ON DYNAMICAL SYSTEMS AND PHASE TRANSITIONS FOR $Q+1$-STATE P-ADIC POTTS MODEL ON THE CAYLEY TREE

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Abstract. In the present paper, we introduce a new kind of p-adic measures for $q+1$-state Potts model, called *p-adic quasi Gibbs measure*. For such a model, we derive a recursive relations with respect to boundary conditions. Note that we consider two mode of interactions: ferromagnetic and antiferromagnetic. In both cases, we investigate a phase transition phenomena from the associated dynamical system point of view. Namely, using the derived recursive relations we define one dimensional fractional p-adic dynamical system. In ferromagnetic case, we establish that if q is divisible by p, then such a dynamical system has two repelling and one attractive fixed points. We find basin of attraction of the fixed point. This allows us to describe all solutions of the nonlinear recursive equations. Moreover, in that case there exists the strong phase transition. If q is not divisible by p, then the fixed points are neutral, and this yields that the existence of the quasi phase transition. In antiferromagnetic case, there are two attractive fixed points, and we find basins of attraction of both fixed points, and describe solutions of the nonlinear recursive equation. In this case, we prove the existence of a quasi phase transition.

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1. introduction

Due to the assumption that p-adic numbers provide a more exact and more adequate description of microworld phenomena, starting the 1980s, various models described in the language of p-adic analysis have been actively studied [8],[21],[47],[63]. The well-known studies in this area are primarily devoted to investigating quantum mechanics models using equations of mathematical physics [7, 64, 62]. Furthermore, numerous applications of the p-adic analysis to mathematical physics have been proposed in [11],[32],[33]. One of the first applications of p-adic numbers in quantum physics appeared in the framework of quantum logic in [12]. This model is especially interesting for us because it could not be described by using conventional real valued probability. Besides, it is also known [33, 41, 47, 55, 61, 62] that a number of p-adic models in physics cannot be described using ordinary Kolmogorov's probability theory. New probability models, namely p-adic ones were investigated in [14],[30],[39]. After that in [40] an abstract p-adic probability theory was developed by means of the theory of non-Archimedean measures [55]. Using that measure theory in [37],[46] the theory of stochastic processes with values in p-adic and more general non-Archimedean fields having probability distributions with non-Archimedean values has been developed. In particular, a non-Archimedean analog of the Kolmogorov theorem was proven (see also [22]). Such a result allows us to construct wide classes of stochastic processes using finite dimensional probability distributions[1]. Therefore,

[1]We point out that stochastic processes on the field $\mathbb{Q}_p$ of p-adic numbers with values of real numbers have been studied by many authors, for example, [2, 3, 4, 15, 42, 65]. In those investigations wide classes of Markov

this result give us a possibility to develop the theory of statistical mechanics in the context of the p-adic theory, since it lies on the basis of the theory of probability and stochastic processes. Note that one of the central problems of such a theory is the study of infinite-volume Gibbs measures corresponding to a given Hamiltonian, and a description of the set of such measures. In most cases such an analysis depend on a specific properties of Hamiltonian, and complete description is often a difficult problem. This problem, in particular, relates to a phase transition of the model (see [23]).

In [34, 35] a notion of ultrametric Markovianity, which describes independence of contributions to random field from different ultrametric balls, has been introduced, and shows that Gaussian random fields on general ultrametric spaces (which were related with hierarchical trees), which were defined as a solution of pseudodifferential stochastic equation (see also [25]), satisfies the Markovianity. In addition, covariation of the defined random field was computed with the help of wavelet analysis on ultrametric spaces (see also [43]). Some applications of the results to replica matrices, related to general ultrametric spaces have been investigated in [36].

The aim of this paper is devoted to the development of p-adic probability theory approaches to study $q+1$-state nearest-neighbor p-adic Potts model on Cayley tree (see [67]). We are especially interested in the construction of p-adic quasi Gibbs measures for the mentioned model, since such measures present more natural concrete examples of p-adic Markov processes (see [37], for definitions). In [50, 51] we have studied p-adic Gibbs measures and existence of phase transitions for the q-state Potts models on the Cayley tree[2]. It was established that a phase transition occurs [3] if q is divisible by p. This shows that the transition depends on the number of spins q.

To investigate phase transitions, a dynamical system approach, in real case, has greatly enhanced our understanding of complex properties of models. The interplay of statistical mechanics with chaos theory has even led to novel conceptual frameworks in different physical settings [18]. On the other hand, the theory p-adic dynamical systems is a rapidly growing topic, there are many papers devoted to this subject (see for example, [39],[57]). We remark that first investigations of non-Archimedean dynamical systems have appeared in [24]. We also point out that intensive development of p-adic (and more general algebraic) dynamical systems has happened few years, (for example, see [1, 9, 10, 13, 19, 20, 27, 53, 60, 66]). More extensive lists may be found in the p-adic dynamics bibliography maintained by Silverman [58] and the algebraic dynamics bibliography of Vivaldi [61].

In the present paper, we are going to investigate a phase transition phenomena from the such a dynamical system point of view. In the paper we introduce a new class of p-adic measures, associated with $q+1$-state Potts model, called *p-adic quasi Gibbs measure*. Note such a class is totaly different from the p-adic Gibbs measures considered in [50, 51]. For the model under consideration, we derive a recursive relations with respect to boundary conditions. Note that we shall consider two mode of interactions: ferromagnetic and antiferromagnetic. Namely, using the derived recursive relations we define one dimensional fractional p-adic dynamical system. In both cases, we are going to investigate a phase transition phenomena from the associated

processes on $\mathbb{Q}_p$ were constructed and studied. In our case the situation is different, since probability measures take their values in $\mathbb{Q}_p$. This leads our investigation to some difficulties. For example, there is no information about the compactness of p-adic values probability measures.

[2]The classical (real value) counterparts of such models were considered in [67]

[3]Here the phase transition means the existence of two distinct p-adic Gibbs measures for the given model.

dynamical system point of view. In ferromagnetic case, we establish that if q is divisible by p, then such a dynamical system has two repelling and one attractive fixed points. We find basin of attraction of the fixed point. This allows us to describe all solutions of the nonlinear recursive equations. Moreover, in that case there exists the strong phase transition. If q is not divisible by p, then the fixed points are neutral, and this yields that the existence of the quasi phase transition. In antiferromagnetic case, there are two attractive fixed points, and we find basins of attraction of both fixed points, and describe solutions of the nonlinear recursive equation. In this case, we prove the existence of a quasi phase transition. Note that the obtained results are totaly different from the results of [50, 51], since when q is divisible by p means that $q+1$ is not divided by p, which according to [50] means that uniqueness and boundedness of p-adic Gibbs measure.

2. Preliminaries

2.1. ***p*-adic numbers.** In what follows p will be a fixed prime number, and $\mathbb{Q}_p$ denotes the field of p-adic filed, formed by completing $\mathbb{Q}$ with respect to the unique absolute value satisfying $|p|_p = 1/p$. The absolute value $|\cdot|_p$, is non- Archimedean, meaning that it satisfies the ultrametric triangle inequality $|x+y|_p \leq \max\{|x|_p, |y|_p\}$.

Any p-adic number $x \in \mathbb{Q}_p$, $x \neq 0$ can be uniquely represented in the form

$$x = p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + ...), \tag{2.1}$$

where $\gamma = \gamma(x) \in \mathbb{Z}$ and x_j are integers, $0 \leq x_j \leq p-1$, $x_0 > 0$, $j = 0, 1, 2, \dots$ In this case $|x|_p = p^{-\gamma(x)}$.

We recall that an integer $a \in \mathbb{Z}$ is called *a quadratic residue modulo* p if the equation $x^2 \equiv a (\mathrm{mod}\ p)$ has a solution $x \in \mathbb{Z}$.

Lemma 2.1. [41] *In order that the equation*

$$x^2 = a, \quad 0 \neq a = p^{\gamma(a)}(a_0 + a_1 p + ...), \quad 0 \leq a_j \leq p-1, \ a_0 > 0$$

has a solution $x \in \mathbb{Q}_p$, it is necessary and sufficient that the following conditions are fulfilled:

(i) *$\gamma(a)$ is even;*

(ii) *a_0 is a quadratic residue modulo p if $p \neq 2$, and moreover $a_1 = a_2 = 0$ if $p = 2$.*

Note the basics of p-adic analysis, p-adic mathematical physics are explained in [41, 48, 56, 55, 62].

2.2. **Dynamical systems in $\mathbb{Q}_p$.** In this subsection we recall some standard terminology of the theory of dynamical systems (see for example [52],[39]).

Given $r, s > 0$ $(r < s)$ and $a \in \mathbb{Q}_p$ denote

$$B_r(a) = \{x \in \mathbb{Q}_p : \ |x-a|_p < r\}, \quad \bar{B}_r(a) = \{x \in \mathbb{Q}_p : \ |x-a|_p \leq r\} \tag{2.2}$$

$$B_{r,s}(a) = \{x \in \mathbb{Q}_p : \ r < |x-a|_p < s\}, \quad S_r(a) = \{x \in \mathbb{Q}_p : \ |x-a|_p = r\}. \tag{2.3}$$

It is clear that $\bar{B}_r(a) = B_r(a) \cup S_r(a)$.

A function $f : B_r(a) \to \mathbb{Q}_p$ is said to be *analytic* if it can be represented by

$$f(x) = \sum_{n=0}^{\infty} f_n (x-a)^n, \quad f_n \in \mathbb{Q}_p,$$

which converges uniformly on the ball $B_r(a)$.

Consider a dynamical system (f, B) in $\mathbb{Q}_p$, where $f : x \in B \to f(x) \in B$ is an analytic function and $B = B_r(a)$ or $\mathbb{Q}_p$. Denote $x^{(n)} = f^n(x^{(0)})$, where $x^0 \in B$ and $f^n(x) = \underbrace{f \circ \cdots \circ f(x)}_{n}$.

If $f(x^{(0)}) = x^{(0)}$ then $x^{(0)}$ is called a *fixed point*. A fixed point $x^{(0)}$ is called an *attractor* if there exists a neighborhood $U(x^{(0)})(\subset B)$ of $x^{(0)}$ such that for all points $y \in U(x^{(0)})$ it holds $\lim_{n\to\infty} y^{(n)} = x^{(0)}$, where $y^{(n)} = f^n(y)$. If $x^{(0)}$ is an attractor then its *basin of attraction* is

$$A(x^{(0)}) = \{y \in \mathbb{Q}_p : \ y^{(n)} \to x^{(0)}, \ n \to \infty\}.$$

A fixed point $x^{(0)}$ is called *repeller* if there exists a neighborhood $U(x^{(0)})$ of $x^{(0)}$ such that $|f(x) - x^{(0)}|_p > |x - x^{(0)}|_p$ for $x \in U(x^{(0)})$, $x \neq x^{(0)}$. For a fixed point $x^{(0)}$ of a function $f(x)$ a ball $B_r(x^{(0)})$ (contained in B) is said to be a *Siegel disc* if each sphere $S_\rho(x^{(0)})$, $\rho < r$ is an invariant sphere of $f(x)$, i.e. if $x \in S_\rho(x^{(0)})$ then all iterated points $x^{(n)} \in S_\rho(x^{(0)})$ for all $n = 1, 2 \dots$. The union of all Siegel discs with the center at $x^{(0)}$ is said to *a maximum Siegel disc* and is denoted by $SI(x^{(0)})$.

Remark 2.1. In non-Archimedean geometry, a center of a disc is nothing but a point which belongs to the disc, therefore, in principle, different fixed points may have the same Siegel disc (see [10]).

Let $x^{(0)}$ be a fixed point of an analytic function $f(x)$. Set

$$\lambda = \frac{d}{dx} f(x^{(0)}).$$

The point $x^{(0)}$ is called *attractive* if $0 \leq |\lambda|_p < 1$, *indifferent* if $|\lambda|_p = 1$, and *repelling* if $|\lambda|_p > 1$.

2.3. ***p*-adic measure.** Let $(X, \mathcal{B})$ be a measurable space, where $\mathcal{B}$ is an algebra of subsets X. A function $\mu : \mathcal{B} \to \mathbb{Q}_p$ is said to be a *p-adic measure* if for any $A_1, \dots, A_n \subset \mathcal{B}$ such that $A_i \cap A_j = \emptyset$ $(i \neq j)$ the equality holds

$$\mu\bigg(\bigcup_{j=1}^{n} A_j\bigg) = \sum_{j=1}^{n} \mu(A_j).$$

A p-adic measure is called a *probability measure* if $\mu(X) = 1$. A p-adic probability measure μ is called *bounded* if $\sup\{|\mu(A)|_p : A \in \mathcal{B}\} < \infty$. Note that in general, a p-adic probability measure need not be bounded [30, 37, 41]. For more detail information about p-adic measures we refer to [30],[39],[55].

2.4. **Cayley tree.** Let $\Gamma^k_+ = (L, E)$ be a semi-infinite Cayley tree of order $k \geq 1$ with the root x^0 (whose each vertex has exactly $k+1$ edges, except for the root x^0, which has k edges). Here L is the set of vertices and E is the set of edges. The vertices x and y are called *nearest neighbors* and they are denoted by $l = < x, y >$ if there exists an edge connecting them. A collection of the pairs $< x, x_1 >, \dots, < x_{d-1}, y >$ is called a *path* from the point x to the point y. The distance $d(x, y), x, y \in V$, on the Cayley tree, is the length of the shortest path from x to y.

Recall a coordinate structure in Γ^k_+: every vertex x (except for x^0) of Γ^k_+ has coordinates $(i_1, \dots, i_n)$, here $i_m \in \{1, \dots, k\}$, $1 \leq m \leq n$ and for the vertex x^0 we put (0). Namely, the

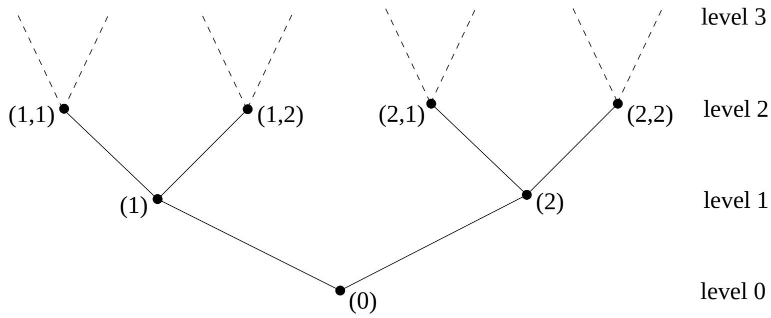

FIGURE 1. The first levels of Γ^2_+

symbol (0) constitutes level 0, and the sites $(i_1,\dots,i_n)$ form level n (i.e. $d(x^0,x)=n$) of the lattice.

Let us set

$$W_n=\{x\in V|d(x,x^0)=n\},\quad V_n=\bigcup_{m=1}^{n}W_m,\quad L_n=\{l=<x,y>\in L|x,y\in V_n\}.$$

For $x\in\Gamma^k_+$, $x=(i_1,\dots,i_n)$ denote

$$S(x)=\{(x,i):\ 1\le i\le k\}, \tag{2.4}$$

here (x,i) means that $(i_1,\dots,i_n,i)$. This set is called a set of *direct successors* of x.

3. p-ADIC POTTS MODEL AND ITS p-ADIC QUASI GIBBS MEASURES

In this section we consider the p-adic Potts model where spin takes values in the set $\Phi=\{0,1,2,\cdots,q\}$, here $q\ge 1$, (Φ is called a *state space*) and is assigned to the vertices of the tree $\Gamma^k=(V,\Lambda)$. A configuration σ on V is then defined as a function $x\in V\to\sigma(x)\in\Phi$; in a similar manner one defines configurations σ_n and ω on V_n and W_n, respectively. The set of all configurations on V (resp. V_n, W_n) coincides with $\Omega=\Phi^V$ (resp. $\Omega_{V_n}=\Phi^{V_n}$, $\Omega_{W_n}=\Phi^{W_n}$). One can see that $\Omega_{V_n}=\Omega_{V_{n-1}}\times\Omega_{W_n}$. Using this, for given configurations $\sigma_{n-1}\in\Omega_{V_{n-1}}$ and $\omega\in\Omega_{W_n}$ we define their concatenations by

$$(\sigma_{n-1}\vee\omega)(x)=\begin{cases}\sigma_{n-1}(x), & \text{if } x\in V_{n-1},\\ \omega(x), & \text{if } x\in W_n.\end{cases}$$

It is clear that $\sigma_{n-1}\vee\omega\in\Omega_{V_n}$.

The Hamiltonian $H_n:\Omega_{V_n}\to\mathbb{Q}_p$ of the p-adic $q+1$-*state Potts* model has a form

$$H_n(\sigma)=N\sum_{<x,y>\in L_n}\delta_{\sigma(x),\sigma(y)},\quad \sigma\in\Omega_{V_n},\ n\in\mathbb{N}, \tag{3.1}$$

where δ is the Kronecker symbol and the coupling constant N ($N\neq 0$), belongs to $\mathbb{Z}$. We call the model *ferromagnetic* if $N>0$, and *antiferromagnetic* if $N<0$.

Note that when $q=1$, then the corresponding model reduces to the p-adic Ising model. Such a model was investigated in [22, 26].

Now let us construct p-adic quasi Gibbs measures corresponding to the model.

Assume that $\mathbf{h}:V\setminus\{x^{(0)}\}\to\mathbb{Q}_p^\Phi$ is a function, i.e. $\mathbf{h}_x=(h_{0,x},h_{1,x},\dots,h_{q,x})$, where $h_{i,x}\in\mathbb{Q}_p$ ($i\in\Phi$) and $x\in V\setminus\{x^{(0)}\}$. Given $n\in\mathbb{N}$, let us consider a p-adic probability measure $\mu^{(n)}_{\mathbf{h}}$ on

Ω_{V_n} defined by

$$\mu_{\mathbf{h}}^{(n)}(\sigma)=\frac{1}{Z_n^{(\mathbf{h})}}p^{H_n(\sigma)}\prod_{x\in W_n}h_{\sigma(x),x} \tag{3.2}$$

Here, $\sigma\in\Omega_{V_n}$, and $Z_n^{(\mathbf{h})}$ is the corresponding normalizing factor called a *partition function* given by

$$Z_n^{(\mathbf{h})}=\sum_{\sigma\in\Omega_{V_n}}p^{H_n(\sigma)}\prod_{x\in W_n}h_{\sigma(x),x}, \tag{3.3}$$

here subscript n and superscript $(\mathbf{h})$ are accorded to the Z, since it depends on n and a function $\mathbf{h}$.

One of the central results of the theory of probability concerns a construction of an infinite volume distribution with given finite-dimensional distributions, which is called *Kolmogorov's Theorem* [59]. Therefore, in this paper we are interested in the same question but in a p-adic context. More exactly, we want to define a p-adic probability measure μ on Ω which is compatible with defined ones $\mu_{\mathbf{h}}^{(n)}$, i.e.

$$\mu(\sigma\in\Omega:\sigma|_{V_n}=\sigma_n)=\mu_{\mathbf{h}}^{(n)}(\sigma_n),\quad \text{for all}\ \ \sigma_n\in\Omega_{V_n},\ n\in\mathbb{N}. \tag{3.4}$$

In general, à priori the existence such a kind of measure μ is not known, since there is not much information on topological properties, such as compactness, of the set of all p-adic measures defined even on compact spaces[4]. Note that certain properties of the set of p-adic measures has been studied in [29], but those properties are not enough to prove the existence of the limiting measure. Therefore, at a moment, we can only use the p-adic Kolmogorov extension Theorem (see [22],[37]) which based on so called *compatibility condition* for the measures $\mu_{\mathbf{h}}^{(n)}$, $n\geq 1$, i.e.

$$\sum_{\omega\in\Omega_{W_n}}\mu_{\mathbf{h}}^{(n)}(\sigma_{n-1}\vee\omega)=\mu_{\mathbf{h}}^{(n-1)}(\sigma_{n-1}), \tag{3.5}$$

for any $\sigma_{n-1}\in\Omega_{V_{n-1}}$. This condition according to the theorem implies the existence of a unique p-adic measure μ defined on Ω with a required condition (3.4). Note that more general theory of p-adic measures has been developed in [28].

So, if for some function $\mathbf{h}$ the measures $\mu_{\mathbf{h}}^{(n)}$ satisfy the compatibility condition, then there is a unique p-adic probability measure, which we denote by $\mu_{\mathbf{h}}$, since it depends on $\mathbf{h}$. Such a measure $\mu_{\mathbf{h}}$ is said to be *a p-adic quasi Gibbs measure* corresponding to the p-adic Potts model. By $Q\mathcal{G}(H)$ we denote the set of all p-adic quasi Gibbs measures associated with functions $\mathbf{h}=\{\mathbf{h}_x,\ x\in V\}$. If there are at least two distinct p-adic quasi Gibbs measures $\mu,\nu\in Q\mathcal{G}(H)$ such that μ is bounded and ν is unbounded, then we say that *a phase transition* occurs. By another words, one can find two different functions $\mathbf{s}$ and $\mathbf{h}$ defined on $\mathbb{N}$ such that there exist the corresponding measures $\mu_{\mathbf{s}}$ and $\mu_{\mathbf{h}}$, for which one is bounded, another one is unbounded. Moreover, if there is a sequence of sets $\{A_n\}$ such that $A_n\in\Omega_{V_n}$ with $|\mu(A_n)|_p\to 0$ and $|\nu(A_n)|_p\to\infty$ as $n\to\infty$, then we say that there occurs a *strong phase transition*. If there are

[4] In the real case, when the state space is compact, then the existence follows from the compactness of the set of all probability measures (i.e. Prohorov's Theorem). When the state space is non-compact, then there is a Dobrushin's Theorem [16, 17] which gives a sufficient condition for the existence of the Gibbs measure for a large class of Hamiltonians.

two different functions $\mathbf{s}$ and $\mathbf{h}$ defined on $\mathbb{N}$ such that there exist the corresponding measures $\mu_{\mathbf{s}}$, $\mu_{\mathbf{h}}$, and they are bounded, then we say there is a *quasi phase transition*.

Remark 3.1. Note that in [50] we considered the following sequence of p-adic measures defined by

$$\mu_{\mathbf{h}}^{(n)}(\sigma) = \frac{1}{\tilde{Z}_n^{(\mathbf{h})}} \exp_p\{H_n(\sigma)\} \prod_{x\in W_n} h_{\sigma(x),x}, \tag{3.6}$$

here as usual $\tilde{Z}_n^{(\mathbf{h})}$ is the corresponding normalizing factor. A limiting p-adic measures generated by (3.6) was called *p-adic Gibbs measure*. Such kind of measures and phase transitions, for Ising and Potts models on Cayley tree, have been studied in [22, 26, 50, 51]. When a state space Φ is countable, the corresponding p-adic Gibbs measures have been investigated in [38, 49].

Now one can ask for what kind of functions $\mathbf{h}$ the measures $\mu_{\mathbf{h}}^{(n)}$ defined by (3.2) would satisfy the compatibility condition (3.5). The following theorem gives an answer to this question.

Theorem 3.1. *The measures $\mu_{\mathbf{h}}^{(n)}$, $n = 1, 2, \dots$ (see (3.2)) satisfy the compatibility condition (3.5) if and only if for any $n \in \mathbb{N}$ the following equation holds:*

$$\hat{h}_x = \prod_{y\in S(x)} \mathbf{F}(\hat{\mathbf{h}}_y; \theta), \tag{3.7}$$

here and below $\theta = p^N$, a vector $\hat{\mathbf{h}} = (\hat{h}_1, \dots, \hat{h}_q) \in \mathbb{Q}_p^q$ is defined by a vector $\mathbf{h} = (h_0, h_1, \dots, h_q) \in \mathbb{Q}_p^{q+1}$ as follows

$$\hat{h}_i = \frac{h_i}{h_0}, \quad i = 1, 2, \dots, q \tag{3.8}$$

and mapping $\mathbf{F} : \mathbb{Q}_p^q \times \mathbb{Q}_p \to \mathbb{Q}_p^q$ is defined by $\mathbf{F}(\mathbf{x}; \theta) = (F_1(\mathbf{x}; \theta), \dots, F_q(\mathbf{x}; \theta))$ with

$$F_i(\mathbf{x}; \theta) = \frac{(\theta - 1)x_i + \sum\limits_{j=1}^{q} x_j + 1}{\sum\limits_{j=1}^{q} x_j + \theta}, \quad \mathbf{x} = \{x_i\} \in \mathbb{Q}_p^q, \quad i = 1, 2, \dots, q. \tag{3.9}$$

The proof consists of checking condition (3.5) for the measures (3.2) (cp. [50, 38]).

Lemma 3.2. *Let $\mathbf{h}$ be a solution of (3.7), and $\mu_{\mathbf{h}}$ be an associated p-adic quasi Gibbs measure. Then for the corresponding partition function $Z_n^{(\mathbf{h})}$ (see (3.3)) the following equality holds*

$$Z_{n+1}^{(\mathbf{h})} = A_{\mathbf{h},n} Z_n^{(\mathbf{h})}, \tag{3.10}$$

where $A_{\mathbf{h},n}$ will be defined below (see (3.13)).

Proof. Since $\mathbf{h}$ is a solution of (3.7), then we conclude that there is a constant $a_{\mathbf{h}}(x) \in \mathbb{Q}_p$ such that

$$\prod_{y\in S(x)} \sum_{j=0}^{q} p^{N\delta_{ij}} h_{j,y} = a_{\mathbf{h}}(x) h_{i,x} \tag{3.11}$$

for any $i \in \{0, \dots, q\}$. From this one gets

$$\prod_{x\in W_n} \prod_{y\in S(x)} \sum_{j=0}^{q} p^{N\delta_{ij}} h_{j,y} = \prod_{x\in W_n} a_{\mathbf{h}}(x) h_{i,x} = A_{\mathbf{h},n} \prod_{x\in W_n} h_{i,x}, \tag{3.12}$$

where

$$A_{\mathbf{h},n}=\prod_{x\in W_n}a_{\mathbf{h}}(x). \tag{3.13}$$

Given $j\in\Phi$, by $\eta^{(j)}\in\Omega_{W_n}$ we denote a configuration on W_n defined as follows: $\eta^{(j)}(x)=j$ for all $x\in W_n$.

Hence, by (3.2),(3.12) we have

$$\begin{aligned}
1 &= \sum_{\sigma\in\Omega_n}\sum_{\omega\in\Omega_{W_n}}\mu^{(n+1)}_{\mathbf{h}}(\sigma\vee\omega)\\
&= \sum_{\sigma\in\Omega_n}\sum_{\omega\in\Omega_{W_n}}\frac{1}{Z^{(\mathbf{h})}_{n+1}}p^{H(\sigma\vee\omega)}\prod_{x\in W_{n+1}}h_{\omega(x),x}\\
&= \frac{1}{Z^{(\mathbf{h})}_{n+1}}\sum_{\sigma\in\Omega_n}p^{H(\sigma)}\prod_{x\in W_n}\prod_{y\in S(x)}\sum_{j=0}^{q}p^{N\delta_{\sigma(x),j}}h_{j,y}\\
&= \frac{A_{\mathbf{h},n}}{Z^{(\mathbf{h})}_{n+1}}\sum_{\sigma\in\Omega_n}p^{H(\sigma)}\prod_{x\in W_n}h_{\sigma(x),x}\\
&= \frac{A_{\mathbf{h},n}}{Z^{(\mathbf{h})}_{n+1}}Z^{(\mathbf{h})}_n
\end{aligned}$$

which implies the required relation. □

4. Dynamical systems and existence of p-adic quasi Gibbs measures

In this section we will establish existence of p-adic quasi Gibbs measures on a Cayley tree of order 2, i.e. $k=2$. To do it, we reduce the equation (3.7) to the fixed point problem for certain dynamical system. This allows us to investigate the existence of p-adic quasi Gibbs measure.

We say that a function $\mathbf{h}=\{\mathbf{h}_x\}_{x\in V\setminus\{x^0\}}$ is called *translation-invariant* if $\mathbf{h}_x=\mathbf{h}_y$ for all $x,y\in V\setminus\{x^0\}$. A p-adic measure $\mu_{\mathbf{h}}$, corresponding to a translation-invariant function $\mathbf{h}$, is called *translation-invariant p-adic quasi Gibbs measure*.

Let us first restrict ourselves to the description of translation-invariant solutions of (3.7), namely $\mathbf{h}_x=\mathbf{h}(=(h_0,h_1,\dots,h_q))$ for all $x\in V$. Then (3.7) can be rewritten as follows

$$\hat{h}_i=\left(\frac{(\theta-1)\hat{h}_i+\sum_{j=1}^{q}\hat{h}_j+1}{\sum_{j=1}^{q}\hat{h}_j+\theta}\right)^2,\quad i=1,2,\dots,q. \tag{4.1}$$

One can see that $(\underbrace{1,\dots,1,h}_{m},1,\dots,1)$ is an invariant line for (4.1) $(m=1,\dots,q)$. On such kind of invariant line equation (4.1) reduces to the following fixed point problem

$$x=f(x), \tag{4.2}$$

where

$$f(x)=\left(\frac{\theta x+q}{x+\theta+q-1}\right)^2. \tag{4.3}$$

A simple calculation shows that (4.2) has a form

$$(x-1)(x^2+(2\theta-\theta^2+2q-1)x+q^2)=0.$$

Hence, $x_0 = 1$ solution defines a p-adic quasi Gibbs measure μ_0.

Now we are interested in finding other solutions of (4.2), which means we need to solve the following one

$$x^2 + (2\theta - \theta^2 + 2q - 1)x + q^2 = 0. \tag{4.4}$$

Observe that the solution of (4.4) can be formally written by

$$x_{1,2} = \frac{-(2\theta - \theta^2 + 2q - 1) \pm (\theta - 1)\sqrt{D(\theta, q)}}{2}, \tag{4.5}$$

where $D(\theta, q) = \theta^2 - 2\theta - 4q + 1$

So, if the defined solutions exist in $\mathbb{Q}_p$, then they define p-adic quasi Gibbs measures μ_1 and μ_2, respectively. Note that to exist such solutions the expression $\sqrt{D(\theta, q)}$ should have a sense in $\mathbb{Q}_p$, since in $\mathbb{Q}_p$ not every quadratic equation has a solution (see Lemma 2.1). Therefore, we are going to check when $\sqrt{D(\theta, q)}$ does exist.

Now consider two distinct cases with respect to N.

4.1. **Ferromagnetic case.** In this case, we assume that $N > 0$, this means $|\theta|_p = p^{-N} < 1$. Now let us consider several cases with respect to q.

Case $q = 1$. Note that this case corresponds to the p-adic Ising model, and $D(\theta, 1) = \theta^2 - 2\theta - 3$.

(i) Let $p = 2$. Then from $-3 = 1 + 2^2 + 2^3 + \cdots$ one has

$$D(\theta, 1) = 1 + 2^2 + 2^3 + 2^4\epsilon - 2\theta + \theta^2,$$

where $\epsilon = 1 + 2 + 2^2 + \cdots$. Hence, due to Lemma 2.1 one can check that for any $N \geq 1$ the $\sqrt{D(\theta, 1)}$ does not exist.

(ii) Let $p = 3$. Then taking into account that $\theta = p^N$ we find

$$D(\theta, 1) = 3(3^{2N-1} - 2 \cdot 3^{N-1} - 1).$$

If $N = 1$ then $D(\theta, 1) = 0$, so $\sqrt{D(\theta, 1)}$ exists. If $N > 1$ then due to Lemma 2.1 we conclude that $\sqrt{D(\theta, 1)}$ does not exist.

(iii) Let $p \geq 5$. Then from the expression

$$-3 = p - 3 + (p - 1)p + (p - 1)p^2 + \cdots$$

we obtain

$$D(\theta, 1) = p - 3 + (p - 1)p\epsilon_1 - 2p^N + p^{2N},$$

where $\epsilon_1 = 1 + p + p^2 + \cdots$. So, according to Lemma 2.1 $\sqrt{D(\theta, 1)}$ exists if and only if the equation $x^2 \equiv p - 3 (\mathrm{mod}\ p)$ has a solution in $\mathbb{Z}$. It is easy to see that the last equation equivalent to $x^2 + 3 \equiv 0 (\mathrm{mod}\ p)$. For example, when $p = 7$ the equation $x^2 + 3 \equiv 0 (\mathrm{mod}\ p)$ has a solution $x = 2$. So, in this case $\sqrt{D(\theta, 1)}$ exists.

Hence, we can formulate the following

Theorem 4.1. *Let $N \geq 1$ and $q = 1$ (ferromagnetic Ising model). Then the following assertions hold true:*

(i) *If $p = 2$, then there is a unique translation-invariant p-adic quasi Gibbs measure μ_0;*

(ii) *Let $p=3$. If $N=1$, then there are three translation-invariant p-adic quasi Gibbs measures μ_0, μ_1 and μ_2, otherwise there is a unique translation-invariant p-adic quasi Gibbs measure μ_0;*

(iii) *Let $p\geq 5$, then there are three translation-invariant p-adic quasi Gibbs measures μ_0, μ_1 and μ_2 if and only if -3 is a quadratic residue of modulo p, otherwise there is a unique translation-invariant p-adic quasi Gibbs measure μ_0;*

Case $q\geq 2$. This case corresponds to $q+1$-state Potts model. Here we shall again consider several cases, but with respect to p.

(i) Let $p=2$. Let us represent q in a 2-adic form, i.e.

$$q=k_0+k_1 2+\cdots+k_s 2^s, \quad s\geq 1.$$

Then we have

$$-4q=2^2\big((2-k_0)+(1-k_1)2+\cdots+(1-k_s)2^s\big).$$

Therefore, one has

$$D(\theta,q)=1+2^2\big((2-k_0)+(1-k_1)2+\cdots+(1-k_s)2^s\big)-2^{N+1}+2^{2N}.$$

Now according to Lemma 2.1 we conclude that $\sqrt{D(\theta,q)}$ exists if and only if $k_0=0$, which is equivalent to $|q|_2\leq 1/2$.

(ii) Let $p=3$. We represent q in a 3-adic form, i.e.

$$q=k_0+k_1 3+\cdots+k_s 3^s, \quad s\geq 0.$$

Then we have

$$\begin{aligned} D(\theta,q) &= 1-q-q\cdot 3-2\cdot 3^N+3^{2N} \\ &= 1+(3-k_0)+(2-k_1)3+\cdots(2-k_s)3^s-q\cdot 3-2\cdot 3^N+3^{2N}. \end{aligned}$$

If $k_0=0$, then from Lemma 2.1 one can see that $\sqrt{D(\theta,1)}$ exists.

If $k_0=2$, then $\sqrt{D(\theta,q)}$ does not exists, since $x^2\equiv 2 (\mathrm{mod}\ 3)$ has no solution in $\mathbb{Z}$.

If $k_0=1$, then this case is more complicated. We cannot provide certain rule to check the existence of $\sqrt{D(\theta,q)}$. But in this case, $\sqrt{D(\theta,q)}$ may exist or may not. For example, if $k_1\neq 2$ then $\sqrt{D(\theta,q)}$ does not exist whenever $N\geq 3$. If $k_1=2$ and $k_2=2$ then $\sqrt{D(\theta,q)}$ exists whenever $N\geq 4$.

(iii) Let $p\geq 5$. Let us represent q in a p-adic expression

$$q=k_0+k_1 p+\cdots+k_s p^s, \quad s\geq 0.$$

Then we have

$$D(\theta,1)=1+4(p-k_0)+4(p-1-k_1)p+\cdots+4(p-1-k_s)p^s-2p^N+p^{2N}.$$

So, according to Lemma 2.1 $\sqrt{D(\theta,q)}$ exists if the equation $x^2\equiv 1-4k_0 (\mathrm{mod}\ p)$ has a solution in $\mathbb{Z}$ whenever $1-4k_0$ is not divided by p. It is clear that if $k_0=0$ then the equation has a solution for any value of p ($p\geq 5$). Note that if $1-4k_0$ is divided by p, then $\sqrt{D(\theta,q)}$ does not exist.

If $p=5$ and $k_0=3$, then one can check that $x^2\equiv -11 (\mathrm{mod}\ 5)$ has a solution $x=5n+2$. So, in this case $\sqrt{D(\theta,q)}$ exists.

So, we have the following

Theorem 4.2. *Let $N \geq 1$ and $q \geq 2$ (ferromagnetic Potts model). Then the following assertions hold true:*

(i) *If $|q|_p < 1$, then there are three translation-invariant p-adic quasi Gibbs measures μ_0, μ_1 and μ_2;*

(ii) *Let $p = 3$. If $|q-2|_p < 1$, then there is a unique translation-invariant p-adic quasi Gibbs measure μ_0; if $|q-1|_p < 1$ there is at least one translation-invariant p-adic quasi Gibbs measures μ_0;*

(iii) *Let $p \geq 5$ and $|4q-1|_p < 1$, then there is a unique translation-invariant p-adic quasi Gibbs measure μ_0;*

4.2. **Antiferromagnetic case.** Now suppose that $N < 0$. Denoting $\bar{N} = -N$, one has $\theta = p^{-\bar{N}}$. Therefore, $D(\theta, q)$ can be represented as follows:

$$D(\theta, q) = p^{-\bar{N}}\big(1 - 2p^{\bar{N}} - 4qp^{2\bar{N}} + p^{2\bar{N}}\big). \tag{4.6}$$

Hence, due to Lemma 2.1 we conclude that $\sqrt{D(\theta, q)}$ exists for all values of prime p, but $\bar{N}$ should be even.

Theorem 4.3. *Let $N < 0$ and $q \geq 1$ (antiferromagnetic Potts model). If $-N$ is even, than, for the model* (3.1)*, there are three translation-invariant p-adic quasi Gibbs measures μ_0, μ_1 and μ_2.*

5. Behavior of the dynamical system (4.3)

In this section we are going to investigate the dynamical system given by (4.3). In the previous section, we have established some conditions for the existence of its fixed points. In the sequel, we are going to describe possible attractors of the system, which allows us to find a relation between behavior of that dynamical system and the phase transitions. In what follows, for the sake of simplicity, we always assume that $p \geq 3$.

From (4.3) we easily find the following auxiliary facts:

$$f'(x) = \left(\frac{\theta x + q}{x + \theta + q - 1}\right)^2 \cdot \frac{2(\theta - 1)(\theta + q)}{(\theta x + q)(x + \theta + q - 1)}; \tag{5.1}$$

$$|f(x) - f(y)|_p = \frac{|\theta - 1|_p |\theta + q|_p |x - y|_p |\eta(\theta, q; x, y)|_p}{|x + \theta + q - 1|_p^2 |y + \theta + q - 1|_p^2}, \tag{5.2}$$

where

$$\eta(\theta, q; x, y) = A\theta(x + y) + 2\theta xy + 2qA + q(x + y), \tag{5.3}$$

here $A = \theta + q - 1$. Furthermore, we assume that $f(x)$ has three fixed points, the existence such points has been investigated in section 4. We denote them as follows x_0, x_1, x_2. Note that $x_0 = 1$. For the fixed points x_1 and x_2, from (4.4) we find that

$$x_1 + x_2 = -2q + 1 + \theta^2 - 2\theta, \quad x_1 \cdot x_2 = q^2. \tag{5.4}$$

To study dynamics of f we shall consider two different settings with respect to ferromagnetic and antiferrimagmetic ones.

5.1. **Ferromagnetic case.** In this setting, we suppose that $|\theta|_p < 1$, moreover $|\theta|_p \le |q|^2$ if $|q|_p < 1$.

Lemma 5.1. *Let x_1 and x_2 be the fixed points of $f(x)$. Then the followings hold true:*

$$|x_1|_p = |q|_p^2, \quad |x_2|_p = 1, \quad \textit{if} \;\; |q|_p < 1, \tag{5.5}$$

$$|x_1|_p = 1, \quad |x_2|_p = 1, \quad \textit{if} \;\; |q|_p = 1. \tag{5.6}$$

$$|\theta x_1 + q|_p = |q|_p, \quad |x_1 + \theta + q - 1|_p = 1, \quad \textit{if} \;\; |q|_p < 1, \tag{5.7}$$

$$|\theta x_2 + q|_p = |q|_p, \quad |x_2 + \theta + q - 1|_p = |q|_p, \quad \textit{if} \;\; |\theta|_p \le |q|_p^2, |q|_p < 1, \tag{5.8}$$

$$|\theta x_i + q|_p = 1, \quad |x_i + \theta + q - 1|_p = 1, \quad \textit{if} \;\; |q|_p = 1, \;\; i = 1, 2. \tag{5.9}$$

Proof. First assume that q is divided by p, i.e. $|q|_p \le 1/p$. Note that, in this case, according to Theorem 4.2 there exist the solutions x_1 and x_2. Hence, from (5.4) we conclude that $|x_1+x_2|_p = 1$ and $|x_1 \cdot x_2|_p = |q^2|_p$. From the last equalities, without loss of generality, it yields that (5.5). Hence, we immediately obtain (5.7). The equality (5.4) implies that

$$x_2 - 1 = \theta^2 - 2\theta - 2q - x_1.$$

This with the strong triangle inequality and (5.5) yields

$$\begin{aligned} &|x_2 + \theta + q - 1|_p = |\theta^2 - \theta - q - x_1|_p = |q|_p, \\ &|\theta x_2 + q|_p = |q|_p, \end{aligned}$$

if $|\theta|_p \le |q|_p^2$.

Now suppose that $|q|_p = 1$, and there exist solutions x_1 and x_2. Note that, in general, the solutions may not exist (see Theorems 4.1 and 4.2). Then from (5.4) we find that

$$|x_1 + x_2|_p \le 1, \tag{5.10}$$

$$|x_1 \cdot x_2|_p = 1. \tag{5.11}$$

In this case, one has $|x_1|_p = 1$, $|x_2|_p = 1$. Indeed, assume that $|x_1|_p < 1$, then the equality (5.11) yields $|x_2|_p > 1$. Due to the strong triangle inequality we get $|x_1 + x_2|_p > 1$ which contradicts to (5.10).

So, due to $|\theta|_p < 1$, we have $|\theta x_i + q|_p = 1$. On the other hand, we know that x_i $(i = 1, 2)$ are solutions (4.2), therefore, from (4.2) one gets

$$|x_i + \theta + q - 1|_p^2 = \frac{|\theta x_i + q|_p^2}{|x_i|_p} = 1.$$

This completes the proof. □

Let us find behavior of the fixed points. From (5.1) we find

$$|f'(x_0)|_p = \left|\frac{\theta - 1}{\theta + q}\right|_p = \begin{cases} 1, & \text{if } |q|_p = 1, \\ 1/|q|_p, & \text{if } |q|_p < 1. \end{cases} \tag{5.12}$$

Let us consider the other fixed points. Again from (5.1) one gets

$$|f'(x_i)|_p = \frac{|x_i|_p |\theta - 1|_p |\theta + q|_p}{|\theta x_i + q|_p |x_i + \theta + q - 1|_p}, \quad i = 1, 2. \tag{5.13}$$

Now taking into account (5.5)–(5.9) we derive

$$|f'(x_1)|_p = \begin{cases} 1, & \text{if } |q|_p = 1, \\ |q|_p^2, & \text{if } |q|_p < 1, \end{cases}$$

$$|f'(x_2)|_p = \begin{cases} 1, & \text{if } |q|_p = 1, \\ 1/|q|_p, & \text{if } |q|_p < 1, \end{cases}$$

Consequently, one has

Proposition 5.2. *Let $|\theta|_p < 1$ and assume that the dynamical system f given by (4.3) has three fixed points x_0,x_1, x_2. Then the following assertions hold true:*

(i) *if $|q|_1 = 1$, then the fixed points are neutral;*
(ii) *if $|q|_p < 1$ and $|\theta|_p \le |q|_p^2$, then x_1 is attractive, and x_0,x_2 are repelling.*

Furthermore, we concentrate ourselves to the case $|q|_p < 1$, which is more interesting.
For a given set $B \subset \mathbb{Q}_p$, let us denote

$$J(B) = \{x \in S_1(0) : \ f^n(x) \in B \ \text{ for some } n \ge 0\}. \tag{5.14}$$

Theorem 5.3. *Let $|q|_p < 1$, and $|\theta|_p \le |q|_p^2$. Then one has*

$$A(x_1) \supset \{x \in \mathbb{Q}_p : \ |x|_p \neq 1\} \cup \{x \in S_1(0) : \ |x-1|_p > |q|_p\} \cup J(B_{|q|_p^2,|q|_p}(x_0)) \cup J(B_{|q|_p^2,|q|_p}(x_2))$$

Proof. Let us consider several cases with respect to $|x|_p$.

(I) Assume that $x \in B_1(0)$, then one finds $|f(x)|_p = |q|_p^2 < 1$, hence $f(B_1(0)) \subset B_1(0)$.

Note that in the considered case we have $|A|_p = 1$, therefore for $x \in B_1(0)$ from (5.3) one immediately gets $|\eta(\theta, q; x, x_1)|_p = |q|_p$. So, (5.2),(5.7) with $|x + \theta + q - 1|_p = 1$ imply that

$$|f(x) - x_1|_p = |q|_p^2 |x - x_1|_p.$$

Hence, f is a contraction of $B_1(0)$, which means $f^n(x) \to x_1$ for every $x \in B_1(0)$, i.e. $B_1(0) \subset A(x_1)$.

Note that $\bar{B}_1(0) \subsetneq A(x_1)$, since $|x_0|_p = |x_2|_p = 1$, i.e. $S_1(0) \subsetneq A(x_1)$.

(II) Assume that $1 < |x|_p \le \frac{|q|_p}{|\theta|_p}$, then $|\theta x + q|_p \le |q|_p$, therefore one finds

$$|f(x)|_p = \left|\frac{\theta x + q}{x + \theta + q - 1}\right|^2 \le \left(\frac{|q|_p}{|x|_p}\right)^2 \le |q|^2 < 1.$$

(III) Now let $|x|_p > \frac{|q|_p}{|\theta|_p}$, then $|\theta x + q|_p = |\theta x|_p$, so we have

$$|f(x)|_p = \frac{|\theta x|_p^2}{|x|_p^2} = |\theta|^2 < 1.$$

Hence, from (II), (III) one concludes that $f(x) \in B_1(0)$, for any x with $|x|_p > 1$, which, due to (I), yields $x \in A(x_1)$.

Consequently, we infer that

$$\{x \in \mathbb{Q}_p : \ |x|_p \neq 1\} \subset A(x_1). \tag{5.15}$$

(IV) Now assume that $|x|_p = 1$, $|x-1|_p > |q|_p$. Then $|x+\theta+q-1|_p = |x-1|_p$, so one finds

$$|f(x)|_p = \frac{|q|_p^2}{|x-1|_p^2} < 1,$$

which, due to (I), implies $x \in A(x_1)$.

(V) Suppose that $|x-1|_p < |q|_p$. Then $|x+\theta+q-1|_p = |q|_p$, and from (5.3) we find $|\eta(\theta,q;x,1)|_p = |q|_p$. Consequently, (5.2) implies

$$|f(x)-1|_p = \frac{|x-1|_p}{|q|_p}. \tag{5.16}$$

Hence, if $|x-1|_p > |q|_p^2$, then $|f(x)-1|_p > |q|_p$, which, due (IV), means $x \in A(x_1)$.

(VI) Consider $J(B_{|q|_p^2,|q|_p}(x_0)$. One can see that $J(B_{|q|_p^2,|q|_p}(x_0) \subset A(x_1)$. Indeed, if $x \in J(B_{|q|_p^2,|q|_p}(x_0)$, then $|q|^2 < |f^{n_0}(x)-1|_p < |q|_p$ for some $n_0 \in \mathbb{N}$. From (5.16) we obtain $|f^{n_0+1}(x)-1|_p > |q|_p$, which with (V) yields $x \in A(x_1)$.

Now look to x_2. From (5.4) one finds

$$|x_2-1|_p = |q|_p, \quad |x_2-1+q|_p = |q|_p. \tag{5.17}$$

Note that the strong triangle inequality implies that $|x-x_2|_p > |q|_p$ if and only if $|x-1|_p > |q|_p$.

(VII) Therefore, assume that $|x-x_2|_p < |q|_p$, which implies that $|x-1|_p = |q|_p$. So, by means of (5.8),(5.17) from (5.3) we derive that $|\eta(\theta,q;x,x_2)|_p = |q|_p^2$. Hence, from (5.2) with (5.8) and $|x+\theta-1+q|_p = |q|_p$ one finds

$$|f(x)-x_2|_p = \frac{|x-x_2|_p}{|q|_p}. \tag{5.18}$$

Now using the same argument as in (V)-(VII) with (5.18) we obtain that $J(B_{|q|_p^2,|q|_p}(x_2)) \subset A(x_1)$. Note that the sets J_1 and J_2 are disjoint. This completes the proof. $\square$

Now we are going to investigate solutions of (3.7) over the invariant line $(1,1,\dots,h,1,\dots,1)$. Let us introduce some notations. If $x \in W_n$, then instead of h_x we use the symbol $h_x^{(n)}$.

Denote

$$g(x) = \frac{\theta x + q}{x+\theta+q-1}. \tag{5.19}$$

Note that $f(x) = (g(x))^2$. Then one can see that

$$|g(x)-g(x)|_p = \frac{|x-y|_p|\theta-1|_p|\theta+q|_p}{|x+\theta+q-1|_p|y+\theta+q-1|_p}, \tag{5.20}$$

$$g^{-1}(x) = \frac{(\theta+q-1)x-q}{\theta-x} \tag{5.21}$$

Moreover, one has the following

Lemma 5.4. *Let $|q|_p < 1$, and $|\theta|_p \le |q|_p^2$. The following assertions hold true:*

(i) *If $|x|_p \ne 1$, then $|g(x)|_p \le \max\{|q|_p, |\theta|_p\}$;*
(ii) *If $|g(x)|_p > 1$, then $|x|_p = 1$.*

Proof. (i). Let $|x|_p < 1$, then from (5.19) we get

$$|g(x)|_p = \left|\frac{\theta x + q}{x+\theta+q-1}\right|_p = |q|_p < 1.$$

Now assume $|x|_p > 1$, then analogously one finds

$$|g(x)|_p \begin{cases} = |\theta|_p, & \text{if } |x|_p > \frac{|q|_p}{|\theta|_p}, \\ \leq |q|_p, & \text{if } 1 < |x|_p \leq \frac{|q|_p}{|\theta|_p}. \end{cases}$$

(ii) Denoting $y = g(x)$, from (5.21) one finds

$$|x|_p = |g^{-1}(y)|_p = \left|\frac{(\theta+q-1)y-q}{\theta-y}\right|_p = \frac{|y|_p}{|y|_p} = 1.$$

□

Theorem 5.5. *Let $|q|_p < 1$, and $|\theta|_p \leq |q|_p^2$. Assume that $\{h_x\}_{x\in V\setminus\{(0)\}}$ is a solution of* (3.7) *such that $|h_x|_p \neq 1$ for all $x \in V \setminus \{(0)\}$. Then $h_x = x_1$ for every x.*

Proof. Let us first show that $|h_x|_p < 1$ for all x. Suppose that $|h_x^{(n_0)}|_p > 1$ for some $n_0 \in \mathbb{N}$ and $x \in W_{n_0}$. Since $\{h_x\}$ is a solution of (3.7), therefore, we have

$$h_x^{(n_0)} = g(h_{(x,1)}^{(n_0+1)})g(h_{(x,2)}^{(n_0+1)}), \tag{5.22}$$

here we have used coordinate structure of the tree.

Now according to $|h_{(x,1)}^{(n_0+1)}|_p \neq 1$, $|h_{(x,2)}^{(n_0+1)}|_p \neq 1$, then Lemma 5.4 (i) implies that $|g(h_{(x,1)}^{(n_0+1)})|_p < 1$, $|g(h_{(x,1)}^{(n_0+1)})|_p < 1$, which with (5.22) means $|h_x^{(n_0)}|_p < 1$. It is a contradiction.

Hence, $|h_x|_p < 1$ for all x. Then from (5.20) we obtain

$$|g(h_x) - g(x_1)|_p = |q|_p|h_x - x_1|_p \tag{5.23}$$

for any $x \in V \setminus \{(0)\}$.

Now denote

$$\|h^{(n)}\|_p = \max\{|h_x^{(n)}|_p : \ x \in W_n\}.$$

Let $\epsilon > 0$ be an arbitrary number. Then from the prof of Lemma 5.4 (i) with (5.23) one finds

$$\begin{aligned} |h_x^{(n)} - x_1|_p &= |g(h_{(x,1)}^{(n+1)})g(h_{(x,2)}^{(n+1)}) - (g(x_1))^2|_p \\ &= \left|g(h_{(x,1)}^{(n+1)})\big(g(h_{(x,2)}^{(n+1)}) - g(x_1)\big) + g(x_1)\big(g(h_{(x,2)}^{(n+1)}) - g(x_1)\big)\right|_p \\ &\leq \max\left\{|g(h_{(x,1)}^{(n+1)})|_p\big|g(h_{(x,2)}^{(n+1)}) - g(x_1)\big|_p, |g(x_1)|_p\big|g(h_{(x,2)}^{(n+1)}) - g(x_1)\big|_p\right\} \qquad (5.24) \\ &\leq |q|_p^2\max\left\{|h_{(x,1)}^{(n+1)} - x_1|_p, |h_{(x,2)}^{(n+1)} - x_1|_p\right\}. \end{aligned}$$

Thus, we derive

$$\|h^{(n)} - x_1\|_p \leq |q|_p^2\|h^{(n+1)} - x_1\|_p.$$

So, iterating the last inequality N times one gets

$$\|h^{(n)} - x_1\|_p \leq |q|_p^{2^N}\|h^{(n+N)} - x_1\|_p. \tag{5.25}$$

Choosing N such that $|q|_p^{2^N} < \epsilon$, from (5.25) we find $\|h^{(n)} - x_1\|_p < \epsilon$. Arbitrariness of ϵ yields that $h_x = x_1$. This completes the proof. □

5.2. **Antiferromagnetic case.** In this subsection we assume that $N < 0$, this means $|\theta|_p = p^{\bar{N}} > 1$, where $\bar{N} = -N$. In this setting equation (4.2) has three solutions x_0 (i.e. $x_0 = 1$), and x_1, x_2. Note that x_1 and x_2 are solutions of (4.4), therefore one gets

$$|x_1 + x_2|_p = |\theta|_p^2, \quad |x_1 \cdot x_2|_p = |q|_p^2 \tag{5.26}$$

$$|x_1|_p = |\theta|_p^2, \quad |x_2|_p = \left|\frac{q}{\theta}\right|_p^2. \tag{5.27}$$

Hence, we obtain

$$|\theta x_1 + q|_p = |\theta|_p^3, \quad |x_1 + \theta + q - 1|_p = |\theta|_p^2, \tag{5.28}$$

$$|\theta x_2 + q|_p = |q|_p, \quad |x_2 + \theta + q - 1|_p = |\theta|_p. \tag{5.29}$$

Proposition 5.6. *Assume that $|\theta|_p > 1$, then a fixed point x_0 is neutral, and the fixed points x_1, x_2 are attractive.*

Proof. From (5.1) we find

$$|f'(x_0)|_p = \left|\frac{\theta - 1}{\theta + q}\right|_p = 1,$$

this means that x_0 is neutral.

Let us consider the other fixed points. Now taking into account (5.13) with (5.26),(5.28),(5.29) one gets

$$|f'(x_1)|_p = \frac{1}{|\theta|_p} < 1, \quad |f'(x_2)|_p = \left|\frac{q}{\theta}\right|_p < 1$$

which is the required assertion. □

Lemma 5.7. *Let $|\theta|_p > 1$ and f be given by (4.3). Then the following assertions hold true:*

(i) *if $|x|_p > |\theta|_p$, then $|f(x)|_p = |\theta|_p^2$. Hence, $|f^n(x)|_p = |\theta|_p^2$ for all $n \in \mathbb{N}$;*
(ii) *if $\frac{|q|_p}{|\theta|_p} < |x|_p < |\theta|_p$, then $|f(x)|_p = |x|_p^2$;*
(iii) *if $|x| \leq \frac{|q|_p}{|\theta|_p}$, then $|f(x)|_p \leq \left(\frac{|q|_p}{|\theta|_p}\right)^2$.*

Proof. (i). Let $|x|_p > |\theta|_p$, then from (4.3) we find

$$|f(x)|_p = \left(\frac{|\theta x|_p}{|x|_p}\right)^2 = |\theta|_p^2.$$

(ii) Let $\frac{|q|_p}{|\theta|_p} < |x|_p < |\theta|_p$, then $|\theta + x + q - 1|_p = |\theta|_p$, $|\theta x|_p > |q|_p$ therefore, one gets

$$|f(x)|_p = \left(\frac{|\theta x|_p}{|\theta|_p}\right)^2 = |x|_p^2.$$

(iii) Let $|x| \leq \frac{|q|_p}{|\theta|_p}$, then $|\theta x|_p \leq |q|_p$, $|\theta + x + q - 1|_p = |\theta|_p$, hence one finds the required equality. □

Now we are going to examine attractors of x_1 and x_2.

Theorem 5.8. *Let $|\theta|_p > 1$ and $f(x)$ is given by (4.3). Then the following assertions holds true:*

(i) *$f(S_1(0)) \subset S_1(0)$;*

(ii) $A(x_2) = B_1(0)$;

(iii) $A(x_1) = \{x \in \mathbb{Q}_p : \ |x|_p > 1\} \setminus \bigcup_{n=0}^{\infty} f^{-n}(1-\theta-q)$.

Proof. (i) Let $|x|_p = 1$, then due to Lemma 5.7 (ii) we find that $|f(x)|_1 = 1$, which means $S_1(0)$ is invariant w.r.t. f.

Now consider (ii). First note that $|x_2|_p = (|q|_p/|\theta|_p)^2 < 1$. Now consider several cases w.r.t. $|x|_p$.

(I_1) Assume that $|x - x_2|_p < \big(\frac{|q|_p}{|\theta|_p}\big)^2$. Then $|x + \theta + q - 1|_p = |\theta|_p$ and $|x + x_2|_p = \big(\frac{|q|_p}{|\theta|_p}\big)^2$. Therefore, from (5.3) one gets that

$$|\eta(\theta, q; x, x_2)|_p = |\theta|_p^2\bigg(\frac{|q|_p}{|\theta|_p}\bigg)^2 = |q|_p^2.$$

Hence, the last equality with (5.2),(5.29) yields that

$$\begin{aligned} |f(x) - x_2|_p &= \frac{|\theta|_p^2|x - x_2|_p|\eta(\theta, q; x, x_2)|_p}{|x + \theta + q - 1|_p^2|\theta|_p^2} \\ &= \frac{|q|_p^2}{|\theta|_p^2}|x - x_2|_p. \end{aligned} \tag{5.30}$$

This means that f maps $B_{\frac{|q|_p^2}{|\theta|_p^2}}(x_2)$ into itself, and it is a contraction. So, for every $x \in B_{\frac{|q|_p^2}{|\theta|_p^2}}(x_2)$, one has $f^n(x) \to x_2$ as $n \to \infty$. Hence, $B_{\frac{|q|_p^2}{|\theta|_p^2}}(x_2) \subset A(x_2)$.

(II_1) Let $|x-x_2|_p = \frac{|q|^2}{|\theta|_p^2}$, i.e. $|x|_p \leq \frac{|q|^2}{|\theta|_p^2}$. Then from (5.3) we find that $|\eta(\theta, q; x, x_2)|_p = |q|_p|\theta|_p$. Hence, from (5.30) we derive

$$|f(x) - x_2|_p = \frac{|x - x_2|_p||q|_p|\theta|_p}{|\theta|_p^2} = \frac{|q|_p^3}{|\theta|_p^3},$$

which implies that $f(x) \in B_{\frac{|q|_p^2}{|\theta|_p^2}}(x_2)$, hence due to ($I_1$) one has $x \in A(x_2)$.

(III_1) Let $|x_2|_p < |x|_p \leq \frac{|q|_p}{|\theta|_p}$, then $|\eta(\theta, q; x, x_2)|_p \leq |q|_p|\theta|_p$, therefore using the same argument as (II_1) we obtain

$$|f(x) - x_2|_p \leq \frac{|q|_p^3}{|\theta|_p^3},$$

which with (I_1) yields $x \in A(x_2)$.

(IV_1) Let $\frac{|q|_p}{|\theta|_p} < |x|_p < 1$, then $|x - x_2|_p = |x|_p$ and $|\eta(\theta, q; x, x_2)|_p = |\theta|_p^2|x|_p$. It follows from (5.30) that

$$|f(x) - x_2|_p = \frac{|x - x_2|_p|\theta|_p^2|x|_p}{|\theta|_p^2} = |x|_p^2. \tag{5.31}$$

If $|x|_p^2 \leq \frac{|q|_p}{|\theta|_p}$, then $f(x)$ falls to (III_1) case, so $x \in A(x_2)$. If $|x|_p^2 > \frac{|q|_p}{|\theta|_p}$, then again repeating (5.31) one gets $|f^2(x) - x_2|_p = |x|_p^4$. Continuing this procedure, we conclude that in any case $f(x)$ falls to (III_1). Hence, $x \in A(x_2)$.

According (i) $S_1(0)$ is invariant w.r.t. f, hence $S_1(0) \cap A(x_2) = \emptyset$. Hence, (ii) is proved.

Now let us prove (iii). From (5.2) with (5.28) we easily obtain

$$|f(x)-x_1|_p = \frac{|\theta|_p^2|x-x_1|_p|\eta(\theta,q;x,x_1)|_p}{|x+\theta+q-1|_p^2|\theta|_p^4}, \tag{5.32}$$

where $\eta(\theta,q;x,x_1)$ is defined in (5.3).

(I$_2$) Let $x\in B_{|\theta|_p^2}(x_1)$ (i.e. $|x-x_1|_p<|\theta^2|_p$). Then $|x|_p=|x_1|_p=|\theta|_p^2$ and $|x+\theta+q-1|_p=|\theta|_p^2$. Therefore, from (5.3) one finds that $|\eta(\theta,q;x,x_1)|_p=|\theta|_p^5$. So, the last ones with (5.32) imply

$$|f(x)-x_1|_p = \frac{|x-x_1|_p}{|\theta|_p}, \tag{5.33}$$

this means that f is a contraction of $B_{|\theta|_p^2}(x_1)$, i.e. for every $x\in B_{|\theta|_p^2}(x_1)$ one has $f^n(x)\to x_1$ as $n\to\infty$. Hence, $B_{|\theta|_p^2}(x_1)\subset A(x_1)$. Note that $S_{|\theta|_p^2}(x_1)\not\subseteq A(x_1)$, since $x_2\in S_{|\theta|_p^2}(x_1)$.

(II$_2$) Let $|x|_p=|\theta|_p^2$, which implies $|x-x_1|_p\leq|\theta|_p^2$. Similarly reasoning as (I$_2$) we find $|\eta(\theta,q;x,x_1)|_p=|\theta|_p^5$, hence (5.33) holds. So, $f(x)\in B_{|\theta|_p^2}(x_1)$, therefore, due to (I$_2$), we get $x\in A(x_1)$.

(III$_2$) Let us assume that $|x-x_1|>|\theta|_p^2$, then $|x|_p>|\theta|_p^2$. This implies $|x-x_1|_p=|x|_p$. So, we have $|\eta(\theta,q;x,x_1)|_p=|\theta|_p^3|x|_p$ and $|x+\theta+q-1|_p=|x|_p$, hence from (5.32) one finds

$$|f(x)-x_1|_p=\frac{|x|_p|\theta|_p^3|x|_p}{|x|_p^2|\theta|_p^2}=|\theta|_p.$$

This implies that $f(x)\in B_{|\theta|_p^2}(x_1)$, which with (I$_2$) means

$$\{x\in\mathbb{Q}_p:\ |x|_p>|\theta|_p^2\}\subset A(x_1).$$

(IV$_2$) Let $|x|=|\theta|_p$ with $x\neq 1-\theta-q$. Denote $\gamma=x+\theta-q+1$, then $|\gamma|_p\leq|\theta|_p$ and $|\gamma|_p\neq 0$. From (4.3) one finds

$$|f(x)|_p=\frac{|\theta|_p^4}{|\gamma|_p^2}\geq|\theta|_p^2. \tag{5.34}$$

Hence due to (II$_2$) and (III$_2$) we conclude that $x\in A(x_1)$.

(V$_2$) Let $|x|_p>|\theta|_p$, then analogously from Lemma 5.7(i) one finds that $|f(x)|_p=|\theta|_p^2$, which, due to (II$_2$), yields that $x\in A(x_1)$.

(VI$_2$) Now assume that $1<|x|<|\theta|_p$. Then $|x+\theta+q-1|_p=|\theta|_p$, $|\eta(\theta,q;x,x_1)|_p=|\theta|_p^4$, hence it follows from (5.32) that

$$|f(x)-x_1|_p=\frac{|x-x_1|_p||\theta|_p^4}{|\theta|_p^2|\theta|_p^2}=|x-x_1|_p. \tag{5.35}$$

On the other hand, according to Lemma 5.7 (ii) we have $|f(x)|_p=|x|_p^2$.

(a) If $|x|^2=|\theta|_p$, and $f(x)\neq 1-\theta-q$, then due to (IV$_2$) one gets $x\in A(x_1)$. If $f(x)=1-\theta-q$, then $x\notin A(x_1)$.

(b) If $|x|^2>|\theta|_p$, then from (V$_2$) we get $|f^2(x)|_p=|\theta|_p^2$, hence $x\in A(x_1)$.

(c) If $|x|^2<|\theta|_p$, then repeating above made argument we find $|f^2(x)-x_1|_p=|x-x_1|_p$ and $|f^2(x)|_p=|x|_p^4$. Therefore, continuing above made procedure, we conclude there can occur either (a) or (b). Hence, $x\in A(x_1)$ if $x\notin\bigcup\limits_{n=0}^{\infty}f^{-n}(1-\theta-q)$. □

To prove our main result, we need the following auxiliary result.

Lemma 5.9. *Let $|\theta|_p > 1$ and $g(x)$ be a function given by (5.19). The following assertions hold true:*

(i) *if $|x|_p \le \frac{1}{|\theta|_p}$, then $|g(x)|_p \le \frac{1}{|\theta|_p}$;*
(ii) *if $\frac{1}{|\theta|_p} < |x|_p < 1$, then $|g(x)|_p = |x|_p$;*
(iii) *if $x, y \in B_1(0)$ then one has*

$$|g(x) - g(x)|_p = |x - y|_p; \tag{5.36}$$

(iv) *if $1 < |x|_p < |\theta|_p$, then $|g(x)|_p = |x|_p$;*
(v) *if $|x|_p \ge |\theta|_p$, then $|g(x)|_p \ge |\theta|_p$. Moreover, if $|x|_p > |\theta|_p$, then $|g(x)|_p = |\theta|_p$;*
(vi) *if $|x|_p, |y|_p \ge |\theta|_p^2$ then one has*

$$|g(x) - g(x)|_p \le \frac{1}{|\theta|_p^2}|x - y|_p; \tag{5.37}$$

Proof. From (5.19) we get

$$|g(x)|_p = \frac{|\theta x + q|_p}{|\theta|_p}\begin{cases} \le \frac{1}{|\theta|_p}, & \text{if } \ |x|_p \le \frac{1}{|\theta|_p}, \\ = |x|_p, & \text{if } \ \frac{1}{|\theta|_p} < |x|_p < 1, \end{cases}$$

which proves (i) and (ii).

The assertions (iii),(vi) immediately follows from (5.20).

From (5.19) we get

$$|g(x)|_p = \frac{|\theta x|_p}{|\theta + q - 1 + x|_p}\begin{cases} = |x|_p, & \text{if } \ 1 < |x|_p < |\theta|_p, \\ \ge |\theta|_p, & \text{if } \ |x|_p \ge |\theta|_p, \end{cases}$$

hence one gets (iv) and (v). □

Now we are going to describe solutions of (3.7).

Theorem 5.10. *Let $|\theta|_p > 0$ and assume that $\{h_x\}_{x\in V\setminus\{(0)\}}$ is a solution of (3.7). Then the following assertions hold ture:*

(i) *if $|h_x|_p < 1$ for all $x \in V \setminus \{(0)\}$, then $h_x = x_2$ for every x;*
(ii) *if $|h_x|_p > 1$ for all $x \in V \setminus \{(0)\}$, then $h_x = x_1$ for every x.*

Proof. Let us prove (i). First we establish that $|h_x|_p \le \frac{1}{|\theta|_p}$ for all $x \in V \setminus \{(0)\}$. Indeed, suppose $|h_x^{(n_0)}|_p > \frac{1}{|\theta|_p}$ for some $n_0 \in \mathbb{N}$. From (3.7) one has

$$|h_x^{(n_0)}|_p = |g(h_{(x,1)}^{(n_0+1)})|_p|g(h_{(x,2)}^{(n_0+1)})|_p. \tag{5.38}$$

Now consider some possible cases.

(a) if $|h_{(x,1)}^{(n_0+1)}|_p \le \frac{1}{|\theta|_p}$, but $|h_{(x,2)}^{(n_0+1)}|_p > \frac{1}{|\theta|_p}$, then due Lemma 5.9(i),(ii) from (5.38) we derive

$$\begin{aligned} \frac{1}{|\theta|_p} < |h_x^{(n_0)}|_p &= |g(h_{(x,1)}^{(n_0+1)})|_p|g(h_{(x,2)}^{(n_0+1)})|_p \\ &\le \frac{1}{|\theta|_p}|h_{(x,2)}^{(n_0+1)}|_p \\ &< \frac{1}{|\theta|_p} \ \ (\text{since } |h_{(x,2)}^{(n_0+1)}|_p < 1), \end{aligned} \tag{5.39}$$

but it is a contradiction.

(b) if $|h^{(n_0+1)}_{(x,2)}|_p \leq \frac{1}{|\theta|_p}$ and $|h^{(n_0+1)}_{(x,1)}|_p > \frac{1}{|\theta|_p}$ similarly as (a) we come to the contradiction.

(c) if $|h^{(n_0+1)}_{(x,1)}|_p \leq \frac{1}{|\theta|_p}$, and $|h^{(n_0+1)}_{(x,2)}|_p \leq \frac{1}{|\theta|_p}$, then again by the same argument one finds a contradiction.

Hence, one has $|h^{(n_0+1)}_{(x,1)}|_p > \frac{1}{|\theta|_p}$, $|h^{(n_0+1)}_{(x,2)}|_p > \frac{1}{|\theta|_p}$. Therefore, assume that $|h^{(n_0+k)}_{(x,i_1\cdots i_k)}|_p > \frac{1}{|\theta|_p}$ for every $k \geq 1$. Then, according to (3.7) with Lemma 5.9(ii) one gets

$$
\begin{aligned}
|h^{(n_0)}_x|_p &= \prod_{i_1=1,2} |h^{(n_0+1)}_{(x,i_1)}|_p \\
&= \prod_{i_1=1,2}\prod_{i_2=1,2} |h^{(n_0+2)}_{(x,i_1i_2)}|_p \\
&\dots \\
&= \prod_{i_1,\dots i_k=1,2} |h^{(n_0+k)}_{(x,i_1\cdots i_k)}|_p. \qquad (5.40)
\end{aligned}
$$

Denote

$$\gamma_k = \max_{i_1,\dots i_k} |h^{(n_0+k)}_{(x,i_1\cdots i_k)}|_p.$$

We know that $|h^{(n_0+k)}_{(x,i_1\cdots i_k)}|_p < 1$ for all $i_1,\dots i_k$, $k \geq 1$. Therefore, due to our assumption one has $\frac{1}{|\theta|_p} < |\gamma_k|_p < 1$ for every $k \in \mathbb{N}$. Hence, from (5.40) one finds

$$|h^{(n_0)}_x|_p \leq |\gamma_k|_p^{2^k}. \qquad (5.41)$$

Thus, when k is large enough, then $|\gamma_k|_p^{2^k} < \frac{1}{|\theta|_p}$. So, from (5.41) we obtain $|h^{(n_0)}_x|_p < \frac{1}{|\theta|_p}$, which is a contradiction.

Take an arbitrary $\epsilon > 0$. Then similarly to (5.24) one has

$$|h^{(n)}_x - x_2|_p \leq \max\left\{|g(h^{(n+1)}_{(x,1)})|_p\big|g(h^{(n+1)}_{(x,2)}) - g(x_1)\big|_p, |g(x_1)|_p\big|g(h^{(n+1)}_{(x,2)}) - g(x_1)\big|_p\right\}. \qquad (5.42)$$

According to $|h_x|_p \leq \frac{1}{|\theta|_p}$, for every x, with Lemma 5.9 (i),(iii) from (5.42) we derive

$$\|h^{(n)} - x_2\|_p \leq \frac{1}{|\theta|_p}\|h^{(n+1)} - x_2\|_p.$$

So, iterating the last inequality M times one gets

$$\|h^{(n)} - x_2\|_p \leq \frac{1}{|\theta|_p^M}\|h^{(n+M)} - x_1\|_p. \qquad (5.43)$$

Choosing M such that $|\theta|_p^{-M} < \epsilon$, from (5.43) we find $\|h^{(n)} - x_2\|_p < \epsilon$. Arbitrariness of ϵ yields that $h_x = x_2$.

Now consider (ii). Let us show that $|h_x|_p \geq |\theta|_p$ for all $x \in V \setminus \{(0)\}$. Assume that from the contrary, $|h^{(n_0)}_x|_p < |\theta|_p$ for some $n_0 \in \mathbb{N}$ and x. Then we have (5.38). Therefore, consider several possible cases.

(a) if $|h^{(n_0+1)}_{(x,1)}|_p \geq |\theta|_p$, and $|h^{(n_0+1)}_{(x,2)}|_p < |\theta|_p$, then due Lemma 5.9(iv),(v) from (5.38) one finds

$$\begin{aligned} |\theta|_p < |h^{(n_0)}_x|_p &= |g(h^{(n_0+1)}_{(x,1)})|_p|g(h^{(n_0+1)}_{(x,2)})|_p \\ &\leq |\theta|_p|h^{(n_0+1)}_{(x,2)}|_p \\ &< |\theta|_p \ \ (\text{since } |h^{(n_0+1)}_{(x,2)}|_p > 1), \end{aligned}$$

but this is a contradiction.

(b) if either $|h^{(n_0+1)}_{(x,2)}|_p \geq |\theta|_p$, $|h^{(n_0+1)}_{(x,1)}|_p < |\theta|_p$ or $|h^{(n_0+1)}_{(x,1)}|_p \geq \frac{1}{|\theta|_p}$, $|h^{(n_0+1)}_{(x,2)}|_p \geq \frac{1}{|\theta|_p}$, then by the same argument as (a) one finds a contradiction.

Hence, we conclude that $|h^{(n_0+1)}_{(x,1)}|_p < |\theta|_p$, $|h^{(n_0+1)}_{(x,2)}|_p < |\theta|_p$. Therefore, assume that $|h^{(n_0+k)}_{(x,i_1\cdots i_k)}|_p > \frac{1}{|\theta|_p}$ for every $k \geq 1$. Then, according to (3.7) with Lemma 5.9(iv) one gets

$$|h^{(n_0)}_x|_p = \prod_{i_1,\dots i_k=1,2} |_p h^{(n_0+k)}_{(x,i_1\cdots i_k)}|_p. \tag{5.44}$$

Denote

$$\delta_k = \min_{i_1,\dots i_k} |h^{(n_0+k)}_{(x,i_1\cdots i_k)}|_p.$$

We know that $|h^{(n_0+k)}_{(x,i_1\cdots i_k)}|_p > 1$ for all $i_1,\dots i_k$, $k \geq 1$. Therefore, from our assumption one has $1 < |\delta_k|_p < |\theta|_p$ for every $k \in \mathbb{N}$. Hence, from (5.44) one finds

$$|h^{(n_0)}_x|_p \geq |\delta_k|_p^{2^k}. \tag{5.45}$$

Thus, when k is large enough, then $|\delta_k|_p^{2^k} \geq |\theta|_p$. So, from (5.45) we obtain $|h^{(n_0)}_x|_p \geq |\theta|_p$, which contradicts to $|h^{(n_0)}_x|_p < |\theta|_p$.

Thus, $|h_x|_p \geq |\theta|_p$ for all x. Then from (3.7) one concludes (see also (5.38)) that $|h_x|_p \geq |\theta|_p^2$. Then taking an arbitrary $\epsilon > 0$ and using (5.42) with Lemma 5.9 (iv),(vi) we obtain

$$\|h^{(n)} - x_1\|_p \leq \frac{1}{|\theta|_p}\|h^{(n+1)} - x_1\|_p.$$

Now the same argument as (i) we get the desired assertion. This completes the proof. ☐

6. Boundedness of p-adic quasi Gibbs measures and phase transitions

From the results of the previous section, we conclude that to investigate the quasi p-adic measure, for us it is enough to study the measures μ_0,μ_1 and μ_2, corresponding to the solutions x_0,x_1 and x_2. In this section we shall study boundedness and unboundedness of the said measures.

Furthermore, we are going to consider the p-adic quasi Gibbs measures corresponding to these solutions. Due to Lemma 3.2 the partition function $Z_{i,n}$ corresponding to the measure μ_i $(i = 1, 2)$ has the following form

$$Z_{i,n} = a_i^{|V_{n-1}|} \tag{6.1}$$

where $a_i = (x_i + \theta + q - 1)^2 h_0$.

For a given configuration $\sigma \in \Omega_{V_n}$ denote

$$\#\sigma = \{x \in W_n : \ \sigma(x) = 1\}.$$

From (3.2),(3.8) and (6.1) we find

$$\begin{aligned} |\mu_1(\sigma)|_p &= \frac{1}{Z_{1,n}}\cdot\frac{1}{p^{H(\sigma)}}\prod_{x\in W_n}\left|\frac{h_{\sigma(x),x}}{h_0}\right|_p |h_0|_p^{|W_n|} \\ &= \frac{|h_0|_p^{|W_n|-|V_{n-1}|}}{|x_1+\theta+q-1|_p^{2|V_{n-1}|}}\cdot\frac{|x_1|_p^{\#\sigma}}{p^{H(\sigma)}} \\ &= \frac{|h_0|_p^{2}}{|x_1+\theta+q-1|_p^{2|V_{n-1}|}}\cdot\frac{|x_1|_p^{\#\sigma}}{p^{H(\sigma)}}, \end{aligned} \tag{6.2}$$

where we have used the equality $|W_n|-|V_{n-1}|=2$.

Similarly, one gets

$$|\mu_2(\sigma)|_p = \frac{|h_0|_p^{2}}{|x_2+\theta+q-1|_p^{2|V_{n-1}|}}\cdot\frac{|x_2|_p^{\#\sigma}}{p^{H(\sigma)}}, \tag{6.3}$$

6.1. **Ferromagnetic case.** Assume that $N>0$, i.e. $|\theta|_p<1$. In this subsection we shall prove the existence of phase transitions. Namely one has the following

Theorem 6.1. *Assume that $|q|_p<1$, $|\theta|_p\leq|q|_p^2$. Then for p-adic quasi Gibbs measures μ_0,μ_1 μ_2 of the ferromagnetic $q+1$-state Potts model* (3.1) *one has: the measure μ_1 is bounded; the measures μ_0 and μ_2 are unbounded. Moreover, there is a strong phase transition.*

Proof. According to Theorem 4.2 the conditions $|q|_p<1$, $|\theta|_p\leq|q|_p^2$ imply the existence of three translation-invariant measures μ_0,μ_1 and μ_2.

Then from (6.2) with (5.5),(5.7) we obtain

$$|\mu_1(\sigma)|_p = \frac{|h_0|_p^2}{p^{H(\sigma)}}\cdot|x_1|_p^{\#\sigma}\leq|h_0|_p^2, \tag{6.4}$$

which implies that the measure μ_1 is bounded.

Similarly, from (6.3) with (5.5),(5.8) we find

$$\begin{aligned} |\mu_2(\sigma)|_p &= \frac{|h_0|_p^2}{|q|_p^{2|V_{n-1}|}}\cdot\frac{1}{p^{H(\sigma)}} \\ &\geq |h_0|_p^2 p^{2|V_{n-1}|-H(\sigma)} \end{aligned} \tag{6.5}$$

Now let us choose $\sigma_{0,n}\in\Omega_{V_n}$ as follows $\sigma_{0,n}(x)=1$ for all $x\in V_n$. Then one can see that $H(\sigma_{0,n})=0$, therefore it follows from (6.5) that

$$|\mu_2(\sigma_{0,n})|_p\geq|h_0|_p^2p^{2|V_{n-1}|}\to\infty \ \text{ as } \ n\to\infty.$$

This yields that the measure μ_2 is not bounded.

Let us consider the measure μ_0. Similarly, we obtain

$$\begin{aligned} |\mu_0(\sigma)|_p &= \frac{|h_0|_p^2}{|\theta+q|_p^{2|V_{n-1}|}}\cdot\frac{1}{p^{H(\sigma)}} \\ &= \frac{|h_0|_p^2}{|q|_p^{2|V_{n-1}|}}\cdot\frac{1}{p^{H(\sigma)}} \\ &\geq |h_0|_p^2p^{2|V_{n-1}|-H(\sigma)} \end{aligned} \tag{6.6}$$

so, we immediately find that $|\mu_0(\sigma_{0,n})|_p\to\infty$ as $n\to\infty$.

It follows from (6.5), (6.6) that

$$\left|\frac{\mu_0(\sigma)}{\mu_2(\sigma)}\right|_p = 1.$$

Now let us compare μ_1 and μ_2. From (6.4),(6.5) with (5.5) one finds

$$\begin{aligned} |\mu_1(\sigma_{0,n})\mu_2(\sigma_{0,n})|_p &= \frac{|h_0|_p^4|x_1|_p^{\#\sigma_{0,n}}}{|q|_p^{2|V_{n-1}|}} \\ &= |h_0|_p^4|q|_p^{2(|W_n|-|V_{n-1}|)} \\ &= |h_0|_p^4|q|_p^4. \end{aligned} \tag{6.7}$$

This implies that $|\mu_1(\sigma_{0,n})|_p \to 0$ as $n \to \infty$. □

Now assume that $|q|_p = 1$. In this case, the solutions x_1 and x_2 may not exists (see Theorems 4.1 and 4.2). Therefore, we suppose the existence of such solutions.

Now taking into account (6.2), (6.3) with (5.6) we derive

$$|\mu_i(\sigma)|_p = \frac{|h_0|_p^2|x_i|_p^{\#\sigma^{(i)}}}{p^{H(\sigma)}} = \frac{|h_0|_p^2}{p^{H(\sigma)}} \leq |h_0|_p^2 \quad (i = 1, 2), \tag{6.8}$$

so the measures μ_1 and μ_2 are bounded.

We would like to compare these measure. Therefore, let us consider the following difference

$$|\mu_0(\sigma) - \mu_i(\sigma)|_p = \frac{|h_0|_p^2}{p^{H(\sigma)}}\left|(\theta + q - 1 + x_i)^{2|V_{n-1}|} - x_i^{\#\sigma}(\theta + q)^{2|V_{n-1}|}\right|_p. \tag{6.9}$$

Denoting

$$x = \theta + q - 1, \quad y = x_i, \quad N = 2|V_{n-1}|, \quad k = \#\sigma$$

and taking into account $|x|_p \leq 1$ and $|y|_p = 1$, the right-hand side of (6.9) can be estimated as follows

$$\begin{aligned} |(x + y)^N - y^k(x + 1)^N|_p &= \left|\sum_{\ell=0}^{N} C_N^\ell x^\ell (y^{N-\ell} - y^k)\right|_p \\ &= \left|\sum_{\ell=0}^{N} C_N^\ell x^\ell y^{\min\{N-\ell,k\}}(1 - y^{M_\ell})\right|_p \\ &= \left|(1 - y)\sum_{\ell=0}^{N} C_N^\ell x^\ell y^{\min\{N-\ell,k\}}\Big(\sum_{j=0}^{M_\ell} y^j\Big)\right|_p \\ &\leq |1 - y|_p \max_{0\leq\ell\leq N}\left\{\left|C_N^\ell x^\ell y^{\min\{N-\ell,k\}}\Big(\sum_{j=0}^{M_\ell} y^j\Big)\right|_p\right\} \\ &\leq |1 - y|_p, \end{aligned} \tag{6.10}$$

here $M_\ell = \max\{N - \ell, k\} - \min\{N - \ell, k\}$.

From (6.10) with (6.9) we immediately find

$$|\mu_0(\sigma) - \mu_i(\sigma)|_p \leq \frac{|h_0|_p^2|1 - x_i|_p}{p^{H(\sigma)}} \quad (i = 1, 2). \tag{6.11}$$

Using the same argument we get

$$|\mu_1(\sigma)-\mu_2(\sigma)|_p \leq \frac{|h_0|_p^2|x_1-x_2|_p}{p^{H(\sigma)}}. \tag{6.12}$$

Consequently, we can formulate the following

Theorem 6.2. *Assume that $|q|_p=1$ and the measures μ_1 μ_2 for p-adic the ferromagnetic $q+1$-state Potts model* (3.1) *exist. Then the measures μ_k $(k=0,1,2)$ are bounded. Moreover, the inequalities* (6.11),(6.12) *hold. In this case, there is a quasi phase transition.*

6.2. **Antiferromagnetic case.** In this case we assume that $N<0$ and $-N$ is even. Then according to Theorem 4.3 there exist the measures μ_0,μ_1 and μ_2.

Now taking into account (5.27),(5.28) from (6.2) one finds

$$\begin{aligned}
|\mu_1(\sigma)|_p &= \frac{|h_0|_p^2}{|x_1|_p^{2|V_{n-1}|}}\cdot\frac{|x_1|_p^{\#\sigma}}{p^{H(\sigma)}}\\
&= \frac{|h_0|_p^2\cdot p^{-2\bar N(2|V_{n-1}|-\#\sigma)}}{p^{H(\sigma)}}\\
&= |h_0|_p^2\cdot p^{-2\bar N\left(2|V_{n-1}|-\#\sigma-\frac12\sum\limits_{<x,y>\in L_n}\delta_{\sigma(x),\sigma(y)}\right)}
\end{aligned} \tag{6.13}$$

Now let us estimate the expression standing inside the brackets. It is clear that

$$0\leq \#\sigma\leq |W_n|, \quad 0\leq \sum_{<x,y>\in L_n}\delta_{\sigma(x),\sigma(y)}\leq |V_n|-1. \tag{6.14}$$

Therefore, from (6.14) with $|W_n|-|V_{n-1}|=2$, $|V_n|=|V_{n-1}|+|W_n|$ we get

$$\begin{aligned}
2|V_{n-1}|-\#\sigma-\frac12\sum_{<x,y>\in L_n}\delta_{\sigma(x),\sigma(y)} &\geq 2|V_{n-1}|-|W_n|-\frac12(|V_n|-1)\\
&= |V_{n-1}|-2+\frac12(1-|V_n|)\\
&= \frac12\big(2|V_{n-1}|-|V_n|-3\big)\\
&= \frac12\big(|V_{n-1}|-|W_n|-3\big)\\
&= -\frac52
\end{aligned} \tag{6.15}$$

Consequently, the last inequality (6.15) with (6.13) implies

$$|\mu_1(\sigma)|_p \leq |h_0|_p^2\cdot p^{5\bar N}=\frac{|h_0|_p^2}{p^{5N}}, \tag{6.16}$$

this means that μ_1 is bounded.

Now consider the measure μ_2. Noting $|x_2|_p=|q|_p^2p^{-2\bar N}$ and $|x_2+\theta+q-1|_p=p^{\bar N}$ (see (5.27),(5.29)), the equality (6.3) yields

$$\begin{aligned}
|\mu_2(\sigma)|_p &= \frac{|q|_p^2|h_0|_p^2}{p^{2\bar N|V_{n-1}|}}\cdot\frac{p^{-2\bar N\#\sigma}}{p^{H(\sigma)}}\\
&= |q|_p^2|h_0|_p^2\cdot p^{-2\bar N\left(|V_{n-1}|-\frac12\sum\limits_{<x,y>\in L_n}\delta_{\sigma(x),\sigma(y)}+\#\sigma\right)}.
\end{aligned} \tag{6.17}$$

Now using the same argument as in (6.15) one gets

$$|V_{n-1}| - \frac{1}{2}\sum_{<x,y>\in L_n}\delta_{\sigma(x),\sigma(y)} + \#\sigma \geq |V_{n-1}| - \frac{1}{2}(|V_n|-1) = -\frac{1}{2}. \tag{6.18}$$

Hence, (6.18) with (6.17) implies

$$|\mu_2(\sigma)|_p \leq \frac{|h_0|_p^2}{p^N}, \tag{6.19}$$

which means that μ_2 is bounded as well.

Let us consider the measure μ_0. From (6.6) we obtain

$$\begin{aligned}
|\mu_0(\sigma)|_p &= \frac{|h_0|_p^2}{|\theta+q|_p^{2|V_{n-1}|}}\cdot\frac{1}{p^{H(\sigma)}}\\
&= |h_0|_p^2 p^{-\bar{N}(2|V_{n-1}|-\sum\limits_{<x,y>\in L_n}\delta_{\sigma(x),\sigma(y)})}\\
&\leq |h_0|_p^2 p^{\bar{N}}\\
&= \frac{|h_0|_p^2}{p^N},
\end{aligned} \tag{6.20}$$

here we have used (see (6.18))

$$2|V_{n-1}| - \sum_{<x,y>\in L_n}\delta_{\sigma(x),\sigma(y)} \geq -1.$$

Hence, μ_0 is bounded too.

Let us consider relations between μ_0 and μ_1,μ_2. From (6.20),(6.13) and (6.17) we find

$$\frac{|\mu_1(\sigma)|_p}{|\mu_0(\sigma)|_p} = p^{-2\bar{N}(|V_{n-1}|-\#\sigma)} \leq p^{4\bar{N}}, \tag{6.21}$$

$$\frac{|\mu_2(\sigma)|_p}{|\mu_0(\sigma)|_p} = |q|_p^2\cdot p^{-\bar{N}(\#\sigma)} \leq |q|_p^2, \tag{6.22}$$

here in (6.21) we have used $|V_{n-1}| - \#\sigma \geq |V_{n-1}| - |W_n| = -2$. Hence, the derived relations imply that

$$|\mu_1(\sigma)|_p \leq p^{4\bar{N}}|\mu_0(\sigma)|_p, \quad |\mu_2(\sigma)|_p \leq |q|_p^2|\mu_0(\sigma)|_p. \tag{6.23}$$

Let us consider relation between μ_1 and μ_2. From (6.13) and (6.17) we find

$$\frac{|\mu_1(\sigma)|_p}{|\mu_2(\sigma)|_p} = \frac{p^{-2\bar{N}(|V_{n-1}|-2\#\sigma)}}{|q|_p^2}. \tag{6.24}$$

Take any configuration σ_n in Ω_{V_n} with $\#\sigma_n = |W_n|$, (for example $\sigma_n(x)=1$ for every $x\in V_n$). Then (6.24) yields

$$\frac{|\mu_1(\sigma_n)|_p}{|\mu_2(\sigma_n)|_p} \geq p^{2\bar{N}(|W_n|-2)} \to \infty \text{ as } n\to\infty. \tag{6.25}$$

Now take any configuration $\tilde{\sigma}_n$ in Ω_{V_n} with $\#\tilde{\sigma}_n = 0$, (for example $\tilde{\sigma}_n(x)=0$ for every $x\in V_n$). Then (6.24) yields

$$\frac{|\mu_1(\tilde{\sigma}_n)|_p}{|\mu_2(\tilde{\sigma}_n)|_p} = \frac{p^{-2\bar{N}|V_{n-1}|}}{|q|_p^2} \to 0 \text{ as } n\to\infty. \tag{6.26}$$

The relations (6.24),(6.25) show that the structure of the measures μ_1 and μ_2 are different even they are bounded.

Consequently, we can formulate the following

Theorem 6.3. *Let $N < 0$ and $-N$ is even. Then the translation-invariant p-adic quasi Gibbs measures μ_0, μ_1 and μ_2 of antiferromagnetic Potts model* (3.1) *are bounded. Moreover, the inequality* (6.23) *holds. In this case, there is a quasi phase transition.*

7. Conclusions

It is known that to investigate phase transitions, a dynamical system approach, in real case, has greatly enhanced our understanding of complex properties of models. The interplay of statistical mechanics with chaos theory has even led to novel conceptual frameworks in different physical settings [18]. Therefore, in the present paper, we have investigated a phase transition phenomena from such a dynamical system point of view. For p-adic quasi Gibbs measures of $q + 1$-state Potts model on a Cayley tree of order two, we derived a recursive relations with respect to the boundary conditions, then we defined one dimensional fractional p-adic dynamical system. In ferromagnetic case, we have established that if q is divisible by p, then such a dynamical system has two repelling and one attractive fixed points. We found basin of attraction of the attractive fixed point, and this allowed us to describe all solutions of the nonlinear recursive equations. Moreover, in that case we prove the existence of the strong phase transition. If q is not divisible by p, then the fixed points are neutral, and the existence of the quasi phase transition has been established. In antiferromagnetic case, there are two attractive and one repelling fixed points. We found basins of attraction of both attractive fixed points, and described solutions of the nonlinear recursive equation. In this case, we proved the existence of a quasi phase transition as well. These investigations show that there are some similarities with the real case, for example, the existence of two repelling fixed points implies the occurrence of the strong phase transition. Moreover, using such a method one can study other p-adic models over trees.

Note that the obtained results are totaly different from the results of [50, 51], since when q is divisible by p means that $q + 1$ is not divided by p, which according to [50] means that uniqueness and boundedness of p-adic Gibbs measure.

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