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DECOMPOSITIONS OF LOOPED CO-H-SPACES AND APPLICATIONS

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Abstract. We prove two homotopy decomposition theorems for the loops on co-H-spaces, including a generalization of the Hilton-Milnor Theorem. These are applied to problems arising in algebra, representation theory, toric topology, and the study of quasi-symmetric functions.

1. Introduction

A central theme in mathematics is to decompose objects into products of simpler ones. The smaller pieces should then be simpler to analyze, and by understanding how the pieces are put back together information is obtained about the original object. In homotopy theory this takes the form of decomposing H-spaces as products of factors or decomposing co-H-spaces as wedges of summands. Powerful decomposition techniques have been developed. Some, such as those in [MNT, CMN] are concerned with decomposing specific spaces as finely as possible, while others, such as those in [SW1, STW2], are concerned with functorial decompositions that are valid for all loop suspensions or looped co-H-spaces.

In this paper we establish two new decomposition theorems that apply to looped co-H-spaces. One is a strong refinement of work in [STW2], and the other is a generalized Hilton-Milnor Theorem. We give four applications which have connections with other areas of study in mathematics: the Poincaré-Birkhoff-Witt Theorem, Lie powers in representation theory, moment-angle complexes in toric topology, and quasi-symmetric functions.

To state specific results, we introduce some notation and context. Let p be an odd prime, and localize all spaces and maps at p. Take homology with mod-p coefficients. Let V be a graded module over $\mathbb{Z}/p\mathbb{Z}$ and let $T(V)$ be the tensor algebra on V. This tensor algebra is given a Hopf algebra structure by declaring that the generators are primitive and extending multiplicatively. In [SW1] it was shown that there is a coalgebra decomposition $T(V) \cong A^{\min}(V) \otimes B^{\max}(V)$ where $A^{\min}(V)$ is the minimal functorial coalgebra retract of $T(V)$ that contains V. One important property of this decomposition is that the primitive elements of $T(V)$ of tensor length not a power of p are all contained in the complement $B^{\max}(V)$. A programme of work ensued to geometrically realize these tensor algebra decompositions, which we now outline.

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By the Bott-Samelson theorem, there is an algebra isomorphism $H_*(\Omega\Sigma X) \cong T(\widetilde{H}_*(X))$. This was generalized in [Be] to the case of a simply-connected co-H space Y: there is an algebra isomorphism $H_*(\Omega Y) \cong T(\Sigma^{-1}\widetilde{H}_*(Y))$, where $\Sigma^{-1}\widetilde{H}_*(Y)$ is the desuspension by one degree of the graded module $\widetilde{H}_*(Y)$. Let $V = \Sigma^{-1}\widetilde{H}_*(Y)$ so $H_*(\Omega Y) \cong T(V)$. The coalgebra decomposition of $T(V)$ suggests that there are spaces $A^{\min}(Y)$ and $B^{\max}(Y)$ such that $\widetilde{H}_*(A^{\min}(Y)) \cong A^{\min}(V)$, $\widetilde{H}_*(B^{\max}(Y)) \cong B^{\max}(V)$, and there is a homotopy decomposition $\Omega Y \simeq A^{\min}(Y) \times B^{\max}(Y)$. Such a decompositions was realized in a succession of papers [SW1, SW2, STW1, STW2] which began with Y being a p-torsion double suspension and ended with the general case of Y being a simply-connected co-H-space.

However, the story does not end there, as the module $B^{\max}(V)$ has a much richer structure. There is a coalgebra decomposition $B^{\max}(V) \cong T(\oplus_{n=2}^{\infty} Q_nB(V))$, where $Q_nB(V)$ is a functorial retract of $V^{\otimes n}$. Ideally, this should be geometrically realized as well. This was proved in [STW1] when Y is a simply-connected, homotopy coassociative co-H space. More precisely, there are spaces $Q_nB(Y)$ for $n \geq 2$ such that $\widetilde{H}_*(Q_nB(Y)) \cong \Sigma Q_nB(V)$, a homotopy fibration sequence $\Omega Y \xrightarrow{*} A^{\min}(Y) \longrightarrow \bigvee_{n=2}^{\infty} Q_nB(Y) \longrightarrow Y$, and a homotopy decomposition $\Omega Y \simeq A^{\min}(Y) \times \Omega(\bigvee_{n=2}^{\infty} Q_nB(Y))$.

In the more general case of a simply-connected co-H-space Y, the geometric realization of $A^{\min}(V)$ in [STW2] produced a homotopy decomposition $\Omega Y \simeq A^{\min}(Y) \times B^{\max}(Y)$ but it did not identify $B^{\max}(Y)$ as a loop space. The first goal of this paper is to do exactly that.

Theorem 1.1. *Let Y be a simply-connected co-H-space and let $V = \Sigma^{-1}\widetilde{H}_*(Y)$. There is a homotopy fibration sequence*

$$\Omega Y \longrightarrow A^{\min}(Y) \longrightarrow \bigvee_{n=2}^{\infty} Q_nB(Y) \longrightarrow Y$$

such that:

1) $\Omega Y \simeq A^{\min}(Y) \times \Omega(\bigvee_{n=2}^{\infty} Q_nB(Y))$;
2) $\widetilde{H}_*(A^{\min}(Y)) \cong A^{\min}(V)$;
3) *for each* $n \geq 2$, $\widetilde{H}_*(Q_nB(Y)) \cong \Sigma Q_nB(V)$.

In fact, Theorem 1.1 is a special case of a more general theorem proved in Section 2 which geometrically realizes any natural coalgebra-split sub-Hopf algebra $B(V)$ of $T(V)$ as a loop space.

The construction of the space $Q_nB(Y)$ exists by a suspension splitting result from [GTW]. To describe this, recall that James [J] proved that there is a homotopy decomposition $\Sigma\Omega\Sigma X \simeq \bigvee_{n=1}^{\infty} \Sigma X^{(n)}$, where $X^{(n)}$ is the n-fold smash of X with itself. Note that $\widetilde{H}_*(\Sigma X^{(n)}) \cong \Sigma\widetilde{H}_*(X)^{\otimes n}$. James' decomposition was generalized in [GTW]. If Y is a simply-connected co-H-space then there is a homotopy decomposition $\Sigma\Omega Y \simeq \bigvee_{n=1}^{\infty}[\Sigma\Omega Y]_n$, where each space $[\Sigma\Omega Y]_n$ is a co-H-space and there is an isomorphism $\widetilde{H}_*([\Sigma\Omega Y]_n) \cong \Sigma(\Sigma^{-1}\widetilde{H}_*(Y))^{\otimes n}$. Succinctly, $[\Omega\Sigma Y]_n$ is an $(n-1)$-fold desuspension of $Y^{(n)}$. A key point is that the space $Q_nB(Y)$ is a retract of the co-H-space $[\Sigma\Omega Y]_n$, so it too is a co-H-space.

Our second result is a generalization of the Hilton-Milnor Theorem, touched upon in [GTW]. Recall that the Hilton-Milnor Theorem states that if $X_1, \ldots, X_m$

are path-connected spaces then there is a homotopy decomposition

$$\Omega(\Sigma X_1 \vee \cdots \vee \Sigma X_m) \simeq \prod_{\alpha \in \mathcal{I}} \Omega(\Sigma X_1^{(\alpha_1)} \wedge \cdots \wedge X_m^{(\alpha_m)})$$

where $\mathcal{I}$ runs over a vector space basis of the free Lie algebra $L\langle x_1, \cdots, x_m \rangle$, and if w_α is the basis element corresponding to α then α_i counts the number of occurances of x_i in w_α. Note that if $\alpha_i = 0$ then, for example, we regard $X_i^{(\alpha_i)} \wedge X_j^{(\alpha_j)}$ as $X_j^{(\alpha_j)}$ rather than as $* \wedge X^{(\alpha_j)} \simeq *$. We generalize the Hilton-Milnor Theorem by replacing each ΣX_i by a simply-connected co-H-space.

Theorem 1.2. *Let $Y_1, \ldots, Y_m$ be simply-connected co-H-spaces. There is a homotopy decomposition*

$$\Omega(Y_1 \vee \cdots \vee Y_m) \simeq \prod_{\alpha \in \mathcal{I}} \Omega M((Y_i, \alpha_i)_{i=1}^m)$$

where $\mathcal{I}$ runs over a vector space basis of the free Lie algebra $L\langle y_1, \ldots, y_m \rangle$ and:

1) *each space $M((Y_i, \alpha_i)_{i=1}^m)$ is a simply-connected co-H-space;*
2) $\widetilde{H}_*(M((Y_i, \alpha_i)_{i=1}^m)) \cong \Sigma \left((\Sigma^{-1} \widetilde{H}_*(Y_1))^{\otimes \alpha_1} \otimes \cdots \otimes (\Sigma^{-1} \widetilde{H}_*(Y_m))^{\otimes \alpha_m} \right)$;
3) *if $Y_i = \Sigma X_i$ for $1 \leq i \leq m$ then $M((Y_i, \alpha_i)_{i=1}^m) \simeq \Sigma X^{(\alpha_1)} \wedge \cdots \wedge X^{(\alpha_m)}$.*

Again, if $\alpha_i = 0$ we interpret $(\Sigma^{-1} \widetilde{H}_*(Y_i))^{\otimes \alpha_i} \otimes (\Sigma^{-1} \widetilde{H}_*(Y_j))^{\otimes \alpha_j})$ as $\Sigma^{-1} \widetilde{H}_*(Y_j))^{\alpha_j}$ rather than 0. Note that Theorem 1.2 (3) is the usual Hilton-Milnor Theorem.

Theorems 1.1 and 1.2 are very useful for producing homotopy decompositions of interesting spaces. In Section 4, we give three examples: a complete decomposition of ΩY into functorially indecomposable factors, a refined decomposition of some generalized moment-angle complexes that arise in toric topology, and a decomposition of $\Omega \Sigma \mathbb{C}P^\infty$ that implies a corresponding algebraic decomposition of the ring of quasi-symmetric functions.

2. Geometric Realization of Natural Coalgebra-Split Sub-Hopf Algebras

In this section we prove Theorem 1.1 as a special case of the more general Theorem 2.3. This gives conditions for when a sub-Hopf algebra of a tensor algebra has a geometric realization as a loop space. Before proving Theorem 2.3 it will be useful to state two results. The first is a geometric realization statement from [STW2]. Recall that p is an odd prime, the ground ring for all algebraic statements is $\mathbb{Z}/p\mathbb{Z}$, and all spaces and maps have been localized at p.

Theorem 2.1. *Let V be a graded module and suppose that $A(V)$ is a functorial coalgebra retract of $T(V)$. Then $A(V)$ has a geometric realization. That is, if Y is a simply-connected co-H-space such that there is an algebra isomorphism $H_*(\Omega Y) \cong T(V)$, then there is a functorial retract $\bar{A}(Y)$ of ΩY with the property that $H_*(\bar{A}(Y)) \cong A(V)$.* □

Second, given a functorial coalgebra retract $A(V)$ of $T(V)$, let $A_n(V)$ be the component of $A(V)$ consisting of homogeneous elements of tensor length n. The following suspension splitting theorem was proved in [GTW].

Theorem 2.2. *Let $A(V)$ be any functorial coalgebra retract of $T(V)$ and let $\bar{A}$ be the functorial geometric realization of A. Then for any simply-connected co-H-space Y of finite type, there is a functorial homotopy decomposition*

$$\Sigma\bar{A}(Y) \simeq \bigvee_{n=1}^{\infty} \bar{A}_n(Y)$$

such that $\bar{A}_n(Y)$ is a functorial retract of $[\Sigma\Omega Y]_n$ and there is a coalgebra isomorphism

$$\widetilde{H}_*(\bar{A}_n(Y)) \cong A_n(\Sigma^{-1}\widetilde{H}_*(Y))$$

for each $n \geq 1$.

Now suppose that $B(V)$ is a sub-Hopf algebra of $T(V)$. We say that $B(V)$ is *coalgebra-split* if the inclusion $B(V) \longrightarrow T(V)$ has a natural coalgebra retraction. Observe that the weaker property of $B(V)$ being a sub-coalgebra of $T(V)$ which splits off $T(V)$ implies by Theorem 2.1 that $B(V)$ has a geometric realization $\overline{B}$. We aim to show that the full force of $B(V)$ being a sub-Hopf algebra of $T(V)$ implies that it has a much more structured geometric realization.

If M is a Hopf algebra, let QM be the set of indecomposable elements of M, and let IM be the augmentation ideal of M. If $B(V)$ is a natural sub-Hopf algebra of $T(V)$ then there is a natural epimorphism $IB(V) \longrightarrow QB(V)$. Let $T_n(V)$ be the component of $T(V)$ consisting of the homogeneous tensor elements of length n, and let $B_n(V) = IB(V) \cap T_n(V)$. Let $Q_nB(V)$ be the quotient of $B_n(V)$ in $QB(V)$. Let **CoH** be the category of simply-connected co-H-spaces and co-H-maps and let $\mathbf{k} = \mathbb{Z}/p\mathbb{Z}$.

Theorem 2.3. *Let $B(V)$ be a natural coalgebra-split sub-Hopf algebra of $T(V)$ and let $\bar{B}$ be its geometric realization. Then there exist functors $\bar{Q}_nB$ from* **CoH** *to spaces such that for any $Y \in$* **CoH***:*

1) *$\bar{Q}_nB(Y)$ is functorial retract of $[\Sigma\Omega Y]_n$;*
2) *there is a functorial coalgebra isomorphism*

$$\Sigma^{-1}\bar{H}_*(\bar{Q}_nB(Y)) \cong Q_nB(\Sigma^{-1}\bar{H}_*(Y));$$

3) *there is a natural homotopy equivalence*

$$\bar{B}(Y) \simeq \Omega\left(\bigvee_{n=1}^{\infty} \bar{Q}_nB(Y)\right).$$

Proof. The proof is to give a geometric construction for the indecomposables of $B(V)$. Let $B^{[n]}(V)$ be the sub-Hopf algebra generated by $Q_iB(V)$ for $i \leq n$. By the method of proof of [LLW, Theorem 1.1], each $B^{[n]}(V)$ is a natural coalgebra-split sub-Hopf algebra of $T(V)$, and there is a natural coalgebra decomposition

$$B^{[n]}(V) \cong B^{[n-1]}(V) \otimes A^{[n]}(V), \tag{2.1}$$

where $A^{[n]}(V) = \mathbf{k} \otimes_{B^{[n-1]}(V)} B^{[n]}(V)$. Note that

$$Q_nB(V) \cong A^{[n]}(V)_n.$$

By Theorem 2.1, the functorial coalgebra splitting in (2.1) has a geometric realization as a natural homotopy decomposition

$$\bar{B}^{[n]}(Y) \simeq \bar{B}^{[n-1]}(Y) \times \bar{A}^{[n]}(Y) \tag{2.2}$$

for some $Y \in \mathbf{CoH}$. This induces a filtered decomposition with respect to the augmentation ideal filtration of $H_*(\Omega Y)$. By Theorem 2.2,

$$\Sigma \bar{A}^{[n]}(Y) \simeq \bigvee_{k=1}^{\infty} \bar{A}_k^{[n]}(Y)$$

where $\bar{A}_k^{[n]}(Y)$ is a functorial retract of $[\Sigma\Omega Y]_n$ and $\bar{A}_k^{[n]}(Y) \simeq *$ for $k < n$ because $A_k^{[n]}(V) = 0$ for $0 < k < n$. Define

$$\bar{Q}_n B(Y) = \bar{A}_n^{[n]}(Y).$$

Let ϕ_n be the composite of inclusions

$$\begin{array}{rcl} \bar{Q}_n B(Y) = \bar{A}_n^{[n]}(Y) & \longrightarrow & \Sigma \bar{A}^{[n]}(Y) \\ & \longrightarrow & \Sigma(\bar{B}^{[n-1]}(Y) \times \bar{A}^{[n]}(Y)) \\ & \xrightarrow{\simeq} & \Sigma \bar{B}(Y) \\ & \longrightarrow & \Sigma\Omega Y. \end{array} \tag{2.3}$$

Consider the composite

$$\Omega\left(\bigvee_{n=1}^{\infty} \bar{Q}_n B(Y)\right) \xrightarrow{\Omega(\bigvee_{n=1}^{\infty}\phi_n)} \Omega\Sigma\Omega Y \xrightarrow{\Omega\sigma} \Omega Y \xrightarrow{r} \bar{B}(Y), \tag{2.4}$$

where σ is the evaluation map and r is the retraction map. We wish to show that this composite induces an isomorphism in homology, implying that it is a homotopy equivalence. The assertions of the theorem would then follow. To show that (2.4) induces an isomorphism in homology it suffices to filter appropriately and show that we obtain an isomorphism of associated graded objects.

Let

$$H_*(\Omega\Sigma\Omega Y) = T(\bar{H}_*(\Omega Y))$$

be filtered by

$$I^n H_*(\Omega\Sigma\Omega Y) = \sum_{t_1 r_1 + \cdots + t_s r_s \geq n} (I^{t_1} H_*(\Omega Y))^{\otimes r_1} \otimes \cdots \otimes (I^{t_s} H_*(\Omega Y))^{\otimes r_s}.$$

Filter $H_*(\Omega Y)$ by the augmentation ideal filtration. Then

$$\Omega\sigma_* \colon H_*(\Omega\Sigma\Omega Y) \longrightarrow H_*(\Omega Y)$$

is a filtered map since $\Omega\sigma_*$ is an algebra map. Let $H_*(\bar{B}(Y))$ be filtered subject to the augmentation ideal filtration of $H_*(\Omega Y)$. Then r_* is a filtered map. Note that as an algebra

$$H_*\left(\Omega\left(\bigvee_{n=1}^{\infty} \bar{Q}_n B(Y)\right)\right) = T\left(\bigoplus_{n=1}^{\infty} \Sigma^{-1}(\bar{H}_*(\bar{Q}_n B(Y)))\right),$$

which is filtered by

$$\sum_{i_1+\cdots+i_t \geq n} \Sigma^{-1}(\bar{H}_*(\bar{Q}_{i_1} B(Y))) \otimes \cdots \otimes \Sigma^{-1}(\bar{H}_*(\bar{Q}_{i_t} B(Y))).$$

Observe that ϕ_{n*} maps $\bar{H}_*(\bar{Q}_n B(Y))$ into $\Sigma I^n H_*(\Omega Y)$ and the composite

$$\bar{H}_*(\bar{Q}_n B(Y)) \xrightarrow{\phi_{n*}} \Sigma I^n H_*(\Omega Y) \twoheadrightarrow \Sigma I^n H_*(\Omega Y)/\Sigma I^{n+1} H_*(\Omega Y) \tag{2.5}$$

is a monomorphism because $\bar{Q}_nB(Y)$ is obtained from the n-homogenous component of $\Sigma\bar{A}^{[n]}(Y)$. Thus $\Omega(\bigvee_{n=1}^{\infty}\phi_n)_*$ is a filtered map and the image of

$$E^0(\Omega\sigma_*\circ\Omega(\bigvee_{n=1}^{\infty}\phi_n)_*)$$

is the sub-Hopf algebra of $E^0H_*(\Omega Y)=T(\Sigma^{-1}\bar{H}_*(Y))$ generated by

$$E^0\phi_{n*}(\Sigma^{-1}\bar{H}_*(\bar{Q}_nB(Y)))$$

for $n\geq 1$. From (2.5),

$$\Sigma^{-1}\bar{H}_*(\bar{Q}_nB(Y))\cong E^0\phi_{n*}(\Sigma^{-1}\bar{H}_*(\bar{Q}_nB(Y))).$$

By the construction of ϕ_n in (2.3), the modules

$$\{E^0\phi_{n*}(\Sigma^{-1}\bar{H}_*(\bar{Q}_nB(Y)))\}$$

are algebraically independent because $\bar{Q}_iB(Y)$ is mapped into $\Sigma B^{[n]}(Y)$ for $i\leq n$ and $\bar{Q}^{[n]}(Y)$ is mapped into $\Sigma\bar{A}^{[n]}(Y)$ which is the complement to $\Sigma\bar{B}^{[n-1]}(Y)$. Since each $\bar{Q}_nB(Y)$ is mapped into $\Sigma\bar{B}(Y)$,

$$\mathrm{Im}(E^0(\Omega\sigma_*\circ\Omega(\bigvee_{n=1}^{\infty}\phi_n)_*))=T(E^0\phi_{n*}(\Sigma^{-1}\bar{H}_*(\bar{Q}_nB(Y))))$$

is a sub-Hopf algebra of $E^0H_*(\bar{B}(Y))\subseteq T(\Sigma^{-1}\bar{H}_*(Y))$. By computing the Poincaré series,

$$\mathrm{Im}(E^0(\Omega\sigma_*\circ\Omega(\bigvee_{n=1}^{\infty}\phi_n)_*))=E^0H_*(\bar{B}(Y)).$$

Since $r\colon\Omega Y\longrightarrow\bar{B}(Y)$ is a retraction map,

$$E^0r_*|_{E^0H_*(\bar{B}(Y))}=\mathrm{id}_{E^0H_*(\bar{B}(Y))}.$$

Therefore the composite

$$E^0r_*\circ E^0(\Omega\sigma_*)\circ E^0\Omega(\bigvee_{n=1}^{\infty}\phi_n)_*$$

of associated graded objects induced by the composition in (2.4) is an isomorphism, as required. □

The proof of Theorem 2.3 does more. Recall the map $\bar{Q}_nB(Y)\longrightarrow\Sigma\Omega Y$ defined in (2.3). Taking the wedge sum for $n\geq 1$ and then evaluating, we obtain a composite

$$\phi\colon\bigvee_{n=1}^{\infty}\bar{Q}_nB(Y)\xrightarrow{\phi_n}\Sigma\Omega Y\xrightarrow{\sigma}Y.$$

The thrust of the proof of Theorem 2.3 was to show that the composite in (2.4) is a homotopy equivalence. That is, the composite $r\circ\Omega\phi$ is a homotopy equivalence. In particular, this implies that $\Omega\phi$ has a functorial retraction. Consequently, if $\bar{A}(Y)$ is the homotopy fiber of ϕ we immediately obtain the following.

Theorem 2.4. *Let $B(V)$ be a natural coalgebra-split sub-Hopf algebra of $T(V)$ and let the functor A be given by $A(V) = \mathbf{k} \otimes_{B(V)} T(V)$. Then there is a homotopy fibration sequence*

$$\Omega\left(\bigvee_{n=1}^{\infty} \bar{Q}_n B(Y)\right) \xrightarrow{\Omega\phi} \Omega Y \longrightarrow \bar{A}(Y) \longrightarrow \bigvee_{n=1}^{\infty} \bar{Q}_n B(Y) \xrightarrow{\phi} Y$$

where $Y \in \mathbf{CoH}$ and a functorial decomposition

$$\Omega Y \simeq \Omega\left(\bigvee_{n=1}^{\infty} \bar{Q}_n B(Y)\right) \times \bar{A}(Y).$$

□

Note that $\bar{A}$ is a geometric realization of A.

Proof of Theorem 1.1. In Theorem 2.4 we can choose $B(V)$ to be $B^{\max}(V)$. The fiber $\bar{A}(Y)$ of ϕ is now, by definition, $A^{\min}(Y)$. The theorem follows immediately. □

3. The generalization of the Hilton-Milnor Theorem

In this section we prove Theorem 1.2. We begin by stating a key general result from [GTW].

Theorem 3.1. *Let Y and Z be simply-connected co-H-spaces. There is a homotopy decomposition*

$$Z \wedge \Omega Y \simeq \bigvee_{n=1}^{\infty} [Z \wedge \Omega Y]_n$$

such that:

1) *each space $[Z \wedge \Omega Y]_n$ is a simply-connected co-H-space;*
2) $\widetilde{H}_*([Z \wedge \Omega Y]_n) \cong \widetilde{H}_*(Z) \otimes (\Sigma^{-1}\widetilde{H}_*(Y))^{\otimes n}$*;*
3) *if $Z = S^1$ and $Y = \Sigma X$ then $[Z \wedge \Omega\Sigma X]_n \simeq \Sigma X^{(n)}$.*

□

In particular, if $Z = S^1$ then we obtain a homotopy decomposition of $\Sigma\Omega Y$ which generalizes James' decomposition of $\Sigma\Omega\Sigma X$, as discussed in the Introduction. The application of Theorem 3.1 that we need is the following.

Proposition 3.2. *Let $Y_1, \ldots, Y_m$ be simply-connected co-H-spaces. There is a homotopy decomposition*

$$\Sigma\Omega Y_1 \wedge \cdots \wedge \Omega Y_m \simeq \bigvee_{n_1,\ldots,n_m=1}^{\infty} M((Y_i, n_i)_{i=1}^m)$$

such that:

1) *each space $M((Y_i, n_i)_{i=1}^m)$ is a simply-connected co-H-space;*
2) $\widetilde{H}_*(M((Y_i, n_i)_{i=1}^m)) \cong \Sigma\left((\Sigma^{-1}\widetilde{H}_*(Y_1))^{\otimes n_1} \otimes \cdots \otimes (\Sigma^{-1}\widetilde{H}_*(Y_m))^{\otimes n_m}\right)$*;*
3) *if $Y_i = \Sigma X_i$ for $1 \leq i \leq m$ then $M((\Sigma X_i, n_i)_{i=1}^m) \simeq \Sigma X^{(n_1)} \wedge \cdots \wedge X^{(n_m)}$.*

Proof. First, consider the special case when $m=1$. We wish to decompose $\Sigma\Omega Y_1$. Applying Theorem 3.1 with $Z=S^1$ and $Y=Y_1$, we obtain a homotopy decomposition

$$\Sigma\Omega Y \simeq \bigvee_{n_1=1}^{\infty} M(Y_1,n_1)$$

where $M(Y_1,n_1)=[\Sigma\Omega Y_1]_{n_1}$. In particular, $M(Y_1,n_1)$ is a simply-connected co-H-space, $\widetilde{H}_*(M(Y_1,n_1))\cong\Sigma(\Sigma^{-1}\widetilde{H}_*(Y_1))^{\otimes n_1}$, and if $Y_1=\Sigma X_1$ then $M(\Sigma X_1,n_1)\simeq\Sigma X_1^{(n_1)}$.

Next, consider the special case when $m=2$. We wish to decompose $\Sigma\Omega Y_1\wedge\Omega Y_2$. From the $m=1$ case we have

$$\Sigma\Omega Y_1\wedge\Omega Y_2\simeq\left(\bigvee_{n_1=1}^{\infty} M(Y_1,n_1)\right)\wedge\Omega Y_2\simeq\bigvee_{n_1=1}^{\infty} M(Y_1,n_1)\wedge\Omega Y_2.$$

Since $M(Y_1,n_1)$ is a co-H-space, for each $n_1\geq 1$ we can apply Theorem 3.1 with $Z=M(Y_1,n_1)$ and $Y=Y_2$ to further decompose $M(Y_1,n_1)\wedge\Omega Y_2$. Collecting these, we obtain a homotopy decomposition

$$\Sigma\Omega Y_1\wedge\Omega Y_2\simeq\bigvee_{n_1,n_2=1}^{\infty} M((Y_i,n_i)_{i=1}^2)$$

where each space $M((Y_i,n_i)_{i=1}^2)$ is a simply-connected co-H-space,

$$\widetilde{H}_*(M((Y_i,n_i)_{i=1}^m))\cong\Sigma\left((\Sigma^{-1}\widetilde{H}_*(Y_1))^{\otimes n_1}\otimes(\Sigma^{-1}\widetilde{H}_*(Y_2))^{\otimes n_2}\right)$$

and if $Y_i=\Sigma X_i$ then $M((Y_i,n_i)_{i=1}^2)\simeq\Sigma X_1^{(n_1)}\wedge X^{(n_2)}$.

More generally, if $m>2$ then the procedure in the previous paragraph is iterated to obtain the homotopy decomposition asserted in the statement of the proposition. □

As a final preliminary result, we state a homotopy decomposition proved in [P]. For a space X and an integer j, let $j\cdot X=\bigvee_{i=1}^{j}X$.

Theorem 3.3. *Let $X_1,\ldots,X_m$ be simply-connected CW-complexes of finite type. Let F be the homotopy fiber of the inclusion $\bigvee_{i=1}^m X_i\longrightarrow\prod_{i=1}^m X_i$. There is a homotopy equivalence*

$$F\simeq\bigvee_{j=2}^{m}\left(\bigvee_{1\leq i_1<\cdots<i_j\leq m}(j-1)\cdot\Sigma\Omega X_{i_1}\wedge\cdots\wedge\Omega X_{i_j}\right).$$

□

Remark 3.4. *A version of Theorem 3.3 holds for an infinite wedge $\bigvee_{i=1}^{\infty}X_i$, provided the spaces X_i can be ordered so that the connectivity of X_i is nondecreasing and tends to infinity. This guarantees that the fiber F of the inclusion $\bigvee_{i=1}^{\infty}X_i\longrightarrow\prod_{i=1}^{\infty}X_i$ is of finite type.*

Proof of Theorem 1.2. We first consider the special case when $Y_i=\Sigma X_i$, that is, the usual Hilton-Milnor Theorem. One way to think of the proof is as follows.

First, including the wedge into the product gives a homotopy fibration

$$F_1 \longrightarrow \bigvee_{i=1}^{m} \Sigma X_i \longrightarrow \prod_{i=1}^{m} \Sigma X_i$$

that defines the space F_1. This fibration splits after looping as

$$\Omega(\bigvee_{i=1}^{m} \Sigma X_i) \simeq \prod_{i=1}^{m} \Omega\Sigma X_i \times \Omega F_1. \tag{3.1}$$

Second, by Theorem 3.3,

$$F_1 \simeq \bigvee_{j=2}^{m} \left(\bigvee_{1\leq i_1<\cdots<i_j\leq m} (j-1)\cdot \Sigma\Omega\Sigma X_{i_1} \wedge \cdots \wedge \Omega\Sigma X_{i_j} \right).$$

Iteratively using James' decomposition $\Sigma\Omega\Sigma X \simeq \bigvee_{n=1}^{\infty} \Sigma X^{(n)}$ we obtain a refined decomposition

$$F_1 \simeq \bigvee_{\alpha_1 \in \mathcal{J}_1} M_{\alpha_1}$$

for some index set $\mathcal{J}_1$, where each M_{α_1} is of the form $\Sigma X_{t_1}^{(r_1)} \wedge \cdots \wedge X_{t_l}^{(r_l)}$ for $l \geq 2$, $1 \leq t_1 < \cdots < t_l \leq m$ and $r_1, \ldots, r_l \geq 1$. Third, including the wedge into the product gives a homotopy fibration

$$F_2 \longrightarrow \bigvee_{\alpha_1 \in \mathcal{J}_1} M_{\alpha_1} \longrightarrow \prod_{\alpha_1 \in \mathcal{J}_1} M_{\alpha_1}$$

which defines the space F_2. This fibration splits after looping so (3.1) refines to a decomposition

$$\Omega(\bigvee_{i=1}^{m} \Sigma X_i) \simeq \prod_{i=1}^{m} \Omega\Sigma X_i \times \prod_{\alpha_1 \in \mathcal{J}_1} \Omega M_{\alpha_1} \times \Omega F_2. \tag{3.2}$$

Observe that since each ΣX_i is simply-connected, the spaces M_{α_1} can be ordered so their connectivity is nondecreasing and tending to infinity. Therefore, Remark 3.4 implies that Theorem 3.3 can be applied to decompose F_2. The process can now be iterated to produce fibers F_k for $k \geq 3$ and a decomposition

$$\Omega(\bigvee_{i=1}^{m} \Sigma X_i) \simeq \prod_{i=1}^{m} \Omega\Sigma X_i \times \left(\prod_{j=1}^{k-1} \prod_{\alpha_j \in \mathcal{J}_j} \Omega M_{\alpha_j} \right) \times \Omega F_k. \tag{3.3}$$

where each M_{α_j} is of the form $\Sigma X_{t_1}^{(r_1)} \wedge \cdots \wedge X_{t_l}^{(r_l)}$ for $l \geq j+1$, $1 \leq t_1 < \cdots < t_l \leq m$ and $r_1, \ldots, r_l \geq 1$. Note that the condition $l \geq j+1$ implies that the connectivity of F_k is strictly increasing with k, and so tends to infinity. Thus, the decompositions of $\Omega(\bigvee_{i=1}^{m} \Sigma X_i)$ stabilize. What remains is a bookkeeping argument that makes explicit the factors $\Sigma X_{t_1}^{(r_1)} \wedge \cdots \wedge X_{t_l}^{(r_l)}$. As stated in the Introduction, this takes the form of an index set determined by a vector space basis of the free Lie algebra $L\langle x_1, \ldots, x_m \rangle$.

We wish to generalize this to the case of $\Omega(\bigvee_{i=1}^{m} Y_i)$ for simply-connected co-H-spaces Y_i. To do so, simply replace the use of James' decomposition above with Proposition 3.1. The rest of the argument goes through verbatim. □

4. Applications

In this section we give four examples to illustrate how Theorems 1.1 and 1.2 can be used to produce useful homotopy decompositions of spaces of interest to other areas of mathematics.

A p-local geometric Poincaré-Birkhoff-Witt Theorem. Combining Theorems 1.1 and 1.2 allows for a complete decomposition of the loops on a co-H-space into a product of functorially indecomposable spaces. That is, Theorem 1.1 states that for a simply-connected co-H-space Y there is a homotopy decomposition

$$\Omega Y \simeq A^{\min}(Y) \times \Omega(\bigvee_{n=2}^{\infty} Q_n B(Y))$$

where $A^{\min}(Y)$ is functorially indecomposable. Since each space $Q_n B(Y)$ is a co-H-space, Theorem 1.2 implies that there is a homotopy decomposition

$$\Omega(\bigvee_{n=2}^{\infty} Q_n B(Y)) \simeq \prod_{\alpha \in \mathcal{I}} \Omega M((Q_n B(Y), \alpha_n)_{n=2}^{\infty})$$

where each space $M((Q_n B(Y), \alpha_n)_{n=2}^{\infty})$ is a simply-connected co-H-space. The homotopy decomposition in Theorem 1.1 can now be applied to each of the factors $\Omega M((Q_n B(Y), \alpha_n)_{n=2}^{\infty})$ to produce an $A^{\min}$ that is functorially indecomposable and a complementary factor which is the loops on a wedge of simply-connected co-H-spaces. Iterating, we obtain a decomposition of ΩY as a product of $A^{\min}$'s.

Theorem 4.1. *Let Y be a simply-connected co-H-space. Then there is a functorial homotopy decomposition*

$$\Omega Y \simeq \prod_{\gamma \in \mathcal{J}} A^{\min}(Y_\gamma)$$

for some index set $\mathcal{J}$, where each Y_γ is a simply-connected co-H-space and each factor $A^{\min}(Y_\gamma)$ is functorially indecomposable. □

Theorem 4.1 is related to the Poincaré-Birkhoff-Witt Theorem. To explain how, let V be a module over a field of characteristic p, and let $T(V)$ be the tensor algebra generated by V. This is made into a Hopf algebra by declaring that the generators are primitive and extending multiplicatively. There is a canonical isomorphism $T(V) \cong UL(V)$, where $L(V)$ is the free Lie algebra on V and $UL(V)$ is its universal enveloping algebra. Let $L_n(V)$ be the homogeneous component of the free Lie algeba $L(V)$ of tensor length n. The Poincaré-Birkhoff-Witt Theorem states that rationally there is a functorial coalgebra isomorphim $T(V) \cong \otimes_{n=1}^{\infty} S(L_n(V))$, where $S(\ )$ is the free symmetric algebra functor. Note that each $S(L_n(V))$ further decomposes as a product of exterior algebras and polynomial algebras on a single generator, with the generators in one-to-one correspondence with the module generators of $L_n(V)$. This can be geometrically realized. If Y is a simply-connected co-H-space such that $H_*(\Omega Y; \mathbb{Z}/p\mathbb{Z}) \cong T(V)$, then there is a rational homotopy equivalence $\Omega Y \simeq \prod_{n=1}^{\infty} S_n$ where S_n is a product of odd dimensional spheres and the loops on odd dimensional spheres, with the property that $H_*(S_n; \mathbb{Q}) \cong S(L_n(V))$. Theorem 4.1 is a p-local analogue, in the sense that it produces a decomposition of ΩY into a product of functorially indecomposable pieces, each of which geometrically realizes a functorially indecomposable factor of $T(\Sigma^{-1}\widetilde{H}_*(Y))$.

Decompositions of Lie powers. As above, let V be a module over a field of characteristic p, regard $T(V)$ as $UL\langle V\rangle$, and let $L_n(V)$ be the homogeneous component of the free Lie algeba $L(V)$ of tensor length n. The module $L_n(V)$ is called the n^{th} *free Lie power* of V. Decompositions of Lie powers $L_n(V)$ over the general linear group $GL(V)$ is a subject of considerable recent activity in modular representation theory (see, for example, [BS, ES]).

Now suppose that Y is a simply-connected co-H-space and $V = \Sigma^{-1}\widetilde{H}_*(Y)$. Then $H_*(\Omega Y) \cong T(V)$. In general, any functorial homotopy decomposition of ΩY determines a functorial coalgebra decomposition of $T(V)$, which in turn determines a module decomposition of $L_n(V)$ over the general linear group for every n. Thus studying homotopy decompositions of ΩY gives information about how Lie powers decompose.

The method developed in [SW1], introducing a functorial Poincaré-Birkhoff-Witt theorem, gives a fundamental connection between the homotopy theory of loops on co-H-spaces and the modular representation theory of Lie powers. It was shown in [LLW] that the modular representation theory of Lie powers over the general linear group is tightly related to functorial coalgebra decompositions of tensor algebras. Through this connection, one can investigate topological applications of new developments in representation theory. Conversely, topological methods such as Hopf invariants and techniques in Hopf algebras provide tools different from traditional methods in representation theory for studying Lie powers. Hopf invariants were obtained in geometry from the suspension splittings of loops on co-H-spaces. The homological behavior of Hopf invariants gives a family of natural coalgebra maps on tensor algebras involving certain important combinatorial information on shuffles [SW1]. By considering Hopf invariants together with techniques in Hopf algebras, [LLW] generalized some important recent results on the representation theory of Lie powers given by Bryant-Schocker [BS].

For example, in [LLW] it was shown that the sub-Hopf algebra $B(V)$ of $T(V)$ generated by the set $\{L_n(V) \mid n \text{ is not a power of } p\}$ is a functorial coalgebra summand of $T(V)$. This was used to construct an explicit decomposition [LLW, 6.3] of $L_m(V)$ when m is not a power of p. For our purposes, observe that $B(V)$ is a coalgebra-split sub-Hopf algebra of $T(V)$. So Theorems 2.3 and 2.4 imply that the Hopf-algebra map $B(V) \longrightarrow T(V)$ can be geometrically realized as a loop map $\Omega(\bigvee_{n=1}^{\infty} \bar{Q}_n B(Y)) \longrightarrow \Omega Y$ for some simply-connected co-H-space Y. The advantage of having a geometric realization is that it is stronger than simply having an algebraic decomposition. The topology of the decomposition may imply additional algebraic information beyond that used in [LLW], which may give further insight into how Lie powers decompose. We leave specific applications of this to later work.

Homotopy types of generalized moment-angle complexes. Let $X_1, \ldots, X_m$ be simply-connected, pointed CW-complexes of finite type. For $1 \leq k \leq m$, define the space T^m_k by

$$T^m_k = \{(x_1, \ldots, x_m) \in \prod_{i=1}^{m} X_i \mid \text{at least } k \text{ of } x_1, \ldots, x_m \text{ are the basepoint}\}.$$

Inclusion into the product gives a map $T_k^m \longrightarrow \prod_{i=1}^m X_i$. Define the space F_k^m by the homotopy fibration

$$F_k^m \longrightarrow T_k^m \longrightarrow \prod_{i=1}^m X_i.$$

In [P] it was shown that there is a homotopy equivalence

$$(4.1) \qquad F_k^m \simeq \bigvee_{j=m-k+1}^{m} \left(\bigvee_{1\leq i_1<\cdots<i_j\leq m} \binom{j-1}{m-k} \Sigma^{m-k}\Omega X_{i_1}\wedge\cdots\wedge\Omega X_{i_j} \right).$$

In particular, Theorem 3.3 is the special case when $k = m-1$.

The spaces F_k^m have received a great deal of attention lately as they are special cases of the generalized moment-angle complexes defined in [BBCG]. The classical moment-angle complexes are given by the special case when each $X_i = \mathbb{C}P^\infty$; they are fundamental objects in toric topology (see [BP1, DJ]). Moreover, classical moment-angle complexes can be identified with complements of complex coordinate subspaces [BP1], which are fundamental objects in combinatorics (see, for example, [Bj]), and a major problem is to determine their homotopy type. It is therefore natural to also try to determine the homotopy type of generalized moment-angle complexes. Progress in this direction has been made in [GT1, GT2].

In the special case of the spaces F_k^m, the decomposition in (4.1) goes a long way towards determining the homotopy type. This can be refined considerably when each X_i is a suspension, $X_i = \Sigma\overline{X}_i$, by iterating James' decomposition $\Sigma\Omega\Sigma X \simeq \bigvee_{n=1}^\infty X^{(n)}$. Doing so, one obtains a homotopy decomposition of F_k^m as a large wedge of spaces of the form $\Sigma^{m-k}\overline{X}_{j_1}^{(n_{j_1})}\wedge\cdots\wedge\overline{X}_{j_l}^{(n_{j_l})}$ where $1\leq j_1<\cdots<j_l\leq m$. Moreover, one obtains a decomposition of ΩF_k^m as the loops on a large wedge of suspensions. The Hilton-Milnor Theorem can now be applied to decompose further.

All of this can now be generalized to the case of F_k^m when each X_i is a simply-connected co-H-space. The iteration of James' decomposition is replaced by the decomposition in Proposition 3.2 and the Hilton-Milnor Theorem is then replaced by its generalization in Theorem 1.2.

Decompositions of the ring of quasi-symmetric functions. The Hopf-algebra of non-symmetric functions **NSymm** is defined as the tensor algebra $T(z_1, z_2, \ldots)$, where $|z_i| = 2i$, the coproduct is given by $\Delta(z_n) = \Sigma_{s+t=n}\, z_s\otimes z_t$, and the antiautomorphism is given by $\chi(z_n) = \Sigma_{\alpha_1+\cdots+\alpha_m=n}\, z_{\alpha_1}\cdots z_{\alpha_m}$. The Hopf algebra of quasi-symmetric functions **QSymm** is defined as the Hopf algebra dual of **NSymm**. In [BR], Baker and Richter observed that there is an integral Hopf-algebra isomorphism $H^*(\Omega\Sigma\mathbb{C}P^\infty;\mathbb{Z})\cong$ **QSymm**. They then used topological properties of $\Omega\Sigma\mathbb{C}P^\infty$ to prove algebraic properties of **QSymm**.

In general, a p-local homotopy decomposition $\Omega\Sigma\mathbb{C}P^\infty \simeq \prod_\alpha A_\alpha$ for some spaces A_α implies that there is a p-local algebra decomposition $\widetilde{H}_*(\Omega\Sigma\mathbb{C}P^\infty;\mathbb{Z}_{(p)}) \cong \otimes_\alpha \widetilde{H}_*(A_\alpha;\mathbb{Z}_{(p)})$. Therefore we obtain a p-local algebra decomposition **QSymm** $\cong \otimes_\alpha \widetilde{H}_*(A_\alpha;\mathbb{Z}_{(p)})$. Baker and Richter gave the following example. Recall that $H^*(\mathbb{C}P^\infty;\mathbb{Z})\cong\mathbb{Z}[x]$, where x is in degree 2. In [MNT] it was shown that that there is a p-local homotopy decomposition $\Sigma\mathbb{C}P^\infty\simeq\bigvee_{i=1}^{p-1} A_i$ where $H^*(A_i)$ consists of those elements in $\Sigma H^*(\mathbb{C}P^\infty)$ in degrees of the form $2i+1+2k(p-1)$ for some $k \geq 0$. We obtain a homotopy decomposition

$$\Omega\Sigma\mathbb{C}P^{\infty} \simeq \Omega(\bigvee_{i=1}^{p-1} A_i). \tag{4.2}$$

Note that each A_i is a simply-connected co-H-space as it is a retract of $\Sigma\mathbb{C}P^{\infty}$. Baker and Richter used this to further decompose $\Omega\Sigma\mathbb{C}P^{\infty}$ by anticipating the generalization of the Hilton-Milnor Theorem in Theorem 1.2.

We give a different p-local homotopy decomposition of $\Omega\Sigma\mathbb{C}P^{\infty}$ which is finer than Baker and Richter's, and which therefore implies a correspondingly finer p-local algebra decomposition of **QSymm**. By [Ga], for any simply-connected space X there is a homotopy fibration $\Sigma\Omega X \wedge \Omega X \longrightarrow \Sigma\Omega X \xrightarrow{ev} X$ which splits after looping as $\Omega\Sigma\Omega X \simeq \Omega X \times \Omega(\Sigma\Omega X \wedge \Omega X)$. In our case, let $B\mathbb{C}P^{\infty}$ be the Eilenberg-MacLane space $K(\mathbb{Z}, 3)$, so $\Omega B\mathbb{C}P^{\infty} \simeq \mathbb{C}P^{\infty}$. Then we obtain a homotopy fibration

$$\Sigma\mathbb{C}P^{\infty} \wedge \mathbb{C}P^{\infty} \longrightarrow \Sigma\mathbb{C}P^{\infty} \xrightarrow{ev} B\mathbb{C}P^{\infty}$$

and a homotopy decomposition

$$\Omega\Sigma\mathbb{C}P^{\infty} \simeq \mathbb{C}P^{\infty} \times \Omega(\Sigma\mathbb{C}P^{\infty} \wedge \mathbb{C}P^{\infty}).$$

Consider three decompositions of the space $\Sigma\mathbb{C}P^{\infty} \wedge \mathbb{C}P^{\infty}$. First, applying the decomposition from [MNT] on the left factor we obtain

$$\Sigma\mathbb{C}P^{\infty} \wedge \mathbb{C}P^{\infty} \simeq (\bigvee_{i=1}^{p-1} A_i) \wedge \mathbb{C}P^{\infty} \simeq \bigvee_{i=1}^{p-1} (A_i \wedge \mathbb{C}P^{\infty}).$$

Second, moving the suspension to the right wedge summand gives a similar homotopy decomposition

$$\mathbb{C}P^{\infty} \wedge \Sigma\mathbb{C}P^{\infty} \simeq \mathbb{C}P^{\infty} \wedge (\bigvee_{i=1}^{p-1} A_i) \simeq \bigvee_{i=1}^{p-1} (\mathbb{C}P^{\infty} \wedge A_i).$$

Third, let $T\colon \mathbb{C}P^{\infty} \wedge \mathbb{C}P^{\infty} \longrightarrow \mathbb{C}P^{\infty} \wedge \mathbb{C}P^{\infty}$ be the map interchanging factors. Define self-maps of $\Sigma\mathbb{C}P^{\infty} \wedge \mathbb{C}P^{\infty}$ by $e_1 = (1 + \Sigma T)/2$ and $e_2 = (1 - \Sigma T)/2$. Observe that $(e_1)_*$ and $(e_2)_*$ are idempotents, $(e_1)_* \circ (e_2)_* = 0$, and $(e_1)_* + (e_2)_* = 1$. Therefore, if E_1 and E_2 are the mapping telescops of e_1 and e_2 respectively, then adding gives a map $e\colon \Sigma\mathbb{C}P^{\infty} \wedge \mathbb{C}P^{\infty} \longrightarrow E_1 \vee E_2$ which is a homology isomorphism. Since $\Sigma\mathbb{C}P^{\infty} \wedge \mathbb{C}P^{\infty}$ is simply-connected, Whitehead's theorem implies that e is a homotopy equivalence, giving a decomposition

$$\Sigma\mathbb{C}P^{\infty} \wedge \mathbb{C}P^{\infty} \simeq \bigvee E_1 \vee E_2.$$

The homotopy theoretic Krull-Schmidt theorem in [Gr] implies that if Y is a simply-connected co-H-space and there are decompositions $f\colon Y \xrightarrow{\simeq} \bigvee_{i=1}^{n} A_i$ and $g\colon Y \xrightarrow{\simeq} \bigvee_{j=1}^{m} B_j$ then g determines a decomposition of each space A_i into a wedge of j summands, and similarly for f with respect to each B_j. In our case, the first two decompositions of $\Sigma\mathbb{C}P^{\infty} \wedge \mathbb{C}P^{\infty}$ above combine to produce a homotopy decomposition

$$\Sigma\mathbb{C}P^{\infty} \wedge \mathbb{C}P^{\infty} \simeq \bigvee_{i,j=1}^{p-1} A_{i,j}$$

where $\widetilde{H}_*(A_{i,j})$ consists of those elements in $H^*(\Sigma\mathbb{C}P^\infty \wedge \mathbb{C}P^\infty)$ in bidegrees of the form $(2i+1+2k(p-1), 2i+2l(p-1))$ for $k, l \geq 0$. Combining this with the third decomposition above gives a refined decomposition

$$\Sigma\mathbb{C}P^\infty \wedge \mathbb{C}P^\infty \simeq \bigvee_{i,j=1}^{p-1} (A^+_{i,j} \vee A^-_{i,j})$$

where $H^*(A^+_{i,j})$ consists of those elements in $H^*(A_{i,j})$ which are also symmetric when considered as elements of $H^*(\Sigma\mathbb{C}P^\infty \wedge \mathbb{C}P^\infty)$ and $H^*(A^-_{i,j})$ is the corresponding complement. Hence there is a homotopy decomposition

$$\Omega\Sigma\mathbb{C}P^\infty \simeq \mathbb{C}P^\infty \times \Omega\left(\bigvee_{i,j=1}^{p-1} (A^+_{i,j} \vee A^-_{i,j})\right). \tag{4.3}$$

Now to give a p-local decomposition of **QSymm**, we can start from the homotopy decomposition in (4.3) involving $p(p-1)$ wedge summands rather than the decomposition in (4.2) involving just $p-1$ summands. As before, Theorem 1.2 can be applied to decompose $\Omega(\bigvee_{i,j=1}^{p-1}(A^+_{i,j} \vee A^-_{i,j}))$ into a product of looped co-H-spaces. If desired, each factor can be even further decomposed using Theorem 4.1.

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