A generalization of the inequality of Audenaert et al.

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Abstract

We extend the inequality of Audenaert et al [ACMMABV] to general von Neumann algebras.

1 Introduction

Let A, B be positive matrices and $0 \leq s \leq 1$. Then an inequality

$$2TrA^sB^{1-s} \geq Tr(A + B - |A - B|) \tag{1}$$

holds. This is a key inequality to prove the upper bound of Chernoff bound, in quantum hypothesis testing theory. This inequality was first proven in [ACMMABV], using an integral representation of the function t^s. Recently, N.Ozawa gave a much simpler proof for the same inequality. In this note, based on his proof, we extend the inequality to general von Neumann algebras. More precisely, we prove the following: Let $\{\mathcal{M}, \mathcal{H}, J, \mathcal{P}\}$ be a standard form associated with a von Neumann algebra $\mathcal{M}$, i.e., $\mathcal{H}$ is a Hilbert space where $\mathcal{M}$ acts on, J is the modular conjugation, and $\mathcal{P}$ is the natural positive cone.(See [T]) Let $\mathcal{M}_{*+}$ be the set of all positive normal linear functionals over $\mathcal{M}$. For each $\varphi \in \mathcal{M}_{*+}$, ξ_φ is the unique element in the natural positive cone $\mathcal{P}$ which satisfies $\varphi(x) = (x\xi_\varphi, \xi_\varphi)$ for all $x \in \mathcal{M}$. We denote the relative modular operator associated with $\varphi, \psi \in \mathcal{M}_{*+}$ by $\Delta_{\varphi\psi}$.(See Appendix.) The main result in this note is the following:

Proposition 1.1 *Let φ, η be positive normal linear functionals on a von Neumann algebra $\mathcal{M}$. Then, for any $0 \leq s \leq 1$,*

$$\eta(1) - (\eta - \varphi)_+(1) \leq \left\| \Delta_{\eta,\varphi}^{\frac{s}{2}} \xi_\varphi \right\|^2. \tag{2}$$

The equality holds iff $\eta = (\eta - \varphi)_+ + \psi$ and $\varphi = (\eta - \varphi)_- + \psi$ for some $\psi \in \mathcal{M}_{*+}$ whose support is orthogonal to the support of $|\eta - \varphi|$.

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As a corollary of this proposition, we obtain a generalization of the inequality of [ACMMABV]:

Corollary 1.1 *Let φ, η be positive normal linear functionals on a von Neumann algebra $\mathcal{M}$. Then, for any $0 \le s \le 1$,*

$$2\left\|\Delta_{\eta\varphi}^{\frac{s}{2}}\xi_{\varphi}\right\|^2 \ge \varphi(1) + \eta(1) - |\varphi - \eta|(1). \tag{3}$$

The equality holds iff $\eta = (\eta-\varphi)_+ + \psi$ and $\varphi = (\eta-\varphi)_- + \psi$ for some $\psi \in \mathcal{M}_{*+}$ whose support is orthogonal to the support of $|\eta - \varphi|$.

If $s = \frac{1}{2}$, this is the Powers-Størmer inequality. Applications of this inequality for hypothesis testing problem can be found in [JOPS].

2 Proof of Proposition 1.1

We first prove the following lemma which we need in the proof of Proposition 1.1:

Lemma 2.1 *Let $\varphi_1, \varphi_2, \psi, \eta$ be faithful normal positive linear functionals over a von Neumann algebra $\mathcal{M}$. Assume that $\varphi_1 \le \varphi_2$ and $\eta \le \psi$. Then for all $0 < s < 1$,*

$$\left\|\Delta_{\varphi_2\eta}^{\frac{s}{2}}\xi_{\eta}\right\|^2 - \left\|\Delta_{\varphi_1\eta}^{\frac{s}{2}}\xi_{\eta}\right\|^2 \le \left\|\Delta_{\varphi_2\psi}^{\frac{s}{2}}\xi_{\psi}\right\|^2 - \left\|\Delta_{\varphi_1\psi}^{\frac{s}{2}}\xi_{\psi}\right\|^2.$$

Proof First we consider the case $\varphi_2 \le \psi$. In this case, by Lemma A.1, $(D\varphi_1 : D\psi)_t, (D\varphi_2 : D\psi)_t, (D\eta : D\psi)_t$ have continuations $(D\varphi_1 : D\psi)_z, (D\varphi_2 : D\psi)_z, (D\eta : D\psi)_z \in \mathcal{M}$, analytic on $I_{-\frac{1}{2}} := \{z \in \mathbb{C} \,:\, -\frac{1}{2} < \Im z < 0\}$ and bounded continuous on $\overline{I_{-\frac{1}{2}}}$, with norm less than or equal to 1.
We define a positive operator

$$T := (D\varphi_2 : D\psi)^*_{-i\frac{s}{2}}(D\varphi_2 : D\psi)_{-i\frac{s}{2}} - (D\varphi_1 : D\psi)^*_{-i\frac{s}{2}}(D\varphi_1 : D\psi)_{-i\frac{s}{2}} \in \mathcal{M}. \tag{4}$$

To see that T is positive, recall from Lemma A.1 that for any $\xi \in D(\Delta_{\psi\psi}^{-\frac{s}{2}})$, we have $\Delta_{\psi\psi}^{-\frac{s}{2}}\xi \in D(\Delta_{\varphi_k\psi}^{\frac{s}{2}})$, and

$$\Delta_{\varphi_k\psi}^{\frac{s}{2}}\Delta_{\psi\psi}^{-\frac{s}{2}}\xi = (D\varphi_k : D\psi)_{-i\frac{s}{2}}\xi, \quad k = 1, 2.$$

From this, we obtain

$$(T\xi, \xi) = \left\|(D\varphi_2 : D\psi)_{-i\frac{s}{2}}\xi\right\|^2 - \left\|(D\varphi_1 : D\psi)_{-i\frac{s}{2}}\xi\right\|^2 = \left\|\Delta_{\varphi_2\psi}^{\frac{s}{2}}\Delta_{\psi\psi}^{-\frac{s}{2}}\xi\right\|^2 - \left\|\Delta_{\varphi_1\psi}^{\frac{s}{2}}\Delta_{\psi\psi}^{-\frac{s}{2}}\xi\right\|^2. \tag{5}$$

As $\Delta_{\psi\psi}^{-\frac{s}{2}}\xi$ is in $D(\Delta_{\varphi_2\psi}^{\frac{s}{2}})$, the last term is positive from Lemma A.2. This proves $T \ge 0$.

Next we define $x' := J((D\eta : D\psi)_{-i\frac{1-s}{2}})J \in \mathcal{M}'$. From $\|x'\| \leq 1$ and $0 \leq T \in \mathcal{M}$, we have

$$\left(x'^* T x' \xi_\psi, \xi_\psi\right) = \left(T^{\frac{1}{2}} x'^* x' T^{\frac{1}{2}} \xi_\psi, \xi_\psi\right) \leq (T\xi_\psi, \xi_\psi)\,. \tag{6}$$

As $\xi_\psi \in D\left(\Delta_{\psi\psi}^{-\frac{1-s}{2}}\right)$, from Lemma A.1, we have

$$x'\xi_\psi = J(D\eta : D\psi)_{-i\frac{1-s}{2}}\xi_\psi = J\Delta_{\eta\psi}^{\frac{1}{2}-\frac{s}{2}}\Delta_{\psi\psi}^{-\frac{1}{2}+\frac{s}{2}}\xi_\psi = J\Delta_{\eta\psi}^{\frac{1}{2}-\frac{s}{2}}\xi_\psi = \Delta_{\psi\eta}^{\frac{s}{2}} J\Delta_{\eta\psi}^{\frac{1}{2}}\xi_\psi = \Delta_{\psi\eta}^{\frac{s}{2}}\xi_\eta \in D(\Delta_{\psi\eta}^{-\frac{s}{2}}). \tag{7}$$

By this and Lemma A.1, we have $\Delta_{\psi\eta}^{-\frac{s}{2}} x'\xi_\psi \in D(\Delta_{\varphi_k\eta}^{\frac{s}{2}})$ and

$$(D\varphi_k : D\psi)_{-i\frac{s}{2}} x'\xi_\psi = \Delta_{\varphi_k\eta}^{\frac{s}{2}}\Delta_{\psi\eta}^{-\frac{s}{2}} x'\xi_\psi = \Delta_{\varphi_k\eta}^{\frac{s}{2}}\xi_\eta \tag{8}$$

for $k = 1, 2$. Hence we obtain

$$\left(x'^* T x'\xi_\psi, \xi_\psi\right) = \left\|\Delta_{\varphi_2\eta}^{\frac{s}{2}}\xi_\eta\right\|^2 - \left\|\Delta_{\varphi_1\eta}^{\frac{s}{2}}\xi_\eta\right\|^2. \tag{9}$$

On the other hand, substituting $\xi = \xi_\psi \in D(\Delta_{\psi\psi}^{-\frac{s}{2}})$ to (5), we have

$$(T\xi_\psi, \xi_\psi) = \left\|\Delta_{\varphi_2\psi}^{\frac{s}{2}}\xi_\psi\right\|^2 - \left\|\Delta_{\varphi_1\psi}^{\frac{s}{2}}\xi_\psi\right\|^2. \tag{10}$$

From (6), (9), (10), we obtain the result for the $\varphi_2 \leq \psi$ case.
To extend the result to a general case, we use Lemma A.3. For any $\varepsilon > 0$, we have

$$\varepsilon\varphi_1 \leq \varepsilon\varphi_2 \leq \psi + \varepsilon\varphi_2, \quad \eta \leq \psi + \varepsilon\varphi_2. \tag{11}$$

Therefore, for any $0 < s < 1$, we have

$$\left\|\Delta_{\varepsilon\varphi_2,\eta}^{\frac{s}{2}}\xi_\eta\right\|^2 - \left\|\Delta_{\varepsilon\varphi_1,\eta}^{\frac{s}{2}}\xi_\eta\right\|^2 \leq \left\|\Delta_{\varepsilon\varphi_2,\psi+\varepsilon\varphi_2}^{\frac{s}{2}}\xi_{\psi+\varepsilon\varphi_2}\right\|^2 - \left\|\Delta_{\varepsilon\varphi_1,\psi+\varepsilon\varphi_2}^{\frac{s}{2}}\xi_{\psi+\varepsilon\varphi_2}\right\|^2.$$

Using relations $\Delta_{\varepsilon\varphi_2,\eta}^{\frac{s}{2}} = \varepsilon^{\frac{s}{2}}\Delta_{\varphi_2,\eta}^{\frac{s}{2}}$ etc, we have

$$\begin{aligned}\left\|\Delta_{\varphi_2,\eta}^{\frac{s}{2}}\xi_\eta\right\|^2 - \left\|\Delta_{\varphi_1,\eta}^{\frac{s}{2}}\xi_\eta\right\|^2 &\leq \left\|\Delta_{\varphi_2,\psi+\varepsilon\varphi_2}^{\frac{s}{2}}\xi_{\psi+\varepsilon\varphi_2}\right\|^2 - \left\|\Delta_{\varphi_1,\psi+\varepsilon\varphi_2}^{\frac{s}{2}}\xi_{\psi+\varepsilon\varphi_2}\right\|^2 \\ &= \left\|\Delta_{\psi+\varepsilon\varphi_2,\varphi_2}^{\frac{1-s}{2}}\xi_{\varphi_2}\right\|^2 - \left\|\Delta_{\psi+\varepsilon\varphi_2,\varphi_1}^{\frac{1-s}{2}}\xi_{\varphi_1}\right\|^2\end{aligned}$$

In the second line we used Lemma A.5. Taking $\varepsilon \to 0$ and applying Lemma A.3 and Lemma A.5, we obtain the result. □

Proof of Proposition 1.1

It is trivial for $s = 0, 1$. We prove the claim for $0 < s < 1$. We first consider faithful φ, η. From $\varphi \leq \varphi + (\eta - \varphi)_+$ and Lemma A.2, we have

$$\left\| \Delta_{\varphi,\varphi}^{\frac{s}{2}} \xi_\varphi \right\|^2 - \left\| \Delta_{\eta,\varphi}^{\frac{s}{2}} \xi_\varphi \right\|^2 \leq \left\| \Delta_{\varphi+(\eta-\varphi)_+,\varphi}^{\frac{s}{2}} \xi_\varphi \right\|^2 - \left\| \Delta_{\eta,\varphi}^{\frac{s}{2}} \xi_\varphi \right\|^2 . \tag{12}$$

By Lemma 2.1 and inequalities $\eta \leq \varphi + (\eta - \varphi)_+,\ \varphi \leq \varphi + (\eta - \varphi)_+$, the last term is bounded as

$$\begin{aligned} &\leq \left\| \Delta_{\varphi+(\eta-\varphi)_+,\varphi+(\eta-\varphi)_+}^{\frac{s}{2}} \xi_{\varphi+(\eta-\varphi)_+} \right\|^2 - \left\| \Delta_{\eta,\varphi+(\eta-\varphi)_+}^{\frac{s}{2}} \xi_{\varphi+(\eta-\varphi)_+} \right\|^2 \\ &= \left\| \xi_{\varphi+(\eta-\varphi)_+} \right\|^2 - \left\| \Delta_{\eta,\varphi+(\eta-\varphi)_+}^{\frac{s}{2}} \xi_{\varphi+(\eta-\varphi)_+} \right\|^2 \\ &= \left\| \xi_{\varphi+(\eta-\varphi)_+} \right\|^2 - \left\| \Delta_{\varphi+(\eta-\varphi)_+,\eta}^{\frac{1-s}{2}} \xi_\eta \right\|^2 . \end{aligned} \tag{13}$$

By $\varphi + (\eta - \varphi)_+ \geq \eta$ and Lemma A.2, we have

$$(13) \leq \left\| \xi_{\varphi+(\eta-\varphi)_+} \right\|^2 - \left\| \Delta_{\eta,\eta}^{\frac{1-s}{2}} \xi_\eta \right\|^2 = \varphi(1) + (\eta - \varphi)_+(1) - \eta(1). \tag{14}$$

Hence we obtain

$$\varphi(1) - \left\| \Delta_{\eta,\varphi}^{\frac{s}{2}} \xi_\varphi \right\|^2 \leq \varphi(1) + (\eta - \varphi)_+(1) - \eta(1), \tag{15}$$

which is equal to

$$\eta(1) - (\eta - \varphi)_+(1) \leq \left\| \Delta_{\eta,\varphi}^{\frac{s}{2}} \xi_\varphi \right\|^2 . \tag{16}$$

We now prove the inequality for general φ, η. By considering a von Neumann algebra $\mathcal{M}_e := e\mathcal{M}e$ with $e := s(\eta) \bigvee s(\varphi)$ instead of $\mathcal{M}$ if it is necessary, we may assume $\varphi + \varepsilon\eta, \eta + \delta\varphi$ are faithful on $\mathcal{M}$ for any $\varepsilon, \delta > 0$. We then have

$$(\eta + \delta\varphi)(1) - (\eta + \delta\varphi - (\varphi + \varepsilon\eta))_+ (1) \leq \left\| \Delta_{\eta+\delta\varphi,\varphi+\varepsilon\eta}^{\frac{s}{2}} \xi_{\varphi+\varepsilon\eta} \right\|^2 . \tag{17}$$

Taking the limit $\varepsilon \to 0$ and then the limit $\delta \to 0$, and using Lemma A.3 and Lemma A.5 we obtain the inequality (2) for general φ, η.
To check the condition for the equality, by approximating φ and η by $\varphi + \varepsilon\eta, \eta + \delta\varphi$ in (12), (13), and (14), just as in (17), and taking the limit $\varepsilon \to 0$ and $\delta \to 0$, we obtain

$$\begin{aligned} &\left\| \Delta_{\varphi,\varphi}^{\frac{s}{2}} \xi_\varphi \right\|^2 - \left\| \Delta_{\eta,\varphi}^{\frac{s}{2}} \xi_\varphi \right\|^2 \leq \left\| \Delta_{\varphi+(\eta-\varphi)_+,\varphi}^{\frac{s}{2}} \xi_\varphi \right\|^2 - \left\| \Delta_{\eta\varphi}^{\frac{s}{2}} \xi_\varphi \right\|^2 \\ &\leq \left\| \Delta_{\varphi+(\eta-\varphi)_+,\varphi+(\eta-\varphi)_+}^{\frac{s}{2}} \xi_{\varphi+(\eta-\varphi)_+} \right\|^2 \quad - \left\| \Delta_{\eta,\varphi+(\eta-\varphi)_+}^{\frac{s}{2}} \xi_{\varphi+(\eta-\varphi)_+} \right\|^2 \\ &= \left\| \xi_{\varphi+(\eta-\varphi)_+} \right\|^2 - \left\| \Delta_{\varphi+(\eta-\varphi)_+,\eta}^{\frac{1-s}{2}} \xi_\eta \right\|^2 \leq \left\| \xi_{\varphi+(\eta-\varphi)_+} \right\|^2 - \left\| \Delta_{\eta,\eta}^{\frac{1-s}{2}} \xi_\eta \right\|^2 \\ &= \varphi(1) + (\eta - \varphi)_+(1) - \eta(1). \end{aligned} \tag{18}$$

By Lemma A.4, the first inequality is an equality iff the support of $(\eta-\varphi)_+$ is orthogonal to φ and the third inequality is an equality iff the support of $(\eta-\varphi)_-$ is orthogonal to η. Therefore, if the equality in (18) holds, then $(\eta-\varphi)_+$ is orthogonal to φ and $(\eta-\varphi)_-$ is orthogonal to η.
Conversely, if $(\eta-\varphi)_+$ is orthogonal to φ and $(\eta-\varphi)_-$ is orthogonal to η. Then we have $\varphi+(\eta-\varphi)_+=\eta+(\eta-\varphi)_-$, where both sides of the equality are sum of orthogonal elements. Therefore, we have

$$\begin{aligned}&\left\|\Delta^{\frac{s}{2}}_{\varphi+(\eta-\varphi)_+,\varphi+(\eta-\varphi)_+}\xi_{\varphi+(\eta-\varphi)_+}\right\|^2-\left\|\Delta^{\frac{s}{2}}_{\eta,\varphi+(\eta-\varphi)_+}\xi_{\varphi+(\eta-\varphi)_+}\right\|^2=\left\|\Delta^{\frac{s}{2}}_{(\eta-\varphi)_-,\varphi+(\eta-\varphi)_+}\xi_{\varphi+(\eta-\varphi)_+}\right\|^2\\&=\left\|\Delta^{\frac{1-s}{2}}_{\varphi+(\eta-\varphi)_+,(\eta-\varphi)_-}\xi_{(\eta-\varphi)_-}\right\|^2=\left\|\Delta^{\frac{1-s}{2}}_{\varphi,(\eta-\varphi)_-}\xi_{(\eta-\varphi)_-}\right\|^2+\left\|\Delta^{\frac{1-s}{2}}_{(\eta-\varphi)_+,(\eta-\varphi)_-}\xi_{(\eta-\varphi)_-}\right\|^2\\&=\left\|\Delta^{\frac{1-s}{2}}_{\varphi,(\eta-\varphi)_-}\xi_{(\eta-\varphi)_-}\right\|^2=\left\|\Delta^{\frac{s}{2}}_{(\eta-\varphi)_-,\varphi}\xi_\varphi\right\|^2.\end{aligned}\tag{19}$$

Furthermore, we have

$$\left\|\Delta^{\frac{s}{2}}_{\varphi+(\eta-\varphi)_+,\varphi}\xi_\varphi\right\|^2-\left\|\Delta^{\frac{s}{2}}_{\eta\varphi}\xi_\varphi\right\|^2=\left\|\Delta^{\frac{s}{2}}_{(\eta-\varphi)_-,\varphi}\xi_\varphi\right\|^2.$$

Hence, the second inequality is an equality in this case. As the first and third inequalities are equalities from the orthogonality of $(\eta-\varphi)_+$ with φ and $(\eta-\varphi)_-$ with η respectively, the equality holds in (18).
Therefore, the equality in (18) holds iff $(\eta-\varphi)_+$ is orthogonal to φ and $(\eta-\varphi)_-$ is orthogonal to η. However, the latter condition means $\eta=(\eta-\varphi)_++\psi$ and $\varphi=(\eta-\varphi)_-+\psi$ for some $\psi\in\mathcal{M}_{*+}$ whose support is orthogonal to the support of $|\eta-\varphi|$. □

Proof of Corollary 1.1
Replacing η,φ,s in (2) with $\varphi,\eta,1-s$ respectively, we obtain

$$\varphi(1)-(\varphi-\eta)_+(1)\leq\left\|\Delta^{\frac{1-s}{2}}_{\varphi,\eta}\xi_\eta\right\|^2=\left\|\Delta^{\frac{s}{2}}_{\eta,\varphi}\xi_\varphi\right\|^2.\tag{20}$$

Summing (2) and (20), we obtain (3). □

A Appendix

Let $\{\mathcal{M},\mathcal{H},J,\mathcal{P}\}$ be a standard form associated with a von Neumann algebra $\mathcal{M}$, i.e., $\mathcal{H}$ is a Hilbert space where $\mathcal{M}$ acts on, J is the modular conjugation, and $\mathcal{P}$ is the natural positive cone. Let $\mathcal{M}_{*+}$ be the set of all positive normal linear functionals over $\mathcal{M}$. For each $\varphi\in\mathcal{M}_{*+}$, ξ_φ is the unique element in the natural positive cone $\mathcal{P}$ which satisfies $\varphi(x)=(x\xi_\varphi,\xi_\varphi)$ for all $x\in\mathcal{M}$. For $\varphi,\psi\in\mathcal{M}_{*+}$, we define an operator $S_{\varphi\psi}$ as the closure of the operator

$$S_{\varphi\psi}\left(x\xi_\psi+(1-j(s(\psi)))\zeta\right):=s(\psi)x^*\xi_\varphi,\quad x\in\mathcal{M},\ \zeta\in\mathcal{H},$$

where $s(\psi)\in\mathcal{M}$ is the support projection of ψ and $j(y):=JyJ$. The polar decomposition of $S_{\varphi\psi}$ is given by $S_{\varphi\psi}=J\Delta^{\frac{1}{2}}_{\varphi\psi}$ where $\Delta_{\varphi\psi}$ is the relative modular operator associated with $\varphi,\psi\in\mathcal{M}_{*+}$. The subspace $\mathcal{M}\xi_\psi+(1-j(s(\psi)))\mathcal{H}$

of $\mathcal{H}$ is a core of $\Delta_{\varphi\psi}^{\frac{1}{2}}$. The support projection of the positive operator $\Delta_{\varphi\psi}$ is $s(\varphi)j(s(\psi))$. For a complex number $z \in \mathbb{C}$, we define a closed operator $\Delta_{\varphi\psi}^{z}$ by

$$\Delta_{\varphi\psi}^{z} := \left(\exp\left[z\left(\log\Delta_{\varphi\psi}\right)s(\varphi)j(s(\psi))\right]\right)s(\varphi)j(s(\psi)).$$

For an operator A on a Hilbert space $\mathcal{H}$, we denote by $D(A)$ its domain.

Lemma A.1 *Let φ, ψ be faithful normal positive linear functionals over a von Neumann algebra $\mathcal{M}$. Suppose that there exists a constant $\lambda > 0$ such that $\lambda\varphi \leq \psi$. Then the cocyle $\mathbb{R} \ni t \mapsto (D\varphi : D\psi)_t \in \mathcal{M}$ has an extension $(D\varphi : D\psi)_z \in \mathcal{M}$ analytic on $I_{-\frac{1}{2}} := \{z \in \mathbb{C} : -\frac{1}{2} < \Im z < 0\}$ and bounded continuous on $\overline{I_{-\frac{1}{2}}}$ with the bound $\|(D\varphi : D\psi)_z\| \leq \lambda^{\Im z}$ for all $z \in \overline{I_{-\frac{1}{2}}}$. Furthermore, for any faithful $\zeta \in \mathcal{M}_{*+}$, $0 < s < \frac{1}{2}$, and any element ξ in $D(\Delta_{\psi\zeta}^{-s})$, $\Delta_{\psi\zeta}^{-s}\xi$ is in the domain of $\Delta_{\varphi\zeta}^{s}$, and*

$$\Delta_{\varphi\zeta}^{s}\Delta_{\psi\zeta}^{-s}\xi = (D\varphi : D\psi)_{-is}\xi. \tag{21}$$

Proof The existence and boundedness of $(D\varphi : D\psi)_z$ is proven in [A1]. To show the latter part of the Lemma, let $\zeta \in \mathcal{M}_{*+}$ be faithful. We define the region I_{-s} in the complex plane by $I_{-s} := \{z \in \mathbb{C} : -s < \Im z < 0\}$ for each $0 < s < \frac{1}{2}$. For any $\xi \in D(\Delta_{\psi\zeta}^{-s})$ and $\xi_1 \in D(\Delta_{\varphi\zeta}^{s})$, we consider two functions on $\overline{I_{-s}}$ by $F(z) := \left(\Delta_{\psi\zeta}^{-iz}\xi, \Delta_{\varphi\zeta}^{-i\bar{z}}\xi_1\right)$, and $G(z) := ((D\varphi : D\psi)_z\xi, \xi_1)$. Both of these functions are bounded continuous on $\overline{I_{-s}}$ and analytic on I_{-s}. Furthermore, they are equal on $\mathbb{R}$:

$$F(t) = \left(\Delta_{\varphi\zeta}^{it}\Delta_{\psi\zeta}^{-it}\xi, \xi_1\right) = ((D\varphi : D\psi)_t\xi, \xi_1) = G(t), \quad \forall t \in \mathbb{R}.$$

This means $F(z) = G(z)$ for all $z \in \overline{I_{-s}}$. In particular, we have $F(-is) = G(-is)$, i.e.,

$$\left(\Delta_{\psi\zeta}^{-s}\xi, \Delta_{\varphi\zeta}^{s}\xi_1\right) = \left((D\varphi : D\psi)_{-is}\xi, \xi_1\right).$$

As this holds for all $\xi_1 \in D(\Delta_{\varphi\zeta}^{s})$, $\Delta_{\psi\zeta}^{-s}\xi$ is in the domain of $\Delta_{\varphi\zeta}^{s}$, and (21) holds. $\square$

Lemma A.2 *Let φ, η, ψ be normal positive linear functionals over a von Neumann algebra $\mathcal{M}$ such that $\varphi \leq \eta$. Then for any $0 \leq s \leq 1$, we have $D(\Delta_{\eta,\psi}^{\frac{s}{2}}) \subset D(\Delta_{\varphi,\psi}^{\frac{s}{2}})$ and*

$$\left\|\Delta_{\varphi,\psi}^{\frac{s}{2}}\xi\right\| \leq \left\|\Delta_{\eta,\psi}^{\frac{s}{2}}\xi\right\|, \quad \forall \xi \in D(\Delta_{\eta,\psi}^{\frac{s}{2}}). \tag{22}$$

Proof This is proven in [AM]. $\square$

Lemma A.3 *Let φ and η be elements in $\mathcal{M}_{*+}$ and φ_n a sequence in $\mathcal{M}_{*+}$ such that $\lim_{n\to\infty}\|\varphi_n - \varphi\| = 0$. Then for any and $0 < s < 1$,*

$$\lim_{n\to\infty}\left\|\Delta_{\varphi_n,\eta}^{\frac{s}{2}}\xi_\eta\right\| = \left\|\Delta_{\varphi\eta}^{\frac{s}{2}}\xi_\eta\right\|.$$

Proof By the integral representation of t^s, we have

$$\left\|\Delta^{\frac{s}{2}}_{\varphi_n,\eta}\xi_\eta\right\|^2 - \left\|\Delta^{\frac{s}{2}}_{\varphi\eta}\xi_\eta\right\|^2$$
$$= \frac{\sin s\pi}{\pi}\int_0^\infty d\lambda\lambda^{s-1}\left(\left(\Delta_{\varphi_n,\eta}\left(\Delta_{\varphi_n,\eta}+\lambda\right)^{-1} - \Delta_{\varphi,\eta}\left(\Delta_{\varphi,\eta}+\lambda\right)^{-1}\right)\xi_\eta,\xi_\eta\right). \tag{23}$$

We denote the term inside of the integral by $f_n(\lambda)$. It is easy to see

$$|f_n(\lambda)| \le \lambda^{s-1}\eta(1),$$
$$|f_n(\lambda)| \le \lambda^{s-2}\left(\left\|\Delta^{\frac{1}{2}}_{\varphi_n,\eta}\xi_\eta\right\|^2 + \left\|\Delta^{\frac{1}{2}}_{\varphi\eta}\xi_\eta\right\|^2\right) \le \lambda^{s-2}\left(\varphi(1) + \sup_n \varphi_n(1)\right).$$

Hence $|f_n(\lambda)|$ is bounded from above by an integrable function independent of n.
Next we show $\lim_{n\to\infty} f_n(\lambda) = 0$ for all $\lambda > 0$. To do so, we first observe that $\Delta^{\frac{1}{2}}_{\varphi_n,\eta}$ converges to $\Delta^{\frac{1}{2}}_{\varphi\eta}$ in the strong resolvent sense: For all $x\xi_\eta + (1 - j(s(\eta)))\zeta \in \mathcal{M}\xi_\eta + (1 - j(s(\eta)))\mathcal{H}$, using Powers-Størmer inequality, we have

$$\left\|\Delta^{\frac{1}{2}}_{\varphi_n,\eta}\left(x\xi_\eta + (1-j(s(\eta)))\zeta\right) - \Delta^{\frac{1}{2}}_{\varphi,\eta}\left(x\xi_\eta + (1-j(s(\eta)))\zeta\right)\right\|^2 = \|s(\eta)x^*\xi_{\varphi_n} - s(\eta)x^*\xi_\varphi\|^2$$
$$\le \|x^*\|^2\|\xi_{\varphi_n} - \xi_\varphi\|^2 \le \|x^*\|^2\|\varphi_n - \varphi\| \to 0, \text{ as } n\to\infty.$$

As $\mathcal{M}\xi_\eta + (1 - j(s(\eta)))\mathcal{H}$ is a common core for all $\Delta^{\frac{1}{2}}_{\varphi_n,\eta}$ and $\Delta^{\frac{1}{2}}_{\varphi\eta}$, this means $\Delta^{\frac{1}{2}}_{\varphi_n,\eta}$ converges to $\Delta^{\frac{1}{2}}_{\varphi\eta}$ in the strong resolvent sense. Therefore, for a bounded continuous function $g(t) = t^2(t^2+\lambda)^{-1}$, $g(\Delta^{\frac{1}{2}}_{\varphi_n,\eta})$ converges to $g(\Delta^{\frac{1}{2}}_{\varphi\eta})$ strongly.
Hence we have $\lim_{n\to\infty} f_n(\lambda) = 0$.
By the Lebesgue's theorem, we obtain the result. □

Lemma A.4 *For any $\varphi, \eta \in \mathcal{M}_{*+}$ with $\varphi \le \eta$ and $0 < s < 1$,*

$$\left\|\Delta^{\frac{s}{2}}_{\eta\varphi}\xi_\varphi\right\| = \|\xi_\varphi\| \tag{24}$$

if and only if $\eta - \varphi$ is orthogonal to φ.

Proof First we prove if $\left\|\Delta^{\frac{s}{2}}_{\eta\varphi}\xi_\varphi\right\| = \|\xi_\varphi\|$, then $\eta - \varphi$ is orthogonal to φ. From Lemma A.2, for any $\zeta \in D(\Delta^{-\frac{s}{2}}_{\eta\varphi})$, $\Delta^{-\frac{s}{2}}_{\eta\varphi}\zeta$ is in $D(\Delta^{\frac{s}{2}}_{\varphi\varphi})$ and

$$\left\|\Delta^{\frac{s}{2}}_{\varphi\varphi}\Delta^{-\frac{s}{2}}_{\eta\varphi}\zeta\right\| \le \left\|\Delta^{\frac{s}{2}}_{\eta\varphi}\Delta^{-\frac{s}{2}}_{\eta\varphi}\zeta\right\| \le \|\zeta\|.$$

Therefore, $\Delta^{\frac{s}{2}}_{\varphi\varphi}\Delta^{-\frac{s}{2}}_{\eta\varphi}$ defined on $D(\Delta^{-\frac{s}{2}}_{\eta\varphi})$ can be uniquely extended to a bounded operator A on $\mathcal{H}$, with norm $\|A\| \le 1$. We define an operator $0 \le T \le 1$ by $T := A^*A$. Note that

$$A\Delta^{\frac{s}{2}}_{\eta\varphi}\xi_\varphi = \Delta^{\frac{s}{2}}_{\varphi\varphi}\Delta^{-\frac{s}{2}}_{\eta\varphi}\Delta^{\frac{s}{2}}_{\eta\varphi}\xi_\varphi = \Delta^{\frac{s}{2}}_{\varphi\varphi}s(\eta)\xi_\varphi = \Delta^{\frac{s}{2}}_{\varphi\varphi}\xi_\varphi = \xi_\varphi.$$

From this, and the assumption, we have

$$\left(T\Delta_{\eta\varphi}^{\frac{s}{2}}\xi_\varphi, \Delta_{\eta\varphi}^{\frac{s}{2}}\xi_\varphi\right) = \left\|A\Delta_{\eta\varphi}^{\frac{s}{2}}\xi_\varphi\right\|^2 = \|\xi_\varphi\|^2 = \left\|\Delta_{\eta\varphi}^{\frac{s}{2}}\xi_\varphi\right\|^2.$$

As the spectrum of T is included in $[0,1]$, this equality means

$$T\Delta_{\eta\varphi}^{\frac{s}{2}}\xi_\varphi = \Delta_{\eta\varphi}^{\frac{s}{2}}\xi_\varphi. \tag{25}$$

For any $\zeta \in D(\Delta_{\eta\varphi}^{\frac{s}{2}})$, we have

$$\left(\Delta_{\eta\varphi}^{\frac{s}{2}}\xi_\varphi, \Delta_{\eta\varphi}^{\frac{s}{2}}\zeta\right) = \left(T\Delta_{\eta\varphi}^{\frac{s}{2}}\xi_\varphi, \Delta_{\eta\varphi}^{\frac{s}{2}}\zeta\right) = \left(A\Delta_{\eta\varphi}^{\frac{s}{2}}\xi_\varphi, A\Delta_{\eta\varphi}^{\frac{s}{2}}\zeta\right) = (\xi_\varphi, \zeta), \tag{26}$$

from (25). Therefore, $\xi_\varphi \in D(\Delta_{\eta\varphi}^s)$ and $\Delta_{\eta\varphi}^s\xi_\varphi = \xi_\varphi$. Hence we obtain $\Delta_{\eta\varphi}^{\frac{1}{2}}\xi_\varphi = \xi_\varphi$. From this, we have

$$s(\varphi)\xi_\eta = J\Delta_{\eta\varphi}^{\frac{1}{2}}\xi_\varphi = \xi_\varphi.$$

We then obtain

$$(\eta - \varphi)\,(s(\varphi)) = 0,$$

i.e., the support of $\eta - \varphi$ is orthogonal to the support of φ.
Conversely, if the support of $\eta - \varphi$ is orthogonal to φ, then we have

$$\left\|\Delta_{\eta\varphi}^{\frac{s}{2}}\xi_\varphi\right\|^2 = \left\|\Delta_{\eta-\varphi,\varphi}^{\frac{s}{2}}\xi_\varphi\right\|^2 + \left\|\Delta_{\varphi\varphi}^{\frac{s}{2}}\xi_\varphi\right\|^2 = \|\xi_\varphi\|^2. \tag{27}$$

□

Lemma A.5 *For all normal positive linear functionals ψ_1, ψ_2 over a von Neumann algebra $\mathcal{M}$, and $0 \le s \le 1$,*

$$\left\|\Delta_{\psi_1,\psi_2}^{\frac{s}{2}}\xi_{\psi_2}\right\| = \left\|\Delta_{\psi_2,\psi_1}^{\frac{1-s}{2}}\xi_{\psi_1}\right\|. \tag{28}$$

Proof Functions $F(z) := \left(\Delta_{\psi_1,\psi_2}^{\frac{z}{2}}\xi_{\psi_2}\Delta_{\psi_1,\psi_2}^{\frac{\bar{z}}{2}}\xi_{\psi_2}\right)$ and $G(z) := \left(\Delta_{\psi_2,\psi_1}^{\frac{1-z}{2}}\xi_{\psi_1}\Delta_{\psi_2,\psi_1}^{\frac{1-\bar{z}}{2}}\xi_{\psi_1}\right)$ are bounded continuous on $0 \le \Re z \le 1$ and analytic on $0 < \Re z < 1$. It is easy to check $F(it) = G(it)$ for $t \in \mathbb{R}$. Hence we obtain $F(z) = G(z)$ on $0 \le \Re z \le 1$.□

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References

[A1] H. Araki: *Relative entropy of states of von Neumann algebras*; Pub. R.I.M.S., Kyoto Univ. **11**, 809-833, (1976).

[AM] H. Araki and T.Masuda: *Positive cones and L_p-spaces for von Neumann algebras*; Pub. R.I.M.S., Kyoto Univ. **18**, 339-411, (1982).

[ACMMABV] K.M.R. Audenaert, J. Calsamiglia, Ll. Masanes, R. Munoz-Tapia, A. Acin, E. Bagan, F. Verstraete: *The Quantum Chernoff Bound*; Phys. Rev. Lett. 98, 160501 (2007)

[JOPS] V. Jakšić Y.Ogata C.-A.Pillet and R. Seiringer In preparation

[T] M. Takesaki: *Theory of Operator Algebras II*; Springer Encyclopedia of Mathematical Sciences Vol 125 (2001).