CHARACTERIZATIONS OF INTRA-REGULAR Γ-AG**-GROUPOIDS BY THE PROPERTIES OF THEIR Γ-IDEALS

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Abstract. We have characterized an intra-regular Γ-AG**-groupoids by using the properties of Γ-ideals (left, right, two-sided), Γ-interior, Γ-quasi, Γ-bi and Γ-generalized bi and Γ-$(1,2)$). We have prove that all the Γ-ideals coincides in an intra-regular Γ-AG**-groupoids. It has been examined that all the Γ-ideals of an intra-regular Γ-AG**-groupoids are Γ-idempotent. In this paper we define all Γ-ideals in Γ-AG**-groupoids and we generalize some results.

Keywords. Γ-AG-groupoid, intra-regular Γ-AG**-groupoid and Γ-$(1,2)$-ideals.

Introduction

The idea of generalization of commutative semigroup was introduced in 1972, they named it as left almost semigroup (LA-semigroup in short)(see [3]). It is also called an Abel-Grassmann's groupoid (AG-groupoid in short) [12]. In this paper we will call it an AG-groupoid.

This structure is closely related with a commutative semigroup because if an AG-groupoid contains a right identity, then it becomes a commutative monoid [8]. A left identity in an AG-groupoid is unique [8]. It is a mid structure between a groupoid and a commutative semigroup with wide range of applications in theory of flocks [11]. Ideals in AG-groupoids have been discussed in [8], [15], [5] and [9]. In 1981. the notion of Γ-semigroups was introduced by M. K. Sen [6] and [7]

In this paper, we have introduced the notion of Γ-AG**-groupoids. Γ-AG-groupoids is the generalization of Γ-AG-groupoids. Here, we explore all basic Γ-ideals, which includes Γ-ideals (left, right,two-sided), Γ-interior, Γ-quasi, Γ-bi, Γ-generalized bi and Γ-$(1,2)$).

Definition 1. *Let S and Γ be two non-empty sets, then S is said to be a Γ-AG-groupoid if there exist a mapping $S\times\Gamma\times S\to S$, written (x,γ,y) as $x\gamma y$, such that S satisfies the left invertive law, that is*

$$(1)\qquad (x\gamma y)\,\delta z=(z\gamma y)\,\delta x,\ \text{for all } x,y,z\in S \text{ and } \gamma,\delta\in\Gamma.$$

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Definition 2. *A Γ-AG-groupoid S is called a Γ-medial if it satisfies the medial law, that is*

$$(x\alpha y)\,\beta\,(l\gamma m) = (x\alpha l)\,\beta\,(y\gamma m)\,,\ \text{for all } x,y,l,m \in S \text{ and } \alpha,\beta,\gamma \in \Gamma \tag{2}$$

Definition 3. *A Γ-AG-groupoid S is called a Γ-AG^{**}-groupoid if it satisfy the following law*

$$a\alpha(b\beta c) = b\alpha(a\beta c),\ \text{for all } a,b,c \in S \text{ and } \alpha,\beta \in \Gamma. \tag{3}$$

Definition 4. *A Γ-AG-groupoid** S is called a Γ-paramedial if it satisfies the paramedial law, that is*

$$(x\alpha y)\,\beta\,(l\gamma m) = (m\alpha l)\,\beta\,(y\gamma x)\,,\ \text{for all } x,y,l,m \in S \text{ and } \alpha,\beta,\gamma \in \Gamma. \tag{4}$$

Definition 5. *Let S be a Γ-AG-groupoid, a non-empty subset A of S is called Γ-AG-subgroupoid if $a\gamma b \in A$ for all a, $b \in A$ and $\gamma \in \Gamma$ or A is called Γ-AG-subgroupoid if $A\Gamma A \subseteq A$.*

Definition 6. *A subset A of a Γ-AG-groupoid S is called left(right) Γ-ideal of S if $S\Gamma A \subseteq A\,(A\Gamma S \subseteq A)$ and A is called Γ-ideal of S if it is both left and right Γ-ideal.*

Definition 7. *A Γ-AG-subgroupoid A of a Γ-AG-groupoid S is called a Γ-bi-ideal of S if $(A\Gamma S)\,\Gamma A \subseteq A$.*

Definition 8. *A Γ-AG-subgroupoid A of a Γ-AG-groupoid S is called a Γ-interior ideal of S if $(S\Gamma A)\,\Gamma S \subseteq A$.*

Definition 9. *A Γ-AG-groupoid A of a Γ-AG-groupoid S is called a Γ-quasi-ideal of S if $S\Gamma A \cap A\Gamma S \subseteq A$.*

Definition 10. *A Γ-AG-subgroupoid A of a Γ-AG-groupoid S is called a Γ-$(1,2)$-ideal of S if $(A\Gamma S)\,\Gamma A^2 \subseteq A$.*

Definition 11. *A Γ-ideal P of a Γ-AG-groupoid S is called Γ-prime(Γ-semiprime) if for any Γ-ideals A and B of S, $A\Gamma B \subseteq P\,(A\Gamma A \subseteq P)$ implies either $A \subseteq P$ or $B \subseteq P\,(A \subseteq P)$.*

Definition 12. *An element a of an Γ-AG-groupoid S is called an intra-regular if there exists $x, y \in S$ and $\beta,\gamma,\delta \in \Gamma$ such that $a = (x\beta(a\delta a))\gamma y$ and S is called an intra-regular Γ-AG-groupoid S, if every element of S is an intra-regular.*

Example 1. *Let S and Γ be two non-empty sets, then S is said to be a Γ-AG-groupoid if there exist a mapping $S \times \Gamma \times S \to S$, written $(x,\ \gamma,\ y)$ as $x\gamma y$, such that $S = S = \{a,b,c,d,e\}$*

.	a	b	c	d	e
a	a	a	a	a	a
b	a	b	c	d	e
c	a	e	b	c	d
d	a	d	e	b	c
e	a	c	d	e	b

Clearly S is an intra-regular because, $a = (a\beta a^2)\gamma a$, $b = (c\beta b^2)\gamma e$, $c = (d\beta c^2)\gamma e$, $d = (c\beta d^2)\gamma c$, $e = (b\beta e^2)\gamma e$.

Note that in a Γ-AG-groupoid S with left identity, $S = S\Gamma S$.

Theorem 1. *A Γ-AG^{**}-groupoid S is an intra-regular Γ-AG^{**}-groupoid if $S\Gamma a = S$ or $a\Gamma S = S$ holds for all $a \in S$.*

Proof. Let S be a Γ-AG**-groupoid such that $S\Gamma a = S$ holds for all $a \in S$, then $S = S\Gamma S$. Let $a \in S$ and therefore, by using (2), we have

$$\begin{aligned} a &\in S = (S\Gamma S)\Gamma S = ((S\Gamma a)\Gamma(S\Gamma a))\Gamma S = ((S\Gamma S)\Gamma(a\Gamma a))\Gamma S \\ &\subseteq (S\Gamma a^2)\Gamma S. \end{aligned}$$

Which shows that S is an intra-regular Γ-AG**-groupoid.

Let $a \in S$ and assume that $a\Gamma S = S$ holds for all $a \in S$, then by using (1), we have

$$a \in S = S\Gamma S = (a\Gamma S)\Gamma S = (S\Gamma S)\Gamma a = S\Gamma a.$$

Thus $S\Gamma a = S$ holds for all $a \in S$ and therefore it follows from above that S is an intra-regular. □

Corollary 1. *If S is a Γ-AG**-groupoid such that $a\Gamma S = S$ holds for all $a \in S$, then $S\Gamma a = S$ holds for all $a \in S$.*

Theorem 2. *If S is an intra-regular Γ-AG**-groupoid, then $(B\Gamma S)\Gamma B = B \cap S$, where B is a Γ-bi-(Γ-generalized bi-) ideal of S.*

Proof. Let S be an intra-regular Γ-AG**-groupoid, then clearly $(B\Gamma S)\Gamma B \subseteq B \cap S$. Now let $b \in B \cap S$ which implies that $b \in B$ and $b \in S$, then since S is an intra-regular Γ-AG**-groupoid so there exists $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $b = (x\alpha(b\beta b))\gamma y$. Now by using (3), (1), (4) and (2), we have

$$\begin{aligned} b &= (x\alpha(b\beta b))\gamma y = (b\alpha(x\beta b))\gamma y = (y\alpha(x\beta b))\gamma b \\ &= (y\alpha(x\beta((x\alpha(b\beta b))\gamma y)))\gamma b = (y\alpha((x\alpha(b\beta b))\beta(x\gamma y)))\gamma b \\ &= ((x\alpha(b\beta b))\alpha(y\beta(x\gamma y)))\gamma b = (((x\gamma y)\alpha y)\alpha((b\beta b)\beta x))\gamma b \\ &= ((b\beta b)\alpha(((x\gamma y)\alpha y)\beta x))\gamma b = ((b\beta b)\alpha((x\alpha y)\beta(x\gamma y)))\gamma b \\ &= ((b\beta b)\alpha((x\alpha x)\beta(y\gamma y)))\gamma b = (((y\gamma y)\beta(x\alpha x))\alpha(b\beta b))\gamma b \\ &= (b\alpha(((y\gamma y)\beta(x\alpha x))\beta b))\gamma b \in (B\Gamma S)\Gamma B. \end{aligned}$$

Which shows that $(B\Gamma S)\Gamma B = B \cap S$. □

Corollary 2. *If S is an intra-regular Γ-AG**-groupoid, then $(B\Gamma S)\Gamma B = B$, where B is a Γ-bi-(Γ-generalized bi-) ideal of S.*

Theorem 3. *If S is an intra-regular Γ-AG**-groupoid, then $(S\Gamma B)\Gamma S = S \cap B$, where B is a Γ-interior ideal of S.*

Proof. Let S be an intra-regular Γ-AG**-groupoid, then clearly $(S\Gamma B)\Gamma S \subseteq S \cap B$. Now let $b \in S \cap B$ which implies that $b \in S$ and $b \in B$, then since S is an intra- regular Γ-AG**-groupoid so there exists $x, y \in S$ and $\alpha, \gamma, \delta \in \Gamma$ such that $b = (x\alpha(b\delta b))\gamma y$. Now by using (3), (1) and (4), we have

$$\begin{aligned} b &= (x\alpha(b\delta b))\gamma y = (b\alpha(x\delta b))\gamma y = (y\alpha(x\delta b))\gamma b \\ &= (y\alpha(x\delta b))\gamma((x\alpha(b\delta b))\gamma y) = (((x\alpha(b\delta b))\gamma y)\alpha(x\delta b))\gamma y \\ &= ((b\gamma x)\alpha(y\delta(x\alpha(b\delta b))))\gamma y = (((y\delta(x\alpha(b\delta b)))\gamma x)\alpha b)\gamma y \in (S\Gamma B)\Gamma S. \end{aligned}$$

Which shows that $(S\Gamma B)\Gamma S = S \cap B$. □

Corollary 3. *If S is an intra-regular Γ-AG**-groupoid, then $(S\Gamma B)\Gamma S = B$, where B is a Γ-interior ideal of S.*

Lemma 1. *If S is an intra-regular regular Γ-AG**-groupoid, then $S = S\Gamma S$.*

Proof. It is simple. □

Lemma 2. *A subset A of an intra-regular Γ-AG^{**}-groupoid S is a left Γ-ideal if and only if it is a right Γ-ideal of S.*

Proof. Let S be an intra-regular Γ-AG**-groupoid and let A be a right Γ-ideal of S, then $A\Gamma S \subseteq A$. Let $a \in A$ and since S is an intra-regular Γ-AG**-groupoid so there exists $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a = (x\beta(a\delta a))\gamma y$. Let $p \in S\Gamma A$ and $\delta \in \Gamma$, then by using (3), (1) and (4), we have

$$\begin{aligned} p &= s\psi a = s\psi((x\beta(a\delta a))\gamma y) = (x\beta(a\delta a))\psi(s\gamma y) = (a\beta(x\delta a))\psi(s\gamma y) \\ &= ((s\gamma y)\beta(x\delta a))\psi a = ((a\gamma x)\beta(y\delta s))\psi a = (((y\delta s)\gamma x)\beta a)\psi a \\ &= (a\beta a)\psi((y\delta s)\gamma x) = (x\beta(y\delta s))\psi(a\gamma a) = a\psi((x\beta(y\delta s))\gamma a) \in A\Gamma S \subseteq A. \end{aligned}$$

Which shows that A is a left Γ-ideal of S.

Let A be a left Γ-ideal of S, then $S\Gamma A \subseteq A$. Let $a \in A$ and since S is an intra-regular Γ-AG**-groupoid so there exists $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a = (x\beta(a\delta a))\gamma y$. Let $p \in A\Gamma S$ and $\delta \in \Gamma$, then by using (1) and (4), we have

$$\begin{aligned} p &= a\psi s = ((x\beta(a\delta a)\gamma y)\psi s = (s\gamma y)\psi(x\beta(a\delta a)) = ((a\delta a)\gamma x)\psi(y\beta s) \\ &= ((y\beta s)\gamma x)\psi(a\delta a) = (a\gamma a)\psi(x\delta(y\beta s)) = ((x\delta(y\beta s))\gamma a)\psi a \in S\Gamma A \subseteq A. \end{aligned}$$

Which shows that A is a right Γ-ideal of S. □

Theorem 4. *In an intra-regular Γ-AG^{**}-groupoid S, the following conditions are equivalent.*

(i) A is a Γ-bi-(Γ-generalized bi-) ideal of S.
(ii) $(A\Gamma S)\Gamma A = A$ and $A\Gamma A = A$.

Proof. $(i) \Longrightarrow (ii)$: Let A be a Γ-bi-ideal of an intra-regular Γ-AG**-groupoid S, then $(A\Gamma S)\Gamma A \subseteq A$. Let $a \in A$, then since S is an intra-regular so there exists x, $y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a = (x\beta(a\delta a))\gamma y$. Now by using (3), (1), (2) and (4), we have

$$\begin{aligned} a &= (x\beta(a\delta a))\gamma y = (a\beta(x\delta a))\gamma y = (y\beta(x\delta a))\gamma a \\ &= (y\beta(x\delta((x\beta(a\delta a))\gamma y)))\gamma a = (y\beta((x\beta(a\delta a))\delta(x\gamma y)))\gamma a \\ &= ((x\beta(a\delta a))\beta(y\delta(x\gamma y)))\gamma a = ((a\beta(x\delta a))\beta(y\delta(x\gamma y)))\gamma a \\ &= ((a\beta y)\beta((x\delta a)\delta(x\gamma y)))\gamma a = ((x\delta a)\beta((a\beta y)\delta(x\gamma y)))\gamma a \\ &= ((x\delta a)\beta((a\beta x)\delta(y\gamma y)))\gamma a = (((y\gamma y)\delta(a\beta x))\beta(a\delta x))\gamma a \\ &= (a\beta(((y\gamma y)\delta(a\beta x))\delta x))\gamma a \in (A\Gamma S)\Gamma A. \end{aligned}$$

Thus $(A\Gamma S)\Gamma A = A$ holds. Now by using (3), (1), (4) and (2), we have

$$\begin{aligned}
a &= (x\beta(a\delta a))\gamma y = (a\beta(x\delta a))\gamma y = (y\beta(x\delta a))\gamma a \\
&= (y\beta(x\delta((x\beta(a\delta a))\gamma y)))\gamma a = (y\beta((x\beta(a\delta a))\delta(x\gamma y)))\gamma a \\
&= ((x\beta(a\delta a))\beta(y\delta(x\gamma y)))\gamma a = ((a\beta(x\delta a))\beta(y\delta(x\gamma y)))\gamma a \\
&= (((y\delta(x\gamma y))\beta(x\delta a))\beta a)\gamma a = (((a\delta x)\beta((x\gamma y)\delta y))\beta a)\gamma a \\
&= (((a\delta x)\beta((y\gamma y)\delta x))\beta a)\gamma a = (((a\delta(y\gamma y))\beta(x\delta x))\beta a)\gamma a \\
&= ((((x\delta x)\delta(y\gamma y))\beta a)\beta a)\gamma a = ((((x\delta x)\delta(y\gamma y))\beta((x\beta(a\delta a))\gamma y))\beta a)\gamma a \\
&= ((((x\delta x)\delta(y\gamma y))\beta((a\beta(x\delta a))\gamma y))\beta a)\gamma a \\
&= ((((x\delta x)\delta(a\beta(x\delta a)))\beta((y\gamma y)\gamma y))\beta a)\gamma a \\
&= (((a\delta((x\delta x)\beta(x\delta a)))\beta((y\gamma y)\gamma y))\beta a)\gamma a \\
&= (((a\delta((a\delta x)\beta(x\delta x)))\beta((y\gamma y)\gamma y))\beta a)\gamma a \\
&= ((((a\delta x)\delta(a\beta(x\delta x)))\beta((y\gamma y)\gamma y))\beta a)\gamma a \\
&= ((((a\delta a)\delta(x\beta(x\delta x)))\beta((y\gamma y)\gamma y))\beta a)\gamma a \\
&= (((((y\gamma y)\gamma y)\delta(x\beta(x\delta x)))\beta(a\delta a))\beta a)\gamma a \\
&= ((a\beta((((y\gamma y)\gamma y)\delta(x\beta(x\delta x)))\delta a))\beta a)\gamma a \subseteq ((A\Gamma S)\Gamma A)\Gamma A \subseteq A\Gamma A.
\end{aligned}$$

Hence $A = A\Gamma A$ holds.
$(ii) \Longrightarrow (i)$ is obvious. □

Theorem 5. *In an intra-regular* Γ-AG^{**}*-groupoid* S*, the following conditions are equivalent.*

(i) A is a Γ-$(1,2)$-ideal of S.
(ii) $(A\Gamma S)\Gamma A^2 = A$ and $A\Gamma A = A$.

Proof. $(i) \Longrightarrow (ii)$: Let A be a Γ-$(1,2)$-ideal of an intra-regular Γ-AG**-groupoid S, then $(A\Gamma S)\Gamma A^2 \subseteq A$ and $A\Gamma A \subseteq A$. Let $a \in A$, then since S is an intra- regular so there exists x, $y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a = (x\beta(a\delta a)\gamma y$. Now by using (3), (1) and (4), we have

$$\begin{aligned}
a &= (x\beta(a\delta a))\gamma y = (a\beta(x\delta a))\gamma y = (y\beta(x\delta a))\gamma a \\
&= (y\beta(x\delta((x\beta(a\delta a))\gamma y)))\gamma a = (y\beta((x\beta(a\delta a))\delta(x\gamma y)))\gamma a \\
&= ((x\beta(a\delta a))\beta(y\delta(x\gamma y)))\gamma a = (((x\gamma y)\beta y)\beta((a\delta a)\delta x))\gamma a \\
&= (((y\gamma y)\beta x)\beta((a\delta a)\delta x))\gamma a = ((a\delta a)\beta(((y\gamma y)\beta x)\delta x))\gamma a \\
&= ((a\delta a)\beta((x\beta x)\delta(y\gamma y)))\gamma a = (a\beta((x\beta x)\delta(y\gamma y)))\gamma(a\delta a) \in (A\Gamma S)\Gamma A\Gamma A.
\end{aligned}$$

Thus $(A\Gamma S)\Gamma A^2 = A$. Now by using (3), (1), (4) and (2), we have

$$\begin{aligned}
a &= (x\beta(a\delta a))\gamma y = (a\beta(x\delta a))\gamma y = (y\beta(x\delta a))\gamma a \\
&= (y\beta(x\delta a))\gamma((x\beta(a\delta a))\gamma y) = (x\beta(a\delta a))\gamma((y\beta(x\delta a))\gamma y) \\
&= (a\beta(x\delta a))\gamma((y\beta(x\delta a))\gamma y) = (((y\beta(x\delta a))\gamma y)\beta(x\delta a))\gamma a \\
&= ((a\gamma x)\beta(y\delta(y\beta(x\delta a))))\gamma a \\
&= ((((x\beta(a\delta a))\gamma y)\gamma x)\beta(y\delta(y\beta(x\delta a))))\gamma a \\
&= (((x\gamma y)\gamma(x\beta(a\delta a)))\beta(y\delta(y\beta(x\delta a))))\gamma a \\
&= (((x\gamma y)\gamma y)\beta((x\beta(a\delta a))\delta(y\beta(x\delta a))))\gamma a \\
&= (((y\gamma y)\gamma x)\beta((x\beta(a\delta a))\delta(y\beta(x\delta a))))\gamma a \\
&= (((y\gamma y)\gamma x)\beta((x\beta y)\delta((a\delta a)\beta(x\delta a))))\gamma a \\
&= (((y\gamma y)\gamma x)\beta((a\delta a)\delta((x\beta y)\beta(x\delta a))))\gamma a \\
&= ((a\delta a)\beta(((y\gamma y)\gamma x)\delta((x\beta y)\beta(x\delta a))))\gamma a \\
&= ((a\delta a)\beta(((y\gamma y)\gamma x)\delta((x\beta x)\beta(y\delta a))))\gamma a \\
&= ((((x\beta x)\beta(y\delta a))\delta((y\gamma y)\gamma x))\beta(a\delta a))\gamma a \\
&= ((((a\beta y)\beta(x\delta x))\delta((y\gamma y)\gamma x))\beta(a\delta a))\gamma a \\
&= (((((x\delta x)\beta y)\beta a)\delta((y\gamma y)\gamma x))\beta(a\delta a))\gamma a \\
&= (((x\beta(y\gamma y))\delta(a\gamma((x\delta x)\beta y)))\beta(a\delta a))\gamma a \\
&= ((a\delta((x\beta(y\gamma y))\gamma((x\delta x)\beta y)))\beta(a\delta a))\gamma a \\
&= ((a\delta((x\beta(x\delta x))\gamma((y\gamma y)\beta y)))\beta(a\delta a))\gamma a \\
&\in ((A\Gamma S)\Gamma A^2)\Gamma A \subseteq A\Gamma A.
\end{aligned}$$

Hence $A\Gamma A = A$.
$(ii) \Longrightarrow (i)$ is obvious. □

Theorem 6. *In an intra-regular Γ-AG^{**}-groupoid S, the following conditions are equivalent.*

(i) A is a Γ-interior ideal of S.
(ii) $(S\Gamma A)\Gamma S = A$.

Proof. $(i) \Longrightarrow (ii)$: Let A be a Γ-interior ideal of an intra-regular Γ-AG**-groupoid S, then $(S\Gamma A)\Gamma S \subseteq A$. Let $a \in A$, then since S is an intra- regular so there exists x, $y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a = (x\beta(a\delta a))\gamma y$. Now by using (3), (1) and (4), we have

$$\begin{aligned}
a &= (x\beta(a\delta a))\gamma y = (a\beta(x\delta a))\gamma y = (y\beta(x\delta a))\gamma a \\
&= (y\beta(x\delta a))\gamma((x\beta(a\delta a))\gamma y) = (((x\beta(a\delta a))\gamma y)\beta(x\delta a))\gamma y \\
&= ((a\gamma x)\beta(y\delta(x\beta(a\delta a))))\gamma y = (((y\delta(x\beta(a\delta a)))\gamma x)\beta a)\delta y \in (S\Gamma A)\Gamma S.
\end{aligned}$$

Thus $(S\Gamma A)\Gamma S = A$.
$(ii) \Longrightarrow (i)$ is obvious. □

Theorem 7. *In an intra-regular Γ-AG^{**}-groupoid S, the following conditions are equivalent.*

(i) A is a Γ-quasi ideal of S.
(ii) $S\Gamma Q \cap Q\Gamma S = Q$.

Proof. $(i) \Longrightarrow (ii)$: Let Q be a Γ-quasi ideal of an intra-regular Γ-AG**-groupoid S, then $S\Gamma Q \cap Q\Gamma S \subseteq Q$. Let $q \in Q$, then since S is an intra- regular so there exists $x,\ y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $q = (x\alpha(q\gamma q))\beta y$. Let $p\delta q \in S\Gamma Q$, for some $\delta \in \Gamma$, then by using (3), (2) and (4), we have

$$\begin{aligned} p\delta q &= p\delta((x\alpha(q\gamma q))\beta y) = (x\alpha(q\gamma q))\delta(p\beta y) = (q\alpha(x\gamma q))\delta(p\beta y) \\ &= (q\alpha p)\delta((x\gamma q)\beta y) = (x\gamma q)\delta((q\alpha p)\beta y) = (y\gamma(q\alpha p))\delta(q\beta x) \\ &= q\delta((y\gamma(q\alpha p))\beta x) \in Q\Gamma S. \end{aligned}$$

Now let $q\delta y \in Q\Gamma S$, then by using (1), (3) and (4), we have

$$\begin{aligned} q\delta p &= ((x\alpha(q\gamma q))\beta y)\delta p = (p\beta y)\delta(x\alpha(q\gamma q)) = x\delta((p\beta y)\alpha(q\gamma q)) \\ &= x\delta((q\beta q)\alpha(y\gamma p)) = (q\beta q)\delta(x\alpha(y\gamma p)) = ((x\alpha(y\gamma p))\beta q)\delta q \in S\Gamma Q. \end{aligned}$$

Hence $Q\Gamma S = S\Gamma Q$. As by using (3) and (1), we have

$$q = (x\alpha(q\gamma q))\beta y = (q\alpha(x\gamma q))\beta y = (y\alpha(x\gamma q))\beta q \in S\Gamma Q.$$

Thus $q \in S\Gamma Q \cap Q\Gamma S$ implies that $S\Gamma Q \cap Q\Gamma S = Q$.
$(ii) \Longrightarrow (i)$ is obvious. □

Theorem 8. *In an intra-regular Γ-AG^{**}-groupoid S, the following conditions are equivalent.*

(i) A is a Γ-$(1,2)$-ideal of S.
(ii) A is a two-sided Γ-ideal of S.

Proof. $(i) \Longrightarrow (ii)$: Let S be an intra-regular Γ-AG**-groupoid and let A be a Γ-$(1,2)$-ideal of S, then $(A\Gamma S)\Gamma A^2 \subseteq A$. Let $a \in A$, then since S is an intra-regular so there exists $x,\ y \in S$ and $\beta, \gamma, \delta \in \Gamma$, such that $a = (x\beta(a\delta a))\gamma y$. Now let $\psi \in \Gamma$, then by using (3), (1) and (4), we have

$$\begin{aligned} s\psi a &= s\psi((x\beta(a\delta a))\gamma y) = (x\beta(a\delta a))\psi(s\gamma y) = (a\beta(x\delta a))\psi(s\gamma y) \\ &= ((s\gamma y)\beta(x\delta a))\psi a = ((s\gamma y)\beta(x\delta a))\psi((x\beta(a\delta a))\gamma y) \\ &= (x\beta(a\delta a))\psi(((s\gamma y)\beta(x\delta a))\gamma y) = (y\beta((s\gamma y)\beta(x\delta a)))\psi((a\delta a)\gamma x) \\ &= (a\delta a)\psi((y\beta((s\gamma y)\beta(x\delta a)))\gamma x) = (x\delta(y\beta((s\gamma y)\beta(x\delta a))))\psi(a\gamma a) \\ &= (x\delta(y\beta((a\gamma x)\beta(y\delta s))))\psi(a\gamma a) = (x\delta((a\gamma x)\beta(y\beta(y\delta s))))\psi(a\gamma a) \\ &= ((a\gamma x)\delta(x\beta(y\beta(y\delta s))))\psi(a\gamma a) \\ &= ((((x\beta(a\delta a))\gamma y)\gamma x)\delta(x\beta(y\beta(y\delta s))))\psi(a\gamma a) \\ &= (((x\gamma y)\gamma(x\beta(a\delta a)))\delta(x\beta(y\beta(y\delta s))))\psi(a\gamma a) \\ &= ((((a\delta a)\gamma x)\gamma(y\beta x))\delta(x\beta(y\beta(y\delta s))))\psi(a\gamma a) \\ &= ((((y\beta x)\gamma x)\gamma(a\delta a))\delta(x\beta(y\beta(y\delta s))))\psi(a\gamma a) \\ &= (((y\beta(y\delta s))\gamma x)\delta((a\delta a)\beta((y\beta x)\gamma x)))\psi(a\gamma a) \\ &= (((y\beta(y\delta s))\gamma x)\delta((a\delta a)\beta((x\beta x)\gamma y)))\psi(a\gamma a) \\ &= ((a\delta a)\delta(((y\beta(y\delta s))\gamma x)\beta((x\beta x)\gamma y)))\psi(a\gamma a) \\ &= ((((x\beta x)\gamma y)\delta((y\beta(y\delta s))\gamma x))\delta(a\beta a))\psi(a\gamma a) \\ &= (a\delta(((x\beta x)\gamma y)\delta(((y\beta(y\delta s))\gamma x)\beta a)))\psi(a\gamma a) \in (A\Gamma S)\Gamma A^2 \subseteq A. \end{aligned}$$

Hence A is a left Γ-ideal of S and by Lemma 2, A is a two-sided Γ-ideal of S.

$(ii) \Longrightarrow (i)$: Let A be a two-sided Γ-ideal of S. Let $y \in (A\Gamma S)\Gamma A^2$, then $y = (a\beta s)\gamma(b\delta b)$ for some $a, b \in A$, $s \in S$ and $\beta, \gamma, \delta \in \Gamma$. Now by using (3), we have

$$y = (a\beta s)\gamma(b\delta b) = b\gamma((a\beta s)\delta b) \in A\Gamma S \subseteq A.$$

Hence $(A\Gamma S)\Gamma A^2 \subseteq A$ and therefore A is a Γ-$(1,2)$-ideal of S. □

Theorem 9. *In an intra-regular Γ-AG^{**}-groupoid S, the following conditions are equivalent.*

(i) A is a Γ-$(1,2)$-ideal of S.
(ii) A is a Γ-interior ideal of S.

Proof. $(i) \Longrightarrow (ii)$: Let A be a Γ-$(1,2)$-ideal of an intra-regular Γ-AG**-groupoid S, then $(A\Gamma S)\Gamma A^2 \subseteq A$. Let $p \in (S\Gamma A)\Gamma S$, then $p = (s\mu a)\psi s'$ for some $a \in A$, $s, s' \in S$ and $\mu, \psi \in \Gamma$. Since S is intra-regular so there exists x, $y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a = (x\beta(a\delta a))\gamma y$. Now by using (3), (1), (2) and (4), we have

$$\begin{aligned}
p &= (s\mu a)\psi s' = (s\mu((x\beta(a\delta a))\gamma y))\psi s' = ((x\beta(a\delta a))\mu(s\gamma y))\psi s' \\
&= (s'\mu(s\gamma y))\psi(x\beta(a\delta a)) = (s'\mu(s\gamma y))\psi(a\beta(x\delta a)) \\
&= a\psi((s'\mu(s\gamma y))\beta(x\delta a)) = ((x\beta(a\delta a))\gamma y)\psi((s'\mu(s\gamma y))\beta(x\delta a)) \\
&= ((a\beta(x\delta a))\gamma y)\psi((s'\mu(s\gamma y))\beta(x\delta a)) \\
&= ((a\beta(x\delta a))\gamma(s'\mu(s\gamma y)))\psi(y\beta(x\delta a)) \\
&= ((a\beta s')\gamma((x\delta a)\mu(s\gamma y)))\psi(y\beta(x\delta a)) \\
&= ((a\beta s')\gamma((y\delta s)\mu(a\gamma x)))\psi(y\beta(x\delta a)) \\
&= ((a\beta s')\gamma(a\mu((y\delta s)\gamma x)))\psi(y\beta(x\delta a)) \\
&= ((a\beta a)\gamma(s'\mu((y\delta s)\gamma x)))\psi(y\beta(x\delta a)) \\
&= ((a\beta a)\gamma((y\delta s)\mu(s'\gamma x)))\psi(y\beta(x\delta a)) \\
&= ((y\beta(x\delta a))\gamma((y\delta s)\mu(s'\gamma x)))\psi(a\beta a) \\
&= ((y\beta(y\delta s))\gamma((x\delta a)\mu(s'\gamma x)))\psi(a\beta a) \\
&= ((y\beta(y\delta s))\gamma((x\delta s')\mu(a\gamma x)))\psi(a\beta a) \\
&= ((y\beta(y\delta s))\gamma(a\mu((x\delta s')\gamma x)))\psi(a\beta a) \\
&= (a\gamma((y\beta(y\delta s))\mu((x\delta s')\gamma x)))\psi(a\beta a) \\
&\in (A\Gamma S)\Gamma A^2 \subseteq A.
\end{aligned}$$

Thus $(S\Gamma A)\Gamma S \subseteq A$. Which shows that A is a Γ-interior ideal of S.

$(ii) \Longrightarrow (i)$: Let A be a Γ-interior ideal of S, then $(S\Gamma A)\Gamma S \subseteq A$. Let $p \in (A\Gamma S)\Gamma A^2$, then $p = (a\mu s)\psi(b\alpha b)$, for some $a, b \in A$, $s \in S$ and $\mu, \psi, \alpha \in \Gamma$. Since S is intra-regular so there exists x, $y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a = (x\beta(a\delta a))\gamma y$. Now by using (1), (3) and (4), we have

$$\begin{aligned}
p &= (a\mu s)\psi(b\alpha b) = ((b\alpha b)\mu s)\psi a = ((b\alpha b)\mu s)\psi((x\beta(a\gamma a))\gamma y) \\
&= (x\beta(a\gamma a))\psi(((b\alpha b)\mu s)\gamma y) = ((((b\alpha b)\mu s)\gamma y)\beta(a\gamma a))\psi x \\
&= ((a\gamma a)\beta(y\delta((b\alpha b)\mu s)))\psi x = (((y\delta((b\alpha b)\mu s))\gamma a)\beta a)\psi x \in (S\Gamma A)\Gamma S \subseteq A.
\end{aligned}$$

Thus $(A\Gamma S)\Gamma A^2 \subseteq A$.

Now by using (3) and (4), we have

$$\begin{aligned} A\Gamma A &\subseteq A\Gamma S = A\Gamma(S\Gamma S) = S\Gamma(A\Gamma S) = (S\Gamma S)\Gamma(A\Gamma S) \\ &= (S\Gamma A)\Gamma(S\Gamma S) = (S\Gamma A)\Gamma S \subseteq A. \end{aligned}$$

Which shows that A is a Γ-$(1,2)$-ideal of S. □

Theorem 10. *In an intra-regular* Γ-AG^{**}*-groupoid* S*, the following conditions are equivalent.*

(i) A is a Γ-bi-ideal of S.
(ii) A is a Γ-interior ideal of S.

Proof. $(i) \implies (ii)$: Let A be a Γ-bi-ideal of an intra-regular Γ-AG**-groupoid S, then $(A\Gamma S)\Gamma A \subseteq A$. Let $p \in (S\Gamma A)\Gamma S$, then $p = (s\mu a)\psi s^{'}$ for some $a \in A$, $s, s^{'} \in S$ and $\mu, \psi \in \Gamma$. Since S is an intra-regular so there exists x, $y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a = (x\beta(a\delta a))\gamma y$. Now by using (3), (1), (4) and (2), we have

$$\begin{aligned} p &= (s\mu a)\psi s^{'} = (s\mu((x\beta(a\delta a))\gamma y))\psi s^{'} = ((x\beta(a\delta a))\mu(s\gamma y))\psi s^{'} \\ &= (s^{'}\mu(s\gamma y))\psi(x\beta(a\delta a)) = ((a\delta a)\mu x)\psi((s\gamma y)\beta s^{'}) \\ &= (((s\gamma y)\beta s^{'})\mu x)\psi(a\delta a) = ((x\beta s^{'})\mu(s\gamma y))\psi(a\delta a) \\ &= (a\mu a)\psi((s\gamma y)\delta(x\beta s^{'})) = (((s\gamma y)\delta(x\beta s^{'}))\mu a)\psi a \\ &= (((s\gamma y)\delta(x\beta s^{'}))\mu((x\beta(a\delta a))\gamma y))\psi a \\ &= (((s\gamma y)\delta(x\beta(a\delta a)))\mu((x\beta s^{'})\gamma y))\psi a \\ &= ((((a\delta a)\gamma x)\delta(y\beta s))\mu((x\beta s^{'})\gamma y))\psi a \\ &= ((((x\beta s^{'})\gamma y)\delta(y\beta s))\mu((a\delta a)\gamma x))\psi a \\ &= ((a\delta a)\mu((((x\beta s^{'})\gamma y)\delta(y\beta s))\gamma x))\psi a \\ &= ((x\delta(((x\beta s^{'})\gamma y)\delta(y\beta s)))\mu(a\gamma a))\psi a \\ &= (a\mu((x\delta(((x\beta s^{'})\gamma y)\delta(y\beta s)))\gamma a))\psi a \\ &\in (A\Gamma S)\Gamma A \subseteq A. \end{aligned}$$

Thus $(S\Gamma A)\Gamma S \subseteq A$. Which shows that A is a Γ-interior ideal of S.

$(ii) \implies (i)$: Let A be a Γ-interior ideal of S, then $(S\Gamma A)\Gamma S \subseteq A$. Let $p \in (A\Gamma S)\Gamma A$, then $p = (a\mu s)\psi b$ for some $a, b \in A$, $s \in S$ and $\mu, \psi \in \Gamma$. Since S is an intra-regular so there exists x, $y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $b = (x\beta(b\delta b))\gamma y$. Now by using (3), (1) and (4), we have

$$\begin{aligned} p &= (a\mu s)\psi b = (a\mu s)\psi((x\beta(b\delta b))\gamma y) = (x\beta(b\delta b))\psi((a\mu s)\gamma y) \\ &= (((a\mu s)\gamma y)\beta(b\delta b))\psi x = ((b\gamma b)\beta(y\delta(a\mu s)))\psi x \\ &= (((y\delta(a\mu s))\gamma b)\beta b)\psi x \in (S\Gamma A)\Gamma S \subseteq A. \end{aligned}$$

Thus $(A\Gamma S)\Gamma A \subseteq A$.

Now

$$\begin{aligned} A\Gamma A &\subseteq A\Gamma S = A\Gamma(S\Gamma S) = S\Gamma(A\Gamma S) = (S\Gamma S)\Gamma(A\Gamma S) \\ &= (S\Gamma A)\Gamma(S\Gamma S) = (S\Gamma A)\Gamma S \subseteq A. \end{aligned}$$

Which shows that A is a Γ-bi-ideal of S. □

Theorem 11. *In an intra-regular* Γ-AG^{**}*-groupoid* S*, the following conditions are equivalent.*

(i) A is a Γ-$(1,2)$-ideal of S.
(ii) A is a Γ-quasi ideal of S.

Proof. $(i) \implies (ii)$: Let A be a Γ-$(1,2)$-ideal of intra-regular Γ-AG**-groupoid S, then $(A\Gamma S)\Gamma(A\Gamma A) \subseteq A$. Now by using (3) and (4), we have

$$S\Gamma A = S\Gamma(A\Gamma A) = S\Gamma((A\Gamma A)\Gamma A) = (A\Gamma A)\Gamma(S\Gamma A) = (A\Gamma S)\Gamma(A\Gamma A) \subseteq A.$$

and by using (1) and (3), we have

$$\begin{aligned} A\Gamma S &= (A\Gamma A)\Gamma S = ((A\Gamma A)\Gamma A)\Gamma S = (S\Gamma A)\Gamma(A\Gamma A) = (S\Gamma(A\Gamma A))\Gamma(A\Gamma A) \\ &= ((S\Gamma S)\Gamma(A\Gamma A))\Gamma(A\Gamma A) = ((A\Gamma A)\Gamma(S\Gamma S))\Gamma(A\Gamma A) = (A\Gamma S)\Gamma(A\Gamma A) \subseteq A. \end{aligned}$$

Hence $(A\Gamma S) \cap (S\Gamma A) \subseteq A$. Which shows that A is a Γ-quasi ideal of S.

$(ii) \implies (i)$: Let A be a Γ-quasi ideal of S, then $(A\Gamma S) \cap (S\Gamma A) \subseteq A$. Now $A\Gamma A \subseteq A\Gamma S$ and $A\Gamma A \subseteq S\Gamma A$. Thus $A\Gamma A \subseteq (A\Gamma S) \cap (S\Gamma A) \subseteq A$.

Now by using (4) and (3), we have

$$(A\Gamma S)\Gamma(A\Gamma A) = (A\Gamma A)\Gamma(S\Gamma A) \subseteq A\Gamma(S\Gamma A) = S\Gamma(A\Gamma A) \subseteq S\Gamma A.$$

and

$$\begin{aligned} (A\Gamma S)\Gamma(A\Gamma A) &= (A\Gamma A)\Gamma(S\Gamma A) \subseteq A\Gamma(S\Gamma A) = S\Gamma(A\Gamma A) \\ &= (S\Gamma S)\Gamma(A\Gamma A) = (A\Gamma A)\Gamma(S\Gamma S) \subseteq A\Gamma S. \end{aligned}$$

Thus $(A\Gamma S)\Gamma(A\Gamma A) \subseteq (A\Gamma S) \cap (S\Gamma A) \subseteq A$. Which shows that A is a Γ-$(1,2)$-ideal of S. □

Lemma 3. *Let A be a subset of an intra-regular Γ-AG^{**}-groupoid S, then A is a two-sided Γ-ideal of S if and only if $A\Gamma S = A$ and $S\Gamma A = A$.*

Proof. It is simple. □

Theorem 12. *For an intra-regular Γ-AG^{**}-groupoid S the following statements are equivalent.*

(i) A is a left Γ-ideal of S.
(ii) A is a right Γ-ideal of S.
(iii) A is a two-sided Γ-ideal of S.
(iv) $A\Gamma S = A$ and $S\Gamma A = A$.
(v) A is a Γ-quasi ideal of S.
(vi) A is a Γ-$(1,2)$-ideal of S.
(vii) A is a Γ-generalized bi-ideal of S.
$(viii)$ A is a Γ-bi-ideal of S.
(ix) A is a Γ-interior ideal of S.

Proof. $(i) \implies (ii)$ and $(ii) \implies (iii)$ are followed by Lemma 2.

$(iii) \implies (iv)$ is followed by Lemma 3, and $(iv) \implies (v)$ is obvious.

$(v) \implies (vi)$ is followed by Theorem 11.

$(vi) \implies (vii)$: Let A be a Γ-$(1,2)$-ideal of an intra-regular Γ-AG**-groupoid S, then $(A\Gamma S)\Gamma A^2 \subseteq A$. Let $p \in (A\Gamma S)\Gamma A$, then $p = (a\mu s)\psi b$ for some $a, b \in A$, $s \in S$ and $\mu, \psi \in \Gamma$. Now since S is an intra-regular so there exists x, $y \in S$ and

$\beta, \gamma, \delta \in \Gamma$ such that such that $b = (x\beta(b\delta b))\gamma y$ then, by using (3) and (4), we have

$$\begin{aligned} p &= (a\mu s)\psi b = (a\mu s)\psi((x\beta(b\delta b))\gamma y) = (x\beta(b\delta b))\psi((a\mu s)\gamma y) \\ &= (y\beta(a\mu s))\psi((b\delta b)\gamma x) = (b\delta b)\psi((y\beta(a\mu s))\gamma x) \\ &= (x\delta(y\beta(a\mu s)))\psi(b\gamma b) = (x\delta(a\beta(y\mu s)))\psi(b\delta b) \\ &= (a\delta(x\beta(y\mu s)))\psi(b\delta b) \in (A\Gamma S)\Gamma A^2 \subseteq A. \end{aligned}$$

Which shows that A is a Γ-generalized bi-ideal of S.
$(vii) \Longrightarrow (viii)$ is simple.
$(viii) \Longrightarrow (ix)$ is followed by Theorem 10.
$(ix) \Longrightarrow (i)$ is followed by Theorems 9 and 8. □

Theorem 13. *In a Γ-AG^{**}-groupoid S, the following conditions are equivalent.*

(i) S is intra-regular.
(ii) Every Γ-bi-ideal of S is Γ-idempotent.

Proof. $(i) \Longrightarrow (ii)$ is obvious by Theorem 4.
$(ii) \Longrightarrow (i)$: Since $S\Gamma a$ is a Γ-bi-ideal of S, and by assumption $S\Gamma a$ is Γ-idempotent, so by using (2), we have

$$\begin{aligned} a &\in (S\Gamma a)\,\Gamma\,(S\Gamma a) = ((S\Gamma a)\,\Gamma\,(S\Gamma a))\,\Gamma\,(S\Gamma a) \\ &= ((S\Gamma S)\,\Gamma\,(a\Gamma a))\,\Gamma\,(S\Gamma a) \subseteq \left(S\Gamma a^2\right)\Gamma\,(S\Gamma S) = \left(S\Gamma a^2\right)\Gamma S. \end{aligned}$$

Hence S is intra-regular. □

Lemma 4. *If I and J are two-sided Γ-ideals of an intra-regular Γ-AG^{**}-groupoid S , then $I \cap J$ is a two-sided Γ-ideal of S.*

Proof. It is simple. □

Lemma 5. *In an intra-regular Γ-AG^{**}-groupoid $I\Gamma J = I \cap J$, for every Γ-ideals I and J in S.*

Proof. Let I and J be any Γ-ideals of S, then obviously $I\Gamma J \subseteq I\cap J$. Since $I\cap J \subseteq I$ and $I \cap J \subseteq J$, then $(I \cap J)^2 \subseteq I\Gamma J$, also by Lemma 4, $I \cap J$ is a Γ-ideal of S, so by Theorem 13, we have $I \cap J = (I \cap J)^2 \subseteq I\Gamma J$. Hence $I\Gamma J = I \cap J$. □

Lemma 6. *Let S be a Γ-AG^{**}-groupoid, then S is an intra-regular if and only if every left Γ-ideal of S is Γ-idempotent.*

Proof. Let S be an intra-regular Γ-AG**-groupoid, then by Theorems 12 and 13, every Γ-ideal of S is Γ-idempotent.

Conversely, assume that every left Γ-ideal of S is Γ-idempotent. Since $S\Gamma a$ is a left Γ-ideal of S, so by using (2), we have

$$\begin{aligned} a &\in S\Gamma a = (S\Gamma a)\,\Gamma\,(S\Gamma a) = ((S\Gamma a)\,\Gamma\,(S\Gamma a))\,\Gamma\,(S\Gamma a) \\ &= ((S\Gamma S)\,\Gamma\,(a\Gamma a))\,\Gamma\,(S\Gamma a) \subseteq (S\Gamma a^2)\Gamma(S\Gamma S) = \left(S\Gamma a^2\right)\Gamma S. \end{aligned}$$

Hence S is intra-regular. □

Lemma 7. *In an AG^{**}-groupoid S, the following conditions are equivalent.*

(i) S is intra-regular.
(ii) $A = (S\Gamma A)^2$, where A is any left Γ-ideal of S.

Proof. $(i) \Longrightarrow (ii)$: Let A be a left Γ-ideal of an intra-regular Γ-AG**-groupoid S, then $S\Gamma A \subseteq A$ and by Lemma 6, $(S\Gamma A)^2 = S\Gamma A \subseteq A$. Now $A = A\Gamma A \subseteq S\Gamma A = (S\Gamma A)^2$, which implies that $A = (S\Gamma A)^2$.

$(ii) \Longrightarrow (i)$: Let A be a left Γ-ideal of S, then $A = (S\Gamma A)^2 \subseteq A\Gamma A$, which implies that A is Γ-idempotent and by using Lemma 6, S is an intra-regular. □

Theorem 14. *For an intra-regular Γ-AG**-groupoid S, the following statements holds.*

(i) Every right Γ-ideal of S is Γ-semiprime.
(ii) Every left Γ-ideal of S is Γ-semiprime.
(iii) Every two-sided Γ-ideal of S is Γ-semiprime

Proof. (i) : Let R be a right Γ-ideal of an intra-regular Γ-AG**-groupoid S. Let $a^2 \in R$ and let $a \in S$. Now since S is an intra-regular so there exists $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a = (x\beta(a\delta a))\gamma y$. Now by using (3), (1) and (2), we have

$$\begin{aligned} a &= (x\beta(a\delta a))\gamma y = (a\beta(x\delta a))\gamma y = (y\beta(x\delta a))\gamma a = (y\beta(x\delta a))\gamma((x\beta(a\delta a))\gamma y) \\ &= (x\beta(a\delta a))\gamma((y\beta(x\delta a))\gamma y) = (x\beta(y\beta(x\delta a)))\gamma((a\delta a)\gamma y) \\ &= (a\delta a)\gamma((x\beta(y\beta(x\delta a)))\gamma y) \in R\Gamma(S\Gamma S) = R\Gamma S \subseteq R. \end{aligned}$$

Which shows that R is Γ-semiprime.

(ii) : Let L be a left Γ-ideal of S. Let $a^2 \in L$ and let $a \in S$ now since S is an intra-regular so there exists $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a = (x\beta(a\delta a))\gamma y$, then by using (3), (1) and (4), we have

$$\begin{aligned} a &= (x\beta(a\delta a))\gamma y = (a\beta(x\delta a))\gamma y = (y\beta(x\delta a))\gamma a \\ &= (y\beta(x\delta a))\gamma((x\beta(a\delta a))\gamma y) = (x\beta(a\delta a))\gamma((y\beta(x\delta a))\gamma y) \\ &= (y\beta(y\beta(x\delta a)))\gamma((a\delta a)\gamma x) = (a\delta a)\gamma((y\beta(y\beta(x\delta a)))\gamma x) \\ &= (x\delta(y\beta(y\beta(x\delta a))))\gamma(a\gamma a) \in S\Gamma L \subseteq L. \end{aligned}$$

Which shows that L is Γ-semiprime.

(iii) is obvious. □

Theorem 15. *In a Γ-AG**-groupoid S, the following statements are equivalent.*

(i) S is intra-regular.
(ii) Every right Γ-ideal of S is Γ-semiprime.

Proof. $(i) \Longrightarrow (ii)$ is obvious by Theorem 14.

$(ii) \Longrightarrow (i)$: Let S be an intra-regular Γ-AG**-groupoid. Let R be any right Γ-ideal of S such that R is Γ-semiprime. Since $a^2\Gamma S$ is a right Γ-ideal of S, therefore $a^2\Gamma S$ is Γ-semiprime. Now clearly $a^2 \in a^2\Gamma S$ so $a \in a^2\Gamma S$. Now let $\delta \in S$, then by using (3) and (2), we have

$$\begin{aligned} a &\in (a\delta a)\Gamma S = (a\delta a)\Gamma(S\Gamma S) = S\Gamma((a\delta a)\Gamma S) = (S\Gamma S)\Gamma((a\delta a)\Gamma S) \\ &= (S\Gamma(a\delta a))\Gamma(S\Gamma S) = (S\Gamma(a\delta a))\Gamma S. \end{aligned}$$

Which shows that S is an intra regular. □

Theorem 16. *A Γ-AG**-groupoid S is intra-regular if and only if $R \cap L = R\Gamma L$, for every Γ-semiprime right Γ-ideal R and every left Γ-ideal L of S.*

Proof. Let S be an intra-regular Γ-AG**-groupoid and R and L be right and left Γ-ideal of S respectively, then by Theorem 2, R and L become Γ-ideals of S, therefore by Lemma 5, $R \cap L = R\Gamma L$, for every Γ-ideal R and L, also by Theorem 14, R is Γ-semiprime.

Conversely, assume that $R \cap L = R\Gamma L$ for every right Γ-ideal R, which is Γ-semiprime and every left Γ-ideal L of S. Since $a^2 \in a^2\Gamma S$, which is a right Γ-ideal of S so is Γ-semiprime which implies that $a \in a^2\Gamma S$. Now clearly $S\Gamma a$ is a left Γ-ideal of S and $a \in S\Gamma a$, therefore let $\gamma \in \Gamma$, then by using (4), (1) and (2), we have

$$\begin{aligned} a &\in ((a\gamma a)\Gamma S) \cap (S\Gamma a) = ((a\gamma a)\Gamma S)\,\Gamma\,(S\Gamma a) \subseteq ((a\gamma a)\Gamma S)\,\Gamma\,(S\Gamma S) \\ &= ((a\gamma a)\Gamma S)\,\Gamma S = ((a\gamma a)\,\Gamma S)\,\Gamma S = (S\Gamma S)\Gamma(a\gamma a) \\ &= (S\Gamma a)\,\Gamma\,(S\Gamma a) = S\Gamma((S\Gamma a)\Gamma a) = (S\Gamma\,(a\gamma a))\,\Gamma S. \end{aligned}$$

Therefore S is an intra-regular. □

Theorem 17. *For a Γ-AG**-groupoid S , the following statements are equivalent.*

(i) *S is intra-regular.*

(ii) *$L \cap R \subseteq L\Gamma R$, for every right Γ-ideal R, which is Γ-semiprime and every left Γ-ideal L of S.*

(iii) *$L \cap R \subseteq (LR)\,L$, for every Γ-semiprime right Γ-ideal R and every left Γ-ideal L.*

Proof. $(i) \Rightarrow (iii)$: Let S be an intra-regular Γ-AG**-groupoid and L, R be any left and right Γ-ideals of S and let $k \in L \cap R$, which implies that $k \in L$ and $k \in R$. Since S is intra-regular so there exist x, y in S, and $\alpha, \beta, \gamma \in \Gamma$ such that $k = (x\alpha(k\gamma k))\,\beta y$, then by using (3), (1) and (4), we have

$$\begin{aligned} k &= (x\alpha\,(k\gamma k))\,\beta y = (k\alpha\,(x\gamma k))\,\beta y = (y\alpha\,(x\gamma k))\,\beta k \\ &= (y\alpha\,(x\gamma\,((x\alpha(k\gamma k))\,\beta y)))\,\beta k = (y\alpha\,((x\alpha(k\gamma k))\,\gamma\,(x\beta y)))\,\beta k \\ &= ((x\alpha\,(k\gamma k))\,\alpha\,(y\gamma\,(x\beta y)))\,\beta k = ((k\alpha\,(x\gamma k))\,\alpha\,(y\gamma\,(x\beta y)))\,\beta k \\ &\in ((R\Gamma\,(S\Gamma L))\Gamma S)\,\Gamma L \subseteq ((R\Gamma L)\,\Gamma S)\,\Gamma L = (L\Gamma S)\,\Gamma\,(R\Gamma L) \\ &= (L\Gamma R)\,\Gamma\,(S\Gamma L) \subseteq (L\Gamma R)\,\Gamma L. \end{aligned}$$

which implies that $L \cap R \subseteq (L\Gamma R)\,\Gamma L$. Also by Theorem 14, L is Γ-semiprime.

$(iii) \Rightarrow (ii)$: Let R and L be any left and right Γ-ideals of S and R is Γ-semiprime, then by assumption (iii) and by using (4), (3) and (1), we have

$$\begin{aligned} R \cap L &\subseteq (R\Gamma L)\,\Gamma R \subseteq (R\Gamma L)\,\Gamma S = (R\Gamma L)\,\Gamma\,(S\Gamma S) = (S\Gamma S)\,\Gamma\,(L\Gamma R) \\ &= L\Gamma\,((S\Gamma S)\,\Gamma R) = L\Gamma\,((R\Gamma S)\,\Gamma S) \subseteq L\Gamma\,(R\Gamma S) \subseteq L\Gamma R. \end{aligned}$$

$(ii) \Rightarrow (i)$: Since $a \in S\Gamma a$, which is a left Γ-ideal of S, and $a^2 \in a^2\Gamma S$, which is a Γ-semiprime right Γ-ideal of S, therefore, $a \in a^2\Gamma S$. Now by using (4), we have

$$\begin{aligned} a &\in (S\Gamma a) \cap \left(a^2\Gamma S\right) \subseteq (S\Gamma a)\,\Gamma(a^2\Gamma S) \subseteq (S\Gamma S)\,\Gamma\,\left(a^2\Gamma S\right) \\ &= \left(S\Gamma a^2\right)\Gamma\,(S\Gamma S) = \left(S\Gamma a^2\right)\Gamma S. \end{aligned}$$

Hence S is intra-regular. □

A Γ-AG**-groupoid S is called Γ-totally ordered under inclusion if P and Q are any Γ-ideals of S such that either $P \subseteq Q$ or $Q \subseteq P$.

A Γ-ideal P of a Γ-AG**-groupoid S is called Γ-strongly irreducible if $A \cap B \subseteq P$ implies either $A \subseteq P$ or $B \subseteq P$, for all Γ-ideals A, B and P of S.

Lemma 8. *Every Γ-ideal of an intra-regular Γ-AG^{**}-groupoid S is Γ-prime if and only if it is Γ-strongly irreducible.*

Proof. It is an easy consequence of Lemma 5. □

Theorem 18. *Every Γ-ideal of an intra-regular Γ-AG^{**}-groupoid S is Γ-prime if and only if S is Γ-totally ordered under inclusion.*

Proof. Assume that every Γ-ideal of S is Γ-prime. Let P and Q be any Γ-ideals of S, so by Lemma 5, $P\Gamma Q = P \cap Q$, and by Lemma 4, $P \cap Q$ is a Γ-ideal of S, so is prime, therefore $P\Gamma Q \subseteq P \cap Q$, which implies that $P \subseteq P \cap Q$ or $Q \subseteq P \cap Q$, which implies that $P \subseteq Q$ or $Q \subseteq P$. Hence S is Γ-totally ordered under inclusion.

Conversely, assume that S is Γ-totally ordered under inclusion. Let I, J and P be any Γ-ideals of S such that $I\Gamma J \subseteq P$. Now without loss of generality assume that $I \subseteq J$ then

$$I = I\Gamma I \subseteq I\Gamma J \subseteq P.$$

Therefore either $I \subseteq P$ or $J \subseteq P$, which implies that P is Γ-prime. □

Theorem 19. *The set of all Γ-ideals I_s of an intra-regular Γ-AG^{**}-groupoid S, forms a Γ-semilattice structure.*

Proof. Let A, $B \in I_s$, since A and B are Γ-ideals of S, then by using (2), we have

$$\begin{aligned} (A\Gamma B)\Gamma S &= (A\Gamma B)\,\Gamma\,(S\Gamma S) = (A\Gamma S)\,\Gamma\,(B\Gamma S) \subseteq A\Gamma B. \\ \text{Also } S\Gamma(A\Gamma B) &= (S\Gamma S)\,\Gamma\,(A\Gamma B) = (S\Gamma A)\,\Gamma(S\Gamma B) \subseteq A\Gamma B. \end{aligned}$$

Thus $A\Gamma B$ is a Γ-ideal of S. Hence I_s is closed. Also using Lemma 5, we have, $A\Gamma B = A \cap B = B \cap A = B\Gamma A$, which implies that I_s is commutative, so is associative. Now by using Theorem 13, $A\Gamma A = A$, for all $A \in I_s$. Hence I_s is Γ-semilattice. □

Theorem 20. *A two-sided Γ-ideal of an intra-regular Γ-AG^{**}-groupoid S is minimal if and only if it is the intersection of two minimal two-sided Γ-ideals.*

Proof. Let S be an intra-regular Γ-AG**-groupoid and Q be a minimal two-sided Γ-ideal of S, let $a \in Q$. As $S\Gamma(S\Gamma a) \subseteq S\Gamma a$ and $S\Gamma(a\Gamma S) \subseteq a\Gamma(S\Gamma S) = a\Gamma S$, which shows that $S\Gamma a$ and $a\Gamma S$ are left Γ-ideals of S so by Lemma 2, $S\Gamma a$ and $a\Gamma S$ are two-sided Γ-ideals of S.

Now

$$\begin{aligned} & S\Gamma(S\Gamma a \cap a\Gamma S) \cap (S\Gamma a \cap a\Gamma S)\Gamma S \\ = \;& S\Gamma(S\Gamma a) \cap S\Gamma(a\Gamma S) \cap (S\Gamma a)\Gamma S \cap (a\Gamma S)\Gamma S \\ \subseteq \;& (S\Gamma a \cap a\Gamma S) \cap (S\Gamma a)\Gamma S \cap S\Gamma a \subseteq S\Gamma a \cap a\Gamma S. \end{aligned}$$

Which implies that $S\Gamma a \cap a\Gamma S$ is a Γ-quasi ideal so by Theorems 8 and 11, $S\Gamma a \cap a\Gamma S$ is a two-sided Γ-ideal.

Also since $a \in Q$, we have

$$S\Gamma a \cap a\Gamma S \subseteq S\Gamma Q \cap Q\Gamma S \subseteq Q \cap Q \subseteq Q.$$

Now since Q is minimal so $S\Gamma a \cap a\Gamma S = Q$, where $S\Gamma a$ and $a\Gamma S$ are minimal two-sided Γ-ideals of S, because let I be a Γ-ideal of S such that $I \subseteq S\Gamma a$, then

$$I \cap a\Gamma S \subseteq S\Gamma a \cap a\Gamma S \subseteq Q,$$

which implies that

$$I \cap a\Gamma S = Q. \text{ Thus } Q \subseteq I.$$

So we have

$$\begin{aligned} S\Gamma a &\subseteq S\Gamma Q \subseteq S\Gamma I \subseteq I, \text{ gives} \\ S\Gamma a &= I. \end{aligned}$$

Thus $S\Gamma a$ is a minimal two-sided Γ-ideal of S. Similarly $a\Gamma S$ is a minimal two-sided Γ-ideal of S.

Conversely, let $Q = I \cap J$ be a two-sided Γ-ideal of S, where I and J are minimal two-sided Γ-ideals of S, then by Theorem 8 and 11, Q is a Γ-quasi ideal of S, that is $S\Gamma Q \cap Q\Gamma S \subseteq Q$.

Let Q' be a two-sided Γ-ideal of S such that $Q' \subseteq Q$, then

$$\begin{aligned} S\Gamma Q' \cap Q'\Gamma S &\subseteq S\Gamma Q \cap Q\Gamma S \subseteq Q, \text{ also } S\Gamma Q' \subseteq S\Gamma I \subseteq I \\ \text{and } Q'\Gamma S &\subseteq J\Gamma S \subseteq J. \end{aligned}$$

Now

$$\begin{aligned} S\Gamma\left(S\Gamma Q'\right) &= (S\Gamma S)\,\Gamma\left(S\Gamma Q'\right) = \left(Q'\Gamma S\right)\Gamma\,(S\Gamma S) \\ &= \left(Q'\Gamma S\right)\Gamma S = (S\Gamma S)\,\Gamma Q' = S\Gamma Q' \end{aligned}$$

implies that $S\Gamma Q'$ is a left Γ-ideal and hence a two-sided Γ-ideal by Lemma 2. Similarly $Q'\Gamma S$ is a two-sided Γ-ideal of S.

But since I and J are minimal two-sided Γ-ideals of S, so

$$S\Gamma Q' = I \text{ and } Q'\Gamma S = J.$$

But $Q = I \cap J$, which implies that,

$$Q = S\Gamma Q' \cap Q'\Gamma S \subseteq Q'.$$

Which give us $Q = Q'$. Hence Q is minimal. □

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