but taken in this form from [11, 4].

Theorem 2.1 *Let E_1, E_2 be two complex vector fields in $TM \otimes \mathbb{C}$ and let e^1, e^2 be the corresponding dual one–forms. The conformal structure defined by*

$$g = e^1 \odot \overline{e}^1 + e^2 \odot \overline{e}^2$$

is ASD if and only if there exists functions f_0, f_1 on $M \times \mathbb{CP}^1$ holomorphic in $\lambda \in \mathbb{CP}^1$ such that the distribution

$$l = \overline{E}_1 - \lambda E_2 + f_0 \frac{\partial}{\partial \lambda}, \quad m = -\overline{E}_2 - \lambda E_1 + f_1 \frac{\partial}{\partial \lambda} \tag{2.8}$$

is Frobenius integrable, that is, $[l, m] = 0$ modulo l and m.

A general ASD conformal structure $[g]$ does not admit a Kähler metric in its conformal class. The existence of such metric is characterised by vanishing of higher order conformal invariants of $[g]$ [6]. In what follows we shall use a simpler twistor characterisation of ASD metrics which are conformal to Kähler.

2.1 Twistors and divisors

Let us complexify (M, g) and regard M as a holomorphic four–manifold with a holomorphic metric g

$$g = \left(V^{00'} \odot V^{11'} - V^{01'} \odot V^{10'}\right), \tag{2.9}$$

where $V^{AA'}, A, A' = 0, 1$ is the null tetrad of one forms written in the two–component spinor notation. The reality conditions can be imposed to recover the Riemannian metric by setting

$$V^{00'} = \overline{e}^1, \quad V^{10'} = -\overline{e}^2, \quad V^{01'} = e^2, \quad V^{11'} = e^1.$$

Let $V_{AA'}$ be the corresponding tetrad of vector fields so that the integrable distribution of Theorem 2.1 is

$$l = V_{00'} - \lambda V_{01'} + f_0 \frac{\partial}{\partial \lambda}, \quad m = V_{10'} - \lambda V_{11'} + f_1 \frac{\partial}{\partial \lambda}. \tag{2.10}$$

A twistor space of (M, g) is the space of two–dimensional totally null surfaces spanned by $V_{00'} - \lambda V_{01'}, V_{10'} - \lambda V_{11'}$ in M. It is a three dimensional complex manifold $\mathcal{Z}$ which arises as a quotient of $M \times \mathbb{CP}^1$ by l, m. The points in M correspond to rational curves (called twistor curves) in $\mathcal{Z}$ with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$, where $\mathcal{O}(n)$ is a line bundle over $\mathbb{CP}^1$ with Chern class n. The holomorphic canonical line bundle κ of $\mathcal{Z}$ restricted to any of these curves is isomorphic to $\mathcal{O}(-4)$. To reconstruct a real four–manifold M, the twistor curves must be invariant under an anti–holomorphic involution τ on $\mathcal{Z}$ which restricts to an antipodal map on each curve.

A theorem of Pontecorvo [16] states that an ASD conformal structure $(M, [g])$ admits a Kähler metric if and only if there exists a section D of the bundle $\mathcal{O}(2) = \kappa^{-1/2}$ over the twistor space $\mathcal{Z}$ of $(M, [g])$, with exactly two distinct zeros on each twistor line. The section must be τ invariant, so the its zeroes lie on the antipodal points on each twistor curve.

Now consider the group G acting on an ASD conformally Kähler manifold M by holomorphic conformal isometries with generically three–dimensional orbits. This gives rise to a complexified group action of $G_{\mathbb{C}}$ on the twistor space $\mathcal{Z}$. Let $\widetilde{R}_j, j = 0, 1, 2$ be holomorphic vector fields on $\mathcal{Z}$ corresponding to the right–invariant conformal Killing vectors R_j on M. The subset of $\mathcal{Z}$ where $\widetilde{R}_j$ are linearly dependent is given by the zero set of $s = vol_{\mathcal{Z}}(\widetilde{R}_0, \widetilde{R}_1, \widetilde{R}_2)$. As the canonical bundle $\kappa = \mathcal{O}(-4)$, the divisor $s = 0$

defines a quartic and vanishes at four points on each twistor line. Hitchin ([8], Proposition 3) showed[1] that s is not identically zero if g is not Ricci–flat and that the divisor when s vanishes is equal to the Pontecorvo's divisor D in the case when $[g]$ is cohomogeneity one and contains a Kähler class. In the double fibration picture

$$M \longleftarrow M \times \mathbb{CP}^1 \longrightarrow \mathcal{Z}$$

the section s pulls back to

$$s = (d\lambda \wedge \nu)(l, m, \widetilde{R}_0, \widetilde{R}_1, \widetilde{R}_2), \tag{2.11}$$

where $\nu = V^{01'} \wedge V^{10'} \wedge V^{11'} \wedge V^{00'}$ is the volume form on M and $\widetilde{R}_j$ are the lifts of the three generators of G to $M \times \mathbb{CP}^1$ such that $[l, \widetilde{R}_j] = 0$, $[m, \widetilde{R}_j] = 0$ modulo $l, m,$. Thus in the proof of Theorem 1.1 we will require that this quartic has two distinct zeros of order two. This will guarantee the existence of a Kähler metric in the cohomogeneity–one conformal class.

Proof of Theorem 1.1. We will work in the complexified category and impose the reality conditions at the end. Let $G_{\mathbb{C}}$ be a three-complex dimensional Lie group (we will eventually take $G_{\mathbb{C}}$ to be a complexification of the Bianchi V group, but the first part of the proof does not depend on the choice of $G_{\mathbb{C}}$). We assume that the orbits of $G_{\mathbb{C}}$ are three–dimensional, and the metric g on $\mathbb{C} \times G_{\mathbb{C}}$ is invariant under the left translations of $G_{\mathbb{C}}$ on itself. One can write the null tetrad $V_{AA'}$ in terms of the vector field ∂_t and three linearly independent vector fields P, Q, R tangent to $G_{\mathbb{C}}$ (in the complexified setting the fields Q and P are independent. Once the reality conditions are imposed at the end of the proof, we shall set $Q = -\overline{P}$) which are t-dependent and invariant under the left translations, as

$$V_{00'} = \partial_t + i\frac{R}{t}, \quad V_{11'} = \partial_t - i\frac{R}{t}, \quad V_{01'} = P, \quad V_{10'} = Q. \tag{2.12}$$

Now, let R_0, R_1, R_2 be the right-invariant vector fields on G corresponding to three generators of the left translations. Since R_j are independent of t,

[1] His result was derived for $G_{\mathbb{C}} = SL(2, \mathbb{C})$ but it remains valid for any three-dimensional group acting transitively on M, as the corresponding action of $G_{\mathbb{C}}$ preserves the points where Pontecorvo's divisor vanishes as long as the group action is holomorphic. Moreover we are allowed to work in a conformal class of g_K, as the twistor equation underlying the Pontecorvo's divisor is conformally invariant [16, 6].

one has

$$[l, R_j] = -R_j(f_0)\partial_\lambda, \quad [m, R_j] = -R_j(f_1)\partial_\lambda.$$

A direct calculation shows that there is no lift of R_j of the form $R_j + \mathcal{Q}_j\partial_\lambda$ for some function $\mathcal{Q}_j$ such that $[l, R_j + \mathcal{Q}_j\partial_\lambda] = 0$, $[m, R_j + \mathcal{Q}_j\partial_\lambda] = 0$ modulo l, m. Hence, we conclude that $[l, R_j], [m, R_j]$, are identically zero. This implies that f_0 and f_1 are constant on G, and hence they are functions of λ and t only.

We claim that the quartic s as defined in (2.11) is proportional to $\lambda f_0 + f_1$, with the proportionality factor given by a function h on M. Indeed, using the fact that $V^{01'}$, $V^{00'} - V^{11'}$, $V^{10'}$ are linearly independent and invariant under the left translations of $G_{\mathbb{C}}$, the right-invariant vector fields R_j can be written in the basis of $P = V_{01'}$, $Q = V_{10'}$, $R = \frac{it}{2}(V_{11'} - V_{00'})$. Thus, performing all the contractions we are left with $s \propto (\det H)\ (\lambda f_0 + f_1)$, where H is the matrix of coefficients of R_0, R_1, R_2 written in the basis of P, Q, R.

As $\det H$ does not depend on λ, the quartic s has two distinct zeros of order two if and only if $\lambda f_0 + f_1$ has two distinct zeros of order two (for the moment we rule out the case where f_0 and f_1 are both zero which corresponds to a hyper-Kähler metric [4]). We shall assume that this is the case so that the ASD conformal structure admits a Kähler metric. It is now possible to use Möbius transformation to put the two zeros at 0 and ∞. The Möbius transformation in λ corresponds to a change of null tetrad by a right rotation $V_{AA'} \to V_{AA'}r^{A'}{}_{B'}$, where r is an $SL(2, \mathbb{C})$-valued function. Since the coefficients of the quartic $\lambda f_0 + f_1$ are functions of t only, the required $r^{A'}{}_{B'}$ will only depend on t. Thus, the rotated tetrad is still $G_{\mathbb{C}}$-invariant.

With the zeros at 0 and ∞, $\lambda f_0 + f_1$ is of the form $a(t)\lambda^2$. Let us first assume that a does not vanish identically. One still has a Möbius degree of freedom that preserves $(0, \infty)$, that is, the multiplication of λ by a function of t. Let us use this freedom to set $a(t) = 2/t$. The current tetrad is some right rotation of the original one. It is possible to use another freedom: a left rotation $V_{AA'} \to l^A{}_B V_{AA'}$, $l \in SL(2, \mathbb{C})$ to keep $V_{00'} - V_{11'}, V_{01'}, V_{10'}$ tangent to $G_{\mathbb{C}}$. The right rotation does not change the quartic $\lambda f_0 + f_1$, and we now have f_0, f_1 of the form

$$f_0 = b(t)\lambda^2 + c(t)\lambda + d(t), \quad f_1 = -b(t)\lambda^3 + \left(\frac{2}{t} - c(t)\right)\lambda^2 - d(t)\lambda \quad (2.13)$$

for some functions $b(t), c(t), d(t)$.

Now, consider a pair of linear combinations of l and m (2.10)

$$\begin{aligned} L &= \frac{\lambda l + m}{\lambda f_0 + f_1} = \frac{\partial}{\partial \lambda} + \frac{2\lambda i R t^{-1} - \lambda^2 P + Q}{\lambda f_0 + f_1} \\ M &= \frac{f_1 l - f_0 m}{\lambda f_0 + f_1} = \frac{\partial}{\partial t} + \frac{(f_1 - \lambda f_0) i R t^{-1} - \lambda f_1 P - f_0 Q}{\lambda f_0 + f_1}. \end{aligned} \tag{2.14}$$

Since the conformal class is ASD, Theorem 2.1 means $[l, m] = (\dots)l + (\dots)m$. This in turn implies that $[L, M] = 0$, modulo L and M. However, one sees that $[L, M]$ does not contain ∂_λ or ∂_t, thus $[L, M]$ must be identically zero. It turns out that $[L, M] = 0$ implies that $b(t) = 0 = d(t)$.

The compatibility conditions $[L, M] = 0$ are then given by

$$\begin{aligned} tP_t - i[R, P] + (tc(t) - 1)P &= 0, \\ 2iR_t - t[P, Q] &= 0, \\ tQ_t + i[R, Q] - (tc(t) - 1)Q &= 0. \end{aligned} \tag{2.15}$$

This shows that a cohomogeneity-one metric (1.2), in the tetrad (2.12) is ASD if the vector fields P, Q, R satisfy the system (2.15), where $c(t)$ is defined in (2.13). Now, let

$$\hat{R} = R, \quad \hat{P} = h(t)P, \quad \hat{Q} = h^{-1}(t)Q, \quad \text{where} \quad h(t) = e^{\int \left(c(t) - \frac{1}{t}\right) dt}. \tag{2.16}$$

The system (2.15) implies that the vector fields $\hat{P}, \hat{Q}, \hat{R}$ satisfy

$$\begin{aligned} tP_t - i[R, P] &= 0, \\ 2iR_t - t[P, Q] &= 0, \\ tQ_t + i[R, Q] &= 0, \end{aligned} \tag{2.17}$$

where we have dropped the hat from the rescaled vector fields. Moreover, the tetrad (2.12) constructed from a solution (P, Q, R) of (2.17) gives the same metric (1.2) as the tetrad determined from $(h^{-1}(t)P,\ h(t)Q,\ R)$ which satisfy (2.15) with $c(t) = \frac{h_t}{h} + \frac{1}{t}$. Neither the metric nor the equations (2.17) depend on h, so we can set it equal to 1. The resulting Lax pair (2.14) is

$$L = \frac{\partial}{\partial \lambda} + \frac{(tQ + 2i\lambda R - \lambda^2 t P)}{2\lambda^2}, \qquad M = \frac{\partial}{\partial t} - \frac{(\lambda Q + \lambda^3 P)}{2\lambda^2}, \tag{2.18}$$

where we again dropped the hat from the rescaled vector fields.

We conclude that any cohomogeneity-one metric (1.2), which belongs to an ASD conformal structure admitting a quartic s defined in (2.11) with two distinct zeros of order two, can be written in terms of a null tetrad (2.12), where the vector fields P, Q, R satisfy the system (2.17).

The three vector fields P, Q, R can be written in the basis of left-invariant vector fields L_0, L_1, L_2 satisfying (1.1) as

$$\begin{aligned} P &= p_0(t)\ L_0 + p_1(t)\ L_1 + p_2(t)\ L_2 \\ Q &= q_0(t)\ L_0 + q_1(t)\ L_1 + q_2(t)\ L_2, \\ R &= r_0(t)\ L_0 + r_1(t)\ L_1 + r_2(t)\ L_2, \end{aligned} \tag{2.19}$$

for some functions $p_j(t), q_j(t), r_j(t)$. Then using the commutation relation (1.1) the system (2.17) implies that p_0, q_0, r_0 are constant and that

$$t(p_j)_t = i(r_0 p_j - p_0 r_j), \quad t(q_j)_t = i(-r_0 q_j + q_0 r_j), \quad 2i(r_j)_t = t(p_0 q_j - q_0 p_j). \tag{2.20}$$

The reality conditions corresponding to Riemannian metrics come down to choosing R real (so that (r_0, r_1, r_2) are all real functions) and $Q = -\overline{P}$ so that $q_j = -\overline{p}_j$). Equations (2.20) then give the linear system (1.3).

Let us now return to the case $a = 0$ which corresponds to f_0 and f_1 vanishing in the distribution (2.10). The resulting conformal structure must then be hyper-Hermitian [4] and (as the vector fields in (2.10) are volume preserving) it is actually conformal to hyper-Kähler. The integrability $[l, m] = 0$ modulo l, m implies the system of Nahm equations

$$P_t - i[R/t, P] = 0, \quad 2i(R/t)_t - [P, Q] = 0, \quad Q_t + i[R/t, Q] = 0. \tag{2.21}$$

Imposing the reality condition that R is real and $Q = -\overline{P}$, (2.21) become

$$\frac{d(\mathbf{r}t^{-1})}{dt} = Im\ (\overline{p}_0 \mathbf{p}), \quad t\frac{d\mathbf{p}}{dt} = i(r_0 \mathbf{p} - p_0 \mathbf{r}), \quad \frac{dp_0}{dt} = 0, \quad \frac{d(r_0 t^{-1})}{dt} = 0.$$

The general solutions for $\mathbf{r}$ and $\mathbf{p}$ are given in terms of trigonometric and exponential functions depending on the constants p_0 and r_0/t, and the resulting metric is conformally flat (in the proof of Proposition 1.2 we shall find the conformal factor which makes it flat).

□

Our derivation of (2.17) and (2.18) did not depend on the choice of the isometry group. The system (2.17) describes the general isomonodromic deformation equations with two double poles. The corresponding Lax pair (2.18) with P, Q, R given by 2×2 matrices was shown by Jimbo and Miwa [9] to give rise to the Painlevé III equation. The same Lax pair also arises as the reduced Lax pair of the ASDYM equation, by the Painlevé III group. It is shown to be the isomonodromic Lax pair for the Painlevé III equation when the gauge group of the ASDYM connection is $SL(2, \mathbb{C})$ [12], or (with certain algebraic constraints on normal forms) $SL(3, \mathbb{C})$ [5].

2.2 Bessel equation

For generic values of the constants (r_0, p_0), the solutions to (1.3) are determined by two linearly independent solutions of the Bessel equation[2].

If $p_0 = 0$ then $\mathbf{r}$ is a constant vector and equations for p_1, p_2 can be easily integrated. The resulting metric is conformally flat.

If $p_0 \neq 0$ we rescale the metric by a constant $|p_0|^2$ and redefine $(\mathbf{r}, p_1, p_2, t)$ to set $p_0 = 1$. Differentiating the first set of equations in (1.3) twice, and using the second set of equations shows that the real functions $f_1 = 2t^{-1}(r_1)_t$ and $f_2 = 2t^{-1}(r_2)_t$ satisfy a pair of Bessel equations

$$t^2 \frac{d^2 f_k}{dt^2} + t \frac{df_k}{dt} + (t^2 + r_0^2) f_k = 0, \quad k = 1, 2. \tag{2.22}$$

If $r_0 \neq 0$ the general solution of the Bessel equation (2.22) is given in terms of Bessel functions of pure imaginary order ir_0

$$f_1 = c_1 J_{ir_0}(t) + c_2 Y_{ir_0}(t), \qquad f_2 = c_3 J_{ir_0}(t) + c_4 Y_{ir_0}(t).$$

The constant complex coefficients c_1, c_2, c_3, c_4 can be chosen so that the functions f_1 and f_2 are real, see for example [7]. Given functionally independent f_1 and f_2 we find r_1, r_2, p_1, p_2 by integrations and algebraic manipulations.

The case $r_0 = 0$ is special. The linear system (1.3) now reduces to a pair of ODEs

$$t^2 \frac{d^2 r_k}{dt^2} - t \frac{dr_k}{dt} + t^2 r_k = 0, \quad k = 1, 2$$

[2]The relation with the Bessel equation is already expected from the result of [14], who classified all reductions of anti–self–dual Yang Mills equations leading (by switch map) to cohomogeneity–one ASD conformal structures without however determining a Kähler class.

whose solutions are given by

$$r_1 = c_1 t J_1(t) + c_2 t Y_1(t), \quad r_2 = c_3 t J_1(t) + c_4 t Y_1(t),$$

where J_α are Y_α are Bessel functions of the first and second kind respectively of order α, and $c_1, \dots, c_4$ are real constants of integrations. In this case p_1, p_2 must be purely imaginary and the recursion relations

$$\partial_t J_0 = -J_1, \quad \partial_t Y_0 = -Y_1, \quad \partial_t(tJ_1) = tJ_0, \quad \partial_t(tY_1) = tY_0 \tag{2.23}$$

imply that

$$p_1 = i(c_1 J_0(t) + c_2 Y_0(t)), \quad p_2 = i(c_3 J_0(t) + c_4 Y_0(t)). \tag{2.24}$$

Performing a linear transformation of the vector fields L_1 and L_2 we can always set $c_1 = 1, c_2 = 0, c_3 = 0, c_4 = 1$ which yields

$$\mathbf{r} = (0, tJ_1(t), tY_1(t)), \quad \mathbf{p} = (1, iJ_0(t), iY_0(t)).$$

To write down the metric we invert the vector fields E_1, E_2 from Theorem 1.1 and use the fact that $2t(Y_0 J_1 - J_0 Y_1) = 4/\pi$. This yields

$$\begin{aligned} g &= \frac{1}{4}(dt^2 + (\lambda^0)^2) + G_{AB}(t)\lambda^A\lambda^B, \quad \text{where} \quad A, B = 1, 2 \\ G(t) &= \frac{\pi^2 t^2}{16}\begin{pmatrix} Y_0^2 + Y_1^2 & -J_0Y_0 - J_1Y_1 \\ -J_0Y_0 - J_1Y_1 & J_0^2 + J_1^2 \end{pmatrix}, \end{aligned}$$

where $\lambda^0, \lambda^1, \lambda^2$ are the left–invariant one forms on G which satisfy (A1).

3 Kähler structure

Proof of Proposition 1.2. The Kähler structure on M can be read off from the divisor (2.11). In the proof of Theorem (1.1) we have moved the double zeros of s to 0 and ∞, which in spinor notation means that $\omega_{A'B'} = o_{(A'}\iota_{B')}$ and $s \approx (\omega_{A'B'}\pi^{A'}\pi^{B'})^2$ where $\pi^{A'}$ are the homogeneous coordinates on $\mathbb{CP}^1$ such that $\lambda = -\pi^{0'}/\pi^{1'}$, and the spinor basis is $o_{A'} = (0, 1), \iota_{A'} = (-1, 0)$. Thus, in the null tetrad (2.12), the Kähler form is proportional to

$$\hat{\omega} = \frac{i}{2}\varepsilon_{AB}\omega_{A'B'}V^{AA'} \wedge V^{BB'} = \frac{i}{2}(e^1 \wedge \bar{e}^1 + e^2 \wedge \bar{e}^2),$$

and the space $T^{1,0}(M)$ of holomorphic vector fields on M is spanned by $V_{11'}, V_{01'}$ (equivalently by the vectors E_1 and E_2 in Theorem 1.1). The Frobenius integrability conditions

$$[T^{1,0}, T^{1,0}] \subset T^{1,0}$$

guaranteeing the vanishing of Nijenhuis torsion follows from the construction, but we can also verify it directly as

$$[E_1, E_2] = \Big(\frac{dp_1}{dt} - \frac{i}{t}r_0p_1 + \frac{i}{t}r_1p_0\Big)L_1 + \Big(\frac{dp_2}{dt} - \frac{i}{t}r_0p_2 + \frac{i}{t}r_2p_0\Big)L_2 = 0,$$

where we have used the constancy of (r_0, p_0) and equations (1.3).

To determine the conformal factor we look for a function $\Omega : M \to \mathbb{R}$ such that $d(\Omega^2\hat{\omega}) = 0$. Once this has been found the Kähler metric g_K and the associated two–form ω will be given by

$$g_K = \Omega^2 g, \quad \omega = \frac{i\Omega^2}{2}(e^1 \wedge \overline{e}^1 + e^2 \wedge \overline{e}^2).$$

It can be verified by explicit calculation that there is no G–invariant Ω (i.e. there is no conformal factor which depends only on t). To demonstrate that $\Omega : G \to \mathbb{R}$ such that (1.5) holds gives the correct conformal factor, consider

$$2\Omega d\Omega \wedge \hat{\omega} + \Omega^2 d\hat{\omega} = 0. \tag{3.25}$$

Since $\partial_t\Omega = 0$, one can write $d\Omega$ in the basis of left-invariant one-forms

$$d\Omega = L_0(\Omega)\lambda^0 + L_1(\Omega)\lambda^1 + L_2(\Omega)\lambda^2.$$

The three-form $d\hat{\omega}$ can be calculated using the expressions for the dual one-forms of E_1 and E_2 in Theorem 1.1 and the Maurer-Cartan's structure equation (A1). Finally, the system (1.3) is used to simplify the LHS of (3.25), and one finds that (3.25) is satisfied if and only if (1.5) holds. This conformal factor is in fact unique - it could have also been read off from the divisor as it is proportional to a power of $\omega_{A'B'}\omega^{A'B'}$ [6].

In the conformally flat case where s in (2.11) is identically zero, which corresponds to the system (2.21), the same conformal factor makes g flat. Moreover we verify by explicit calculation that g_K given by (1.4) where $s \neq 0$ is Ricci flat if and only if it is flat, which proves the last part the Proposition.

□

The conformal Killing vectors generating the group action on g_K are given by the right–invariant vector fields on G. If the coordinates are chosen for the group, these vectors are given by (A3).
Example. The simplest explicit example of the scalar–flat Kähler metric corresponds to $r_0 = 0$ in Theorem 1.1 and is given by (1.6). The determinant of the metric (1.6) given by

$$\det g_K = \frac{\pi^2\rho^2 t^2}{1024}.$$

Since by definition $\rho \neq 0$, g_K may be degenerate only at $t = 0$ or $t = \infty$. The Ricci scalar R is identically zero because the Kähler metric is ASD. The remaining curvature invariants are

$$R_{abcd}R^{abcd} = \frac{256}{\rho^4 t^2}, \qquad W_{abcd}W^{abcd} = \frac{128}{\rho^4 t^2}$$

which indicates that $t = 0$ is a singularity. Rescaling the metric by a conformal factor $\rho^{-2}(tf(t))^{-1}$ gives $W_{abcd}W^{abcd} = 128 f^2$, which needs to be regular if the conformal class contains a complete metric, but the regularity of the conformal factor requires that $tf(t)$ is also regular and non-zero. Thus the norm of the Weyl tensor blows up at 0. The asymptotic behaviour of g_K for large t is

$$g_K = \frac{1}{4}\Big(d\rho^2 + \rho^2 dt^2\Big) + \frac{\pi t}{8}(d(x^1)^2 + d(x^2)^2).$$

4 $SU(\infty)$ Toda equation

LeBrun [10] has shown that any Kähler metric g_K with symmetry preserving the Kähler form admits a local coordinate system $\{\tau, x, y, z\}$ such that

$$g_K = Wh + \frac{1}{W}(d\tau + \theta)^2, \quad \omega = We^u dx \wedge dy + dz \wedge (d\tau + \theta), \qquad (4.26)$$

where

$$h = e^u(dx^2 + dy^2) + dz^2. \qquad (4.27)$$

Here τ is a coordinate along the orbits of the Killing vector $K = \partial_\tau$, $\{x, y, z\}$ are coordinates on the space of orbits, and (u, W) and θ are functions and a one–form on the space of orbits such that u satisfies the $SU(\infty)$ Toda equation (1.7), W satisfies the so–called monopole equation

$$W_{xx} + W_{yy} + (We^u)_{zz} = 0 \qquad (4.28)$$

and θ is determined by W together with the condition $d\omega = 0$.

The ansatz (4.26) can be understood as follows. Given that $K = \partial_\tau$ is a Killing vector, the metric necessarily takes the form (4.26), where $\frac{1}{W} = g_K(K, K)$ and h is a metric on the three–dimensional space of orbits. Now, since the Kähler form ω Lie derives along K, we have

$$K \lrcorner\, \omega = dz \tag{4.29}$$

for some function z on the space of orbits of K. The isothermal coordinates x, y on the orthogonal complement of the space spanned by K and $I(K)$ (where I is the complex structure) can be used together with z to parametrise the space of orbits. The metric then takes the form (4.26) with h given by (4.27) for some $u = u(x, y, z)$. The integrability of the complex structure and the closure of the ω imply (4.28). The scalar-flat condition gives (1.7).

4.1 Bessel solutions to $SU(\infty)$ Toda equation

The scalar–flat Kähler metric (1.6) has two Killing symmetries $\partial/\partial x^1$ and $\partial/\partial x^2$ preserving the Kähler form. We can follow the algorithm described above and find the solution u of the $SU(\infty)$ Toda equation and the associated monopole W corresponding to a linear combination

$$K = c_1\partial/\partial x^1 + c_2\partial/\partial x^2.$$

We shall set $c_2 = 0$ for simplicity. Set $\tau = x^1$ so that the vector field $K = \partial_\tau$ and $dx^1 = d\tau$. Then the metric (1.6) takes the form (4.26), where

$$\begin{aligned} h &= \frac{1}{4W}\Big(\rho^2 dt^2 + d\rho^2\Big) + \frac{t^2\pi^2}{64}\, d(x^2)^2, \\ W &= \frac{16}{t^2\pi^2(Y_0^2 + Y_1^2)}, \quad \text{and} \quad \theta = -\frac{(J_0Y_0 + J_1Y_1)}{Y_0^2 + Y_1^2}\, d(x^2). \end{aligned} \tag{4.30}$$

Using identities (2.23) and contracting the Kähler form of (1.6) with $\partial/\partial x^1$, one recovers the equations (4.29) for z. The other two coordinates are[3]

$$\tau = x^1, \quad x = -\frac{\pi x^2}{8}, \quad y = -\frac{\pi\rho Y_0}{8}, \quad z = \frac{\pi\rho t Y_1}{8}. \tag{4.31}$$

[3]For a Killing vector given by a general linear combination of $\partial/\partial x^1$ and $\partial/\partial x^2$, a linear combination of Bessel functions appears in the final formula, and in particular

$$\frac{z}{y} = -t\left(\frac{c_2J_1 - c_1Y_1}{c_2J_0 - c_1Y_0}\right).$$

The solution to $SU(\infty)$ Toda equation is now implicitly given by

$$e^u = t^2. \tag{4.32}$$

Formulae (4.31) imply that $u = u(v)$, where $v = z/y$. Thus u is constant on the plane $y\, v(u) - z = 0$ (compare [18] where solutions constant on quadrics were constructed) and is invariant under a two–dimensional group of Lie point symmetries generated by vector fields

$$\partial/\partial x, \qquad x\partial_x + y\partial_y + z\partial_z \tag{4.33}$$

We shall now show that the existence of these symmetries uniquely characterises (4.32). Any solution u of (1.7) which is invariant under symmetries generated by the vector fields (4.33) is a function $u = u(v)$. Then (1.7) becomes an ODE

$$(v^2 + e^u)u_{vv} + 2vu_v + u_v^2 e^u = 0.$$

Equivalently, interchanging the dependent and independent variables we have

$$(v^2 + e^u)^2 \partial_u \left(\frac{v_u}{v^2 + e^u} \right) = 0,$$

which integrates to

$$v_u = c(v^2 + e^u) \tag{4.34}$$

for some constant c. Now we shall argue that this constant c can always be set to $-1/2$, provided that we consider solutions to the $SU(\infty)$ Toda equation as equivalent if they determine the conformally equivalent metrics (4.27). First, note that a transformation

$$u \longrightarrow u + 2\beta, \qquad z \longrightarrow \pm e^{\beta} z, \tag{4.35}$$

where β is a constant is a symmetry of (1.7) which rescalles the three–metric by a constant factor. Now, under (4.35), the variable v transforms as $v \longrightarrow \pm e^{\beta} v$. This can be used to set $c = -1/2$.

The equation (4.34) with $c = -1/2$ is equivalent to the Bessel equation of order 0

$$t^2 Y_{tt} + tY_t + t^2 Y = 0. \tag{4.36}$$

To see it use $t^2 = e^u$ and $v = 2\partial_u \ln Y$. The four–dimensional metric (4.26) resulting from the solution $u = u(z/y)$ together with the monopole (4.30) admits a conformal action of the Bianchi V group generated by

$$R_0 = \tau\partial_\tau + x\partial_x + y\partial_y + z\partial_z, \qquad R_1 = \partial_\tau, \qquad R_2 = \partial_x. \tag{4.37}$$

We shall now establish that given a solution $u = u(z/y)$ to the $SU(\infty)$ Toda equation, the function W given by (4.30) is (up to gauge transformation) the only solution to the linearised $SU(\infty)$ Toda equation (4.28) such that the resulting metric (4.26) in four dimensions defines a cohomogeneity–one Bianchi V conformal class. To show it, observe that the metric h in (4.27) is invariant under R_1, R_2 and conformally invariant $h \to c^2 h$ under the one parameter group of transformations generated by R_0. This implies that the function W must be invariant under the Bianchi V group G and the one-form θ is invariant under the translations generated by R_1, R_2, but transforms as $\theta \to c\,\theta$ under the action generated by R_0. Thus the metric (4.26) is conformally invariant under G if and only if the function W and the components of the one-form θ are functions of t only. Therefore the PDE (4.28) for W becomes a second order ODE

$$W_{tt}v_t(v^2+t^2) + 2W_t(v_t(2t+vv_t) - v_{tt}(v^2+t^2)) + 2W(v_t - tv_{tt}) = 0.$$

The invariance properties of θ lead to one further constraint

$$W_t(t^2+v^2) + 2tW = 0. \tag{4.38}$$

The relation (4.31) yields $v = -t\frac{Y_1}{Y_0}$ and using (2.23) we find the general solution W of (4.38)

$$W = \frac{k}{t^2(Y_0^2+Y_1^2)}, \quad k = \text{const.} \tag{4.39}$$

Changing the proportionality constant k in W amounts to changing the coordinate $\tau \longrightarrow \tau/k$ in (4.26) and rescaling the metric by the constant k. Hence we can always set $k = 64/\pi^2$, which gives the monopole W in (4.31).

Example: ASD Einstein metric with symmetry. Here we shall present an example of a metric (4.26) which is obtained from a monopole W different from (4.39).

It was shown in [20] that any ASD Einstein metric with symmetry and non-zero scalar curvature can be written as

$$g_E = \frac{W}{z^2}\left[e^u(dx^2+dy^2)+dz^2\right] + \frac{1}{Wz^2}(d\tau+\theta)^2, \tag{4.40}$$

where $W = \text{const}\ (zu_z - 2)$, and u is a solution to the $SU(\infty)$ Toda equation. Take u given by (4.32) and the Einstein monopole W with const$= 1/2$. Then

$$W = \frac{Y_0Y_1}{t(Y_0^2+Y_1^2)} - 1, \quad \theta = \left[\frac{1}{2}\left(\frac{Y_0^2-Y_1^2}{Y_0^2+Y_1^2}\right) + \ln\left(\frac{\pi\rho}{8}\right)\right]\, dx.$$

The resulting Einstein metric (4.40) has negative scalar curvature and is non-conformally flat. Note that the metric is not conformal to a Bianchi V metric. It only admits a two–dimensional group of symmetries.

Appendix

The real three–dimensional Lie algebra of Bianchi type V is defined by commutation relations

$$[X_0, X_1] = X_1, \quad [X_0, X_2] = X_2, \quad [X_1, X_2] = 0.$$

We can choose its representation by 3×3 matrices

$$X_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The corresponding Lie group G is the multiplicative group of real matrices of the form

$$\mathsf{g} = \begin{pmatrix} \rho & x^1 & x^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{where} \quad \rho \in \mathbb{R}^+, (x^1, x^2) \in \mathbb{R}^2.$$

The left-invariant one-forms $\{\lambda^j, j = 0, 1, 2\}$ corresponding to a basis $\{X_j\}$ of a Lie algebra are given by

$$\mathsf{g}^{-1} d\mathsf{g} = \lambda^j X_j, \quad \text{where} \quad \mathsf{g} \in G.$$

Hence

$$\lambda^0 = \rho^{-1} d\rho, \quad \lambda^1 = \rho^{-1} dx^1, \quad \lambda^2 = \rho^{-1} dx^2,$$

and

$$d\lambda^0 = 0, \quad d\lambda^1 = \lambda^1 \wedge \lambda^0, \quad d\lambda^2 = \lambda^2 \wedge \lambda^0. \tag{A1}$$

The left invariant vector fields defined by $L_j \lrcorner \lambda^k = \delta_j{}^k$ are found to be

$$L_0 = \rho \frac{\partial}{\partial \rho}, \quad L_1 = \rho \frac{\partial}{\partial x^1}, \quad L_2 = \rho \frac{\partial}{\partial x^2}. \tag{A2}$$

The right–invariant one forms and vector fields can be found analogously from $d\mathrm{g}\mathrm{g}^{-1}$. The algebra of isometries of metrics from Theorem 1.1 is spanned by the right–invariant vector fields, which in our coordinate basis are given by

$$R_0 = \rho\frac{\partial}{\partial\rho} + x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}, \quad R_1 = \frac{\partial}{\partial x^1}, \quad R_2 = \frac{\partial}{\partial x^2}. \qquad \text{(A3)}$$

The left and right invariant vector fields satisfy the commutation relations

$$[L_0, L_1] = L_1, \quad [L_0, L_2] = L_2, \quad [L_1, L_2] = 0,$$
$$[R_0, R_1] = -R_1, \quad [R_0, R_2] = -R_2, \quad [R_1, R_2] = 0,$$
$$[R_j, L_k] = 0.$$

References

[1] Bianchi, L. (1897) Sugli spazii a tre dimensioni che ammettono un gruppo continuo di movimenti, Soc. Ital. Sci. Mem. di Mat. **11**, 267.

[2] Dancer, A. S. and Strachan, I. A. B. (1995) *Cohomogeneity one Kähler metrics.* Twistor Theory. Ed. S. Huggett. Lect. Notes Pure Appl. Math., **169**, Marcel Dekker.

[3] Dancer, A. S. (1996) Scalar–flat Kähler metrics with $SU(2)$ symmetry. J. Reine. Angew. Math. **479**, 99-120

[4] Dunajski, M. (2009) *Solitons, Instantons & Twistors.* Oxford Graduate Texts in Mathematics **19**, Oxford University Press.

[5] Dunajski, M. and Plansangkate, P. (2009) Strominger–Yau–Zaslow geometry, Affine Spheres and Painleve III , Commun. Math. Phys. **290**, 997-1024.

[6] Dunajski, M. and Tod, K. P. (2010) Four Dimensional Metrics Conformal to Kähler, Math. Proc. Camb. Phil. Soc. 148, 485-503.

[7] Dunster, T. M. (1990) Bessel functions of purely imaginary order, with an application to second-order linear differential equations having a large parameter, SIAM J. Math. Anal. **21**, 995–1018.

[8] Hitchin, N. J. (1995) Twistor spaces, Einstein metrics and isomonodromic deformations, J. Differential Geom. **42**, 30–112.

[9] Jimbo, M. and Miwa, T. (1981) Monodromy preserving deformations of linear ordinary differential equations with rational coefficients. II and III, Physica, **2D, 4D** 407–448, 26–46.

[10] LeBrun, C. (1991) Explicit self-dual metrics on $C\mathrm{P}_2\#\cdots\#C\mathrm{P}_2$, J. Differential Geom. **1**, 223–253.

[11] Mason, L. J. and Woodhouse, N. M. J. (1996) *Integrability, selfduality, and twistor theory.* Oxford, UK: Clarendon (LMS monographs, new series: 15).

[12] Mason, L. J. and Woodhouse, N. M. J. (1993) Self-duality and the Painlevé transcendents, Nonlinearity **6**, 569–581.

[13] Maszczyk, R., Mason, L. J. and Woodhouse, N. M. J. (1994) Selfdual Bianchi metrics and the Painleve transcendents, Class. Quant. Grav. **11**, 65–71.

[14] Maszczyk, R. (1996) The classification of self-dual Bianchi metrics, Class. Quant. Grav. **13**, 513–527.

[15] Penrose, R. (1976) Nonlinear gravitons and curved twistor theory, Gen. Rel. Grav. **7**, 31–52.

[16] Pontecorvo, M. (1992) On twistor spaces of anti-self-dual Hermitian surfaces, Trans. Amer. Math. Soc. **331**, 653–661.

[17] Tod, K. P. (1994) Self-dual Einstein metrics from the Painlevé VI, Phys. Lett. A **190**, 221–224.

[18] Tod, K. P. (1995) Scalar-flat Kähler and hyper-Kähler metrics from Painlevé III, Class. Quantum Grav. **12**, 1535–1547.

[19] Tod, K. P. (1995) *Cohomogeneity-one metrics with self-dual Weyl tensor.* Twistor Theory. Ed. S. Huggett. Lect. Notes Pure Appl. Math., **169**, Marcel Dekker.

[20] Tod, K. P. (1997) *The $SU(\infty)$-Toda field equation and special four-dimensional metrics.* Geometry and Physics, Dekker, New York, 307–312.