MAXIMAL THEOREMS AND SQUARE FUNCTIONS FOR ANALYTIC OPERATORS ON L^p-SPACES

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Abstract. Let $T\colon L^p(\Omega)\to L^p(\Omega)$ be a contraction, with $1<p<\infty$, and assume that T is analytic, that is, $\sup_{n\geq 1} n\|T^n - T^{n-1}\| < \infty$. Under the assumption that T is positive (or contractively regular), we establish the boundedness of various Littlewood-Paley square functions associated with T. In particular we show that T satisfies an estimate $\bigl\|\bigl(\sum_{n=1}^\infty n^{2m-1}\bigl|T^n(T-I)^m(x)\bigr|^2\bigr)^{\frac{1}{2}}\bigr\|_p \lesssim \|x\|_p$ for any integer $m\geq 1$. As a consequence we show maximal inequalities of the form $\bigl\|\sup_{n\geq 0}(n+1)^m\bigl|T^n(T-I)^m(x)\bigr|\bigr\|_p \lesssim \|x\|_p$, for any integer $m\geq 0$. We prove similar results in the context of noncommutative L^p-spaces. We also give analogs of these maximal inequalities for bounded analytic semigroups, as well as applications to R-boundedness properties.

2000 Mathematics Subject Classification : 47B38, 46L52, 46A60.

1. Introduction.

Let (Ω,μ) be a measure space, let $1<p<\infty$ and let $T\colon L^p(\Omega)\to L^p(\Omega)$ be a positive contraction. Then Akcoglu's Theorem [1] asserts that T satisfies a maximal ergodic inequality,

$$\Bigl\|\sup_{n\geq 0}\frac{1}{n+1}\Bigl|\sum_{k=0}^{n} T^k(x)\Bigr|\Bigr\|_p \lesssim \|x\|_p,\qquad x\in L^p(\Omega). \tag{1.1}$$

A well-known question is to determine which operators satisfy a stronger maximal inequality,

$$\Bigl\|\sup_{n\geq 0}|T^n(x)|\Bigr\|_p \lesssim \|x\|_p,\qquad x\in L^p(\Omega). \tag{1.2}$$

In this paper we show that this holds true provided that T is analytic, that is, there exists a constant $K\geq 0$ such that

$$n\|T^n - T^{n-1}\|\leq K$$

for any $n\geq 1$ (see Section 2 for some background). More generally, we show that for any integer $m\geq 0$, analytic positive contractions $T\colon L^p(\Omega)\to L^p(\Omega)$ satisfy a maximal inequality

$$\Bigl\|\sup_{n\geq 0}(n+1)^m\bigl|T^n(T-I)^m(x)\bigr|\Bigr\|_p \lesssim \|x\|_p,\qquad x\in L^p(\Omega). \tag{1.3}$$

Note that for any $m\geq 1$, the sequence of operators $(T^n(T-I)^m)_{n\geq 0}$ appearing here is the m-th order discrete derivative of the original sequence $(T^n)_{n\geq 0}$. The proofs of these

Date: November 8, 2010.

The authors are both supported by the research program ANR-06-BLAN-0015.

inequalities rely on the boundedness of certain discrete Littlewood-Paley square functions of independent interest that we establish in Section 3. In particular we will show that for T as above, we have an estimate

$$\Big\|\Big(\sum_{n=1}^{\infty} n\left|T^n(x)-T^{n-1}(x)\right|^2\Big)^{\frac{1}{2}}\Big\|_p \lesssim \|x\|_p, \qquad x\in L^p(\Omega). \tag{1.4}$$

These maximal theorems and square function estimates extend Stein's famous results [35, 36] which show that (1.2), (1.3) and (1.4) hold true in the case when T acts as a contraction $L^q(\Omega)\to L^q(\Omega)$ for any $1\leq q\leq\infty$ and its L^2-realization is a positive selfadjoint operator.

Let M be a von Neumann algebra equipped with a normal semifinite faithful trace and for any $1\leq p\leq\infty$, let $L^p(M)$ be the associated noncommutative L^p-space. Let $T\colon M\to M$ be a positive contraction whose restriction to $L^1(M)\cap M$ extends to a contraction $T\colon L^1(M)\to L^1(M)$. Recall that in this case, T actually extends to a contraction $L^q(M)\to L^q(M)$ for any $1\leq q\leq\infty$. It is shown in [17] that T satisfies a noncommutative analog of (1.1). In the latter paper, a large part of Stein's work mentioned above is also transfered to the noncommutative setting. Indeed it is shown that if the L^2-realization $T\colon L^2(M)\to L^2(M)$ is a positive selfadjoint operator, then for any $1<p<\infty$, T satisfies noncommutative analogs of (1.2) and (1.3). This is generalized in [3] under an appropriate condition on the numerical range of $T\colon L^2(M)\to L^2(M)$. We extend these results by showing that for any $1<p<\infty$, the noncommutative analogs of (1.2) and (1.3) hold true provided that $T\colon L^p(M)\to L^p(M)$ is merely analytic (which is a much weaker assumption).

Besides investigating the behaviour of operators and their powers (discrete semigroups), we consider continuous semigroups $(T_t)_{t\geq 0}$, both in the commutative and in the noncommutative settings. The continuous analog of the maximal inequality (1.2) reads as follows:

$$\Big\|\sup_{t>0}\big|T_t(x)\big|\Big\|_p \lesssim \|x\|_p. \tag{1.5}$$

We prove that such an estimate holds true whenever $(T_t)_{t\geq 0}$ is a bounded analytic semigroup on $L^p(\Omega)$ (with $1<p<\infty$) such that $T_t\colon L^p(\Omega)\to L^p(\Omega)$ is a positive contraction for any $t\geq 0$. Likewise we show that the noncommutative analog of (1.5) holds true whenever $(T_t)_{t\geq 0}$ is a semigroup of positive contractions on $L^q(M)$ for any $1\leq q\leq\infty$ and $(T_t)_{t\geq 0}$ is a bounded analytic semigroup on $L^p(M)$ (with $1<p<\infty$). These results both extend Stein's classical maximal theorem [35, 36] for semigroups and its recent noncommutative counterpart from [17]. Finally we extend some results from [18, Chapter 5] concerning R-boundedness in the noncommutative setting.

In the above presentation and later on in the paper, $\lesssim$ stands for an inequality up to a constant which may depend on T and m, but not on x.

2. Preliminaries.

An operator $T\colon L^p(\Omega)\to L^p(\Omega)$ is called regular if there is a constant $C\geq 0$ such that

$$\bigl\Vert\sup_{k\geq 1}|T(x_k)|\bigr\Vert_p\,\leq\, C\bigl\Vert\sup_{k\geq 1}|x_k|\bigr\Vert_p$$

for any finite sequence $(x_k)_{k\geq 1}$ in $L^p(\Omega)$. Then we let $\Vert T\Vert_r$ denote the smallest C for which this holds. The set of all regular operators on $L^p(\Omega)$ is a vector space on which $\Vert\ \Vert_r$ is a norm. We say that T is contractively regular if $\Vert T\Vert_r\leq 1$. Clearly any positive operator T is regular and $\Vert T\Vert_r=\Vert T\Vert$ in this case. Thus all statements given for contractively regular operators apply to positive contractions. It is well-known that conversely, T is regular with $\Vert T\Vert_r\leq C$ if and only if there is a positive operator $S\colon L^p(\Omega)\to L^p(\Omega)$ with $\Vert S\Vert\leq C$, such that $|T(x)|\leq S(|x|)$ for any $x\in L^p(\Omega)$ (see [27, Chap. 1]). Furthermore, T is contractively regular if T acts as a contraction $L^q(\Omega)\to L^q(\Omega)$ for any $1\leq q\leq\infty$.

We recall some definitions and simple facts about sectorial operators and analyticity. Throughout we let X denote an arbitrary (complex) Banach space and we let $B(X)$ denote the algebra of all bounded operators on X. Next for any angle $\omega\in(0,\pi)$, we introduce

$$\Sigma_\omega\,=\,\bigl\{z\in\mathbb{C}^*\,:\,|\text{Arg}(z)|<\omega\bigr\},$$

the open sector of angle 2ω around $(0,\infty)$.

Let $A\colon D(A)\subset X\to X$ be a (possibly unbounded) closed linear operator, with dense domain $D(A)$. We let $\sigma(A)$ denote the spectrum of A and for any $\lambda\in\mathbb{C}\setminus\sigma(A)$, we let $R(\lambda,A)=(\lambda-A)^{-1}$ denote the corresponding resolvent operator. We say that A is sectorial if there exists an angle $\theta\in(0,\pi)$ such that $\sigma(A)$ is contained in the closed sector $\overline{\Sigma_\theta}$ and

$$(S)_\theta\qquad\qquad \exists\, K\geq 0\ \ \Big|\ \ |\lambda|\Vert R(\lambda,A)\Vert\leq K,\qquad \lambda\in\mathbb{C}\backslash\overline{\Sigma_\theta}.$$

Then we let $\omega(A)$ be the infimum of all θ such that $(S)_\theta$ holds, and this real number is called the type of A. It is well-known that if $(S)_\theta$ holds true for some $\theta\in(0,\pi)$, then there exists $\varepsilon>0$ such that $(S)_{\theta-\varepsilon}$ holds true as well. Thus,

$$(S)_\theta\iff\omega(A)<\theta. \tag{2.1}$$

Let $(T_t)_{t\geq 0}$ be a bounded strongly continuous semigroup on X. We call it a bounded analytic semigroup if there exists a positive angle $\alpha\in\bigl(0,\frac{\pi}{2}\bigr)$ and a bounded analytic family $z\in\Sigma_\alpha\mapsto T_z\in B(X)$ extending $(T_t)_{t>0}$. Let $-A$ be the infinitesimal generator of $(T_t)_{t\geq 0}$. Analyticity has two classical characterizations in terms of that operator. First, $(T_t)_{t\geq 0}$ is a bounded analytic semigroup if and only if $T_t(X)\subset D(A)$ for any $t>0$ and there exists a constant $K\geq 0$ such that $\Vert tAT_t\Vert\leq K$ for any $t>0$. Note here that since $T_t=e^{-tA}$, we have

$$tAT_t\,=\,-t\,\frac{\partial}{\partial t}\bigl(T_t\bigr),\qquad t>0. \tag{2.2}$$

Second, $(T_t)_{t\geq 0}$ is a bounded analytic semigroup if and only if A is sectorial and $\omega(A)<\frac{\pi}{2}$. According to (2.1), this is also equivalent to saying that A satisfies $(S)_{\frac{\pi}{2}}$. We refer e.g. to [15, 30] for proofs and complements on semigroups.

We will make a crucial use of H^∞-calculus and square functions for sectorial operators. Here are the basic notions and results which will be needed. For more information, we refer e.g. to [11, 19, 21, 23].

For any $\theta \in (0, 2\pi)$, we define

$$H^\infty(\Sigma_\theta) = \{f \colon \Sigma_\theta \to \mathbb{C} \,|\, f \text{ is analytic and bounded}\}.$$

This is a Banach algebra with the norm

$$\|f\|_{H^\infty(\Sigma_\theta)} = \sup\bigl\{|f(\lambda)| \,:\, \lambda \in \Sigma_\theta\bigr\}.$$

Then let $H_0^\infty(\Sigma_\theta) \subset H^\infty(\Sigma_\theta)$ be the subalgebra of all f for which there exist two constants $s, C > 0$ such that

$$|f(\lambda)| \leq C\, \min\bigl\{|\lambda|^s,\, |\lambda|^{-s}\bigr\}, \qquad \lambda \in \Sigma_\theta.$$

For any sectorial operator A, for any $\theta \in (\omega(A), \pi)$ and for any $f \in H_0^\infty(\Sigma_\theta)$, we define

$$f(A) \,=\, \frac{1}{2\pi i}\int_{\Gamma_\gamma} f(\lambda) R(\lambda, A)\, d\lambda,$$

where $\omega(A) < \gamma < \theta$ and Γ_γ is the boundary $\partial\Sigma_\gamma$ oriented counterclockwise. This integral is well-defined, its definition does not depend on γ and the resulting mapping $f \mapsto f(A)$ is an algebra homomorphism from $H_0^\infty(\Sigma_\theta)$ into $B(X)$. We say that A has a bounded $H^\infty(\Sigma_\theta)$ functional calculus if the latter homomorphism is bounded, that is, there exists a constant $C > 0$ such that

$$\|f(A)\| \leq C\|f\|_{H^\infty(\Sigma_\theta)}, \qquad f \in H_0^\infty(\Sigma_\theta).$$

Consider now the specific case when $X = L^p(\Omega)$, with $1 < p < \infty$. On such a space, Cowling, Doust, McIntosh and Yagi have proved a remarkable equivalence result between the boundedness of H^∞ functional calculus and certain square function estimates. In particular they established the following key result.

Proposition 2.1. [11] *Let A be a sectorial operator on $L^p(\Omega)$ and assume that there exists $\theta_0 \in (0, \pi)$ such that A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta \in (\theta_0, \pi)$. Then for any $\theta \in (\theta_0, \pi)$ and any $\varphi \in H_0^\infty(\Sigma_\theta)$, there exists a constant $C \geq 0$ such that*

$$\Bigl\|\Bigl(\int_0^\infty \bigl|\varphi(tA)x\bigr|^2\, \frac{dt}{t}\Bigr)^{\frac{1}{2}}\Bigr\|_p \,\leq\, C\|x\|_p, \qquad x \in L^p(\Omega). \tag{2.3}$$

Let us now turn to discrete semigroups. Let $T \in B(X)$. We say that T is power bounded if the set

$$\mathcal{P}_T = \{T^n \,:\, n \geq 0\} \tag{2.4}$$

is bounded. Then we say that T is analytic if moreover the set

$$\mathcal{A}_T = \bigl\{n(T^n - T^{n-1}) \,:\, n \geq 1\bigr\} \tag{2.5}$$

is bounded. This notion of discrete analyticity goes back to [10]. Since $(T^n - T^{n-1})_{n\geq 1}$ is the 'discrete derivative' of the sequence $(T^n)_{n\geq 0}$, we can regard $n(T^n - T^{n-1})$ as a discrete analog of $t\frac{\partial}{\partial t}(T_t)$. In view of (2.2), the boundedness of (2.5) is therefore a natutal discrete analog of the boundedness of $\{tAT_t \,:\, t > 0\}$.

The most important result concerning discrete analyticity is perhaps the following characterization: an operator $T\colon X\to X$ is power bounded and analytic if and only if

$$\sigma(T)\subset\overline{\mathbb{D}}\qquad\text{and}\qquad\bigl\{(\lambda-1)R(\lambda,T)\,:\,|\lambda|>1\bigr\}\text{ is bounded.}\tag{2.6}$$

This property is called the 'Ritt condition'. The key argument for this characterization is due to O. Nevanlinna [29], however we refer to [25, 28] for a complete proof and complements. Let us gather a few observations which will be used later on in the paper. First we note that (2.6) implies that

$$\sigma(T)\subset\mathbb{D}\cup\{1\}.\tag{2.7}$$

Indeed, $\|R(\lambda,T)\|\geq d(\lambda,\sigma(T))^{-1}$ for any $\lambda\notin\sigma(T)$. Second, (2.6) implies the existence of a constant $K\geq 0$ such that $|\lambda-1|\|R(\lambda,T)\|\leq K$ whenever $\mathrm{Re}(\lambda)>1$. This means that

$$A=I-T$$

satisfies $(S)_{\frac{\pi}{2}}$. According to (2.1), this implies that A is a sectorial operator of type $<\frac{\pi}{2}$. Hence

$$\exists\,\theta\in\bigl(0,\tfrac{\pi}{2}\bigr)\;\big|\;\sigma(T)\subset\,1-\overline{\Sigma_\theta}.\tag{2.8}$$

In this case, the bounded analytic semigroup $(T_t)_{t\geq 0}$ generated by $-A$ is given by

$$T_t=e^{-t}e^{tT},\qquad t\geq 0.\tag{2.9}$$

We now recall the definition of R-boundedness (see [4, 7]). Let $(\varepsilon_k)_{k\geq 1}$ be a sequence of independent Rademacher variables on some probability space Ω_0. Let $\mathrm{Rad}(X)\subset L^2(\Omega_0;X)$ be the closure of $\mathrm{Span}\{\varepsilon_k\otimes x\,:\,k\geq 1,\ x\in X\}$ in the Bochner space $L^2(\Omega_0;X)$. Thus for any finite family $x_1,\ldots,x_n$ in X, we have

$$\Bigl\|\sum_k\varepsilon_k\otimes x_k\Bigr\|_{\mathrm{Rad}(X)}=\Bigl(\int_{\Omega_0}\Bigl\|\sum_k\varepsilon_k(s)\,x_k\Bigr\|_X^2\,ds\Bigr)^{\frac{1}{2}}.$$

By definition, a set $\mathcal{F}\subset B(X)$ is R-bounded if there is a constant $C\geq 0$ such that for any finite families $T_1,\ldots,T_n$ in $\mathcal{F}$, and any $x_1,\ldots,x_n$ in X, we have

$$\Bigl\|\sum_k\varepsilon_k\otimes T_k(x_k)\Bigr\|_{\mathrm{Rad}(X)}\,\leq\,C\,\Bigl\|\sum_k\varepsilon_k\otimes x_k\Bigr\|_{\mathrm{Rad}(X)}.$$

Obviously any R-bounded set is bounded and if X is isomorphic to a Hilbert space, then all bounded subsets of $B(X)$ are automatically R-bounded. However if X is not isomorphic to a Hilbert space, then $B(X)$ contains bounded subsets which are not R-bounded [2, Prop. 1.13].

Let (Ω,μ) be a measure space and let $1<p<\infty$. Then $\mathrm{Rad}(L^p(\Omega))\approx L^p(\Omega;\ell^2)$. Hence a set $\mathcal{F}\subset B(L^p(\Omega))$ is R-bounded if and only if we have an estimate

$$\Bigl\|\Bigl(\sum_k|T_k(x_k)|^2\Bigr)^{\frac{1}{2}}\Bigr\|_p\,\leq\,C\,\Bigl\|\Bigl(\sum_k|x_k|^2\Bigr)^{\frac{1}{2}}\Bigr\|_p$$

for finite families $(T_k)_k$ in $\mathcal{F}$ and $(x_k)_k$ in X.

We shall now consider these general definitions for specific sets of operators. Let $(T_t)_{t\geq 0}$ be a bounded analytic semigroup on X. We say that this is an R-bounded analytic semigroup if there exists a positive angle $\alpha > 0$ such that $\{T_z \,:\, z \in \Sigma_\alpha\}$ is R-bounded. It was observed in [37] that this holds true if and only if the two sets

$$\big\{T_t \,:\, t > 0\big\} \qquad \text{and} \qquad \big\{tAT_t \,:\, t > 0\big\}$$

are R-bounded.

Accordingly we will say that an operator $T \in B(X)$ is an R-analytic power bounded operator if the two sets $\mathcal{P}_T$ and $\mathcal{A}_T$ from (2.4) and (2.5) are R-bounded.

The above notions of R-analyticity were introduced by Weis [37] for the continuous case and Blunck [5] for the discrete one. In both cases they played a crucial role in the solution of maximal regularity problems on UMD Banach spaces, see the above papers for more information. R-boundedness for sectorial operators is also a key tool for various questions regarding H^∞ functional calculus, see in particular [19, 21, 18].

The next result is well-known to specialists.

Proposition 2.2. *Let $(T_t)_{t\geq 0}$ be a bounded analytic semigroup on $L^p(\Omega)$, with $1 < p < \infty$, and assume that $\|T_t\|_r \leq 1$ for any $t \geq 0$. Let $-A$ be the generator of $(T_t)_{t\geq 0}$. Then there exists $\theta \in \big(0, \frac{\pi}{2}\big)$ such that A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus.*

Proof. By [13] (see also [23, Thm. 4.13]), the operator A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > \frac{\pi}{2}$. On the other hand, it follows from [38, Section 4] that $(T_t)_{t\geq 0}$ is an R-bounded analytic semigroup. Applying [19, Prop. 5.1] we deduce the result. □

We end this section with a few notation. For any complex number a and any $r > 0$, we will let $D(a, r)$ denote the open disc of center a and radius r. We let $\mathbb{D} = D(0, 1)$ be the usual unit disc. Also we let $\mathcal{P}$ denote the algebra of complex polynomials in one variable.

3. Square functions on $L^p(\Omega)$.

Throughout the next two sections we let (Ω, μ) be a measure space and we fix some $1 < p < \infty$. We will establish general square function estimates for analytic contractively regular operators on $L^p(\Omega)$ (see Theorem 3.3 below).

We will need the following elementary fact.

Lemma 3.1. *Let $\mathcal{U} \subset \mathbb{C}$ be an open set and let $\Gamma \subset \mathcal{U}$ be a compact C^1-curve. Let $\varphi \colon \mathcal{U} \to B(L^p(\Omega))$ be an analytic function. Then there exists a contant $C \geq 0$ such that*

$$\Big\|\Big(\int_\Gamma \big|\varphi(\lambda)x\big|^2\,|d\lambda|\Big)^{\frac{1}{2}}\Big\|_p \leq C\|x\|_p$$

for any $x \in L^p(\Omega)$.

Proof. Let $0 < r \leq d(\Gamma, \mathcal{U}^c)/3$. Write Γ as the juxtaposition of C^1-curves $\Gamma_1, \ldots, \Gamma_N$ of length $< r$. Then for each $j = 1, \ldots, N$, choose $\lambda_j \in \Gamma_j$ and set

$$C_j = \sup\{\|\varphi(\lambda)\| \,:\, \lambda \in D(\lambda_j, 2r)\}.$$

Let

$$\varphi(\lambda) \,=\, \sum_{k=0}^{\infty} c_{jk}\,(\lambda - \lambda_j)^k \tag{3.1}$$

be the Taylor expansion of φ about λ_j. Then $\|c_{jk}\| \leq C_j/(2r)^k$ by Cauchy's inequalities. Any $\lambda \in \Gamma_j$ satisfies (3.1) hence we have

$$\Big\|\Big(\int_{\Gamma_j} \big|\varphi(\lambda)x\big|^2\,|d\lambda|\Big)^{\frac{1}{2}}\Big\|_p \leq \sum_{k=0}^{\infty}\Big\|\Big(\int_{\Gamma_j} \big|c_{jk}(x)\,(\lambda-\lambda_j)^k\big|^2\,|d\lambda|\Big)^{\frac{1}{2}}\Big\|_p.$$

However for any $k \geq 0$, we have

$$\Big\|\Big(\int_{\Gamma_j} \big|c_{jk}(x)\,(\lambda-\lambda_j)^k\big|^2\,|d\lambda|\Big)^{\frac{1}{2}}\Big\|_p = \big\|c_{jk}(x)\big\|_p\,\Big(\int_{\Gamma_j} |\lambda-\lambda_j|^{2k}\,|d\lambda|\Big)^{\frac{1}{2}}$$

and $|\lambda-\lambda_j| \leq r$ for any $\lambda \in \Gamma_j$. Thus

$$\Big\|\Big(\int_{\Gamma_j} \big|c_{jk}(x)\,(\lambda-\lambda_j)^k\big|^2\,|d\lambda|\Big)^{\frac{1}{2}}\Big\|_p \leq \|c_{jk}\|\,\|x\|_p\,|\Gamma_j| r^k \leq \|x\|_p\,|\Gamma_j|\,\frac{C_j}{2^k}\,.$$

Consequently,

$$\Big\|\Big(\int_{\Gamma_j} \big|\varphi(\lambda)x\big|^2\,|d\lambda|\Big)^{\frac{1}{2}}\Big\|_p \leq 2C_j\|x\|_p|\Gamma_j|.$$

Since

$$\Big\|\Big(\int_{\Gamma} \big|\varphi(\lambda)x\big|^2\,|d\lambda|\Big)^{\frac{1}{2}}\Big\|_p = \sum_{j=1}^{N}\Big\|\Big(\int_{\Gamma_j} \big|\varphi(\lambda)x\big|^2\,|d\lambda|\Big)^{\frac{1}{2}}\Big\|_p,$$

we obtain the result with $C = 2\max\{C_1,\ldots,C_N\}\,|\Gamma|$. □

For any $\gamma \in \big(0,\frac{\pi}{2}\big)$, let

$$B_\gamma = \Big\{z \in \big(1+\Sigma_{\pi-\gamma}\big)^c \,:\, |z| \leq \sin\gamma \text{ or } \mathrm{Re}(z) \geq \sin^2\gamma\Big\}.$$

Alternatively, B_γ is the convex hull of 1 and the disc $D(0,\sin\gamma)$.

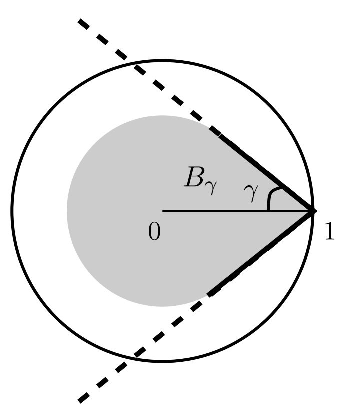

FIGURE 1.

Following usual terminology, these sets will be called 'Stolz domains' in the sequel. We will use the fact that for any $\gamma \in \big(0, \frac{\pi}{2}\big)$, there exists a constant C_γ such that

$$\frac{|1-z|}{1-|z|} \leq C_\gamma, \qquad z \in B_\gamma. \tag{3.2}$$

Let $N \geq 1$ be an integer and let $[F_{ij}]$ be an $N \times N$ matrix of polynomials, that is, $F_{i,j}$ belongs to $\mathcal{P}$ for any $1 \leq i, j \leq N$. Then for any $\gamma \in \big(0, \frac{\pi}{2}\big)$, we set

$$\big\|[F_{ij}]\big\|_\gamma = \sup\Big\{\big\|[F_{ij}(z)]\big\|_{M_N} \,:\, z \in B_\gamma\Big\}.$$

Proposition 3.2. *Let $T\colon L^p(\Omega) \to L^p(\Omega)$ be any analytic contractively regular operator. Then there exists an angle $\gamma \in \big(0, \frac{\pi}{2}\big)$ and a constant $C \geq 1$ satisfying the following property. For any $N \geq 1$, for any $N \times N$ matrix $[F_{ij}]$ of polynomials and for any $x_1, \ldots, x_N$ in $L^p(\Omega)$, we have*

$$\Big\|\Big(\sum_{i=1}^N \Big|\sum_{j=1}^N F_{ij}(T)x_j\Big|^2\Big)^{\frac{1}{2}}\Big\|_p \leq C\big\|[F_{ij}]\big\|_\gamma \,\Big\|\Big(\sum_{j=1}^N |x_j|^2\Big)^{\frac{1}{2}}\Big\|_p. \tag{3.3}$$

Proof. We let $p' = p/(p-1)$ be the conjugate number of p. Let $A = I - T$ and let $(T_t)_{t\geq 0}$ be the semigroup defined by (2.9), whose generator is $-A$. We noticed in Section 2 that this is a bounded analytic semigroup. Furthermore for any $t \geq 0$, we have

$$\|T_t\|_r = e^{-t}\big\|e^{tT}\big\|_r \leq e^{-t}e^{t\|T\|_r} \leq 1.$$

Hence by Proposition 2.2, A admits a bounded $H^\infty(\Sigma_{\theta_0})$ functional calculus for some $\theta_0 < \frac{\pi}{2}$. By (2.7) and (2.8), there exists $\gamma_0 \in \big[\theta_0, \frac{\pi}{2}\big)$ such that $\sigma(T) \subset B_{\gamma_0}$. Equivalently,

$$\sigma(A) = 1 - \sigma(T) \subset 1 - B_{\gamma_0}.$$

We now fix $\gamma \in \big(\gamma_0, \frac{\pi}{2}\big)$. Then we let L_γ be the boundary of $1 - B_\gamma$ oriented counterclockwise.

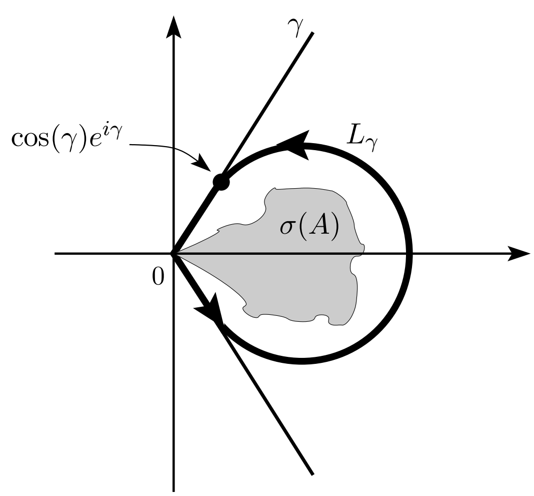

FIGURE 2.

We claim that we have estimates

$$\Big\|\Big(\int_{L_\gamma}\big|A^{\frac12}(\lambda-A)^{-1}x\big|^2\,|d\lambda|\Big)^{\frac12}\Big\|_p\lesssim\|x\|_p,\qquad x\in L^p(\Omega),\tag{3.4}$$

and

$$\Big\|\Big(\int_{L_\gamma}\big|A^{*\frac12}(\lambda+A^*)^{-1}y\big|^2\,|d\lambda|\Big)^{\frac12}\Big\|_{p'}\lesssim\|y\|_{p'},\qquad y\in L^{p'}(\Omega).\tag{3.5}$$

Recall that we let Γ_γ denote the boundary of Σ_γ oriented counterclockwise. Thus the contour L_γ is the juxtaposition of a part $L_{\gamma,1}$ of Γ_γ and the curve $L_{\gamma,2}$ going from $\cos(\gamma)e^{-i\gamma}$ to $\cos(\gamma)e^{i\gamma}$ counterclockwise along the circle of center 1 and radius $\sin\gamma$. Obviously we have

$$\begin{aligned}\Big\|\Big(\int_{L_\gamma}\big|A^{\frac12}(\lambda-A)^{-1}x\big|^2\,|d\lambda|\Big)^{\frac12}\Big\|_p &= \Big\|\Big(\int_{L_{\gamma,1}}\big|A^{\frac12}(\lambda-A)^{-1}x\big|^2\,|d\lambda|\Big)^{\frac12}\Big\|_p\\ &\quad+\Big\|\Big(\int_{L_{\gamma,2}}\big|A^{\frac12}(\lambda-A)^{-1}x\big|^2\,|d\lambda|\Big)^{\frac12}\Big\|_p\\ &\le \Big\|\Big(\int_{\Gamma_\gamma}\big|A^{\frac12}(\lambda-A)^{-1}x\big|^2\,|d\lambda|\Big)^{\frac12}\Big\|_p\\ &\quad+\Big\|\Big(\int_{L_{\gamma,2}}\big|A^{\frac12}(\lambda-A)^{-1}x\big|^2\,|d\lambda|\Big)^{\frac12}\Big\|_p\end{aligned}$$

Since $L_{\gamma,2}\cap\sigma(A)=\emptyset$, Lemma 3.1 ensures that we can control the last integral by a constant times $\|x\|_p$. Hence to prove (3.4), it suffices to prove an estimate

$$\Big\|\Big(\int_{\Gamma_\gamma}\big|A^{\frac12}(\lambda-A)^{-1}x\big|^2\,|d\lambda|\Big)^{\frac12}\Big\|_p\lesssim\|x\|_p,\qquad x\in L^p(\Omega).\tag{3.6}$$

Likewise, to prove (3.5), it suffices to prove an estimate

$$\Big\|\Big(\int_{\Gamma_\gamma}\big|A^{*\frac12}(\lambda+A^*)^{-1}y\big|^2\,|d\lambda|\Big)^{\frac12}\Big\|_{p'}\lesssim\|y\|_{p'},\qquad y\in L^{p'}(\Omega).\tag{3.7}$$

Consider $\theta_0<\theta<\gamma<\frac{\pi}{2}$, and define two functions $\varphi,\psi\in H_0^\infty(\Sigma_\theta)$ by letting

$$\varphi(z)=\frac{z^{\frac12}}{e^{i\gamma}-z}\qquad\text{and}\qquad\psi(z)=\frac{z^{\frac12}}{e^{-i\gamma}-z}\,.$$

For any $x\in L^p(\Omega)$, we have

$$\begin{aligned}\Big\|\Big(\int_0^\infty\big|\varphi(tA)x\big|^2\,\frac{dt}{t}\Big)^{\frac12}\Big\|_p &= \Big\|\Big(\int_0^\infty\big|A^{\frac12}(e^{i\gamma}-tA)^{-1}x\big|^2\,dt\Big)^{\frac12}\Big\|_p\\ &= \Big\|\Big(\int_0^\infty\big|A^{\frac12}(te^{i\gamma}-A)^{-1}x\big|^2\,dt\Big)^{\frac12}\Big\|_p.\end{aligned}$$

Likewise,

$$\Big\|\Big(\int_0^\infty\big|\psi(tA)x\big|^2\,\frac{dt}{t}\Big)^{\frac12}\Big\|_p=\Big\|\Big(\int_0^\infty\big|A^{\frac12}(te^{-i\gamma}-A)^{-1}x\big|^2\,dt\Big)^{\frac12}\Big\|_p.$$

Hence

$$\Big\|\Big(\int_{\Gamma_\gamma}\big|A^{\frac12}(\lambda-A)^{-1}x\big|^2\,|d\lambda|\Big)^{\frac12}\Big\|_p=\Big\|\Big(\int_0^\infty\big|\varphi(tA)x\big|^2\,\frac{dt}{t}\Big)^{\frac12}\Big\|_p+\Big\|\Big(\int_0^\infty\big|\psi(tA)x\big|^2\,\frac{dt}{t}\Big)^{\frac12}\Big\|_p.$$

Applying Proposition 2.1 to φ and ψ, we deduce the estimate (3.4). Now note that A^* also admits a bounded $H^\infty(\Sigma_{\theta_0})$ functional calculus (see e.g. [11] for this duality principle). Hence arguing as above with the two functions

$$z\mapsto\frac{z^{\frac12}}{e^{i\gamma}+z}\qquad\text{and}\qquad z\mapsto\frac{z^{\frac12}}{e^{-i\gamma}+z}\,,$$

we get (3.5).

The estimates (3.4) and (3.5) can be formally strengthened as follows. There is a constant $C\geq 0$ such that for any integer $N\geq 1$, we have

$$\Big\|\Big(\int_{L_\gamma}\sum_{j=1}^N\big|A^{\frac12}(\lambda-A)^{-1}x_j\big|^2\,|d\lambda|\Big)^{\frac12}\Big\|_p\leq C\,\Big\|\Big(\sum_{j=1}^N|x_j|^2\Big)^{\frac12}\Big\|_p \tag{3.8}$$

for any $x_1,\ldots,x_N$ in $L^p(\Omega)$ and similarly,

$$\Big\|\Big(\int_{L_\gamma}\sum_{j=1}^N\big|A^{*\frac12}(\lambda+A^*)^{-1}y_i\big|^2\,|d\lambda|\Big)^{\frac12}\Big\|_{p'}\leq C\,\Big\|\Big(\sum_{i=1}^N|y_i|^2\Big)^{\frac12}\Big\|_{p'} \tag{3.9}$$

for any $y_1,\ldots,y_N$ in $L^{p'}(\Omega)$. Indeed (3.8) (resp. (3.9)) can be deduced from (3.4) (resp. (3.5)) by applying Khintchine's inequality and Fubini's Theorem. The argument is similar to the one in the proof of [22, Lemma 5.4] so we omit it.

In the sequel, we let $\mathcal{P}_0\subset\mathcal{P}$ be the space of polynomials vanishing at 0. The function $\lambda\mapsto\lambda(\lambda-A)^{-1}$ is well-defined and bounded on $L_\gamma\setminus\{0\}$, and the same is true for $\lambda\mapsto f(\lambda)(\lambda-A)^{-1}$ whenever $f\in\mathcal{P}_0$. It therefore follows from the Dunford functional calculus that

$$f(A)=\frac{1}{2\pi i}\int_{L_\gamma}f(\lambda)(\lambda-A)^{-1}\,d\lambda$$

for any $f\in\mathcal{P}_0$. Likewise,

$$0=\frac{1}{2\pi i}\int_{L_\gamma}f(\lambda)(\lambda+A)^{-1}\,d\lambda$$

for any $f\in\mathcal{P}_0$. Hence

$$f(A)=\frac{1}{2\pi i}\int_{L_\gamma}f(\lambda)\big((\lambda-A)^{-1}-(\lambda+A)^{-1}\big)\,d\lambda\,,$$

that is,

$$f(A)=\frac{1}{\pi i}\int_{L_\gamma}f(\lambda)A(\lambda-A)^{-1}(\lambda+A)^{-1}\,d\lambda\,. \tag{3.10}$$

Let $N\geq 1$ be an integer, let $[F_{ij}]$ be an $N\times N$ matrix of polynomials, and let $x_1,\ldots,x_N$ be in $L^p(\Omega)$. For any $i,j=1,\ldots,N$, we set $f_{ij}(\lambda)=F_{ij}(1-\lambda)$, so that $F_{ij}(T)=f_{ij}(A)$. Also we assume that $F_{ij}(1)=0$, so that $f_{ij}\in\mathcal{P}_0$. For any $y_1,\dots,y_N$ in $L^{p'}(\Omega)$, we have

$$\begin{aligned}\sum_{i,j}\big\langle f_{ij}(A)x_j,y_i\big\rangle &= \frac{1}{\pi i}\int_{L_\gamma}\sum_{i,j} f_{ij}(\lambda)\big\langle A(\lambda-A)^{-1}(\lambda+A)^{-1}x_j,y_i\big\rangle\,d\lambda\\ &= \frac{1}{\pi i}\int_{L_\gamma}\sum_{i,j} f_{ij}(\lambda)\big\langle A^{\frac12}(\lambda-A)^{-1}x_j,A^{*\frac12}(\lambda+A^*)^{-1}y_i\big\rangle\,d\lambda\end{aligned}$$

by (3.10). Applying Cauchy-Schwarz and Hölder's inequalities, we deduce that

$$\begin{aligned}\Big|\sum_{i,j}\big\langle f_{ij}(A)x_j,y_i\big\rangle\Big| \leq \frac{1}{\pi}\,&\Big\|\Big(\int_{L_\gamma}\sum_i\Big|\sum_j f_{ij}(\lambda)A^{\frac12}(\lambda-A)^{-1}x_j\Big|^2\,|d\lambda|\Big)^{\frac12}\Big\|_p\\ &\times\Big\|\Big(\int_{L_\gamma}\sum_i\big|A^{*\frac12}(\lambda+A^*)^{-1}y_i\big|^2\,|d\lambda|\Big)^{\frac12}\Big\|_{p'}.\end{aligned}$$

Furthermore,

$$\Big\|\Big(\int_{L_\gamma}\sum_i\Big|\sum_j f_{ij}(\lambda)A^{\frac12}(\lambda-A)^{-1}x_j\Big|^2\,|d\lambda|\Big)^{\frac12}\Big\|_p$$

is less than or equal to

$$\Big\|\Big(\int_{L_\gamma}\big\|[f_{ij}(\lambda)]\big\|_{M_N}^2\sum_j\big|A^{\frac12}(\lambda-A)^{-1}x_j\big|^2\,|d\lambda|\Big)^{\frac12}\Big\|_p,$$

which in turn is less than or equal to

$$\sup\Big\{\big\|[f_{ij}(\lambda)]\big\|_{M_N}\,:\,\lambda\in L_\gamma\Big\}\,\Big\|\Big(\int_{L_\gamma}\sum_j\big|A^{\frac12}(\lambda-A)^{-1}x_j\big|^2\,|d\lambda|\Big)^{\frac12}\Big\|_p.$$

Now recall that $F_{ij}(T)=f_{ij}(A)$ and note that $\sup\big\{\|[f_{ij}(\lambda)]\|_{M_N}\,:\,\lambda\in L_\gamma\big\}$ is less than or equal to $\|[F_{ij}\|_\gamma$. Appealing to (3.8) and (3.9), we therefore obtain an estimate

$$\Big|\sum_{i,j}\big\langle F_{ij}(T)x_j,y_i\big\rangle\Big| \lesssim \big\|[F_{ij}\big\|_\gamma\,\Big\|\Big(\sum_j|x_j|^2\Big)^{\frac12}\Big\|_p\,\Big\|\Big(\sum_i|y_i|^2\Big)^{\frac12}\Big\|_{p'}.$$

Passing to the supremum over all $y_1,\dots,y_N$ in $L^{p'}(\Omega)$ such that $\big\|\big(\sum_i|y_i|^2\big)^{\frac12}\big\|_{p'}\leq 1$, we finally obtain (3.3) in the case when all F_{ij}'s vanish at 1.

The general case follows at once. Indeed for an arbitrary matrix $[F_{ij}]$ of polynomials, write $\widetilde{F}_{ij}=F_{ij}-F_{ij}(1)$. Then

$$\big\|[F_{ij}(1)]\big\|_{M_N}\leq\big\|[F_{ij}]\big\|_\gamma\qquad\text{and}\qquad\big\|[\widetilde{F}_{ij}]\big\|_\gamma\leq 2\big\|[F_{ij}]\big\|_\gamma.$$

Thus if (3.3) holds true for $[\widetilde{F}_{ij}]$ and a certain constant C, we deduce that

$$\Bigl\|\Bigl(\sum_{i=1}^{N}\Bigl|\sum_{j=1}^{N}F_{ij}(T)x_j\Bigr|^2\Bigr)^{\frac12}\Bigr\|_p \leq \Bigl\|\Bigl(\sum_{i=1}^{N}\Bigl|\sum_{j=1}^{N}\widetilde{F}_{ij}(T)x_j\Bigr|^2\Bigr)^{\frac12}\Bigr\|_p + \Bigl\|\Bigl(\sum_{i=1}^{N}\Bigl|\sum_{j=1}^{N}F_{ij}(1)x_j\Bigr|^2\Bigr)^{\frac12}\Bigr\|_p$$
$$\leq \bigl(2C+1\bigr)\bigl\|[F_{ij}]\bigr\|_\gamma \Bigl\|\Bigl(\sum_{j=1}^{N}|x_j|^2\Bigr)^{\frac12}\Bigr\|_p.$$

□

Theorem 3.3. *Let $T\colon L^p(\Omega)\to L^p(\Omega)$ be an analytic contractively regular operator.*

(1) *There exists an angle $\gamma\in\bigl(0,\frac{\pi}{2}\bigr)$ and a constant $C\geq 1$ such that for any sequence $(F_n)_{n\geq 1}$ of polynomials and any $x\in L^p(\Omega)$,*

$$\Bigl\|\Bigl(\sum_{n=1}^{\infty}\bigl|F_n(T)x\bigr|^2\Bigr)^{\frac12}\Bigr\|_p \leq C\,\|x\|_p\,\sup\Bigl\{\Bigl(\sum_{n=1}^{\infty}\bigl|F_n(z)\bigr|^2\Bigr)^{\frac12}\,:\, z\in B_\gamma\Bigr\}. \tag{3.11}$$

(2) *For any integer $m\geq 1$, there is an estimate*

$$\Bigl\|\Bigl(\sum_{n=0}^{\infty}(n+1)^{2m-1}\bigl|T^n(T-I)^m(x)\bigr|^2\Bigr)^{\frac12}\Bigr\|_p \lesssim \|x\|_p. \tag{3.12}$$

Proof. We apply Proposition 3.2 to T and we thus obtain $\gamma\in\bigl(0,\frac{\pi}{2}\bigr)$ for which (3.3) holds true. Let $(F_n)_{n\geq 1}$ be any sequence of polynomials. We get (3.11) by applying (3.3) to the column matrix

$$\begin{bmatrix} F_1 & 0 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ F_N & 0 & \cdots & 0\end{bmatrix}$$

for any $N\geq 1$ and then by passing to the limit when $N\to\infty$.

To prove part (2), we fix $m\geq 1$, we set

$$F_n(z) = n^{m-\frac12}z^{n-1}(z-1)^m$$

for any $n\geq 1$, and we aim at applying (3.11) to this sequence. For any $z\in\mathbb{D}$, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\bigl|F_n(z)\bigr|^2 &= \sum_{n=1}^{\infty}n^{2m-1}|z|^{2(n-1)}|z-1|^{2m}\\ &\leq |1-z|^{2m}\sum_{n=0}^{\infty}(n+1)(n+2)\cdots(n+2m-1)|z|^{2n}\\ &\leq |1-z|^{2m}\,\frac{1}{\bigl(1-|z|^2\bigr)^{2m}}\\ &\leq \Bigl(\frac{|1-z|}{1-|z|}\Bigr)^{2m}.\end{aligned}$$

This upper bound is bounded on B_γ by (3.2) hence (3.12) now follows from part (1). □

Note that (1.4) corresponds to (3.12) for $m = 1$.

Remark 3.4. Consider T as in Theorem 3.3. We will establish additional estimates, which are all consequences of the above theorem.

(1) By the Mean Ergodic Theorem, we have a direct sum decomposition

$$L^p(\Omega) \,=\, N(I-T) \oplus \overline{R(I-T)},$$

where $N(\cdot\,)$ and $R(\cdot\,)$ denote the kernel and the range, respectively. Let $P\colon L^p(\Omega) \to L^p(\Omega)$ be the projection onto $\overline{R(I-T)}$ with respect to this decomposition. Then for any $m \geq 1$, we have an estimate

$$\|P(x)\|_p \lesssim \Bigl\|\Bigl(\sum_{n=0}^{\infty}(n+1)^{2m-1}\bigl|T^n(T-I)^m(x)\bigr|^2\Bigr)^{\frac{1}{2}}\Bigr\|_p \tag{3.13}$$

on $L^p(\Omega)$. In other words, the estimate (3.12) can be reversed on $\overline{R(I-T)}$.

Let us prove (3.13) for $m = 1$, the other cases being similar. We start from the identity

$$\sum_{n=0}^{\infty}(n+1)z^{2n}(1-z^2)^2 \,=\, 1, \qquad z \in \mathbb{D}.$$

It implies that for any $0 < r < 1$, we have

$$\sum_{n=0}^{\infty}(n+1)(rT)^{2n}(rT+I)^2(rT-I)^2 \,=\, I.$$

Let $x \in L^p(\Omega)$ and $y \in L^{p'}(\Omega)$. Set $y_r = (rT^* + I)^2 y$ for any r. From the above identity, we get

$$\langle x, y\rangle \,=\, \sum_{n=0}^{\infty}(n+1)\bigl\langle (rT)^n(rT-I)x, (rT^*)^n(rT^*-I)y_r\bigr\rangle.$$

Hence

$$\bigl|\langle x, y\rangle\bigr| \,\leq\, \Bigl\|\Bigl(\sum_{n=0}^{\infty}(n+1)\bigl|(rT)^n(rT-I)x\bigr|^2\Bigr)^{\frac{1}{2}}\Bigr\|_p \Bigl\|\Bigl(\sum_{n=0}^{\infty}(n+1)\bigl|(rT^*)^n(rT^*-I)y_r\bigr|^2\Bigr)^{\frac{1}{2}}\Bigr\|_{p'}.$$

The operator $T^*\colon L^{p'}(\Omega) \to L^{p'}(\Omega)$ is analytic and contractively regular, hence satisfies the first part of Theorem 3.3. Moreover $\|y_r\|_{p'} \leq 4\|y\|_{p'}$ for any r. Hence we can control the second factor in the right handside of the above inequality by $\|y\|_{p'}$, up to a constant not depending on r. We deduce that

$$\bigl|\langle x, y\rangle\bigr| \,\lesssim\, \Bigl\|\Bigl(\sum_{n=0}^{\infty}(n+1)\bigl|(rT)^n(rT-I)x\bigr|^2\Bigr)^{\frac{1}{2}}\Bigr\|_p \|y\|_{p'}$$

uniformly in r. Taking the supremum over all $y \in L^{p'}(\Omega)$ with $\|y\|_{p'} \leq 1$, we obtain a uniform estimate

$$\|x\|_p \,\lesssim\, \Bigl\|\Bigl(\sum_{n=0}^{\infty}(n+1)r^{2n}\bigl|T^n(rT-I)x\bigr|^2\Bigr)^{\frac{1}{2}}\Bigr\|_p, \qquad x \in L^p(\Omega),\ 0 < r < 1.$$

Now assume that $x \in R(I-T)$, i.e. $x = (T-I)\widetilde{x}$ for some $\widetilde{x}$ in $L^p(\Omega)$. Applying (3.12) to $\widetilde{x}$ (with $m=1$), we see that the sequence $\big((n+1)^{\frac{1}{2}}T^n(x)\big)_{n\geq 0}$ belongs to $L^p(\ell^2)$. Consequently, the sequence $\big((n+1)^{\frac{1}{2}}(rT-I)T^n(x)\big)_{n\geq 0}$ belongs to $L^p(\ell^2)$ as well for any $0<r<1$ and this family of sequences tends to $\big((n+1)^{\frac{1}{2}}(T-I)T^n(x)\big)_{n\geq 0}$ when $r\to 1$. We deduce that

$$\Big\|\Big(\sum_{n=0}^{\infty}(n+1)\big|T^n(rT-I)x\big|^2\Big)^{\frac{1}{2}}\Big\|_p \longrightarrow \Big\|\Big(\sum_{n=0}^{\infty}(n+1)\big|T^n(T-I)x\big|^2\Big)^{\frac{1}{2}}\Big\|_p$$

when $r\to 1$, and hence that

$$\|x\|_p \lesssim \Big\|\Big(\sum_{n=0}^{\infty}(n+1)\big|T^n(T-I)x\big|^2\Big)^{\frac{1}{2}}\Big\|_p. \tag{3.14}$$

This establishes (3.13) for the elements of $R(I-T)$.

To complete the proof, set

$$\Lambda_m \,=\, \frac{1}{m+1}\sum_{k=1}^{m}(I-T^k)$$

for any integer $m\geq 0$. Then $\Lambda_m\to P$ pointwise when $m\to\infty$. Let x be an arbitrary element of $L^p(\Omega)$. Applying (3.14) with $\Lambda_m(x)$ in the place of x and letting $m\to\infty$, we obtain the desired estimate (3.13).

(2) For any $m\geq 1$, T satisfies the following estimate

$$\Big\|\Big(\sum_{n=1}^{\infty} n\,\big|(n+1)^m T^n(T-I)^m(x)\,-\,n^m T^{n-1}(T-I)^m(x)\big|^2\Big)^{\frac{1}{2}}\Big\|_p \lesssim \|x\|_p, \tag{3.15}$$

that we record here for further use in Section 4.

For its proof it will be convenient to set

$$\Delta_n^m = T^n(T-I)^m \qquad\text{and}\qquad B_n^m = (n+1)^m\Delta_n^m \tag{3.16}$$

for any integers $m,n\geq 0$. We fix some $m\geq 1$ and $x\in L^p(\Omega)$. Then we have

$$\begin{aligned} B_n^m(x) - B_{n-1}^m(x) \,&=\, \big((n+1)^m T - n^m\big)T^{n-1}(T-I)^m x\\ &=\, (n+1)^m T^{n-1}(T-I)^{m+1}x \,+\, \big((n+1)^m - n^m\big)T^{n-1}(T-I)^m x \end{aligned}$$

for any $n\geq 1$. Consequently,

$$\begin{aligned} \big|B_n^m(x) - B_{n-1}^m(x)\big|^2 \,&\leq\, 2\Big((n+1)^{2m}\big|\Delta_{n-1}^{m+1}(x)\big|^2 \,+\, \big((n+1)^m - n^m\big)^2\big|\Delta_{n-1}^m(x)\big|^2\Big)\\ &\lesssim\, n^{2m}\big|\Delta_{n-1}^{m+1}(x)\big|^2 \,+\, n^{2(m-1)}\big|\Delta_{n-1}^m(x)\big|^2. \end{aligned}$$

Summing up, we obtain that

$$\sum_{n=1}^{\infty} n\,\big|B_n^m(x) - B_{n-1}^m(x)\big|^2 \lesssim \sum_{n=1}^{\infty} n^{2m+1}\big|\Delta_{n-1}^{m+1}(x)\big|^2 \,+\, \sum_{n=1}^{\infty} n^{2m-1}\big|\Delta_{n-1}^m(x)\big|^2\,.$$

Applying (3.12) twice, with m and $m+1$, we deduce the estimate (3.15).

(3) Set

$$M_n(T) = \frac{1}{n+1}\sum_{k=0}^{n} T^k \tag{3.17}$$

for any $n \geq 0$. Then we have

$$\Big\| \Big(\sum_{n=0}^{\infty} (n+1) \big| M_{n+1}(T)(x) - M_n(T)(x) \big|^2 \Big)^{\frac{1}{2}} \Big\|_p \lesssim \|x\|_p \tag{3.18}$$

for $x \in L^p(\Omega)$. By an entirely classical averaging argument, one obtains this estimate as a consequence of (3.12). We skip the details.

Inequality (3.18) plays a key role in [36, Section 5], where it is shown in the case when T acts as a contraction $L^q(\Omega) \to L^q(\Omega)$ for any $1 \leq q \leq \infty$ and its L^2-realization is a positive selfadjoint operator.

Remark 3.5. For any $\gamma \in \big(0, \frac{\pi}{2}\big)$, let $\mathcal{P}_\gamma \subset C(B_\gamma)$ be the algebra $\mathcal{P}$ regarded as a subspace of $C(B_\gamma)$, the commutative C^*-algebra of all complex valued continuous functions on the compact set B_γ. Let $u_\gamma \colon \mathcal{P}_\gamma \to B(L^p(\Omega))$ be the natural functional calculus map, defined by

$$u_\gamma(F) = F(T).$$

(a) Proposition 3.2 means that for some $\gamma \in \big(0, \frac{\pi}{2}\big)$, the map u_γ is ℓ_2-completely bounded in the sense of [34] (see also [20, Section 4]). In the case $p = 2$, this means that u_γ is completely bounded.

(b) If we restrict (3.3) to diagonal matrices, we readily obtain that whenever $(F_n)_{n\geq 1}$ is a bounded sequence of $\mathcal{P}_\gamma$, then the set $\{F_n(T) \,:\, n \geq 1\}$ is R-bounded. Applying this property to the two sequences

$$z \mapsto z^n \qquad \text{and} \qquad z \mapsto n(z^n - z^{n-1}),$$

we deduce that any analytic contractively regular $T \colon L^p(\Omega) \to L^p(\Omega)$ is an R-analytic power bounded operator (in the sense of Section 2). This result is due to Blunck (see [5, Thm. 1.1 and Thm. 1.2]).

4. Maximal theorems on $L^p(\Omega)$.

The general maximal theorem we aim at proving is the following. The case $m = 0$, which gives (1.2), is of particular interest.

Theorem 4.1. *Let $T \colon L^p(\Omega) \to L^p(\Omega)$ be an analytic contractively regular operator. Then for any integer $m \geq 0$, there is a constant $C \geq 0$ such that*

$$\Big\| \sup_{n\geq 0}\, (n+1)^m \big| T^n (T-I)^m (x) \big| \Big\|_p \leq C\, \|x\|_p, \qquad x \in L^p(\Omega). \tag{4.1}$$

Proof. We will use classical 'integration by parts' arguments and induction. Recall the notation from (3.16). For any $m \geq 1$, let us consider the estimate

$$\Big\|\sup_{n\geq 0}\frac{1}{n+1}\Big|\sum_{k=0}^{n} B_k^m(x)\Big|\Big\|_p \lesssim \|x\|_p, \qquad x\in L^p(\Omega). \tag{4.2}$$

This is clearly weaker than (4.1), however we will need to use it explicitly later on. For clarity we will write $(4.1)_m$ and $(4.2)_m$ instead of (4.1) and (4.2) in this proof.

For any $n \geq 1$, we have

$$\sum_{k=1}^{n} k\big(T^k - T^{k-1}\big) = \sum_{k=1}^{n} kT^k - \sum_{k=0}^{n-1}(k+1)T^k = nT^n - \sum_{k=0}^{n-1} T^k,$$

hence

$$T^n = \frac{1}{n}\sum_{k=0}^{n-1} T^k + \frac{1}{n}\sum_{k=1}^{n} k\big(T^k - T^{k-1}\big). \tag{4.3}$$

By Cauchy-Schwarz, we deduce that for any $x \in L^p(\Omega)$,

$$\big|T^n(x)\big| \leq \frac{1}{n}\Big|\sum_{k=0}^{n-1} T^k(x)\Big| + \Big(\sum_{k=1}^{n} k\big|T^k(x) - T^{k-1}(x)\big|^2\Big)^{\frac12}.$$

According to [31] or [8] (which generalized Akcoglu's Theorem to contractively regular operators), T satisfies (1.1). Hence applying (3.12) with $m = 1$, we obtain $(4.1)_0$. Appealing to (4.3) again, we immediatly deduce that $(4.2)_1$ holds true as well.

Now let $m \geq 1$. Arguing as above we have

$$B_n^m = \frac{1}{n}\sum_{k=0}^{n-1} B_k^m + \frac{1}{n}\sum_{k=1}^{n} k\big(B_k^m - B_{k-1}^m\big). \tag{4.4}$$

Also we have

$$\begin{aligned}\sum_{k=0}^{n} B_k^{m+1} &= (T-I)^m \sum_{k=0}^{n}(k+1)^{m+1}\big(T^{k+1} - T^k\big)\\ &= (T-I)^m\Big((n+1)^{m+1}T^{n+1} - \sum_{k=0}^{n}\big((k+1)^{m+1} - k^{m+1}\big)T^k\Big),\end{aligned}$$

hence

$$\frac{1}{n+1}\sum_{k=0}^{n} B_k^{m+1} = \Big(\frac{n+1}{n+2}\Big)^{m+1} B_{n+1}^m - \frac{1}{n+1}\sum_{k=0}^{n}\big((k+1)^{m+1} - k^{m+1}\big)\Delta_k^m. \tag{4.5}$$

By Cauchy-Schwarz,

$$\frac{1}{n}\Big|\sum_{k=1}^{n} k\big(B_k^m(x) - B_{k-1}^m(x)\big)\Big| \leq \Big(\sum_{k=1}^{n} k\big|B_k^m(x) - B_{k-1}^m(x)\big|^2\Big)^{\frac12},$$

hence $(4.2)_m$ implies $(4.1)_m$ by (4.4) and Remark 3.4. Likewise,

$$\frac{1}{n+1}\Big|\sum_{k=0}^{n}\big((k+1)^{m+1}-k^{m+1}\big)\Delta_k^m(x)\Big|$$
$$\leq \frac{1}{n+1}\bigg(\sum_{k=0}^{n}\frac{\big((k+1)^{m+1}-k^{m+1}\big)^2}{(k+1)^{2m-1}}\bigg)^{\frac12}\bigg(\sum_{k=0}^{n}(k+1)^{2m-1}\big|\Delta_k^m(x)\big|^2\bigg)^{\frac12}$$
$$\lesssim \bigg(\sum_{k=0}^{n}(k+1)^{2m-1}\big|\Delta_k^m(x)\big|^2\bigg)^{\frac12},$$

hence $(4.2)_{m+1}$ and $(4.1)_m$ are equivalent by (4.5) and (3.12). Thus $(4.1)_m$ holds true for any $m \geq 0$ by induction. □

Theorem 4.1 is a generalization of [35]. In that paper, (4.1) is established for an operator T which is a positive contraction $L^q(\Omega) \to L^q(\Omega)$ for any $1 \leq q \leq \infty$ whose L^2-realization is a positive selfadjoint operator. Clearly the L^p-realization of such an operator satisfies the assumptions of Theorem 4.1. Indeed if $T\colon L^2(\Omega) \to L^2(\Omega)$ is a positive selfadjoint operator, then it is analytic by spectral representation. Hence $T\colon L^p(\Omega) \to L^p(\Omega)$ is analytic for any $1 < p < \infty$ by [6, Thm 1.1].

The following is an analog of Theorem 4.1 for continuous semigroups.

Corollary 4.2. *Let $(T_t)_{t\geq 0}$ be a bounded analytic semigroup on $L^p(\Omega)$, and assume that $\|T_t\|_r \leq 1$ for any $t \geq 0$. Then for any integer $m \geq 0$, we have an estimate*

$$\Big\|\sup_{t>0} t^m\Big|\frac{\partial^m}{\partial t^m}\big(T_t(x)\big)\Big|\Big\|_p \lesssim C\|x\|_p, \qquad x \in L^p(\Omega). \tag{4.6}$$

Proof. Let $-A$ be the generator of $(T_t)_{t\geq 0}$. According to Proposition 2.2, it admits a bounded $H^\infty(\Sigma_{\theta_0})$ functional calculus for some $\theta_0 < \frac{\pi}{2}$. Let $\theta \in \big(\theta_0, \frac{\pi}{2}\big)$. Arguing as in Proposition 3.2 and Theorem 3.3 (1), we obtain the existence of a constant $C \geq 1$ such that for any sequence $(f_n)_{n\geq 1}$ of functions in $H_0^\infty(\Sigma_{\theta_0})$ and any $x \in L^p(\Omega)$, we have

$$\Big\|\Big(\sum_{n=1}^{\infty}\big|f_n(A)x\big|^2\Big)^{\frac12}\Big\|_p \leq C\,\|x\|_p \sup\Big\{\Big(\sum_{n=1}^{\infty}\big|f_n(z)\big|^2\Big)^{\frac12} \,:\, z \in \Sigma_\theta\Big\}.$$

Then arguing as in Theorem 3.3 (2), we deduce that for any $m \geq 1$, there is a constant $C_m \geq 1$ such that for any $t > 0$ and for any $x \in L^p(\Omega)$,

$$\Big\|\Big(\sum_{n=0}^{\infty}(n+1)^{2m-1}\big|T_t^n(T_t-I)^m(x)\big|^2\Big)^{\frac12}\Big\|_p \leq C_m\,\|x\|_p.$$

In other words, the operators T_t satisfy (3.12) uniformly. The above proof of Theorem 4.1 therefore shows that they satisfy (4.1) uniformly.

Let $t_1, t_2, \ldots, t_N$ be positive real numbers. For any $j = 1, \ldots, N$ and $k \geq 1$, let n_{jk} be the integral part of $kt_j + 1$ and let $t_{jk} = n_{jk}/k$, so that $t_{kj} \geq t_j$ and $t_{kj} \to t_j$ when $k \to \infty$. It follows from above that we have an estimate

$$\Big\|\Big((n_{jk}+1)^m T_{\frac1k}^{n_{jk}}\big(T_{\frac1k}-I\big)^m(x)\Big)_{1\leq j\leq N}\Big\|_{L^p(\Omega;\ell_N^\infty)} \leq K\|x\|_p,$$

for some constant $K\geq 1$ neither depending on x, k or the t_j's. Letting $k\to\infty$, we deduce that

$$\Bigl\Vert\Bigl(t_j^m(-A)^mT_j(x)\Bigr)_{1\leq j\leq N}\Bigr\Vert_{L^p(\Omega;\ell^\infty_N)}\,\leq\,K\Vert x\Vert_p.$$

Clearly this uniform estimate implies (4.6). □

Remark 4.3. Here is an alternative proof of Corollary 4.2 not using the discrete case. For any real $t>0$, consider the average operator $M_t\in B(L^p(\Omega))$ defined by letting

$$M_t(x)\,=\,\frac{1}{t}\int_0^t T_u(x)\,du$$

for any $x\in L^p(\Omega)$. Since $\Vert T_t\Vert_r\leq 1$ for any $t\geq 1$, it follows from [14] that we have an estimate

$$\bigl\Vert\sup_{t>0}\bigl|M_t(x)\bigr|\bigr\Vert_p\,\lesssim\,\Vert x\Vert_p,\qquad x\in L^p(\Omega). \tag{4.7}$$

For any integer $m\geq 1$, let φ_m be the analytic function defined by $\varphi_m(z)=z^me^{-z}$. Then φ_m belongs to $H^\infty_0(\Sigma_\theta)$ for any $\theta\in\bigl(0,\frac{\pi}{2}\bigr)$. Hence according to Propositions 2.1 and 2.2, the square function estimate (2.3) holds for $\varphi=\varphi_m$. For any real $t>0$, we have

$$\varphi_m(tA)(x)=t^mA^me^{-tA}(x)=(-1)^mt^m\,\frac{\partial^m}{\partial t^m}\,\bigl(T_t(x)\bigr).$$

Hence we obtain estimates

$$\Bigl\Vert\Bigl(\int_0^\infty t^{2m-1}\Bigl|\frac{\partial^m}{\partial t^m}\,(T_t(x))\Bigr|^2\,\frac{dt}{t}\,\Bigr)^{\frac12}\Bigr\Vert_p\lesssim\Vert x\Vert_p$$

for any $m\geq 1$. Then Stein's arguments in [36, pp. 73-76] show that (4.7) together with these estimates imply that (4.6) holds true for any $m\geq 0$.

5. Maximal theorems on noncommutative L^p-spaces

In this section we will partly extend the results established in the previous one, in the light of the recent work [17]. We start with a few preliminaries on semifinite noncommutative L^p-spaces.

Let M be a von Neumann algebra equipped with a normal semifinite faithful trace τ. Let M_+ be the set of all positive elements of M and let S_+ be the set of all x in M_+ such that $\tau(x)<\infty$. Then let S be the linear span of S_+. For any $1\leq p<\infty$, define

$$\Vert x\Vert_p\,=\,\bigl(\tau(|x|^p)\bigr)^{\frac1p},\qquad x\in S,$$

where $|x|\,=\,(x^*x)^{\frac12}$ is the modulus of x. Then $(S,\Vert\ \Vert_p)$ is a normed space. The corresponding completion is the noncommutative L^p-space associated with (M,τ) and is denoted by $L^p(M)$. By convention we set $L^\infty(M)=M$, equipped with the operator norm. The elements of $L^p(M)$ can also be described as measurable operators with respect to (M,τ). Further multiplication of measurable operators leads to contractive bilinear maps $L^p(M)\times L^q(M)\,\to\,L^r(M)$ for any p,q,r such that $p^{-1}+q^{-1}=r^{-1}$ (noncommutative

Hölder's inequality). Using trace duality, we then have $L^p(M)^* = L^{p'}(M)$ isometrically for any $1 \leq p < \infty$. Moreover, complex interpolation yields

$$L^p(M) = [L^\infty(M), L^1(M)]_{\frac{1}{p}} \tag{5.1}$$

for any $1 \leq p < \infty$. We refer the reader to [33] for details and complements.

Maximal functions in the noncommutative setting require a specific definition. Indeed, $\sup_n |x_n|$ does not make any sense for a sequence $(x_n)_n$ of operators. This difficulty is overcome by considering the spaces $L^p(M; \ell^\infty)$, which are the noncommutative analogs of the usual Bochner spaces $L^p(\Omega; \ell^\infty)$. Given $1 \leq p < \infty$, $L^p(M; \ell^\infty)$ is defined as the space of all sequences $(x_n)_{n\geq 0}$ in $L^p(M)$ for which there exist $a, b \in L^{2p}(M)$ and a bounded sequence $(z_n)_{n\geq 0}$ in M such that

$$x_n = a z_n b, \qquad n \geq 0. \tag{5.2}$$

For such a sequence, set

$$\big\|(x_n)_{n\geq 0}\big\|_{L^p(M;\ell^\infty)} = \inf\big\{\|a\|_{2p} \sup_n \|z_n\| \|b\|_{2p}\big\},$$

where the infimum runs over all possible factorizations of $(x_n)_{n\geq 0}$ in the form (5.2). This is a norm and $L^p(M; \ell^\infty)$ is a Banach space. These spaces were first introduced by Pisier [32] in the case when M is hyperfinite and by Junge [16] in the general case. We will adopt the convention in [17] that the norm $\|(x_n)_{n\geq 0}\|_{L^p(M;\ell^\infty)}$ is denoted by

$$\big\|{\sup_{n\geq 0}}^+ x_n\big\|_p. \tag{5.3}$$

We warn the reader that this suggestive notation should be treated with care. It is used for possibly non positive operators and $\big\|{\sup_{n\geq 0}}^+ x_n\big\|_p \neq \big\|{\sup_{n\geq 0}}^+ |x_n|\big\|_p$ in general. However it has an intuitive description in the positive case, as observed in [17, p. 392]: a positive sequence $(x_n)_{n\geq 0}$ of $L^p(M)$ belongs to $L^p(M; \ell^\infty)$ if and only if there exists a positive $a \in L^p(M)$ such that $x_n \leq a$ for any $n \geq 0$ and in this case,

$$\big\|{\sup_{n\geq 0}}^+ x_n\big\|_p = \inf\big\{\|a\|_p \,:\, a \in L^p(M),\ a \geq 0 \quad\text{and}\quad x_n \leq a \text{ for any } n \geq 0\big\}. \tag{5.4}$$

Let $T\colon M \to M$ be a contraction We say that it is an absolute contraction if its restriction to $L^1(M) \cap M$ extends to a contraction $L^1(M) \to L^1(M)$. In this case, it extends (by interpolation) to a contraction on $L^p(M)$ for any $1 \leq p \leq \infty$. We let $T_p\colon L^p(M) \to L^p(M)$ denote the resulting operator.

Lemma 5.1. *Let $1 < p, q < \infty$. The operator T_p is analytic if and only if T_q is analytic.*

Proof. This result was proved by Blunck in the commutative setting [6, Thm. 1.1], using interpolation. His arguments apply as well to the noncommutative setting, using (5.1). ☐

In accordance with this lemma we will say that an absolute contraction $T\colon M \to M$ is analytic if T_p is analytic for one (equivalently for all) $1 < p < \infty$.

We say that $T\colon M \to M$ is positive if $T(x) \geq 0$ for any $x \in M_+$. If T is an absolute contraction, then $T_p(x) \geq 0$ for any $x \in L^p(M)_+$ and any p.

Theorem 5.2. *Let T be a positive analytic absolute contraction. Then for any $1 < p < \infty$ and any integer $m \geq 0$, we have an estimate*

$$\Bigl\|{\sup_{n\geq 0}}^{+}(n+1)^m T^n(T-I)^m(x)\Bigr\|_p \lesssim \|x\|_p, \qquad x \in L^p(M). \tag{5.5}$$

In particular we obtain a maximal inequality

$$\Bigl\|{\sup_{n\geq 0}}^{+} T^n(x)\Bigr\|_p \lesssim \|x\|_p$$

for any T as above.

These maximal theorems were proved in [17] under the assumption that the Hilbertian operator $T_2\colon L^2(M) \to L^2(M)$ is selfadoint and positive in the sense that $\sigma(T_2) \subset [0,1]$. This was recently extended by Bekjan [3] to the case when the numerical range of T_2 is included in a Stolz domain B_γ for some $\gamma \in \bigl(0, \frac{\pi}{2}\bigr)$. These results are covered by Theorem 5.2. Indeed it is easy to see that the latter numerical range condition implies that T_2 is analytic.

A key step in proving Theorem 5.2 is the following series of square function estimates.

Proposition 5.3. *Let $T\colon M \to M$ be an analytic absolute contraction. Then for any integer $m \geq 1$, we have an estimate*

$$\Bigl(\sum_{n=0}^{\infty}(n+1)^{2m-1}\bigl\|T^n(T-I)^m(x)\bigr\|_2^2\Bigr)^{\frac12} \lesssim \|x\|_2, \qquad x \in L^2(M). \tag{5.6}$$

Proof. The argument is entirely similar to the one devised to prove (3.12). We use the assumption that T_2 is analytic. We let $A = I - T_2$ and we let $(T_t)_{\geq 0}$ be the semigroup generated by $-A$ on $L^2(M)$. This is a bounded analytic semigroup and since T_2 is a contraction, we have $\|T_t\| \leq 1$ for any $t \geq 0$. Hence by [26] (see also [24]), A admits a bounded $H^\infty(\Sigma_{\theta_0})$ for some $\theta_0 < \frac{\pi}{2}$ and hence, for every $\theta \in \bigl(\theta_0, \frac{\pi}{2}\bigr)$ and for any $\varphi \in H_0^\infty(\Sigma_\theta)$, there exists a constant $C \geq 0$ such that

$$\Bigl(\int_0^\infty \bigl\|\varphi(tA)x\bigr\|_2^2\,\frac{dt}{t}\Bigr)^{\frac12} \leq C\|x\|_2, \qquad x \in L^2(M). \tag{5.7}$$

Arguing as in the proof of Proposition 3.2 and using (5.7) in place of Proposition 2.1, we obtain that there exists an angle $\gamma \in \bigl(0, \frac{\pi}{2}\bigr)$ such that the natutal functional calculus

$$u_\gamma\colon \mathcal{P}_\gamma \longrightarrow B(L^2(M)), \qquad u_\gamma(F) = F(T_2),$$

is completely bounded. That is, there exists a constant $C \geq 0$ such that for any $N \geq 1$, for any $N \times N$ matrix $[F_{ij}]$ of polynomials and for any $x_1, \ldots, x_N$ in $L^2(M)$,

$$\Bigl(\sum_{i=1}^{N}\Bigl\|\sum_{j=1}^{N} F_{ij}(T)x_j\Bigr\|_2^2\Bigr)^{\frac12} \leq C\bigl\|[F_{ij}]\bigr\|_\gamma \Bigl(\sum_{j=1}^{N}\|x_j\|_2^2\Bigr)^{\frac12}.$$

Then the argument in the proof of Theorem 3.3 yields the result. □

Proof of Theorem 5.2. Once we have the estimates (5.6) in hands, one can deduce Theorem 5.2 by repeating the arguments of [17, Section 5] (see also [3]). □

Remark 5.4. Let T be as in Theorem 5.2 and for any complex number α, let $M_n^\alpha(T)$ be defined as in [17, p. 409]. ($M_n^1(\cdot\,)$ is equal to the average $M_n(\cdot\,)$ given by (3.17).) Then the argument in [17, Section 5] shows that for any $\alpha \in \mathbb{C}$ and any $1 < p < \infty$, there is an estimate

$$\Big\|\sup_{n\geq 0}{}^+ M_n^\alpha(T)x\Big\|_p \lesssim \|x\|_p, \qquad x \in L^p(M).$$

The estimate (5.5) corresponds to $\alpha = -m$.

A similar comment applies to Theorem 4.1.

Following [17, Rem. 2.4], the definition of $L^p(M, \ell^\infty)$ can be extended to arbitrary index sets. For any set I and any $1 \leq p < \infty$, $L^p(M; \ell_I^\infty)$ is defined as the space of all families $(x_i)_{i\in I}$ of $L^p(M)$ which can be factorized as $x_i = az_ib$, where $a, b \in L^{2p}(M)$ and $(z_i)_{i\in I}$ belongs to $\ell_I^\infty(M)$. Moreover the norm of $(x_i)_{i\in I}$ in $L^p(M; \ell_I^\infty)$ is defined as the infimum of all $\|a\|_{2p} \sup_i \|z_i\| \|b\|_{2p}$ running over all such factorizations. We let $\big\|\sup_i{}^+ x_i\big\|_p$ denote the norm of an element $(x_i)_{i\in I}$ of $L^p(M; \ell_I^\infty)$. The analog of (5.4) holds in this general case, that is, a positive family $(x_i)_{i\in I}$ belongs to $L^p(M; \ell_I^\infty)$ if and only if there exists a positive $a \in L^p(M)$ such that $x_i \leq a$ for any $i \in I$ and moreover,

$$\big\|\sup_i{}^+ x_i\big\|_p = \inf\big\{\|a\|_p \,:\, a \in L^p(M),\ a \geq 0 \quad \text{and} \quad x_i \leq a \text{ for any } i \in I\big\}. \tag{5.8}$$

In the sequel we will deal with semigroups and apply the above facts with $I = \mathbb{R}_+$.

Let $(T_t)_{t\geq 0}$ be a semigroup of operators on M. Assume that for any $t \geq 0$, T_t is an absolute contraction and that for any $1 < p < \infty$, $(T_t)_{t\geq 0}$ is strongly continuous on $L^p(M)$. (By [12, Prop. 1.23], this holds true for example if for any $x \in M$, $T_t(x) \to x$ in the w^*-topology of M when $t \to 0^+$.) We let $-A_p$ denote the generator of $(T_t)_{t\geq 0}$ acting on $L^p(M)$.

Given any two indices $1 < p, q < \infty$, A_p is sectotial of type $< \frac{\pi}{2}$ if and only if A_q is sectotial of type $< \frac{\pi}{2}$. In other words $(T_t)_{t\geq 0}$ being a bounded analytic semigroup on $L^p(M)$ does not depend on $1 < p < \infty$. This is a continuous analog of Lemma 5.1, whose proof is identical to the one of [18, Prop. 5.4]. We skip the details.

Theorem 5.5. *Let $(T_t)_{t\geq 1}$ be a semigroup on M as above. Assume that for any $t \geq 0$, T_t is positive and that for one $1 < p < \infty$ (equivalently, for all $1 < p < \infty$), $(T_t)_{t\geq 1}$ is analytic on $L^p(M)$. Then for any $1 < p < \infty$ and any integer $m \geq 0$, we have an estimate*

$$\Big\|\sup_{t>0}{}^+ t^m \frac{\partial^m}{\partial t^m}\big(T_t(x)\big)\Big\|_p \lesssim \|x\|_p, \qquad x \in L^p(M).$$

Proof. Fix p and $m \geq 0$. According to [17, Prop. 2.1 and Rem. 2.4], it suffices to find a constant $C \geq 0$ such that for any finite family $t_1, \ldots, t_N$ of positive real numbers,

$$\Big\|\sup_k{}^+ t_k^m \frac{\partial^m}{\partial t^m}\big(T_t(x)\big)_{\big|t=t_k}\Big\|_p \leq C\,\|x\|_p$$

for any $x \in L^p(M)$. This follows from Theorem 5.2, using the same approximation argument as in the proof of Corollary 4.2. □

We end this section with applications to R-analyticity (see Section 2 for terminology and background). We recall Weis's Theorem [38] that if $(T_t)_{t\geq 0}$ is a bounded analytic semigroup on some commutative L^p-space (with $1 < p < \infty$) such that each T_t is contractively regular, then $(T_t)_{t\geq 0}$ is actually an R-bounded analytic semigroup. (This result was used in the proof of Proposition 2.2 in the present paper.) The next corollary is an analog of that result in our noncommutative setting. In the selfadjoint case, it was established in [18, Thm. 5.6]. The proof in the analytic case follows a similar scheme so we will be brief.

Corollary 5.6. *Let $(T_t)_{t\geq 1}$ be as in Theorem 5.5. Then for any $1 < p < \infty$, the realization of $(T_t)_{t\geq 0}$ on $L^p(M)$ is an R-bounded analytic semigroup.*

Proof. We first observe that the dual semigroup $(T_t^*)_{t\geq 0}$ satisfies the assumptions of Theorem 5.5. Let $1 < r < \infty$. Applying the latter theorem for $m = 0$ and (5.8), we find a constant $C_r > 0$ such that for any $y \in L^r(M)_+$, there exists $a \in L^r(M)_+$ such that

$$\|a\|_r \leq C_r \|y\|_r \qquad \text{and} \qquad T_t^*(y) \leq a \quad \text{for any} \quad t \geq 0.$$

Then the argument in the proof of [18, Thm. 5.6] shows that for any $2 \leq q < \infty$, the set

$$F_q = \bigl\{T_t \colon L^q(M) \longrightarrow L^q(M) \ : \ t \geq 0\bigr\} \tag{5.9}$$

is R-bounded.

The analyticity assumption ensures the existence of an angle $\nu \in \bigl(0, \frac{\pi}{2}\bigr)$ such that the realization of $(T_t)_{t\geq 0}$ on $L^2(M)$ extends to a bounded family $(T_z)_{z\in\overline{\Sigma_\nu}}$ of opertors on $L^2(M)$, whose restriction to Σ_ν is analytic. Since boundedness is equivalent to R-boundedness on Hilbert spaces, this immediately implies that the sets

$$\bigl\{T_{te^{i\nu}} \colon L^2(M) \to L^2(M) \ : \ t \geq 0\bigr\} \qquad \text{and} \qquad \bigl\{T_{te^{-i\nu}} \colon L^2(M) \to L^2(M) \ : \ t \geq 0\bigr\} \tag{5.10}$$

are R-bounded.

Let $2 < p < \infty$, let $q > p$ be a finite number and let $\alpha = 2(q-2)^{-1}\bigl(\frac{q}{p} - 1\bigr)$. In accordance with (5.1), this number is chosen so that $L^p(M) = [L^q(M), L^2(M)]_\alpha$. As is well-known, this implies that

$$\text{Rad}\bigl(L^p(M)\bigr) \ = \ \bigl[\text{Rad}\bigl(L^q(M)\bigr), \text{Rad}\bigl(L^2(M)\bigr)\bigr]_\alpha$$

isomorphically. Applying Stein's interpolation principle as in the proof of [18, Thm. 5.6] and the R-boundedness of the sets in (5.10), we deduce that

$$\bigl\{T_z \colon L^p(M) \longrightarrow L^p(M) \ : \ z \in \Sigma_{\alpha\nu}\bigr\}$$

is R-bounded. This shows that $(T_t)_{t\geq 0}$ is an R-bounded analytic semigroup on $L^p(M)$.

The case $1 < p < 2$ easily follows by duality. □

Let us finally come back to the discrete case. Blunck [5, Thm. 1.1 and Thm. 1.2] showed that any analytic contractively regular operator on a commutative L^p-space (with $1 < p < \infty$) is an R-analytic power bounded operator (see Remark 3.5 (b) in the present paper for a proof of this result). This is a discrete analog of Weis's Theorem. Here is a noncommutative version.

Proposition 5.7. *Let $T \colon M \to M$ be an absolute contraction and assume that T is positive. Let $1 < p < \infty$. If T is analytic, then $T_p \colon L^p(M) \to L^p(M)$ is an R-analytic power bounded operator for any $1 < p < \infty$.*

Proof. Let $(T_t)_{t\geq 0}$ be defined by (2.9). Then for any $t \geq 0$, T_t is a positive absolute contraction. Moreover for any $1 < p < \infty$, $(T_t)_{t\geq 0}$ is analytic on $L^p(M)$. Hence by Proposition 5.6, $(T_t)_{t\geq 0}$ is actually an R-bounded analytic semigoup on $L^p(M)$. By [5, Thm. 1.1], this implies that T_p is an R-analytic power bounded operator. $\square$

Remark 5.8. Consider the notions of column boundedness and row boundedness as defined in [18, Section 4.A] and let us state Col-bounded and Row-bounded versions of Theorem 5.5 and Proposition 5.7. Let $T\colon L^p(M) \to L^p(M)$ and let us say that T is Col-analytic (resp. Row-analytic) power bounded it the two sets $\mathcal{P}_T$ and $\mathcal{A}_T$ from (2.4) and (2.5) are both Col-bounded (resp. Row-bounded). Likewise, let us say that a semigroup $(T_t)_{t\geq 0}$ on $L^p(M)$ is a Col-bounded (resp. Row-bounded) analytic semigroup if the two sets $\{T_t \,:\, t > 0\}$ and $\{tAT_t \,:\, t > 0\}$ are both Col-bounded (resp. Row-bounded).

Let $(T_t)_{t\geq 0}$ be a semigroup on M as in Theorem 5.5 and assume that $T_t\colon M \to M$ is 2-positive for any $t \geq 0$. Then as in [18, Thm. 5.6], one can show that for any $1 < p < \infty$, the realization of $(T_t)_{t\geq 0}$ on $L^p(M)$ is both a Col-bounded and a Row-bounded analytic semigroup.

Likewise, if $T\colon M \to M$ is a 2-positive and analytic absolute contraction, then for any $1 < p < \infty$, $T_p\colon L^p(M) \to L^p(M)$ is both a Col-analytic and a Row-analytic power bounded operator. More concretely, this implies in particular that we have estimates

$$\Bigl\Vert\Bigl(\sum_n T^n(x_n)^*T^n(x_n)\Bigr)^{\frac{1}{2}}\Bigr\Vert_p \lesssim \Bigl\Vert\Bigl(\sum_n x_n^*x_n\Bigr)^{\frac{1}{2}}\Bigr\Vert_p$$

and

$$\Bigl\Vert\Bigl(\sum_n T^n(x_n)T^n(x_n)^*\Bigr)^{\frac{1}{2}}\Bigr\Vert_p \lesssim \Bigl\Vert\Bigl(\sum_n x_nx_n^*\Bigr)^{\frac{1}{2}}\Bigr\Vert_p$$

for any $1 < p < \infty$.

References

[1] M. Akcoglu, and L. Sucheston, *Dilations of positive contractions on L_p spaces*, Canad. Math. Bull. 20 (1977), 285-292.

[2] W. Arendt, and S. Bu, *The operator-valued Marcinkiewicz multiplier theorem and maximal regularity*, Math. Z. 240 (2002), 311-343.

[3] T. Bekjan, *Noncommutative maximal ergodic theorems for positive contractions*, J. Funct. Anal. 254 (2008), 2401-2418.

[4] E. Berkson, and T. A. Gillespie, *Spectral decompositions and harmonic analysis on UMD Banach spaces*, Studia Math. 112 (1994), 13-49.

[5] S. Blunck, *Maximal regularity of discrete and continuous time evolution equations*, Studia Math. 146 (2001), no. 2, 157-176.

[6] S. Blunck, *Analyticity and discrete maximal regularity on L_p-spaces*, J. Funct. Anal. 183 (2001), 211-230.

[7] P. Clément, B. de Pagter, F. A. Sukochev, and H. Witvliet, *Schauder decompositions and multiplier theorems*, Studia Math. 138 (2000), 135-163.

[8] R. Coifman, R. Rochberg, and G. Weiss, *Applications of transference: the L_p version of von Neumann's inequality and the Littlewood-Paley-Stein theory*, pp. 53-67 in "Linear spaces and Approximation", Birkhäuser, Basel, 1978.

[9] R. R. Coifman, and G. Weiss, *Transference methods in analysis*, CBMS 31, Amer. Math. Soc., 1977.

[10] T. Coulhon, and L. Saloff-Coste, *Puissances d'un opérateur régularisant*, Ann. Inst. H. Poincaré Probab. Statist. 26 (1990), no. 3, 419-436

[11] M. Cowling, I. Doust, A. McIntosh, and A. Yagi, *Banach space operators with a bounded H^∞ functional calculus*, J. Aust. Math. Soc., Ser. A 60 (1996), 51-89.

[12] E. B. Davies, *One-parameter semigroups*, L.M.S. Monographs 15, Academic Press, 1980.

[13] X. T. Duong, *H^∞ functional calculus of second order elliptic partial differential operators on L^p spaces*, Proc. CMA Canberra 24 (1989), 91-102.

[14] G. Fendler, *Dilations of one parameter semigroups of positive contractions on L_p-spaces*, Canad. J. Math. 49 (1997), 736-748.

[15] J. A. Goldstein, *Semigroups of linear operators and applications*, Oxford University Press, New-York, 1985.

[16] M. Junge, *Doob's inequality for non-commutative martingales*, J. Reine Angew. Math. 549 (2002), 149-190.

[17] M. Junge, and Q. Xu, *Noncommutative maximal ergodic theorems*, J. Amer. Math. Soc. 20 (2007), 385-439.

[18] M. Junge, C. Le Merdy, and Q. Xu, *H^∞ functional calculus and square functions on noncommutative L^p-spaces*, Soc. Math. France, Astérisque 305, 2006.

[19] N. J. Kalton, and L. Weis, *The H^∞-calculus and sums of closed operators*, Math. Ann. 321 (2001), 319-345.

[20] C. Kriegler, and C. Le Merdy, *Tensor extension properties of $C(K)$-representations and applications to unconditionality*, J. Aust. Math. Soc. 88 (2010), 205-230.

[21] P. C. Kunstmann, and L. Weis, *Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus*, pp. 65-311 in "Functional analytic methods for evolution equations", Lect. Notes in Math. 1855, Springer, 2004.

[22] F. Lancien, G. Lancien, and C. Le Merdy, *A joint functional calculus for sectorial operators with commuting resolvents*, Proc. London Math. Soc. 77 (1998), 387-414.

[23] C. Le Merdy, *H^∞-functional calculus and applications to maximal regularity*, Publ. Math. Besançon 16 (1998), 41-77.

[24] C. Le Merdy, *The similarity problem for bounded analytic semigroups on Hilbert space*, Semigroup Forum 56 (1998), 205-224.

[25] Yu. Lyubich, *Spectral localization, power boundedness and invariant subspaces under Ritt's type condition*, Studia Math. 134 (1999), 153-167.

[26] A. McIntosh, *Operators which have an H^∞ functional calculus*, Proc. CMA Canberra 14 (1986), 210-231.

[27] P. Meyer-Nieberg, *Banach lattices*, Springer, Berlin-Heidelberg-NewYork, 1991.

[28] B. Nagy, and J. Zemanek, *A resolvent condition implying power boundedness*, Studia Math. 134 (1999), 143-151.

[29] O. Nevanlinna, *Convergence of iterations for linear equations*, Birkhaüser, Basel, 1993.

[30] A. Pazy, *Semigroups of linear operators and applicatios to partial differential equations*, Springer, 1983.

[31] V. Peller, *An analogue of J. von Neumann's inequality for the space L^p (Russian)*, Dokl. Akad. Nauk SSSR 231 (1976), no. 3, 539-542.

[32] G. Pisier, *Non-commutative vector valued L_p-spaces and completely p-summing maps*, Soc. Math. France, Astérisque 247, 1998.

[33] G. Pisier, and Q. Xu, *Non-commutative L^p-spaces*, pp. 1459-1517 in "Handbook of the Geometry of Banach Spaces", Vol. II, edited by W.B. Johnson and J. Lindenstrauss, Elsevier, 2003.

[34] A. Simard, *Factorization of sectorial operators with bounded H^∞-functional calculus*, Houston J. Math. 25 (1999), 351-370.

[35] E.M. Stein, *On the maximal ergodic theorem*, Proc. Nat. Acad. Sci. USA 47 (1961), 1894-1897.

[36] *Topics in harmonic analysis related to the Littlewood-Paley theory*, Ann. Math. Studies, Princeton, University Press, 1970.

[37] L. Weis, *Operator valued Fourier multiplier theorems and maximal regularity*, Math. Annalen 319 (2001), 735-758.

[38] L. Weis, *A new approach to maximal L_p-regularity*, pp. 195-214 in "Proc. of the 6th International Conference on Evolution Equations 1998", edited by G. Lumer and L. Weis, Marcel Dekker, 2000.

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