MEANDER GRAPHS AND FROBENIUS SEAWEED LIE ALGEBRAS

VINCENT COLL, ANTHONY GIAQUINTO, AND COLTON MAGNANT

Abstract. The index of a seaweed Lie algebra can be computed from its associated meander graph. We examine this graph in several ways with a goal of determining families of Frobenius (index zero) seaweed algebras. Our analysis gives two new families of Frobenius seaweed algebras as well as elementary proofs of known families of such Lie algebras.
2000 MSC: 17B05, 17B08

1. Introduction

Let $\mathfrak{L}$ be a Lie algebra over a field of characteristic zero. For any functional $F \in \mathfrak{L}^*$ there is an associated skew bilinear form B_F on $\mathfrak{L}$ defined by $B_F(x, y) = F([x, y])$ for $x, y \in \mathfrak{L}$. The index of $\mathfrak{L}$ is defined to be

$$\operatorname{ind} \mathfrak{L} = \min_{F \in \mathfrak{L}^*} \dim(\ker(B_F)).$$

The Lie algebra $\mathfrak{L}$ is *Frobenius* if dim$=0$; equivalently, if there is a functional $F \in \mathfrak{L}^*$ such that $B_F(-, -)$ is non-degenerate.

Frobenius Lie algebras were first studied by Ooms in [8] where he proved that the universal enveloping algebra $U\mathfrak{L}$ is primitive (i.e. admits a faithful simple module) provided that $\mathfrak{L}$ is Frobenius and that the converse holds when $\mathfrak{L}$ is algebraic. The relevance of Frobenius Lie algebras to deformation and quantum group theory stems from their relation to the classical Yang-Baxter equation (CYBE). Suppose $B_F(-, -)$ is non-degenerate and let M be the matrix of $B_F(-.-)$ relative to some basis $\{x_1, \ldots, x_n\}$ of $\mathfrak{L}$. Belavin and Drinfeld showed that $r = \sum_{i,j}(M^{-1})_{ij} x_i \wedge x_j$ is a (constant) solution of the CYBE, see [1]. Thus, each pair consisting of a Lie algebra $\mathfrak{L}$ together with functional $F \in \mathfrak{L}^*$ such that B_F is non-degenerate provides a solution to the CYBE, see [5] and [6] for examples.

The index of a semisimple Lie algebra $\mathfrak{g}$ is equal to its rank and thus such algebras can never be Frobenius. However, there always exist subalgebras of $\mathfrak{g}$ which are Frobenius. In particular, many amongst the class of *biparabolic* subalgebras of $\mathfrak{g}$ are Frobenius. A biparabolic subalgebra is the intersection of two parabolic subalgebras whose sum is $\mathfrak{g}$. They were first introduced in the case $\mathfrak{g} = \mathfrak{sl}(n)$ by Dergachev and Kirillov in [2] where they were called Lie algebras of *seaweed* type. Associated to each seaweed algebra is a certain graph called the *meander*. One of the main reults of [2] is that the algebra's index is determined by graph-theoretical properties of its meander, see Section 3 for details.

Using different methods, Panyushev developed an inductive procedure for computing the index of seaweed subalgebras, see [9]. In the same paper, he exhibits a closed form for the index of a biparabolic subalgebra of $\mathfrak{sp}(n)$.

Tauvel and Tu found in [10] an upper bound for the index of a biparabolic subalgebra of an arbitrary semisimple Lie algebra, and they conjectured that this was an equality. Joseph proved the Tauvel-Tu conjecture in [7].

The methods of [2], [9], [10], and [7] are all combinatorial in nature. Yet even with the this theory available, it is difficult in practice to implement this theory to find families of Frobenius biparabolic Lie algebras. In contrast, for many cases it is known explicitly which biparabolic algebras have the maximum possible index. For example, the only biparabolics in $\mathfrak{sl}(n)$ and $\mathfrak{sp}(n)$ which have maximal index are the Levi subalgebras. In contrast, the problem of determining the biparabolics of minimal index is an open question in all cases.

Our focus in this note is on the seaweed Lie algebras – these are the biparabolic subalgebras of $\mathfrak{sl}(n)$. The only known families of Frobenius seaweed Lie algebras that seem to be in the literature will be outlined in Section 4, although the unpublished preprint [3] may offer more examples. We shall examine these families using the meander graphs of Dergachev and Kirillov. Our methodology provides new proofs that these algebras are indeed Frobenius. We shall also exhibit two new infinite families of Frobenius seaweed Lie algebras in Sections 4.3 and 4.4.

2. Seaweed Lie algebras

In this section we introduce the seaweed Lie algebras of [2]. Recall that a composition of a positive integer n is an unordered partition $\underline{x} = (a_1, \ldots, a_m)$. That is, each $a_i \geq 0$ and $\sum a_i = n$.

Definition 2.1. Let V be an n-dimensional vector space with a basis $e_1, \ldots, e_n$. Let $\underline{x} = (a_1, \ldots, a_m)$ and $\underline{y} = (b_1, \ldots, b_t)$ be two compositions of n and consider the flags

$$\{0\} \subset V_1 \subset \cdots \subset V_{m-1} \subset V_m = V \quad \text{and} \quad V = W_0 \supset W_1 \supset \cdots \supset W_t = \{0\}$$

where $V_i = \text{span}\{e_1, \ldots, e_{a_1+\cdots+a_i}\}$ and $W_j = \text{span}\{e_{b_1+\cdots+e_{b_j}+1}, \ldots e_n\}$. The subalgebra of $\mathfrak{sl}(n)$ preserving these flags is called a seaweed Lie algebra and is denoted $\mathfrak{p}(\underline{x} \mid \underline{y})$.

A basis-free definition is available but is not necessary for the present discussion. The name seaweed Lie algebra was chosen due to their suggestive shape when exhibited in matrix form. For example, the algebra $\mathfrak{p}(3, 1, 3, 2 \mid 4, 2, 3)$ consists of traceless matrices of the form

$$\begin{bmatrix}
* & * & * & * & \cdot & \cdot & \cdot & \cdot & \cdot \\
* & * & * & * & \cdot & \cdot & \cdot & \cdot & \cdot \\
* & * & * & * & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & * & * & * & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & * & * & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & * & * & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & * & * & * & * & * \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * & * & * \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * & * & *
\end{bmatrix}$$

where the entries marked by the dots are zero.

Many important subalgebras of $\mathfrak{sl}(n)$ are of seaweed type, as illustrated in the following example.

Example 2.2.

- The entire algebra $\mathfrak{sl}(n) = \mathfrak{p}(n \mid n)$ has index $n - 1$.
- The Cartan subalgebra of traceless diagonal matrices is $\mathfrak{p}(\underline{1} \mid \underline{1})$, where $\underline{1} = (1, 1, \ldots, 1)$ and has index $n - 1$.
- The Borel subalgebra is $\mathfrak{p}(\underline{1} \mid n)$ and has index $\lfloor (n + 1)/2 \rfloor$.
- A maximal parabolic subalgebra is of the form $\mathfrak{p}(a, b \mid n)$. Elashvili proved in [4] that its index is $\gcd(a, n) - 1$.

The only explicitly known Frobenius examples in the above list are the maximal parabolic algebras $\mathfrak{p}(a, b \mid n)$ with a and n relatively prime. Of course, another infinite family of Frobenius seaweed algebras occurs when $\underline{a} = (2, \ldots, 2, 1)$, $\underline{b} = (1, 2, \ldots 2)$, and n is odd. A similar case is $\underline{a} = (1, 2 \ldots, 2, 1)$, $\underline{b} = (2, \ldots, 2)$, and n is even. These two families are detailed in [9].

A tantalizing question is how to classify which seaweed algebras are Frobenius, especially given their importance in the general theory of Lie algebras and applications to deformations and quantum groups.

3. Meanders

As stated earlier, Dergachev and Kirillov have developed a combinatorial algorithm to compute the index of an arbitrary $\mathfrak{p}(\underline{x} \mid \underline{y})$ from its associated *meander* graph $M(\underline{x} \mid \underline{y})$ determined by the compositions $\underline{x}$ and $\underline{y}$. The vertices of $M(\underline{x} \mid \underline{y})$ consist of n ordered points on a horizontal line, which can be called $1, 2, \ldots, n$. The edges are arcs above and below the line connecting pairs of different vertices.

More specifically, the composition $\underline{x} = (a_1, \ldots, a_m)$ determines arcs above the line which we will call the top edges. The component a_1 of $\underline{x}$ determines $\lfloor a_1/2 \rfloor$ arcs above vertices $1, \ldots, a_1$. The arcs are obtained by connecting vertex 1 to vertex a_1, vertex 2 to vertex $a_1 - 1$, and so on. If a_1 is odd then vertex $a_{\lceil a_1/2 \rceil}$ has no arc above it. For the component a_2 of $\underline{a}$, we do the same procedure over vertices $a_1 + 1, \ldots, a_1 + a_2$, and continue with the higher a_i.

The arcs corresponding to $\underline{y} = (b_1, \ldots, b_t)$ are drawn with the same rule but are under the line containing the vertices. These are called the bottom edges.

It is easy to see that every meander consists of a disjoint union of cycles, paths, and isolated points, but not all of these are necessarily present in any given meander.

Theorem 3.1 (Dergachev-Kirillov)**.** *The index of the Lie algebra of seaweed type $\mathfrak{p}(\underline{a} \mid \underline{b})$ is equal to the number of connected components in the meander plus the number of closed cycles minus 1.*

Remark 3.2. The presence of the minus one in the theorem is due to our use of seaweed subalgebras of $\mathfrak{sl}(n)$ rather than of $\mathfrak{gl}(n)$ as used by Dergachev and Kirillov [2]. The index drops by one by the restriction to $\mathfrak{sl}(n)$ from $\mathfrak{gl}(n)$.

Example 3.3. The following is the meander $M(\underline{x} \mid \underline{y})$ corresponding to the compositions $\underline{x} = (5, 2, 2)$ and $\underline{y} = (2, 4, 3)$.

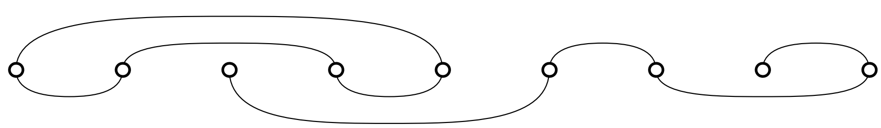

Figure 1. $M(5, 2, 2 \mid 2, 4, 3)$.

We see that there is a single path and a single cycle. Using the theorem above, the index is $2 + 1 - 1 = 2$. Hence, $\mathfrak{p}(5, 2, 2 \mid 2, 4, 3)$ is not a Frobenius algebra.

It is easy to see that to obtain a Frobenius algebra, the only possibility for the meander is that it consist of a single path with no cycles and no isolated points. The following illustrates this point.

Example 3.4. Consider the algebra $\mathfrak{p}(3, 2, 2 \mid 2, 5)$. Its meander is given in Figure 2.

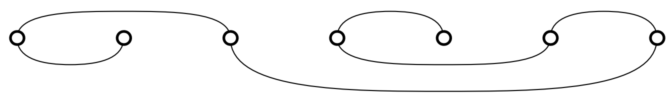

FIGURE 2. $M(3, 2, 2 \mid 2, 5)$.

Labeling the vertices with $\{1, 2, \ldots, n\}$ from left to right, notice that $M(3, 2, 2 \mid 2, 5)$ is the single path $2, 1, 3, 7, 6, 4, 5$ (if we start with 2) or its reversal $5, 4, 6, 7, 3, 1, 2$ if we start with 5. In particular, the index is $1 - 1 = 0$ and so this is a Frobenius algebra.

Question 1. What are the conditions on the compositions $\underline{x}$ and $\underline{y}$ so that the meander $M(\underline{x} \mid \underline{y})$ consists of a single path with no cycles or isolated points?

As stated, this seems to be an elementary question involving nothing more that the basics of graph theory. However, the apparent simplicity of the question is misleading since an answer would provide a complete classification of Frobenius seaweed algebras - a difficult problem. Even so, it is easy to give some necessary conditions on $\underline{x} = (a_1, \ldots, a_m)$ and $\underline{y} = (b_1, \ldots, b_t)$ for $M(\underline{x} \mid \underline{y})$ to be a single path. For example, exactly two elements of the set $\{a_1, \ldots, a_m, b_1, \ldots, b_t)$ must be odd. This is because a path must have a starting and ending point, and these corresponds to vertices of degree one. A vertex of degree one is either missing a top edge or bottom edge connecting to it, and this happens only if some a_i or b_j is odd.

Another necessary condition for $M(\underline{x} \mid \underline{y})$ to be a single path is that $a_1 \neq b_1$. In this case,

$$\mathfrak{p}(\underline{x} \mid \underline{y}) \simeq \mathfrak{sl}(a_1) \bigoplus \mathfrak{p}(a_2, \ldots, a_m \mid b_2, \ldots, b_t)$$

and thus $\mathfrak{p}(\underline{x} \mid \underline{y})$ is not Frobenius since the index is additive for direct sums of Lie algebras. More generally, if $\sum_{i=1}^{r} a_i = \sum_{j=1}^{r} b_j$ for some $r \leq \min\{m, t\}$ then the meander is not a single path. Other necessary conditions can be given, but none seems to shed light on what is sufficient.

4. Families of Frobenius seaweed algebras

In this section we revisit some known families of Frobenius seaweed algebras in terms of meanders. At the end we also provide two new families.

First consider Panyushev's example with $\underline{x} = (2, \ldots, 2, 1)$, $\underline{y} = (1, 2, \ldots 2)$, and n is odd. Again, numbering as in Example 3.4, the top edges connect 2 to 4, 4 to 6, etc. while the bottom edges connect 1 to 3, 3 to 5, etc. Hence, the meander consists of the single path $1, 2, \ldots, n$. A similar argument verifies that the meander for $\underline{x} = (1, 2 \ldots, 2, 1)$ and $\underline{y} = (2, \ldots, 2)$ with n even is also the path $1, 2, \ldots, n$.

To analyze some other cases it is convenient to modify the definition of the meander $M(\underline{x} \mid \underline{y})$.

Definition 4.1. Suppose $\underline{x}$ and $\underline{y}$ are compositions of n. The modified meander $M'(\underline{x} \mid \underline{y})$ is the graph $M(\underline{x} \mid \underline{y})$ appended with a loop corresponding to each odd a_i and b_j. Specifically, for all odd a_i, add a loop connecting $a_1 + \cdots + a_{i-1} + \lceil a_i/2 \rceil$ to itself. Similarly, for all odd b_j, add a bottom loop connecting $b_1 + \cdots + b_{j-1} + \lceil b_j/2 \rceil$ to itself.

Note that in $M'(\underline{x} \mid \underline{y})$ each vertex is incident with exactly one top and one bottom edge or loop.

Example 4.2. Below is the modified meander $M'(5, 2, 2 \mid 2, 4, 3)$. Compare with the meander $M(5, 2, 2 \mid 2, 4, 3)$ given in Example 3.3.

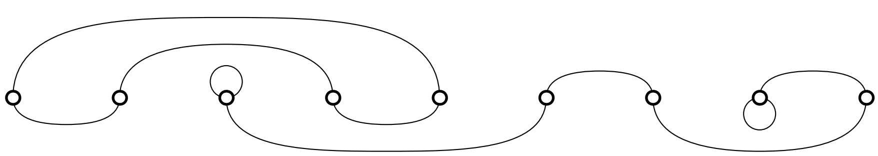

Figure 3. $M(5, 2, 2 \mid 2, 4, 3)$ with loops.

4.1. **The top and bottom bijections.** Each modified meander determines two bijections of $S = \{1, 2, \ldots, n\}$ to itself. Define a "top" bijection t of S by $t(i) = i$, where j is the unique vertex incident with the same top edge as i. If i is joined to itself by a top loop, then $t(i) = 1$. In a similar way, define a "bottom" bijection b of S by $b(i) = j$, where j is the unique vertex incident with the same bottom edge as j. If i is joined to itself by a bottom loop, then $b(i) = 1$. Clearly the maps t and b are well-defined. For instance, in Example 4.2, we have $t(3) = 3$ and $b(3) = 6$.

Definition 4.3. Let $\underline{x}$ and $\underline{y}$ be compositions of n. The meander permutation $\sigma_{\underline{x},\underline{y}} \in S_n$ is the permutation $t \circ b$ of S. That is, $\sigma_{\underline{x},\underline{y}}(i) = t(b(i))$.

Example 4.4. Consider the meander permutation $\sigma_{\underline{x},\underline{y}}$ with $\underline{x}$ and $\underline{y}$ as in Example 4.2. We can write $\sigma_{\underline{x},\underline{y}}$ as a product of disjoint cycles in S_n: $(1,4)(2,5)(3,7,8,9,6)$ (note the different use of the term "cycle").

Theorem 4.5. *Suppose $\underline{x}$ and $\underline{y}$ are compositions of n. Then the meander $M(\underline{x} \mid \underline{y})$ is a single path if and only if the meander permutation $\sigma_{\underline{x},\underline{y}}$ is an n-cycle in S_n.*

Proof. Suppose the meander $M(\underline{x} \mid \underline{y})$ is the single path $a_1, a_2, \ldots, a_n$. By switching $\underline{x}$ and $\underline{y}$ if necessary, we can assume that $b(a_1) = a_2$. Then the meander permutation is the n-cycle $(a_1, a_3, \ldots a_{n-1}, a_n, a_{n-2}, \ldots a_2)$ if n is even and if n is odd it is the n-cycle if $(a_1, a_3, \ldots, a_n, a_{n-1}, a_{n-3}, \ldots, a_2)$.

Conversely suppose $\sigma_{\underline{x},\underline{y}}$ is an n-cycle but $M(\underline{x} \mid \underline{y})$ is not a single path. Then $M(\underline{x} \mid \underline{y})$ contains either an isolated point, a path of length less than n, or a cycle. We shall show that each of these possibilities leads to a contradiction.

If i is an isolated point of $M(\underline{x} \mid \underline{y})$, then it is a fixed point of $\sigma_{\underline{x},\underline{y}}$ which therefore can not be an n-cycle.

If $a_1, \ldots a_k$ is a path in $M(\underline{x} \mid \underline{y})$ with $k < n$ then, depending on whether k is even or odd, either the $(a_1, a_3, \ldots a_{k-1}, a_k, a_{k-2}, \ldots a_2)$ or $(a_1, a_3, \ldots, a_k, a_{k-1}, a_{k-3}, \ldots, a_2)$ appears in the cycle decomposition of $\sigma_{\underline{x},\underline{y}}$. Since $k < n$ we conclude that $\sigma_{\underline{x},\underline{y}}$ is not an n-cycle.

Now if $M(\underline{x} \mid \underline{y})$ contains a cycle $a_1, a_2, \ldots, a_k, a_1$, then the meander permutation contains either the $k/2$ cycle $(a_1, a_3, \ldots, a_{n-1})$ if n is even or the k-cycle $(a_1, a_3, \ldots, a_n, a_2, a_4, \ldots, a_{n-1})$ if n is odd. If $k < n$ then $\sigma_{\underline{x},\underline{y}}$ is not an n-cycle. If $k = n$ is even, then the same argument shows that $\sigma_{\underline{x},\underline{y}}$ is not an n-cycle. The remaining case is that $k = n$ is odd. If this happens though, we must have $M(\underline{x} \mid \underline{y}) = M'(\underline{x} \mid \underline{y})$, and consequently all components a_i and b_j are even. Since $\sum a_i = n$ we have a contradiction. Thus, in all cases when $M(\underline{x} \mid \underline{y})$ is not a single path, the meander permutation $\sigma_{\underline{x},\underline{y}}$ is not an n-cycle, which is a contradiction. The proof is complete. □

4.2. **Maximal Parabolic Subalgebras.** To generate more examples of Frobenius Lie algebras, we consider maximal parabolic seaweed subalgebras of $\mathfrak{sl}(n)$ which are necessarily of the form $\mathfrak{p}(a, b \mid n)$.

Lemma 4.6. *Consider the compositions $\underline{x} = (a, b)$ and $\underline{y} = n$. The meander permutation $\sigma_{\underline{x},\underline{y}}$ is the map sending i to $i + a \mod n$ for all i.*

Proof. By definition of the top and bottom maps, we have

$$b(i) = n + 1 - i \quad \text{and} \quad t(i) = \begin{cases} a + 1 & \text{if} \quad 1 \leq i \leq a \\ n + a + 1 - i & \text{if} \quad a + 1 \leq i \leq n \end{cases}$$

and thus

$$t(b(i)) = \begin{cases} a - n + i & \text{if} \quad 1 \leq b(i) \leq a \\ a + i & \text{if} \quad a + 1 \leq b(i) \leq n \end{cases}.$$

Therefore $\sigma_{\underline{x},\underline{y}}(i) = t(b(i)) = i + a \mod n$. □

Recall Elashvili's result asserting that the maximal parabolic algebra $\mathfrak{p}(a, b \,|n)$ is Frobenius if and only if $\gcd(a, n) = 1$. An immediate corollary of the previous lemma gives a new simple proof of Elashvili's result.

Corollary 4.7. *The maximal parabolic algebra $\mathfrak{p}(a, b \mid n)$ is Frobenius if and only if $\gcd(a, n) = 1$.*

Proof. By Theorem 4.5 it suffices to show that the meander permutation is an n-cycle. According to Lemma 4.6, $\sigma_{\underline{x},\underline{y}}(i) = i + a \mod n$ for all i. Thus, the meander permutation is an n-cycle if and only if the sequence $i, i + a, i + 2a, \ldots, i + (n - 1)a$ forms a complete residue system modulo n. This occurs precisely when $\gcd(a, n) = 1$. The proof is complete. □

4.3. **Opposite maximal parabolic subalgebras.** We now use the same ideas to present a new family of Frobenius seaweed algebras each of which is an intersection of a positive and negative maximal parabolic algebra. Such algebras are of the form $\mathfrak{p}(a, b \mid c, d)$ and are called *opposite maximal parabolic subalgebras.*

Lemma 4.8. *Let $\underline{x} = (a, b)$ and $\underline{y} = (c, d)$ be compositions of n. The permutation meander $\sigma_{\underline{x},\underline{y}}$ is the map sending i to $a - c \mod n$ for all i.*

Proof. The bottom and top maps are given by

$$b(i) = \begin{cases} c + 1 - i & \text{if} \quad 1 \leq i \leq c \\ n + c + 1 - i & \text{if} \quad c + 1 \leq i \leq n \end{cases},$$

and

$$t(i) = \begin{cases} a+1-i & \text{if} \quad 1 \le i \le a \\ n+a+1-i & \text{if} \quad a+1 \le i \le n \end{cases}.$$

There are four possible compositions $t(b(i))$, depending on and whether $i \le c$ or $i > c$ and whether $b(i) \le a$ or $b(i) > a$. It is an easy calculation to see that in each case $t(b(i)) = a - c + i \mod n$. □

An immediate consequence is the following result.

Corollary 4.9. *The opposite maximal parabolic seaweed algebra* $\mathfrak{p}(a, b \mid c, d)$ *is Frobenius if and only if* $\gcd(a - c, n) = 1$.

Proof. The argument is exactly as that used in Corollary 4.7. Namely, that the meander permutation is an n-cycle if and only if the sequence $i, i+(a-c), i+2(a-c), \ldots, i+(n-1)(a-c)$ is a complete residue system modulo n, and this is the case if and only if $\gcd(a - c, n) = 1$. □

The corollary produces a hitherto undiscovered infinite family of seaweed algebras. For example, the Lie algebra $\mathfrak{p}(2, 3 \mid 4, 1)$ is Frobenius since $2 - 4 = -2$ is relatively prime to 7.

At this time, the above line of reasoning does not easily extend to compositions $\underline{x}$ and $\underline{y}$ with more than two components. However, some calculations offer hope of producing more families of Frobenius Lie algebras using methods similar to those above.

4.4. **Submaximal parabolic algebras.** We conclude with another new family of Frobenius algebras. These are of the form $\mathfrak{p}(a, b, c \mid n)$, so they are parabolic algebras omitting exactly two simple roots. We use a different technique than for maximal or opposite maximal algebras to analyze this family. Our result is the following classification theorem.

Theorem 4.10. *The submaximal parabolic algebra* $\mathfrak{p}(a, b, c \mid n)$ *is Frobenius if and only if* $\gcd(a + b, b + c) = 1$.

We first establish some conditions on the degrees of the vertices $\{v_1, v_2, \ldots, v_n\}$ of the meander $M = M(a, b, c \mid n)$. Since the vertices of M are viewed as the numbers $\{1, 2, \ldots, n\}$ on a line the *interval* between vertices v_i and v_{i+1} makes sense.

Lemma 4.11. *Suppose* $\gcd(a + b, b + c) = 1$. *Then there are exactly two vertices of degree* 1 *in* M *and all other vertices have degree* 2.

Proof. Suppose for a moment that there exists a vertex v of degree 0. This vertex must have no bottom edge, meaning that n is odd and $v = v_{(n+1)/2}$. We also know v has no top edge so b is odd and v is halfway between v_{a+1} and v_{a+b}. This implies that $a = c$ so $a+b = b+c$, a contradiction. Hence, we get exactly one vertex of degree 1 for each integer in $\{a, b, c, n\}$ which is odd.

If n is odd, the vertex $v_{(n+1)/2}$ has degree 1. If all three of a, b and c are odd, then $a + b$ and $b + c$ are both even, meaning they have a common factor of 2, a contradiction. This implies that exactly one of a, b or c must be odd. Then there is exactly one other vertex of degree one as desired.

If n is even, the bottom edges form a perfect matching. If all three of a, b and c are even, then $a+b$ and $b+c$ are again even, a contradiction. This implies that exactly two of a, b or c are odd, meaning there are two vertices of degree 1 as desired. $\square$

By Lemma 4.11, one component of M must be a path and there are possibly more components which are all cycles. Let P be this path and suppose P has $a' \leq a$ vertices in the first part of the partition, and $b' \leq b$ and $c' \leq c$ vertices in the other parts respectively. Note that one of a', b' or c' may be zero. Label the vertices of P with $u_1, u_2, \ldots, u_{n'}$ where $n' = |P|$ following the inherited order (the order of the labels v_i) of the vertices. Notice that the path P forms a meander graph on its own. This means that, by the proof of Lemma 4.11, we know that exactly two of the integers in $\{a', b', c', n'\}$ are even and two are odd.

Now suppose there exists at least one component of M that is a cycle. Let C be the set of all vertices in cycles of M. Suppose C has d vertices in the interval between u_i and u_{i+1}. For the moment, let us suppose that $i \neq \frac{n'}{2}$. Following the bottom edges, this means that C must also have d vertices in the interval between $u_{n'-i}$ and $u_{n'-i+1}$. Using this argument, we will show that C has d vertices in almost every interval.

Define a *dead end* in M to be an interval u_i to u_{i+1} such that M contains an edge joining u_i and u_{i+1}. In particular, if n' is even, then the interval between $u_{n'/2}$ and $u_{n'/2+1}$ is a dead end.

Lemma 4.12. *Suppose* $\gcd(a + b, b + c) = 1$*. Then there are exactly two dead ends in* M*.*

Proof. A dead end is formed by two consecutive vertices of P which are adjacent. Each occurrence of a dead end coincides with one of a', b', c' or n' being even, and we know that exactly two of these are even. Thus, there are exactly two dead ends and the proof is complete. $\square$

Proof of Theorem 4.10. Call an interval a *partition interval* if it is the meeting point of two parts of our partition. Namely, the partition intervals are from $u_{a'}$ to $u_{a'+1}$ and from $u_{a'+b'}$ to $u_{a'+b'+1}$. Now suppose C

has d vertices in the interval from u_i to u_{i+1}. For the moment we suppose this interval is not a dead end. As mentioned before, this means that, by following bottom edges, C must also have d vertices in the interval from $u_{n'-i}$ to $u_{n'-i+1}$. Also, by following top edges, C must have d vertices in another interval (depending where the top edges go).

If the interval from u_i to u_{i+1} happens to be one of the two partition intervals (for example suppose $i = a'$) then this means C must have d_1 vertices in the interval outside u_1 and at least d_2 vertices in the interval from $u_{a'+b'}$ to $u_{a'+b'+1}$ where $d_1+d_2 = d$. This then implies that C has d_1 vertices in the interval beyond u_n (following bottom edges) and another d_1 vertices in the interval from $u_{a'+b'}$ to $u_{a'+b'+1}$ (following top edges) for a total of d vertices in the interval $u_{a'+b'}$ to $u_{a'+b'+1}$. See Figure 4 for an example. In this figure, the dark lines represent the edges of P while light lines represent edges of C. The unlabeled light lines represent d edges each. Here $n' = 7$, $a' = 2$, $b' = 2$ and $c' = 3$.

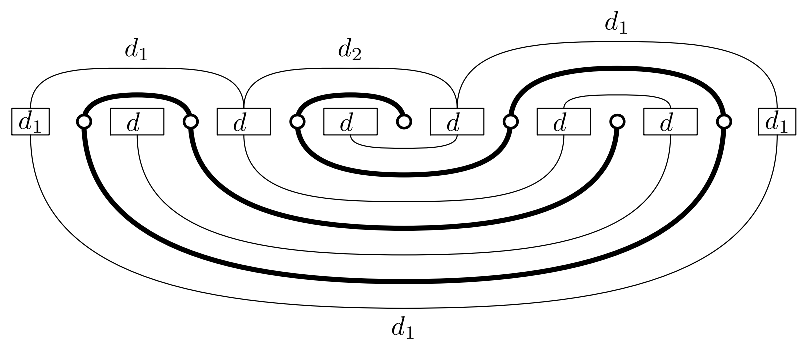

FIGURE 4. $M(2, 2, 3 \mid 7)$ with inserted cycle.

Alternating following top and bottom edges, we see that the cycle C has exactly d vertices in every interval between vertices and possibly $d_1 \leq d$ vertices on each end beyond u_1 and beyond $u_{n'}$. Carefully counting, we see that the first part of our partition has $a = a' + 2d_1 + (a'-1)d$ vertices. Similarly, the second part has $b = b' + 2d_2 + (b' - 1)d$ and the third part has $c = c' + 2d_1 + (c' - 1)d$. This means that $a + b = (a' + b')(d + 1)$ and $b + c = (b' + c')(d + 1)$ and these have a common factor of $d + 1$, a contradiction. This shows that C must be empty so G is simply the path P. $\square$

The following is an example to show that this argument does not work when we break n into more pieces. Consider the meander $M = M(3, 2, 2, 2 \mid 9)$ pictured in Figure 5.

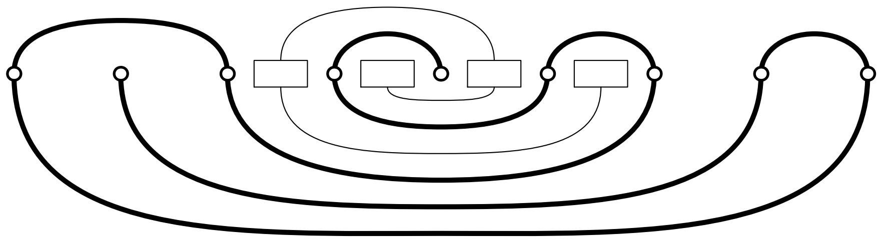

FIGURE 5. $M(3,2,2,2 \mid 9)$ with inserted cycle.

Here we have broken the top into 4 pieces while leaving the bottom in one piece. Notice that we can add a cycle to this structure which does not pass through all the intervals. This happens because, as the number of pieces we have increases, the number of dead ends also increases, allowing more flexibility in the placement of the cycles.

The above illustrates the complexity of the meander graphs $M(\underline{x} \mid \underline{y})$ as the number of parts of $\underline{x}$ and $\underline{y}$ grow. At the moment, the problem of classifying all Frobenius seaweed Lie algebras seems to be out of reach. Of late, there has been a great deal of interest in Frobenius Lie algebras. Perhaps these recent developments will be instrumental in the development of a classification theory.

REFERENCES

[1] A. A. Belavin and V. G. Drinfel'd. Solutions of the classical Yang–Baxter equations for simple Lie algebras. *Funct. Anal. Appl.*, 16:159–180, 1982.

[2] V. Dergachev and A. Kirillov. Index of Lie algebras of seaweed type. *J. Lie Theory*, 10:331–343, 2000.

[3] A. G. Elashvili. On the index of parabolic subalgebras of semisimple Lie algebras. preprint.

[4] A. G. Elashvili. Frobenius Lie algebras. *Funktsional. Anal. i Prilozhen.*, 16:94–95, 1982.

[5] M. Gerstenhaber and A. Giaquinto. Boundary solutions of the classical Yang-Baxter equation. *Letters Math. Phys.*, 40:337–353, 1997.

[6] M. Gerstenhaber and A. Giaquinto. Graphs, Frobenius functionals, and the classical Yang-Baxter equation. arXiv:0808.2423v1 [math.QA], August, 2008.

[7] A. Joseph. On semi-invariants and index for biparabolic (seaweed)algebras, I. *J. of Algebra*, 305:487–515, 2006.

[8] A. I. Ooms. On Lie algebras having a primitive universal enveloping algebra. *J. Algebra*, 32:488–500, 1974.

[9] D. Panyushev. Inductive formulas for the index of seaweed Lie algebras. *Mosc. Math. Journal*, 2:2001, 221-241.

[10] P. Tauvel and R. Tu. Sur l'indice de certaines algebres de Lie. *Ann. Inst. Fourier (Grenoble)*, 54:2004, 1793-1810.

Department of Mathematics, Lehigh University, Bethlehem, PA 18015

E-mail address: vec208@lehigh.edu

Department of Mathematics and Statistics, Loyola University Chicago, Chicago, Illinois 60626 USA

E-mail address: tonyg@math.luc.edu

Department of Mathematics, Lehigh University, Bethlehem, PA 18015

E-mail address: dr.colton.magnant@gmail.com