From Sturm-Liouville problems to fractional and anomalous diffusions

Mirko D'Ovidio
Sapienza University of Rome

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Abstract Some fractional and anomalous diffusions are driven by equations involving fractional derivatives in both time and space. Such diffusions are processes with randomly varying times. In representing the solutions to those diffusions, the explicit laws of certain stable processes turn out to be fundamental. This paper directs one's efforts towards the explicit representation of solutions to fractional and anomalous diffusions related to Sturm-Liouville problems of fractional order associated to fractional power function spaces. Furthermore, we study a new version of the Bochner's subordination rule and we establish some connections between subordination and space-fractional operator.

Keywords: Anomalous diffusion, Sturm-Liouville problem, Stable subordinator, Mellin convolution.

Contents

1 Introduction

In recent years, many researchers have shown their interest in fractional and anomalous diffusions. The term fractional is achieved by replacing standard derivatives w.r.t. time t with fractional derivatives, for instance, those of Riemann-Liouville or Dzhrbashyan-Caputo. Anomalous diffusion occurs, according to most of the significant literature, when the mean square displacement (or time-dependent variance) is stretched by some index, say $\alpha \neq 1$ or, in other words proportional to a power α of time, for instance t^α. Such anomalous feature can be found in transport phenomena in complex systems, e.g. in random fractal structures (see Giona and Roman [16]).

Fractional diffusions have been studied by several authors. Wyss [48], Schneider and Wyss [45] and later Hilfer [20] studied the solutions to the heat-type fractional diffusion equation and presented such solutions in terms of Fox's functions. For the same equation, up to some scaling constant, Beghin and Orsingher [4]; Orsingher and Beghin [40] represented the solutions by means of stable densities and found the explicit representations only in some cases. Different boundary value problems have also been studied by Metzler and Klafter [36]; Beghin and Orsingher [3]. In the papers by Mainardi et al. [27, 29, 30] the authors presented the solutions to space-time fractional equations by means of Wright functions or Mellin-Barnes integral representations, that is Fox's functions. See also Mainardi et al. [31] for a review on the Mainardi-Wright function.

For a general operator $\mathscr{A}$ acting on space, several results can also be listed. Nigmatullin [39] gave a physical interpretation when $\mathscr{A}$ is the generator of a Markov process whereas Kochubei [23, 24] first introduced a mathematical approach. Zaslavsky [49] introduced the fractional kinetic equation for Hamiltonian chaos. Baeumer and Meerschaert [1] studied the problem when $\mathscr{A}$ is an infinitely divisible generator on a finite dimensional space. For a short survey of these results see Nane [38].

In general, the stochastic solutions to fractional diffusion equations can be realized through subordination. Indeed, for a guiding process $X(t)$ with generator $\mathscr{A}$ we have that $X(V(t))$ is governed by $\partial_t^\beta u = \mathscr{A} u$ where the process $V(t)$, $t > 0$ is an inverse or hitting time process to a β-stable subordinator (see Baeumer and Meerschaert [1]). Thus, explicit representations of stable densities are fundamental in finding explicit solutions to fractional problems. The time-fractional derivative comes from the fact that $X(V(t))$ can be viewed as a scaling limit of continuous time random walk where the iid jumps are separated by iid power law waiting times (see Meerschaert and Scheffler [34]).

In this paper we will study some Sturm-Liouville problems of fractional order associated with fractional power function spaces. Such a study leads to the generalized gamma densities and, as a special case, to the 1-dimensional laws of some well-known processes (as the Bessel process and its squared version). We will deal with Fourier and Mellin convolutions of generalized gamma densities in order to find out the explicit representations of the solutions to some fractional equations which belong to the class of stable densities. Those representations turn out to be useful in representing the solutions to time/space (or time-space) fractional diffusion equations including the equation whose stochastic solution appears as a new version of the Bochner's subordination rule. Here, we study a fractional operator acting on space and associated with fractional power function spaces which is the infinitesimal generator of a subordinated squared Bessel process.

In Section 2 we introduce and study the operator

$$\mathcal{G}^* f(x) = \frac{1}{\gamma^2}\frac{\partial}{\partial x}\left(x^{\gamma\mu-\gamma+1}\frac{\partial}{\partial x}\left(x^{1-\gamma\mu}f(x)\right)\right)$$

where $x, t > 0$, $\gamma \neq 0$ and $\mu > 0$. In particular, we study the solutions to the p.d.e.

$$\frac{\partial}{\partial t} g^{\gamma}_{\mu}(x,t) = \mathcal{G}^* g^{\gamma}_{\mu}(x,t). \tag{1.1}$$

Such solutions belong to the family of generalized gamma densities (or generalized reciprocal gamma densities). Throughout the paper we will often refer to $G^{\gamma}_{\mu}(t)$, $t > 0$ (for some $\gamma \neq 0$, $\mu > 0$) as the generalized gamma process. We are aware that we are making some abuse of language by considering a process without its covariance structure. For our purpose, this assumption will be useful in better handling the mathematical tools we will deal with. Moreover, we will consider only one-dimensional processes and thus, only one-dimensional marginals are involved. Thus, the stochastic solutions to (1.1) are represented by $G^{\gamma}_{\mu}(t)$, $t > 0$. In particular, for $\gamma = 1$, we have the squared Bessel process $S_\mu = G^1_\mu$, satisfying the stochastic differential equation

$$dS_\mu(t) = \mu\, dt + 2\sqrt{S_\mu(t)}\, dB_1(t)$$

whereas, for $\gamma = -1$, the stochastic solution to (1.1) is the process satisfying the stochastic equation

$$dE_\mu(t) = -\left(E_\mu(t) - \frac{1}{\mu-1}\right) dt + \sqrt{\frac{2\,|E_\mu(t)|^2}{\mu-1}}\, dB_2(t)$$

where $B_1(t)$, $t > 0$ is a Brownian motion with variance $t/2$ and $B_2(t)$, $t > 0$ is a standard Brownian motion. The process $E_\mu = G^{-1}_\mu$, is the reciprocal gamma process which also emerges, in this paper, as the inverse process to a μ-dimensional squared Bessel process S_μ. Indeed, between the processes S_μ and E_μ there exists the relation

$$Pr\{S_\mu(x) > t\} = Pr\{E_\mu(t) < x\}.$$

For $\gamma = 2$ in (1.1), we obtain the governing equation of $R_{2\mu} = S^{1/2}_{2\mu}$ which is the 2μ-dimensional Bessel process starting from the origin.

In Section 3 we show the role of g^{γ}_{μ}, $\gamma \neq 0$ in representing the laws of stable subordinators $\tilde{\tau}^{\nu}_t$, $t > 0$, say h_ν, $\nu \in (0,1)$ and those of inverse stable subordinators L^{ν}_t, $t > 0$, say l_ν, $\nu \in (0,1)$. In particular we exploit the Mellin convolution formula in order to write both h_ν and l_ν as integrals of modified Bessel functions of the second kind (Macdonald's functions). Such convolutions are invariant under permutation of the characterizing parameters. In other words, for $\bar{\mu} = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}^n_+$, the convolution involving n functions $g^{\gamma}_{\mu_j}$, $j = 1, 2, \ldots, n$ is invariant under different choices of $\bar{\mu}$ if and only if $\prod_{j=1}^n \mu_j = const$. To be clear, for $\nu = 1/(n+1)$, $n \in \mathbb{N}$, we show that

$$h_\nu(x,t) = g^{-1,\star n}_{\bar{\mu}}(x, \varphi_{n+1}(t)), \quad \text{and} \quad l_\nu(x,t) = g^{(n+1),\star n}_{\bar{\mu}}(x, \psi_{n+1}(t))$$

for some φ and ψ (function of t) such that $\varphi = \psi^{-1}$ and $\forall \bar{\mu} \in \mathscr{P}^n_{n+1}(n!)$ where the set $\mathscr{P}^m_\kappa(\varrho)$, for $m, \kappa, \varrho \in \mathbb{N}$, is defined in (3.28), see Theorem 2. Here, we have used the Mellin convolution $f_{\bar{\mu}}^{\star n} = f_{\mu_1} \star \ldots \star f_{\mu_n}$, see Definition 3. The law g_μ^{-1} is the one-dimensional marginal of $E_\mu(t)$, $t>0$ and thus, the stable subordinator $\tilde{\tau}_t^\nu$, $t>0$ can be written as the composition of n reciprocal gamma processes,

$$\tilde{\tau}_t^\nu \overset{law}{=} E_{\mu_1}(E_{\mu_2}(\ldots E_{\mu_n}(t)\ldots)), \quad \bar{\mu} = (\mu_1, \ldots, \mu_n) \in \mathscr{P}_{n+1}^n(n!)$$

where E_{μ_j}, $j=1,2,\ldots,n$ are independent copies of E_μ.

In Section 4 we study the equation (1.1) in the half one-dimensional space and, in a bounded domain $\Omega_a = (0,a)$, $a>0$, when the time derivative is replaced by the Riemann-Liouville fractional derivative. Moreover, we show that the solution to (1.1) with $\gamma = 1$ coincides with the solution to

$$D_{0+,t}^\nu\, g_\mu^1(x,t) = D_{0-,x}^\nu \Big(x^{\mu-1+\nu} D_{0-,x}^\nu \Big(x^{1-\mu}\, g_\mu^1(x,t) \Big) \Big), \quad \forall\, \nu \in (0,1] \tag{1.2}$$

on Ω_∞, whose stochastic solution is the squared Bessel process S_μ and $D_{0\pm,x}^\nu$ are the Riemann-Liouville fractional derivatives to be defined below. For the fractional equation

$$-D_{0+,t}^\beta \mathbf{g}_\mu^{\nu,\beta}(x,t) = D_{0+,x}^\nu \Big(x^{\mu-1+\nu} D_{0-,x}^\nu \Big(x^{1-\mu}\, \mathbf{g}_\mu^{\nu,\beta}(x,t) \Big) \Big), \quad \beta, \nu \in (0,1] \tag{1.3}$$

on Ω_∞, we show that the solutions to (1.3) can be written in terms of H functions as

$$\mathbf{g}_\mu^{\nu,\beta}(x,t) = \frac{1}{t^{\beta/\nu}} \mathbf{G}_\mu^{\nu,\beta} \left(\frac{x}{t^{\beta/\nu}} \right) \tag{1.4}$$

where

$$\mathbf{G}_\mu^{\nu,\beta}(x) = \frac{1}{x} H_{3,3}^{2,1} \left[x \middle| \begin{matrix} (1,\frac{1}{\nu}); & (1,\frac{\beta}{\nu}); & (\mu,0) \\ (\mu,1); & (1,\frac{1}{\nu}); & (0,1) \end{matrix} \right], \quad x>0.$$

Furthermore, we present the explicit representations of (1.4) as the laws of the compositions $S_\mu(\tilde{\tau}_{L_t^\beta}^\nu)$, $t>0$ where S_μ is the μ-dimensional squared Bessel process and $\tilde{\tau}_{L_t^\beta}^\nu$, $t>0$, $\nu,\beta \in (0,1)$, is governed by

$$(D_{0+,t}^\beta + D_{0+,x}^\nu) \mathfrak{f}_{\nu,\beta}(x,t) = \delta(x) t^{-\beta}/\Gamma(1-\beta), \quad x \geq 0,\, t>0.$$

The novelty here is the connection between subordination and space-fractional operator which extends the results given by D'Ovidio and Orsingher [13]. In that paper, stable subordinators are leading to higher-order derivatives in time. Furthermore, this result is strictly related to the Bochner's subordination rule for which

$$\frac{\partial u}{\partial t} = \frac{\partial^{2\nu} u}{\partial |x|^{2\nu}}, \quad x \in \mathbb{R},\ t>0,\ \nu \in (0,1)$$

is the governing equation of the subordinated process $B(\tilde{\tau}_t^\nu)$, $t>0$. Indeed, we show that a fractional version of the operator $\mathcal{G}^*$ is the generator of the subordinated process $S_\mu(\tilde{\tau}_t^\nu)$, $t>0$, which is the stochastic solution to (1.3) for $\beta = 1$.

2 The generalized gamma functions

2.1 Preliminaries

Throughout the paper we will use the Mellin machinery and the strictly related Fox's H-functions. For this reason we give here a short introduction to those arguments.

Definition 1. *For $-\infty < a < b < \infty$, we define the space*

$$\mathbb{M}_a^b = \left\{ f : \mathbb{R}_+ \mapsto \mathbb{C} \,\middle|\, x^{\eta-1} f(x) \in L^1(\mathbb{R}_+), \forall \eta \in \mathbb{H}_a^b \right\}$$

where $\mathbb{H}_a^b = \{\zeta : \zeta \in \mathbb{C}, a < \Re\{\zeta\} < b\}$.

The Mellin transform of $f \in \mathbb{M}_a^b$ is defined as

$$\mathcal{M}[f(\cdot)](\eta) = \int_0^\infty x^{\eta-1} f(x) dx, \quad \eta \in \mathbb{H}_a^b.$$

We say that $f \in \widetilde{\mathbb{M}}_k$ if $f \in \mathbb{M}_a^b$ and is a rapidly decreasing function such that

$$\exists \mathfrak{a} \in \mathbb{R} \; s.t. \; \lim_{x \to +\infty} x^{\mathfrak{a}-k-1} \frac{d^k}{dx^k} f(x) = 0, \quad k = 0, 1, \dots, n-1, \quad n \in \mathbb{N}, \; x \in \mathbb{R}_+$$

and

$$\exists \mathfrak{b} \in \mathbb{R} \; s.t. \; \lim_{x \to 0^+} x^{\mathfrak{b}-k-1} \frac{d^k}{dx^k} f(x) = 0, \quad k = 0, 1, \dots, n-1, \quad n \in \mathbb{N}, \; x \in \mathbb{R}_+.$$

For $f \in \widetilde{\mathbb{M}}_{n-1}$ and $n \in \mathbb{N}$ we have that

$$\begin{aligned} \mathcal{M}\left[\frac{d^n}{dx^n} f(\cdot)\right](\eta) =& (-1)^n \frac{\Gamma(\eta)}{\Gamma(\eta-n)} \mathcal{M}\left[f(\cdot)\right](\eta-n) \\ =& \frac{\Gamma(1+n-\eta)}{\Gamma(1-\eta)} \mathcal{M}\left[f(\cdot)\right](\eta-n). \end{aligned} \tag{2.1}$$

Let us point out some useful operational rules that will be useful throughout the paper: for some $-\infty < a < b < \infty$ and $\mathfrak{b} > 0$, $f, f_1, f_2 \in \mathbb{M}_a^b$:

$$\int_0^\infty x^{\eta-1} f(\mathfrak{b}x) dx = \mathfrak{b}^{-\eta} \mathcal{M}\left[f(\cdot)\right](\eta), \tag{2.2}$$

$$\mathcal{M}\left[x^{\mathfrak{b}} f(\cdot)\right](\eta) = \mathcal{M}\left[f(\cdot)\right](\eta+\mathfrak{b}), \tag{2.3}$$

$$\mathcal{M}\left[\int_0^\infty f_1\left(\frac{\cdot}{s}\right) f_2(s) \frac{ds}{s}\right](\eta) = \mathcal{M}\left[f_1(\cdot)\right](\eta) \times \mathcal{M}\left[f_2(\cdot)\right](\eta), \tag{2.4}$$

$$\mathcal{M}[I(\cdot)](\eta) = \eta^{-1} \mathcal{M}[f(\cdot)](\eta+1), \tag{2.5}$$

where

$$I(x) = \int_x^\infty f(s) ds, \quad x > 0, \tag{2.6}$$

see e.g. Glaeske et al. [17]. The formula (2.4) is the well-known Mellin convolution formula which turns out to be very useful in the study of the product of random variables.

The Fox functions, also referred to as Fox's H-functions, H-functions, generalized Mellin-Barnes functions, or generalized Meijer's G-functions, were introduced by Fox [15] in 1996. Here, the Fox's H-functions will be recalled as the class of functions uniquely identified by their Mellin transforms. A function $f \in \mathbb{M}_a^b$ can be written in terms of H-functions by observing that

$$\int_0^\infty x^\eta H_{p,q}^{m,n}\left[x\middle|\begin{array}{l}(a_i,\alpha_i)_{i=1,..,p}\\(b_j,\beta_j)_{j=1,..,q}\end{array}\right]\frac{dx}{x} = \mathcal{M}_{p,q}^{m,n}\left[\eta\middle|\begin{array}{l}(a_i,\alpha_i)_{i=1,..,p}\\(b_j,\beta_j)_{j=1,..,q}\end{array}\right], \quad \eta\in\mathbb{H}_a^b \tag{2.7}$$

where

$$\mathcal{M}_{p,q}^{m,n}\left[\eta\middle|\begin{array}{l}(a_i,\alpha_i)_{i=1,..,p}\\(b_j,\beta_j)_{j=1,..,q}\end{array}\right] = \frac{\prod_{j=1}^m\Gamma(b_j+\eta\beta_j)\prod_{i=1}^n\Gamma(1-a_i-\eta\alpha_i)}{\prod_{j=m+1}^q\Gamma(1-b_j-\eta\beta_j)\prod_{i=n+1}^p\Gamma(a_i+\eta\alpha_i)}. \tag{2.8}$$

Thus, according to a standard notation, the Fox H-function can be defined as follows

$$H_{p,q}^{m,n}\left[x\middle|\begin{array}{l}(a_i,\alpha_i)_{i=1,..,p}\\(b_j,\beta_j)_{j=1,..,q}\end{array}\right] = \frac{1}{2\pi i}\int_{\mathcal{P}(\mathbb{H}_a^b)}\mathcal{M}_{p,q}^{m,n}(\eta)x^{-\eta}d\eta$$

where $\mathcal{P}(\mathbb{H}_a^b)$ is a suitable path in the complex plane $\mathbb{C}$ depending on the fundamental strip $(\mathbb{H}_a^b)$ such that the integral (2.7) converges. For an extensive discussion on this functions see Fox [15]; Kilbas et al. [22]; Mathai and Saxena [32].

We recall the Modified Bessel function (of imaginary argument) K_α also known as Macdonald's function and the Bessel function of the first kind J_α. In particular

$$K_\alpha(z) = \frac{\pi}{2}\frac{I_{-\alpha}(z)-I_\alpha(z)}{\sin\nu\pi}, \quad \alpha \text{ not integer} \tag{2.9}$$

(see [19, formula 8.485]) where

$$I_\alpha(z) = \sum_{k\geq 0}\frac{(z/2)^{\alpha+2k}}{k!\,\Gamma(\alpha+k+1)}, \quad |z|<\infty, \quad |\arg z|<\pi \tag{2.10}$$

is the modified Bessel function of the first kind (see [19, formula 8.445]). The Bessel function of the first kind writes

$$J_\alpha(z) = \sum_{k\geq 0}\frac{(-1)^k\,(z/2)^{\alpha+2k}}{k!\,\Gamma(\alpha+k+1)}, \quad |z|<\infty, \quad |\arg z|<\pi \tag{2.11}$$

(see [26, p. 102]). The functions K_α and I_α are two linearly independent solutions of the Bessel equation

$$x^2\frac{d^2Z_\alpha(x)}{dx^2}+x\frac{dZ_\alpha(x)}{dx}-x^2Z_\alpha(x)=0 \tag{2.12}$$

whereas, the functions J_α and Y_α (see [26] for definition) are linearly independent solutions to

$$x^2\frac{d^2Z_\alpha(x)}{dx^2}+x\frac{dZ_\alpha(x)}{dx}+x^2Z_\alpha(x)=0 \tag{2.13}$$

(see [26, pp. 105 - 110]). Furthermore, (see Lebedev [26, pp. 102])

$$I_\beta(x) = e^{i\frac{\pi\beta}{2}}J_\beta\left(x\,e^{-i\frac{\pi}{2}}\right), \quad -\pi < arg\, x < \pi/2.$$

2.2 The operators $\mathcal{G}$ and $\mathcal{G}^*$

The operators we deal with are given by

$$\begin{aligned}\mathcal{G}^*\, f_2 =& \frac{1}{\gamma^2}\left(\frac{\partial}{\partial x}x^{2-\gamma}\frac{\partial}{\partial x} - (\gamma\mu-1)\frac{\partial}{\partial x}x^{1-\gamma}\right) f_2 \\ =&\frac{1}{\gamma^2}\frac{\partial}{\partial x}\left(x^{\gamma\mu-\gamma+1}\frac{\partial}{\partial x}\left(\frac{1}{\mathfrak{w}(x)}\,f_2\right)\right), \quad f_2\in D(\mathcal{G}^*)\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}\mathcal{G}\, f_1 =& \frac{x^{1-\gamma}}{\gamma^2}\left(x\frac{\partial^2}{\partial x^2} + (\gamma\mu-\gamma+1)\frac{\partial}{\partial x}\right) f_1 \\ =&\frac{1}{\gamma^2\,\mathfrak{w}(x)}\frac{d}{dx}\left(x^{\gamma\mu-\gamma+1}\frac{d}{dx}f_1\right), \quad f_1\in D(\mathcal{G})\end{aligned} \tag{2.15}$$

where $\mathfrak{w}(x) = x^{\gamma\mu-1}$ is the weight function. We shall refer to $\mathcal{G}^*$ as the adjont of $\mathcal{G}$. Indeed, as a straightforward check shows, we have that $\mathcal{G}^*\,\mathfrak{w}\,f_1 = \mathfrak{w}\,\mathcal{G}\,f_1$ and the Lagrange's identity

$$f_2\,\mathcal{G}\,f_1 - f_1\,\mathcal{G}^*\,f_2 = 0 \tag{2.16}$$

immediately follows. Thus, by observing that

$$D(\mathcal{G}^*) = \{f\in\tilde{\mathbb{M}}_1 : f = \mathfrak{w}\,f_1,\ f_1\in D(\mathcal{G})\},$$

we obtain that $\langle\mathcal{G}f_1,\, f_2\rangle = \langle f_1,\, \mathcal{G}^* f_2\rangle$ holds true $\forall f_1\in D(\mathcal{G})$ and $\forall f_2\in D(\mathcal{G}^*)$.

Lemma 1. *The following hold true:*

i) For the operator appearing in (2.15) we have that

$$\mathcal{G}\,\psi_\kappa = (\kappa/2)^2\,\psi_\kappa \tag{2.17}$$

where

$$\psi_\kappa(x) = x^{\frac{\gamma}{2}(1-\mu)}K_{\mu-1}\left(\kappa\,x^{\gamma/2}\right), \quad \kappa>0,\ x>0,\ \gamma\neq 0 \tag{2.18}$$

and K_α is the Macdonald's function (2.9).

ii) For the operator appearing in (2.15) we have that

$$\mathcal{G}\,\bar{\psi}_\kappa = -(\kappa/2)^2\,\bar{\psi}_\kappa \tag{2.19}$$

where

$$\bar{\psi}_\kappa(x) = x^{\frac{\gamma}{2}(1-\mu)}J_{\mu-1}\left(\kappa\,x^{\gamma/2}\right), \quad \kappa>0,\ x>0,\ \gamma\neq 0 \tag{2.20}$$

and J_α is the Bessel function of the first kind (2.11).

Proof. We first recall some properties of the Macdonald's function (2.9). In particular we will use the fact that $K_{-\alpha} = K_\alpha$ and

$$\frac{d}{dz}K_\alpha(z) = -K_{\alpha-1}(z) - \frac{\alpha}{z}K_\alpha(z). \tag{2.21}$$

(see [26, p. 110]). By performing the first and the second derivative with respect to x of the function $\psi_\kappa = \psi_\kappa(x)$ we obtain

$$\begin{aligned}\psi'_\kappa =& \frac{\gamma}{2}(1-\mu)\frac{1}{x}\psi_\kappa + x^{\frac{\gamma}{2}(1-\mu)}\frac{\gamma\kappa}{2x}x^{\gamma/2}\left[-K_{-\mu} - \frac{1-\mu}{\kappa x^{\gamma/2}}K_{1-\mu}\right]\\ =& \frac{\gamma}{2}(1-\mu)\frac{1}{x}\psi_\kappa - \frac{\gamma\kappa}{2x}x^{\frac{\gamma}{2}(2-\mu)}K_{-\mu} - \frac{\gamma}{2}(1-\mu)\frac{1}{x}\psi_\kappa = -\frac{\gamma\kappa}{2x}x^{\frac{\gamma}{2}(2-\mu)}K_{-\mu}\end{aligned}$$

and

$$\begin{aligned}\psi''_\kappa =& \left(\frac{\gamma}{2}(2-\mu)-1\right)\frac{1}{x}\psi'_\kappa + \frac{\gamma\kappa}{2}x^{\frac{\gamma}{2}(2-\mu)-1}\frac{\gamma\kappa}{2x}x^{\gamma/2}\left[-K_{\mu-1} - \frac{\mu}{\kappa x^{\gamma/2}}K_\mu\right]\\ =& \left(\frac{\gamma}{2}(2-\mu)-1\right)\frac{1}{x}\psi'_\kappa - \left(\frac{\gamma\kappa}{2}\right)^2\frac{x^{\frac{\gamma}{2}(1-\mu)+\gamma}}{x^2}K_{\mu-1} - \frac{\gamma\mu}{2x}\psi'_\kappa.\end{aligned}$$

By keeping in mind the operator $\mathcal{G}$, from the fact that

$$x\psi''_\kappa + (\gamma\mu - \gamma + 1)\psi'_\kappa = x^{\gamma-1}\frac{\gamma^2\kappa^2}{2^2}\psi_\kappa \tag{2.22}$$

the relation (2.17) is obtained. The equation (2.22) can be rewritten as

$$x^2\,\psi''_\kappa + (\gamma\mu - \gamma + 1)\,x\,\psi'_\kappa - \gamma^2\,(\kappa/2)^2\,x^\gamma\,\psi_\kappa = 0 \tag{2.23}$$

which is related to the formula (2.12) whereas, a slightly modified version of (2.23), that is

$$x^2\,\bar\psi''_\kappa + (\gamma\mu - \gamma + 1)\,x\,\bar\psi'_\kappa + \gamma^2\,(\kappa/2)^2\,x^\gamma\,\bar\psi_\kappa = 0 \tag{2.24}$$

is related to the formula (2.13). The equation (2.19) can be written as formula (2.24) or equivalently as (2.25). After some algebra, from (2.13), we have at once that

$$\bar\psi_\kappa(x) = x^{\frac{\gamma}{2}(1-\mu)}J_{\mu-1}\left(\kappa\,x^{\gamma/2}\right)$$

as announced in the statement of the Lemma. □

The formula (2.24) can be put into the Sturm-Liouville form as follows

$$\left(x^{\gamma\mu-\gamma+1}\bar\psi'_\kappa\right)' + \gamma^2\,(\kappa/2)^2\,\mathfrak{w}(x)\,\bar\psi_\kappa = 0. \tag{2.25}$$

According to the Sturm-Liouville theory and formula (2.25), we have obtained an orthogonal system $\{\bar\psi_{\kappa_i}\}$ such that

$$\mathcal{G}\,\bar\psi_{\kappa_i} = -(\kappa_i/2)^2\,\bar\psi_{\kappa_i}, \tag{2.26}$$

where κ_i are the zeros of J_α. In particular, from the formulas (2.26) and (2.16), we have that $\mathcal{G}$ is a Hermitian linear operator and then

$$\int_0^1 \bar\psi_{\kappa_i}(x)\bar\psi_{\kappa_j}(x)\mathfrak{w}(x)dx = 0, \quad \text{if } i \neq j \tag{2.27}$$

where $\mathfrak{w}(x) = x^{\gamma\mu-1}$ is the weight function. In a more general setting we have

$$\int_0^\infty \bar\psi_{\kappa_i}(x)\bar\psi_{\kappa_j}(x)\mathfrak{w}(x)dx = \delta(\kappa_i - \kappa_j)/\kappa_j \tag{2.28}$$

which leads to the Hankel transform of a well-defined function f, that is

$$(\mathcal{H}\,f)(\rho) = \int_0^\infty xJ_\nu(\rho x)f(x)dx \quad \text{and} \quad f(x) = \int_0^\infty \rho J_\nu(\rho x)(\mathcal{H}\,f)(\rho)d\rho.$$

2.3 The generalized gamma densities

The well known generalized gamma density of the r.v. G^γ_μ is given by

$$Q^\gamma_\mu(z) = \gamma \frac{z^{\gamma\mu-1}}{\Gamma(\mu)} \exp\{-z^\gamma\}, \quad z>0,\, \gamma>0,\, \mu>0$$

by means of which we define the distributions we are going to investigate, that are

$$g^\gamma_\mu(x,t) = \mathrm{sign}(\gamma)\frac{1}{t}Q^\gamma\left(\frac{x}{t}\right) \quad \text{and} \quad \tilde{g}^\gamma_\mu(x,t) = g^\gamma_\mu(x, t^{1/\gamma}). \tag{2.29}$$

We begin by stating the following result.

Lemma 2. *For $\mu>0$, $\gamma \neq 0$, the density law*

$$\tilde{g}^\gamma_\mu(x,t) = \mathfrak{w}(x)\tilde{k}^\gamma_\mu(x,t) = |\gamma| \frac{\mathfrak{w}(x)}{t^\mu \Gamma(\mu)} \exp\left\{-\frac{x^\gamma}{t}\right\}, \quad x,t>0, \tag{2.30}$$

say $\tilde{g}^\gamma_\mu = \tilde{g}^\gamma_\mu(x,t)$, satisfies the following p.d.e.

$$\frac{\partial}{\partial t}\tilde{g}^\gamma_\mu = \mathcal{G}^*\, \tilde{g}^\gamma_\mu, \quad x>0,\ t>0 \tag{2.31}$$

where $\mathcal{G}^*$ is given in (2.14) and $\mathfrak{w}(x) = x^{\gamma\mu-1}$ is the weight function.

Proof. Consider $\gamma>0$. The Mellin transform of the function (2.30) reads

$$\Psi_t(\eta) = \mathcal{M}[\tilde{g}^\gamma_\mu(\cdot,t)](\eta) = \Gamma\left(\frac{\eta-1}{\gamma}+\mu\right)\frac{t^{\frac{\eta-1}{\gamma}}}{\Gamma(\mu)}, \quad \eta \in \mathbb{H}^\infty_{1-\gamma\mu}. \tag{2.32}$$

We perform the time derivative of the formula (2.32) and we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\Psi_t(\eta) =& \frac{\eta-1}{\gamma}\Gamma\left(\frac{\eta-1}{\gamma}+\mu\right)t^{\frac{\eta-\gamma-1}{\gamma}}\\
=& \frac{\eta-1}{\gamma}\left(\frac{\eta-\gamma-1+\gamma\mu}{\gamma}\right)\Gamma\left(\frac{\eta-\gamma-1}{\gamma}+\mu\right)t^{\frac{\eta-\gamma-1}{\gamma}}\\
=& \frac{1}{\gamma^2}(\eta-1)(\eta-\gamma-1+\gamma\mu)\Psi_t(\eta-\gamma)\\
=& \frac{1}{\gamma^2}(\eta-1)(\eta-\gamma)\Psi_t(\eta-\gamma) + \frac{1}{\gamma^2}(\eta-1)(\gamma\mu-1)\Psi_t(\eta-\gamma)\\
=& \frac{1}{\gamma^2}\mathcal{M}\left\{\frac{\partial}{\partial x}x^{2-\gamma}\frac{\partial}{\partial x}\tilde{g}^\gamma_\mu\right\}(\eta) - \frac{(\gamma\mu-1)}{\gamma^2}\mathcal{M}\left\{\frac{\partial}{\partial x}x^{1-\gamma}\tilde{g}^\gamma_\mu\right\}(\eta).
\end{aligned}$$

From the fact that $\tilde{g}^\gamma_\mu \in \tilde{\mathbb{M}}_1$ and according to the properties (2.1), (2.2) and (2.3), the inverse Mellin transform yields the claimed result. Similar calculations must be done for $\gamma<0$ for the proof to be completed. □

We observe that

$$\frac{\partial}{\partial t}\tilde{g}^\gamma_\mu(x,t) = \mathcal{G}^*\, \tilde{g}^\gamma_\mu(x,t), \qquad \Leftrightarrow \qquad \frac{\partial}{\partial t}\tilde{k}^\gamma_\mu(x,t) = \mathcal{G}\, \tilde{k}^\gamma_\mu(x,t) \tag{2.33}$$

where the kernel $\tilde{k}^\gamma_\mu(x,t)$ is such that $\tilde{g}^\gamma_\mu(x,t) = \mathfrak{w}(x)\,\tilde{k}^\gamma_\mu(x,t)$. Moreover, from (2.32) and by direct inspection of (2.8) we obtain

$$g^\gamma_\mu(x,t) = \frac{|\gamma|}{x} H^{1,0}_{1,1}\left[\frac{x^\gamma}{t^\gamma} \middle| \begin{array}{c} (\mu,0) \\ (\mu,1) \end{array} \right], \quad x,t>0,\ \gamma \neq 0. \tag{2.34}$$

The r.v. G^γ_μ possesses law $g^\gamma_\mu(x,1)$, $x>0$ and a straightforward check shows that $E^\gamma_\mu = G^{-\gamma}_\mu \stackrel{law}{=} 1/G^\gamma_\mu$ is distributed as $e^\gamma_\mu = g^{-\gamma}_\mu$. Thus, we can refer to $e^\gamma_\mu = e^\gamma_\mu(x,1)$, $x>0$, as the reciprocal generalized gamma density. Let us write $e^\gamma_\mu(x,t)$, $x,t>0$ as the one-dimensional law of the reciprocal process $E^\gamma_\mu(t)$, $t>0$. We observe that $E^\gamma_\mu(t)$ can be also viewed as the inverse of $G^\gamma_\mu(t)$, see [11] for details. Indeed, from $Pr\{G^\gamma_\mu(x)>t\} = Pr\{E^\gamma_\mu(t)<x\}$, we obtain

$$e^\gamma_\mu(x,t) = Pr\{E^\gamma_\mu(t) \in dx\}/dx = \frac{\partial}{\partial x}\int_t^\infty g^\gamma_\mu(s,x)\,ds, \quad x>0,\, t>0$$

and, from (2.5) we have that

$$\begin{aligned} \mathcal{M}[e^\gamma_\mu(x,\cdot)](\eta) =& \frac{\partial}{\partial x}\frac{1}{\eta}\mathcal{M}[g^\gamma_\mu(\cdot,x)](\eta+1) \\ =& \frac{\Gamma(\eta/\gamma+\mu)}{\Gamma(\mu)}x^{\eta-1}, \quad \eta \in \mathbb{M}^\infty_{\gamma\mu}. \end{aligned} \tag{2.35}$$

From (2.35) and (2.8) we obtain

$$e^\gamma_\mu(x,t) = \frac{\gamma}{x} H^{1,0}_{1,1}\left[\frac{t^\gamma}{x^\gamma} \middle| \begin{array}{c} (\mu,0) \\ (\mu,1) \end{array} \right], \quad x,t>0,\ \gamma>0$$

which coincides with (2.34) for $\gamma=-1$. Thus, the reciprocal gamma process $E^\gamma_\mu = G^{-\gamma}_\mu$ is the inverse to a gamma process G^γ_μ. The inverse gamma process $E_\mu = E^1_\mu$, for $\mu = 1/2$, can be viewed as

$$E_{1/2}(t) = \inf\{s;\, B(s) = \sqrt{2t}\}$$

where B is a standard Brownian motion. Thus, $E_{1/2}$ can be interpreted as the first-passage time of a standard Brownian motion through the level $\sqrt{2t}$. As a direct consequense of these facts, for $\mu>0$, we obtain

$$Pr\{S_\mu(x)>t\} = Pr\{E_\mu(t)<x\}, \quad x,t>0$$

and

$$E_\mu(t) \stackrel{law}{=} 1/S_\mu(t), \quad t>0$$

where $S_\mu = G^1_\mu$ is the squared Bessel process starting from the origin. The process S_μ is a non-negative diffusion satisfying the stochastic differential equation

$$dS_\mu(t) = \mu\,dt + 2\sqrt{S_\mu(t)}\,dB_1(t) \tag{2.36}$$

where $B_1(t)$, $t > 0$ is a Brownian motion with variance $t/2$. The reciprocal gamma law $e_\mu = g_\mu^{-1}$, $\mu > 0$, represents the 1-dimensional marginal law of the process satisfying the stochastic equation

$$dE_\mu(t) = -\left(E_\mu(t) - \frac{1}{\mu-1}\right) dt + \sqrt{\frac{2\,|E_\mu(t)|^2}{\mu-1}}\, dB_2(t) \tag{2.37}$$

where $B_2(t)$, $t > 0$ is a standard Brownian motion (see e.g. Bibby et al. [7], Peškir [41]). Due to the global Lipschitz condition on both coefficients, the stochastic equation (2.37) has a unique solution which is a strong Markov process. The process E_μ also appears by considering the integral of a geometric Brownian motion with drift μ, that is

$$\frac{1}{2} E_\mu \stackrel{law}{=} \int_0^\infty \exp\left(2B(s) + 2\mu s\right) ds \tag{2.38}$$

see Dufresne [14]; Pollack and Siegmund [42]. The process G_μ^γ, for $\gamma = 2$, becomes the 2μ-dimensional Bessel process $R_{2\mu} = S_{2\mu}^{1/2}$. Consider now the Rodrigues's formula

$$\mathbb{L}_n^{\gamma,\mu}(x) = \frac{e^{x^\gamma}}{\mathfrak{w}(x)} \frac{d^n}{dx^n}\left(x^{2n} \frac{\mathfrak{w}(x)}{e^{x^\gamma}}\right), \quad \gamma \neq 0,\ \mu > 0. \tag{2.39}$$

The polynomials $\{\mathbb{L}_n^{\gamma,\mu}(x),\, :\, n \in \mathbb{N}\}$ are orthogonal under the generalized gamma density $\mathfrak{w}(x) \exp(-x^\gamma)$, $\gamma \neq 0$. Indeed, we have that

$$\begin{aligned}
\langle \mathbb{L}_n^{\gamma,\mu}(\cdot),\, \mathbb{L}_m^{\gamma,\mu}(\cdot)\rangle_{g_\mu^\gamma} =& \int_0^\infty \mathbb{L}_n^{\gamma,\mu}(x)\, \mathbb{L}_m^{\gamma,\mu}(x)\, g_\mu^\gamma(x,1)\, dx \\
=& \frac{\gamma}{\Gamma(\mu)} \int_0^\infty \frac{d^n}{dx^n}\left(x^{2n} \frac{\mathfrak{w}(x)}{e^{x^\gamma}}\right) \frac{d^m}{dx^m}\left(x^{2m} \frac{\mathfrak{w}(x)}{e^{x^\gamma}}\right) \frac{e^{x^\gamma}}{\mathfrak{w}(x)}\, dx \\
=& \begin{cases} 0, & n \neq m \\ c(n), & n = m \end{cases}
\end{aligned}$$

where

$$\begin{aligned}
&\frac{d^0}{dx^0}\left(x^0 \frac{\mathfrak{w}(x)}{e^{x^\gamma}}\right) = \frac{\mathfrak{w}(x)}{e^{x^\gamma}}, \qquad \frac{d}{dx}\left(x^2 \frac{\mathfrak{w}(x)}{e^{x^\gamma}}\right) = \left[(\gamma\mu+1)x - \gamma x^{\gamma+1}\right] \frac{\mathfrak{w}(x)}{e^{x^\gamma}} \\
&\frac{d^2}{dx^2}\left(x^4 \frac{\mathfrak{w}(x)}{e^{x^\gamma}}\right) = \left[(\gamma\mu+2)(\gamma\mu+3)x^2 - \gamma(2\gamma\mu+\gamma+5)x^{\gamma+2} + \gamma^2 x^{2\gamma+2}\right] \frac{\mathfrak{w}(x)}{e^{x^\gamma}}.
\end{aligned}$$

We notice that $\mathbb{L}_n^{1,\mu}(x) = L_n^\mu(x)$ are the well-known Laguerre polynomials. As a direct consequence of the discussion made so far we obtain, for $n \neq m$:

$$\begin{aligned}
&E[\mathbb{L}_n^{1,\mu}(S_\mu)\, \mathbb{L}_m^{1,\mu}(S_\mu)] = 0, \\
&E[\mathbb{L}_n^{-1,\mu}(E_\mu)\, \mathbb{L}_m^{-1,\mu}(E_\mu)] = 0, \\
&E[\mathbb{L}_n^{2,\mu}(R_{2\mu})\, \mathbb{L}_m^{2,\mu}(R_{2\mu})] = 0.
\end{aligned}$$

3 Convolutions of generalized gamma densities

3.1 Introductory remarks and notations

In this Section we study the convolution of generalized gamma densities which leads to useful representations of solutions to fractional partial differential equations. Here we present the basic set up and introduce some notation.

Definition 2. *We define the class of processes*

$$\mathbb{P}_\alpha = \left\{ Y(t),\, t > 0 \,:\, \exists\, \eta \in \mathbb{H}^b_a \text{ s.t. } E\left| Y(t^\alpha)/t \right|^{\eta-1} = \mathcal{P}_Y(\eta) \right\}, \quad \alpha \in \mathbb{R}$$

and, the class of functions $\mathbb{F}_\alpha = \left\{ f \,\middle|\, Y \sim f,\, Y \in \mathbb{P}_\alpha \right\}$, $\mathbb{F}_\alpha \subset \mathbb{M}^b_a$, *where* $Y \sim f$ *means that the process* Y *possesses the density law* f.

We point out that for a composition involving the processes $Y^{\sigma_j} = (X^{\sigma_j})^\alpha$, $X^{\sigma_j} \in \mathbb{P}_\alpha$, $j = 1, 2, \ldots n$ we have $Y^{\sigma_j} \in \mathbb{P}_1$, $\forall j$ and this implies that

$$Y^{\sigma_1}(Y^{\sigma_2}(\ldots Y^{\sigma_n}(t)\ldots)) \stackrel{law}{=} Y^{\sigma_1}(t^{1/n}) Y^{\sigma_2}(t^{1/n}) \cdots Y^{\sigma_n}(t^{1/n}) \tag{3.1}$$

for all possible permutations of $\{\sigma_j\}$, $j = 1, 2, \ldots, n$. This can be easily carried out by observing that $\mathcal{P}_Y(\eta) = \mathcal{P}_X(\eta\alpha - \alpha + 1)$.

For the density law $g^\gamma_\mu \in \mathbb{M}^\infty_{1-\gamma\mu}$ we observe that $g^\gamma_\mu \in \mathbb{F}_1$ and $\tilde{g}^\gamma_\mu \in \mathbb{F}_\gamma$ (see (2.29)). Indeed, we have that

$$\mathcal{M}[g^\gamma_\mu(\cdot, t)](\eta) = \frac{\Gamma\left(\mu + \eta/\gamma - 1/\gamma\right)}{\Gamma(\mu)} t^{\eta-1}, \quad \eta \in \mathbb{H}^\infty_{1-\gamma\mu},\, \gamma \neq 0,\, t > 0. \tag{3.2}$$

We now introduce the convolution

$$g^{\gamma_1}_{\mu_1} \star g^{\gamma_2}_{\mu_2}(x,t) = \langle g^{\gamma_1}_{\mu_1}(x, \cdot),\, g^{\gamma_2}_{\mu_2}(\cdot, t) \rangle = \mathrm{sign}(\gamma_1\gamma_2) \frac{1}{t} \int_0^\infty Q^{\gamma_1}_{\mu_1}(x/s) Q^{\gamma_2}_{\mu_2}(s/t) \frac{ds}{s} \tag{3.3}$$

which is a Mellin convolution in the sense that

$$\mathcal{M}[g^{\gamma_1}_{\mu_1} \star g^{\gamma_2}_{\mu_2}(\cdot, t)](\eta) = \mathcal{M}[g^{\gamma_1}_{\mu_1}(\cdot, t^{1/2})](\eta) \times \mathcal{M}[g^{\gamma_2}_{\mu_2}(\cdot, t^{1/2})](\eta)$$

(see (2.4)). The Mellin convolution is the principle tool we will now utilize. For this reason we focus on the following definition and main properties.

Definition 3. *Fix* $\bar{\gamma} = (\gamma_1, \ldots, \gamma_n)$, $\gamma_j \neq 0$, $j = 1, \ldots, n$, $\bar{\mu} = (\mu_1, \ldots, \mu_n)$, $\mu_j > 0$, $j = 1, \ldots, n$. *From the formula* (3.3), $\forall t > 0$, *we define the Mellin convolution*

$$g^{\bar{\gamma}, \star n}_{\bar{\mu}}(x,t) = g^{\gamma_1}_{\mu_1} \star \ldots \star g^{\gamma_n}_{\mu_n}(x,t) \tag{3.4}$$

with Mellin transform

$$\begin{aligned} \mathcal{M}[g^{\bar{\gamma}, \star n}_{\bar{\mu}}(\cdot, t)](\eta) &= \prod_{j=1}^n \mathcal{M}[g^{\gamma_j}_{\mu_j}(\cdot, t^{1/n})](\eta) \\ &= t^{\eta-1} \prod_{j=1}^n \frac{\Gamma((\eta-1)/\gamma_j + \mu_j)}{\Gamma(\mu_j)}, \quad \eta \in \mathbb{H}^1_a, \end{aligned} \tag{3.5}$$

where $a = 1 - \min_j\{\gamma_j \mu_j\}$.

We will refer to the integral $f_1 \circ f_2(x,t) = \langle f_1(x,\cdot)\, f_2(\cdot,t)\rangle$ (for a well-defined functions $f_1,\ f_2$) as the general composition. Roughly speaking, for two positive processes $X^1(t)$, $t>0$ and $X^2(t)$, $t>0$, with distributions f_1 and f_2, we will write $f_1 \circ f_2$ to mean the distribution of the composition $X^1(X^2(t))$, $t>0$. Thus, the Mellin convolution $f_1 \star f_2$ is a special composition where the relation (2.4) occurs. In a general context, for $f_1 \in \mathbb{F}_1$ and a well-behaved function f_2 we have $f_1 \circ f_2 \neq f_2 \circ f_1$ whereas, for $f_1,\ f_2 \in \mathbb{F}_1$, we can write $f_1 \star f_2$ instead of $f_2 \star f_1$. The simplest convolutions we deal with are written below: for $x,t>0$ and $\gamma \neq 0$, $\mu_1,\ \mu_2 > 0$,

$$g^{\gamma}_{\mu_1} \star g^{-\gamma}_{\mu_2}(x,t) = \frac{|\gamma|}{B(\mu_1,\mu_2)} \frac{x^{\gamma\mu_1 - 1} t^{\gamma\mu_2}}{(t^\gamma + x^\gamma)^{\mu_1+\mu_2}} \tag{3.6}$$

where $B(\cdot,\cdot)$ is the Beta function (see e.g. Gradshteyn and Ryzhik [19, formula 8.384]) and,

$$g^{\gamma}_{\mu_1} \star g^{\gamma}_{\mu_2}(x,t) = \frac{2|\gamma|\ (x^\gamma/t^\gamma)^{\frac{\mu_1+\mu_2}{2}}}{x\,\Gamma(\mu_1)\Gamma(\mu_2)} K_{\mu_2-\mu_1}\left(2\sqrt{(x^\gamma/t^\gamma)}\right), \tag{3.7}$$

where K_α is the modified Bessel function of the second kind.

For $X \in \mathbb{P}_1$ with law $f_X \in \mathbb{F}_1$ and $Y \in \mathbb{P}_\alpha$ with law $f_Y \in \mathbb{F}_\alpha$ we have that

$$\mathcal{M}[f_X(\cdot)](\eta) = \mathcal{P}_X(\eta)\, t^{\eta-1} \quad \text{and} \quad \mathcal{M}[f_Y(\cdot)](\eta) = \mathcal{P}_Y(\eta)\, t^{\frac{\eta-1}{\alpha}}.$$

We observe that

$$\begin{aligned} \mathcal{P}_X(\eta)\mathcal{P}_Y(\eta)\, t^{\frac{\eta-1}{\alpha}} &= \int_0^\infty x^{\eta-1}\left[\int_0^\infty f_X(x,s)\, f_Y(s,t)\, ds\right] dx \\ &= \int_0^\infty y^{\eta-1}\left[\int_0^\infty f_Y(y,s^\alpha)\, f_X(s,t^{1/\alpha})\, ds\right] dy \end{aligned} \tag{3.8}$$

which means that $X(Y(t)) \stackrel{law}{=} Y(X(t^{1/\alpha})^\alpha)$. We also introduce the well-known Fourier convolution $f_X * f_Y$ which is the law of $X+Y$. Indeed,

$$\mathcal{F}[f_X * f_Y(\cdot,t)](\xi) = \mathcal{F}[f_X(\cdot,t)](\xi) \times \mathcal{F}[f_Y(\cdot,t)](\xi)$$

where $\mathcal{F}[f(\cdot)](\xi)$ is the Fourier transform of f. We notice that $f_{X^\alpha} \star f_{Y^\beta} = f_{Y^\beta} \star f_{X^\alpha}$ is the law of $X^\alpha \cdot Y^\beta$ if $X \in \mathbb{P}_\alpha$ and $Y \in \mathbb{P}_\beta$.

3.2 Stable densities

Let us introduce the fractional derivatives and their connection with the stable densities. Consider the α-stable process $\tau^{\alpha,\theta}_t \sim s^\theta_\nu$, $s^\theta_\nu = s^\theta_\nu(x,t)$, $x \in \mathbb{R}$, $t>0$ with characteristic function given by

$$E \exp\left\{i\,\beta\,\tau^{\alpha,\theta}_t\right\} = \exp\left\{-t|\xi|^\alpha\left[1 - i\theta\frac{\xi}{|\xi|}\tan\left(\frac{\alpha\pi}{2}\right)\right]\right\}, \quad \xi \in \mathbb{R} \tag{3.9}$$

with $\alpha \in (0,1)\cup(1,2]$ and $\theta \in [-1,1]$ (see Zolotarev [51]). When $\theta = 0$, we have a symmetric process with $E\exp\{i\beta\tau^\alpha_t\} = \exp\{-t|\beta|^\alpha\}$, $\alpha \in (0,2]$ and distribution $s_\nu = s_\nu(x,t)$, $x \in \mathbb{R}$, $t>0$. For the sake of simplicity we will write τ^α_t instead of $\tau^{\alpha,0}_t$. Let us consider the functions s_ν^-, s_ν^+ where $s_\nu^-(x,\cdot) = s_\nu(x,\cdot)$, $x \in \mathbb{R}_-$ and $s_\nu^+(x,\cdot) = s_\nu(x,\cdot)$, $x \in \mathbb{R}_+$. We notice that for all $t > 0$, $s_\nu^+(-x,t) = s_\nu^-(x,t)$ because of the symmetry of τ_t^α, $t > 0$. Moreover, we will refer to $\tilde{\tau}_t^\nu = \tau_t^{\nu,1}$, $t > 0$ as the totally skewed process which is also named stable subordinator. We notice that

$$\tau_t^\nu \in \mathbb{P}_\nu \;\Leftrightarrow\; \tau_t^\nu \stackrel{law}{=} t^{1/\nu}\,\tau_1^\nu.$$

For $n-1 < \alpha < n$, according to Kilbas et al. [22]; Samko et al. [44], we define

$$(D_-^\alpha f)(x) = -\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_x^\infty (s-x)^{n-\alpha-1} f(s)\,ds, \quad x \in \mathbb{R}$$

and

$$(D_+^\alpha f)(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_{-\infty}^x (x-s)^{n-\alpha-1} f(s)\,ds, \quad x \in \mathbb{R}$$

which are the left and right Riemann-Liouville fractional derivatives. The governing equation of $\tau_t^{\alpha,\theta}$, $t > 0$, $\alpha \in (0,1) \cup (1,2]$, is given by

$$\frac{\partial s_\alpha}{\partial t}(x,t) = {}_\theta D_{|x|}^\alpha s_\alpha(x,t), \quad x \in \mathbb{R},\ t > 0 \tag{3.10}$$

where

$${}_\theta D_{|x|}^\alpha s_\alpha(x,t) = -\frac{1}{\cos \alpha\pi/2}\left[\kappa(D_-^\alpha s_\alpha(\cdot,t))(x) + (1-\kappa)(D_+^\alpha s_\alpha(\cdot,t))(x)\right]$$

and $0 \le \kappa = \kappa(\theta) \le 1$ (see e.g. Benson et al. [5]; Chaves [9]; Mainardi et al. [27]). The Riesz operator

$$\frac{\partial^\alpha s_\alpha}{\partial |x|^\alpha}(x,t) = {}_0 D_{|x|}^\alpha s_\alpha(x,t)$$

where $\kappa = \kappa(0) = 1/2$ is the governing equation of the symmetric process τ_t^α, $t > 0$ or Lèvy flights. We also consider the fractional derivatives $(D_{0-}^\alpha f)(x) = (D_-^\alpha f)(x)$, $x \in \mathbb{R}_+$ and

$$(D_{0+}^\alpha f)(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-s)^{n-\alpha-1} f(s)\,ds, \quad x \in \mathbb{R}_+. \tag{3.11}$$

As a generalized version of the integer derivative (2.1) we introduce the Mellin transform of fractional derivatives. For a given $f \in \mathbb{M}_a^b$ and $0 < \alpha < 1$, if $\Re\{\eta\} > 0$,

$$\mathcal{M}[(D_{0-}^\alpha f)(\cdot)](\eta) = \frac{\Gamma(\eta)}{\Gamma(\eta-\alpha)}\mathcal{M}[f(\cdot)](\eta-\alpha) + (\mathcal{T}I_{0-}^{1-\alpha} f)(\eta) \tag{3.12}$$

whereas, if $\Re\{\eta\} < \alpha + 1$,

$$\mathcal{M}[(D_{0+}^\alpha f)(\cdot)](\eta) = \frac{\Gamma(1+\alpha-\eta)}{\Gamma(1-\eta)}\mathcal{M}\left[f(\cdot)\right](\eta-\alpha) + (\mathcal{T}I_{0+}^{1-\alpha} f)(\eta) \tag{3.13}$$

where

$$(\mathcal{T}I_{0\pm}^{1-\alpha} f)(\eta) = \frac{\Gamma(\alpha-\eta)}{\Gamma(1-\eta)}\left[x^{\eta-1}(I_{0\pm}^{1-\alpha} f)(x)\right]_{x=0}^{x=\infty}$$

(see Kilbas et al. [22]; Samko et al. [44] for details). For $\mu > 0$, $\alpha \in (0,1)$, we observe that

$$\exists \eta \in \mathbb{H}_0^\infty \text{ s.t. } (\mathcal{T} I_{0-}^{1-\alpha} k_\mu^1)(\eta) = 0 \tag{3.14}$$

where $g_\mu^\gamma(x,t) = \mathfrak{w}(x)\, k_\mu^\gamma(x,t)$ and $k_\mu^\gamma(x,t) = |\gamma|/\Gamma(\mu) \exp\{-(x/t)^\gamma\}/t^{\gamma\mu}$. Indeed, being

$$(I_{0-}^{1-\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (s-x)^{\alpha-1} f(s)\, ds, \quad x > 0$$

(see Kilbas et al. [22]) we obtain

$$(I_{0-}^{1-\alpha} k_\mu^1(\cdot,t))(x) = t^{\alpha-1}\, k_\mu^1(x,t)$$

and (3.14) immediately follows.

The Dzhrbashyan-Caputo fractional derivative is written as

$$\frac{d^\alpha f}{dx^\alpha}(x) = ({}^C D_{0+}^\alpha f)(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-s)^{n-\alpha-1} \frac{d^n f}{ds^n}(s)\, ds \tag{3.15}$$

with $n-1 < \alpha < n$, $n \in \mathbb{N}$ and the connection between (3.15) and (3.11) is given by

$$(D_{0+,x}^\alpha f)(x) = \frac{d^\alpha f}{dx^\alpha}(x) + \sum_{k=0}^{n-1} \frac{d^k}{dx^k} f(x)\bigg|_{x=0^+} \frac{x^{k-\alpha}}{\Gamma(k-\alpha+1)}, \tag{3.16}$$

see Gorenflo and Mainardi [18] and Kilbas et al. [22]. We refer to Kilbas et al. [22]; Samko et al. [44] for a close examination of this topic.

From the relation $(L_t^\nu < x) = (\tilde{\tau}_x^\nu > t)$, according to [1; 11; 33; 35], we define the inverse process L_t^ν, $t > 0$ with law $l_\nu = l_\nu(x,t)$, $x, t > 0$. As already mentioned before, $\tilde{\tau}_t^\nu$, $t > 0$ is the ν-stable subordinator, $\nu \in (0,1)$ with law, say, $h_\nu = h_\nu(x,t)$, $x,t > 0$. The process $\tilde{\tau}_t^\nu$, $t > 0$ is a process with non-negative, independent and homogeneous increments, see Bertoin [6]. The stable subordinator and its inverse process are characterized by the Laplace transforms

$$E \exp\{-\lambda \tilde{\tau}_t^\nu\} = \exp\{-t\lambda^\nu\}, \qquad E \exp\{-\lambda L_t^\nu\} = E_\nu(-\lambda t^\nu) \tag{3.17}$$

and

$$\mathcal{L}[h_\nu(x,\cdot)](\lambda) = x^{\nu-1} E_{\nu,\nu}(-\lambda x^\nu), \qquad \mathcal{L}[l_\nu(x,\cdot)](\lambda) = \lambda^{\nu-1} \exp\{-x\lambda^\nu\}. \tag{3.18}$$

Here, the entire function

$$E_{\alpha,\beta}(z) = \sum_{k\geq 0} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{H}_0^\infty,\ \alpha \in \mathbb{H}_0^\infty,\ \beta \in \mathbb{H}_0^\infty \tag{3.19}$$

is the generalized Mittag-Leffler function for which

$$\int_0^\infty e^{-\lambda z} z^{\beta-1} E_{\alpha,\beta}(-\mathfrak{c} z^\alpha)\, dz = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha + \mathfrak{c}}, \quad \lambda \in \mathbb{H}_{|\mathfrak{c}|^{1/\alpha}}^\infty,\ \mathfrak{c} \in \mathbb{H}_0^\infty$$

and $E_\alpha(z) = E_{\alpha,1}(z)$ is the Mittag-Leffler function. From (3.17) we immediately verify that the law h_ν satisfies the fractional equation $-\frac{\partial}{\partial t} h_\nu(x,t) = (D^\nu_{0+,x} h_\nu(\cdot,t))(x)$ whereas, for the law of L^ν_t, from (3.18) we have that $(D^\nu_{0+,t} l_\nu(x,\cdot)(t) = -\frac{\partial}{\partial x} l_\nu(x,t)$. The following Mellin transforms turn out to be useful further on:

$$\mathcal{M}[h_\nu(\cdot,t)](\eta) = \Gamma\left(\frac{1-\eta}{\nu}\right) \frac{t^{\frac{\eta-1}{\nu}}}{\nu\,\Gamma(1-\eta)}, \quad \mathcal{M}[l_\nu(\cdot,t)](\eta) = \frac{\Gamma(\eta)\, t^{\nu(\eta-1)}}{\Gamma(\eta\nu - \nu + 1)}, \tag{3.20}$$

see e.g. [11; 27; 28]. From (2.8) we obtain

$$h_\nu(x,t) = \frac{1}{\nu t^{1/\nu}} H^{0,1}_{1,1}\left[\frac{x}{t^{1/\nu}} \middle| \begin{matrix} (1-\frac{1}{\nu}, \frac{1}{\nu}) \\ (0,1) \end{matrix} \right], \;\; l_\nu(x,t) = \frac{1}{t^\nu} H^{1,0}_{1,1}\left[\frac{x}{t^\nu} \middle| \begin{matrix} (1-\nu, \nu) \\ (0,1) \end{matrix} \right]$$

for $x, t > 0$, $\nu \in (0,1)$. We notice that in general

$$l_\nu(x,t) = t\nu^{-1} x^{-1-1/\nu} h_\nu(tx^{-1/\nu}, 1),$$

see e.g. [34, Corollary 3.1] or [11; 51]. The inverse process L^ν_t, $t > 0$ has non-negative, non-stationary and non-independent increments (see Meerschaert and Scheffler [34]). For this reason we will refer to a subordinated process only as the process where $\tilde{\tau}^\nu_t$ is taken as a replacement for time. Thus, from our point of view, subordination is a special composition of processes.

We recall the following fact.

Proposition 1. *For $\beta \in (0,2)$, $\nu \in (0,1)$, the solution to (3.10) with $\theta = 0$ and $\alpha = \beta\nu$, is given by $s_{\beta\nu}(x,t) = s_\beta \circ h_\nu(x,t)$.*

Proof. The proposition can be ascertained by noticing that

$$E \exp\{i\xi \tau^\beta_{\tilde{\tau}^\nu_t}\} = E \exp\{-|\xi|^\beta \tilde{\tau}^\nu_t\} = \exp\{-t|\xi|^{\beta\nu}\}$$

which coincides with (3.9) with $\theta = 0$ and $\alpha = \beta\nu$. Thus, we have that $\tau^\beta_{\tilde{\tau}^\nu_t} \stackrel{law}{=} \tau^{\beta\nu}_t$. □

For $\alpha = 1$, the process τ^1_t, $t > 0$, is the Cauchy process $C(t)$, $t > 0$, with distribution s_1. We observe that $C \in \mathbb{P}_1$ with $\mathcal{P}_C(\eta) = 1/\sin\frac{\pi}{2}\eta$, $\eta \in \mathbb{H}^1_0$. Indeed, $E|C(t)|^{\eta-1} = t^{\eta-1}/\sin\frac{\pi}{2}\eta = 2\,\mathcal{M}[s^+_1(\cdot,t)](\eta)$. Moreover, $2\,s^+_1(x,t) = g^2_{1/2} \star g^{-2}_{1/2}(x,t)$ as formula (3.6) shows. From the Proposition 1 and the fact that $g^{\pm 2}_{1/2} \in \mathbb{F}_1$ we can write

$$s^+_\nu(x,t) = 2\,(g^2_{1/2} \star g^{-2}_{1/2}) \circ h_\nu(x,t) = 2\,(g^{-2}_{1/2} \star g^2_{1/2}) \circ h_\nu(x,t). \tag{3.21}$$

For a symmetric α-stable process we have that $\tau^\alpha \in \mathbb{P}_\alpha$ with

$$\mathcal{P}_{\tau^\alpha}(\eta) = 2\frac{\Gamma\left(\frac{1-\eta}{\alpha}\right)}{\alpha\,\Gamma(1-\eta)} \frac{(c)^{\frac{\eta-1}{\alpha}}}{\sin\frac{\pi}{2}\eta}, \quad \eta \in \mathbb{H}^1_0. \tag{3.22}$$

Indeed, from the Proposition 1 we have that $\tau^\nu_t \sim s_1 \circ h_\nu(x,t)$, $x \in \mathbb{R}$, $t > 0$, and thus, by considering that $|C(t)| \sim 2s^+_1(x,t)$ and $\tilde{\tau}^\nu_t \sim h_\nu(x,t)$, we obtain that $|C(\tilde{\tau}^\nu_t)| \sim 2s^+_1 \circ h_\nu(x,t)$. From the fact that $s^+_1 \in \mathbb{F}_1$ we have that

$$\mathcal{M}[s^+_1 \circ h_\nu(\cdot,t)](\eta) = \frac{\mathcal{M}[h_\nu(\cdot,t)](\eta)}{\sin\frac{\pi}{2}\eta} = \frac{\Gamma\left(\frac{1-\eta}{\alpha}\right)}{\alpha\,\Gamma(1-\eta)} \frac{(ct)^{\frac{\eta-1}{\alpha}}}{\sin\frac{\pi}{2}\eta}, \quad \eta \in \mathbb{H}^1_0.$$

We give an alternative proof of the claimed result. From (3.9) with $\theta = 0$, we have that the Mellin transform of the function s_ν^+ reads

$$\begin{aligned}\mathcal{M}[s_\nu^+(\cdot,t)](\eta) =& \int_0^\infty x^{\eta-1}\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\beta x - c|\beta|^\alpha t} d\beta\, dx\\ =&\frac{1}{2\pi}\int_{\mathbb{R}} e^{-c|\beta|^\alpha t}\int_0^\infty x^{\eta-1}e^{-i\beta x}dx\, d\beta\\ =&\frac{1}{\pi}\int_{\mathbb{R}} e^{-c|\beta|^\alpha t}\Gamma(\eta)|\beta|^{\eta-1}\cos(\frac{\pi}{2}\eta)d\beta\\ =&\frac{1}{\pi}\Gamma(\eta)\cos(\frac{\pi}{2}\eta)\, 2\int_0^\infty \beta^{-\eta}e^{-c\beta^\alpha t}d\beta\\ =&\frac{2}{\alpha\pi}\Gamma(\eta)\cos(\frac{\pi}{2}\eta)\Gamma\left(\frac{1-\eta}{\alpha}\right)(ct)^{\frac{\eta-1}{\alpha}}\end{aligned}$$

where

$$\cos(\frac{\pi}{2}\eta) = \frac{1}{2}\left(e^{i\frac{\pi}{2}\eta} + e^{-i\frac{\pi}{2}\eta}\right) = \frac{1}{2}\frac{\left(e^{i\pi\eta} - e^{-i\pi\eta}\right)}{\left(e^{i\frac{\pi}{2}\eta} - e^{-i\frac{\pi}{2}\eta}\right)} = \frac{\sin(\pi\eta)}{2\,\sin(\frac{\pi}{2}\eta)}.$$

From the Euler's reflection formula $\Gamma(\eta)\Gamma(1-\eta) = \pi/\sin(\pi\eta)$, we obtain

$$\mathcal{M}[s_\nu^+(\cdot,t)](\eta) =\frac{\Gamma\left(\frac{1-\eta}{\alpha}\right)}{\alpha\,\Gamma(1-\eta)}\frac{(ct)^{\frac{\eta-1}{\alpha}}}{\sin\frac{\pi}{2}\eta},\quad \eta\in\mathbb{H}_0^1.$$

3.3 Representations via convolutions

We can rewrite the convolution (3.4) in terms of Fox's functions.

Theorem 1. *The following representation in terms of H-functions holds*

$$g_{\bar\mu}^{\bar\gamma,\star n}(x,t) = \frac{1}{xt}H_{n,n}^{n,0}\left[\frac{x}{t}\left|\begin{matrix}(\mu_1,0); & (\mu_2,0); & \ldots; & (\mu_n,0)\\ (\mu_1,\frac{1}{\gamma_1}); & (\mu_2,\frac{1}{\gamma_2}); & \ldots; & (\mu_n,\frac{1}{\gamma_n})\end{matrix}\right.\right],\quad x,t>0. \tag{3.23}$$

Proof. From (3.5) follows that

$$\mathcal{M}[g_{\bar\mu}^{\bar\gamma,\star n}(\cdot,t)](\eta) = t^{\eta-1}\mathcal{M}_{n,n}^{n,0}\left[\eta\left|\begin{matrix}(\mu_1,0); & \ldots; & (\mu_n,0)\\ (\mu_1-\frac{1}{\gamma_1},\frac{1}{\gamma_1}); & \ldots; & (\mu_n-\frac{1}{\gamma_n},\frac{1}{\gamma_n})\end{matrix}\right.\right].$$

Set $f(x,t) = t^{-1}w(xt^{-1})$ for a well-behaved function $w:\mathbb{R}_+\mapsto\mathbb{R}_+$, from the fact that $\mathcal{M}[f(\cdot,t)](\eta) = t^{\eta-1}\mathcal{M}[w(\cdot)](\eta)$ and by direct inspection of the formula (2.8) we obtain

$$g_{\bar\mu}^{\bar\gamma,\star n}(x,t) = \frac{1}{t}H_{n,n}^{n,0}\left[\frac{x}{t}\left|\begin{matrix}(\mu_1,0); & \ldots; & (\mu_n,0)\\ (\mu_1-1/\gamma_1,1/\gamma_1); & \ldots; & (\mu_n-1/\gamma_n,1/\gamma_n)\end{matrix}\right.\right].$$

From the property of the H functions

$$H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_i,\alpha_i)_{i=1,..,p}\\ (b_j,\beta_j)_{j=1,..,q}\end{matrix}\right.\right] = \frac{1}{x^c}\,H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_i+c\alpha_i,\alpha_i)_{i=1,..,p}\\ (b_j+c\beta_j,\beta_j)_{j=1,..,q}\end{matrix}\right.\right] \tag{3.24}$$

for all $c\in\mathbb{R}$ (see Mathai and Saxena [32]), we obtain, for $c = 1$ the claimed result. □

The following facts will be useful later on.

Lemma 3. *For $g_\mu^\gamma = g_\mu^\gamma(x,t)$, $x \in \mathbb{R}_+$, $t > 0$, $\mu > 0$ the following hold:*

i) *$\star$-commutativity: $g_{\mu_1}^{\gamma_1} \star g_{\mu_2}^{\gamma_2} = g_{\mu_2}^{\gamma_2} \star g_{\mu_1}^{\gamma_1}$ for all $\gamma_1, \gamma_2 \neq 0$.*

ii) *$*$-commutativity: $g_{\mu_1}^{\gamma_1} * g_{\mu_2}^{\gamma_2} = g_{\mu_2}^{\gamma_2} * g_{\mu_1}^{\gamma_1}$ for all $\gamma_1, \gamma_2 \neq 0$. Furthermore, for $\gamma = 1, 2$,*

$$g_{\mu_1}^\gamma * g_{\mu_2}^\gamma = g_{\mu_1+\mu_2}^\gamma.$$

iii) *$(\star,*)$-distributivity: when $\star$- and $*$- commutativity hold, we have*

$$g_{\mu_1}^\gamma \star (g_{\mu_2}^\gamma * g_{\mu_3}^\gamma) = (g_{\mu_1}^\gamma \star g_{\mu_2}^\gamma) * (g_{\mu_1}^\gamma \star g_{\mu_3}^\gamma).$$

iv) *for $\mu_1, \mu_2, c \in \mathbb{N}$*

$$g_{\mu_1\cdot\mu_2}^1 = \circledast_{j_1=1}^{\mu_1} g_{\mu_2}^1 = \circledast_{j_2=1}^{\mu_2} g_{\mu_1}^1 \qquad \textit{and} \qquad \circledast_{j_1=1}^{\mu_1 \pm c}\, g_{\mu_2}^1 = g_{\mu_1\cdot\mu_2\cdot c^{\pm 1}}^1 \tag{3.25}$$

where $\circledast_{j=1}^n f = f_1 * f_2 *, \dots, * f_n$.

Proof. The point $i)$ comes directly from the formula (3.1) and the fact that $g_{\mu_j}^{\gamma_j} \in \mathbb{F}_1$, $\forall \gamma_j \neq 0$, and $\mu_j > 0$, $j = 1, 2$. We show that $ii)$ holds. For $\gamma = 1$, $\forall\, t > 0$, g_μ^γ is the gamma density with Laplace transform $\mathcal{L}[g_\mu^1(\cdot, t)](\lambda) = 1/(1+\lambda t)^\mu$ and the statement follows easily. This is a well-known result. The case $\gamma = 2$ is considered in Shiga and Watanabe [46] being g_μ^2 the semigroup for a Bessel process $R_{2\mu} = S_{2\mu}^{1/2}$ where S_μ satisfies the stochastic equation (2.36). The result in $iii)$ can be obtained by considering, $\forall\, t > 0$, the independent r.v.'s $G_{\mu_j}^{\gamma_j}(t)$, $j = 1, 2, 3$ with densities $g_{\mu_j}^{\gamma_j} = g_{\mu_j}^{\gamma_j}(x,t)$, $j = 1, 2, 3$, $x \in \mathbb{R}_+$. From the fact that $g_{\mu_j}^{\gamma_j} \in \mathbb{F}_1$, $j = 1, 2, 3$ we have that $G_{\mu_1}^{\gamma_1}(G_{\mu_2}^{\gamma_2}(t)) \stackrel{law}{=} G_{\mu_2}^{\gamma_2}(G_{\mu_1}^{\gamma_1}(t))$, that is, $\forall\, j$, $g_{\mu_j}^{\gamma_j}$ are commutative under $\star$. For this reason and the $*$-commutativity, $\forall\, t > 0$, we can write

$$\begin{aligned} G_{\mu_1}^1(G_{\mu_2}^1(t) + G_{\mu_3}^1(t)) =& G_{\mu_1}^1(G_{\mu_2+\mu_3}^1(t)) = G_{\mu_2+\mu_3}^1(G_{\mu_1}^1(t)) \\ =& G_{\mu_2}^1(G_{\mu_1}^1(t)) + G_{\mu_3}^1(G_{\mu_1}^1(t)). \end{aligned}$$

In the last calculation we have used the fact that

$$E\Big[\exp\left(-\lambda[G_{\mu_1}(s) + G_{\mu_2}(s)]\right)\Big| s = X_t\Big] = E\Big[\exp\left(-\lambda G_{\mu_1+\mu_2}(s)\right)\Big| s = X_t\Big].$$

The same result can be achieved for $\gamma = 2$. In order to prove $iv)$ we proceed as follows: first of all we observe that $ii)$ imply $g_{\mu_1\cdot\mu_2}^1 = \circledast_{j_1=1}^{\mu_1} g_{\mu_2}^1 = \circledast_{j_2=1}^{\mu_2} g_{\mu_1}^1$. Second of all we show that

$$\circledast_{j_1=1}^{\mu_1+c} g_{\mu_2}^1 = \circledast_{j_c=1}^{c} \circledast_{j_1=1}^{\mu_1} g_{\mu_2}^1 = g_{\mu_1\cdot\mu_2\cdot c}^1$$

whereas

$$\circledast_{j_c=1}^{c} \circledast_{j_1=1}^{\mu_1 - c} g_{\mu_2}^1 = \circledast_{j_c=1}^{c} g_{\frac{\mu_1\cdot\mu_2}{c}}^1 = g_{\mu_1\cdot\mu_2}^1$$

and this concludes the proof. ∎

When $\gamma_j = \gamma$ for all $j = 1, \dots, n$, we will write $g_{\bar{\mu}}^{\gamma,\star n}$ instead of $g_{\bar{\mu}}^{\bar{\gamma},\star n}$. Moreover, we observe the following fact.

Proposition 2. *The following holds true*

$$g_{\bar{\mu}}^{\gamma,\star n}(x,t) = \tilde{g}_{\mu_j}^{\gamma} \circ g_{\bar{\mu}\setminus\{\mu_j\}}^{1,\star(n-1)}(x,t^{\gamma}), \quad \forall \mu_j \in \bar{\mu},\ j=1,2,\dots,n \tag{3.26}$$

where $\tilde{g}_{\mu}^{\gamma}$ *is that in* (2.29)

Proof. Fix $n=3$. $\forall\, t>0$, it is enough to consider the r.v.'s G_{μ}^{γ}, $\tilde{G}_{\mu}^{\gamma}$ and their density laws g_{μ}^{γ}, $\tilde{g}_{\mu}^{\gamma}$ where $\tilde{g}_{\mu}^{\gamma}(x,t) = g_{\mu}^{\gamma}(x,t^{1/\gamma})$ or equivalently $G_{\mu}^{\gamma}(t) \stackrel{law}{=} \tilde{G}_{\mu}^{\gamma}(t^{\gamma})$. In this setting, we have that $X(t) = G_{\mu_1}^{\gamma}(G_{\mu_2}^{\gamma}(G_{\mu_3}^{\gamma}(t)))$ can be written as $X(t) \stackrel{law}{=} \tilde{G}_{\mu_1}^{\gamma}(G_{\mu_2}^{1}(G_{\mu_3}^{1}(t^{\gamma})))$ thank to the fact that $\left(G_{\mu}^{\gamma}\right)^{\gamma} \stackrel{law}{=} G_{\mu}^{1}$. Thus, we can write

$$g_{\bar{\mu}}^{\gamma,\star 3}(x,t) = \tilde{g}_{\mu_1}^{\gamma} \circ g_{(\mu_2,\mu_3)}^{1,\star 2}(x,t^{\gamma}) = \tilde{g}_{\mu_1}^{\gamma} \circ (g_{\mu_2}^{1} \star g_{\mu_3}^{1})(x,t^{\gamma}).$$

Thanks to the $\star$-commutativity we have that $g_{\mu_2}^{1} \star g_{\mu_3}^{1} = g_{\mu_3}^{1} \star g_{\mu_2}^{1}$ and also that

$$g_{\bar{\mu}}^{\gamma,\star 3}(x,t) = \tilde{g}_{\mu_2}^{\gamma} \circ g_{(\mu_1,\mu_3)}^{1,\star 2}(x,t^{\gamma}) = \tilde{g}_{\mu_2}^{\gamma} \circ (g_{\mu_1}^{1} \star g_{\mu_3}^{1})(x,t^{\gamma}).$$

By considering n processes, the formula (3.26) immediately appears. □

For $m, \kappa, \varrho \in \mathbb{N}$, we introduce the sets

$$\mathscr{S}_{\kappa}^{n}(\varsigma) = \left\{ \wp \in \mathbb{R}_{+}^{n} \,:\, \wp = \frac{\bar{\upsilon}}{\kappa},\ \bar{\upsilon} = (\upsilon_1,\dots,\upsilon_n) \in \mathbb{N}^n,\ \sum_{j=1}^{n} \upsilon_j = \varsigma \right\} \tag{3.27}$$

and

$$\mathscr{P}_{\kappa}^{n}(\varrho) = \left\{ \bar{\wp} \in \mathbb{R}_{+}^{n} \,:\, \bar{\wp} = \frac{\bar{\upsilon}}{\kappa},\ \bar{\upsilon} = (\upsilon_1,\dots,\upsilon_n) \in \mathbb{N}^n,\ \prod_{j=1}^{n} \upsilon_j = \varrho \right\}. \tag{3.28}$$

For $\gamma = 1,2$ and a fixed $\bar{\mu} = (\mu_1,\dots,\mu_n) \in \mathscr{S}_{\kappa}^{n}(\varsigma)$, we have that

$$g_{\mu_1}^{\gamma} * \cdots * g_{\mu_n}^{\gamma}(x,t) = g_{\theta_1}^{\gamma} * \cdots * g_{\theta_n}^{\gamma}(x,t)$$

for all $\bar{\theta} = (\theta_1,\dots,\theta_n) \in \mathscr{S}_{\kappa}^{n}(\varsigma)$. This fact follows easily from the semigroup property ($*$-commutativity) of the law g_{μ}^{γ} shown in *ii*), Lemma 3. We present the following relevant result about the $\star$-commutativity of the semigroup g_{μ}^{γ}.

Theorem 2. *Fix* $\bar{\mu} \in \mathscr{P}_{\kappa}^{n}(\varrho)$. *Then,* $g_{\bar{\mu}}^{1,\star n} = g_{\bar{\vartheta}}^{1,\star n}$, *for all* $\bar{\vartheta} \in \mathscr{P}_{\kappa}^{n}(\varrho)$.

Proof. Fix κ, $\varrho \in \mathbb{N}$. We have $g_{\bar{\mu}}^{1,\star n} = g_{\mu_1}^{1} \star \ldots \star g_{\mu_n}^{1}$, $\bar{\mu} = (\mu_1,\ldots\mu_n) \in \mathscr{P}_{\kappa}^{n}(\varrho)$. From (3.28) we can write $\bar{\mu} = \frac{1}{\kappa}(\tilde{\mu}_1,\dots,\tilde{\mu}_n)$. Let us first consider $n=2$. We recall that $g_{(\mu_1,\mu_2)}^{1,\star 2} = g_{(\mu_2,\mu_1)}^{1,\star 2}$ from the $\star$-commutativity. Thus, form the properties *i*) and *ii*) of the Lemma 3, we have that

$$\begin{aligned} g_{(\frac{1}{\kappa}\tilde{\mu}_1,\frac{1}{\kappa}\tilde{\mu}_2)}^{1,\star 2} &= \divideontimes_{j_1=1}^{\tilde{\mu}_1} g_{(\frac{1}{\kappa},\frac{1}{\kappa}\tilde{\mu}_2)}^{1,\star 2} = \divideontimes_{j_1=1}^{\tilde{\mu}_1} g_{(\frac{1}{\kappa}\tilde{\mu}_2,\frac{1}{\kappa})}^{1,\star 2} \\ &= \divideontimes_{j_1=1}^{\tilde{\mu}_1} \divideontimes_{j_2=1}^{\tilde{\mu}_2} g_{(\frac{1}{\kappa},\frac{1}{\kappa})}^{1,\star 2} = \divideontimes_{j_1=1}^{\tilde{\mu}_1} \divideontimes_{j_2=1}^{\tilde{\mu}_2} g_{\frac{1}{\kappa}(1,1)}^{1,\star 2}. \end{aligned}$$

For $n \in \mathbb{N}$, $\bar{\vartheta}_0 = \frac{1}{\kappa}(1,\dots,1) \in \mathbb{R}^n_+$, we can write

$$g^{1,\star n}_{\bar{\mu}} = g^{1,\star n}_{\frac{1}{\kappa}(\tilde{\mu}_1,\dots,\tilde{\mu}_n)} = \circledast^{\tilde{\mu}_1}_{j_1=1} \cdots \circledast^{\tilde{\mu}_n}_{j_n=1} g^{1,\star n}_{\bar{\vartheta}_0}.$$

We shall refer to $\bar{\vartheta}_0$ as the 0-configuration. We first observe that

$$g^{1,\star n}_{\bar{\mu}} = \circledast^{\tilde{\mu}_1}_{j_1=1} \cdots \circledast^{\tilde{\mu}_n}_{j_n=1} g^{1,\star n}_{\bar{\vartheta}_0} = g^{1,\star n}_{\frac{1}{\kappa}(\varrho,1,\dots,1)}, \quad \varrho = \prod_{j=1}^{n} \tilde{\mu}_j,$$

or equivalently

$$g^{1,\star n}_{\bar{\mu}} = \circledast^{\tilde{\mu}_1}_{j_1=1} \cdots \circledast^{\tilde{\mu}_n}_{j_n=1} g^{1,\star n}_{\bar{\vartheta}_0} = \circledast^{\tilde{\mu}_i}_{j_i=1} g^{1,\star n}_{\frac{1}{\kappa}(\varrho/\tilde{\mu}_i,1,\dots,1)} = g^{1,\star n}_{\frac{1}{\kappa}(\varrho_i,\tilde{\mu}_i,1,\dots,1)}$$

$\varrho_i = \varrho/\tilde{\mu}_i$, for all $i = 1,2,\dots,n$. The last identity comes from the $\star$-commutativity. By exploiting the $*$-commutativity and the $\star$-commutativity we have that $g^{1,\star n}_{\bar{\mu}} = g^{1,\star n}_{\bar{\theta}}$ for all $\bar{\theta}$ such that

$$\mathbb{R}^n_+ \ni \bar{\theta} = \frac{1}{\kappa}(\varrho_{\mathbf{i}}, \tilde{\mu}_{\mathbf{i}}, \mathbf{1}), \quad \varrho_{\mathbf{i}} = \varrho / \prod_{s_j=1}^{|\mathbf{i}|} \tilde{\mu}_{s_j}$$

where $dim(\mathbf{1}) = n - |\mathbf{i}| - 1$, $\tilde{\mu}_{\mathbf{i}} = (\tilde{\mu}_{s_1}, \dots, \tilde{\mu}_{s_{|\mathbf{i}|}}) \in \mathbb{N}^{|\mathbf{i}|}_+$, $s_j \in \mathbf{i}$, $j = 1,2,\dots|\mathbf{i}|$ and $|\mathbf{i}| < n$ is the cardinality of $\mathbf{i}$. A further configuration is given by $\bar{\bar{\theta}} = (\varrho_{\mathbf{i}}, \tilde{\mu}_{\mathbf{i}})/\kappa$ where $|\mathbf{i}| = n-1$. In this case, $\bar{\bar{\theta}} \in \mathscr{P}^n_\kappa(\varrho)$ is obtainable by $n!$ permutation of the elements of $\bar{\mu}$. In a more general setting, for $\bar{\alpha} = (\alpha_1 \cdot \beta, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$, $\beta \in \mathbb{N}$, $c \in \mathbb{N}$ the following rules hold

$$g^{1,\star n}_{\bar{\alpha}} = \circledast^{\beta}_{j=1} g^{1,\star n}_{(\alpha_1,\alpha_2,\dots,\alpha_n)} \tag{3.29}$$

(see iv), Lemma 3) and, for $c \geq 1$,

$$\circledast^{\beta \pm c}_{j=1} g^{1,\star n}_{(\alpha_1,\alpha_2,\dots,\alpha_n)} = \circledast^{\beta \pm c}_{j=1} g^{1,\star n}_{(\alpha_{\sigma_1},\alpha_{\sigma_2},\dots,\alpha_{\sigma_n})} = g^{1,\star n}_{(\alpha_{\sigma_1} \cdot \beta \cdot c^{\pm 1},\alpha_{\sigma_2},\dots,\alpha_{\sigma_n})} \tag{3.30}$$

for all permutation of $\{\sigma_j\}$, $j = 1,2,\dots,n$. We recall that

$$\circledast^{\beta+c}_{j=1} g_\alpha = \circledast^{c}_{j_1=1} \circledast^{\beta}_{j_2=1} g_\alpha = g_{\alpha\beta c},$$

for $\alpha, \beta, c \in \mathbb{N}$. By making use of the properties i), ii) and iii) of the Lemma 3 we can obtain all possible configurations of $\bar{\vartheta} \in \mathbb{R}^n_+$ starting from the 0-configuration $\bar{\vartheta}_0$. All different configurations of $\bar{\vartheta}$ are included in $\mathscr{P}^n_\kappa(\varrho)$. From (3.30), for all $\bar{\vartheta} = (\vartheta_1, \dots, \vartheta_n) \in \mathscr{P}^n_\kappa(\varrho)$ we have that $\prod_{j=1}^n \vartheta_j = \varrho$. This concludes the proof. □

Example Let the previous setting prevail. Fix $n = 2$. For $c_1, c_2 \in \mathbb{N}$ we have that

$$g^{1,\star 2}_{(c_1,c_2)} = \circledast^{c_1}_{j_1=1} \circledast^{c_2}_{j_2=1} g^{1,\star 2}_{(1,1)}.$$

Let us take $k \in \mathbb{N}$, $k < c_1$. The property ii) of the Lemma 3 says that

$$g^{1,\star 2}_{(c_1,c_2)} = \circledast^{c_1}_{j_1=1} \circledast^{c_2}_{j_2=1} g^{1,\star 2}_{(1,1)} = \circledast^{c_1}_{j_1=1} \circledast^{c_2-k}_{j_2=1} g^{1,\star 2}_{(k,1)}$$

Thank to the $\star$-commutativity we have that

$$g^{1,\star 2}_{(c_1,c_2)} = \divideontimes_{j_1=1}^{c_1} \divideontimes_{j_2=1}^{c_2-k} g^{1,\star 2}_{(1,k)}.$$

Finally, we obtain $g^{1,\star 2}_{(c_1,c_2)} = g^{1,\star 2}_{(\frac{c_1\cdot c_2}{k},k)}$.

We observe that $\aleph = |\mathscr{P}^m_\kappa| < |\mathbb{N}|$ is the cardinality of $\mathscr{P}^m_\kappa$, thus $\mathscr{P}^m_\kappa$ is a finite set. Furthermore, $\forall \varrho \in \mathbb{N}$ and a fixed $\bar{\mu} \in \mathscr{P}^m_\kappa(\varrho)$, we have that

$$\mathcal{M}[g^{\gamma,\star n}_{\bar{\mu}}(\cdot,t)](\eta) = \mathcal{M}[g^{\gamma,\star n}_{\bar{\theta}}(\cdot,t)](\eta), \quad \forall \bar{\theta} \in \mathscr{P}^m_\kappa(\varrho) \tag{3.31}$$

whereas, for $\bar{\mu} \in \mathscr{S}^m_\kappa(\varsigma)$ and $\gamma = 1,2$, we have that

$$\mathcal{F}[g^{\gamma,*n}_{\bar{\mu}}(\cdot,t)](\xi) = \mathcal{F}[g^{\gamma,*n}_{\bar{\theta}}(\cdot,t)](\xi), \quad \forall \bar{\theta} \in \mathscr{S}^m_\kappa(\varsigma) \tag{3.32}$$

where we used, the familiar notation, $f^{*n}_{\bar{\mu}} = f_{\mu_1} * \cdots * f_{\mu_n}$.

Stable subordinators and their inverse processes are fundamental in studying fractional and anomalous diffusion as we will discuss in the next Section. Hereafter, we extend the result given in [11] (Lemmas 4 and 5) and show how the Mellin convolution turns out to be very useful in order to explicitly write the distributions of both stable subordinators and their inverse processes. Let us consider the time-stretching functions $\psi_m(s) = m\, s^{1/m}$, $s \in (0,\infty)$, $m \in \mathbb{N}$ and φ_m such that $\psi_m = \varphi_m^{-1}$.

Lemma 4. *[11, Lemma 2] The Mellin convolution $e^{\star n}_{\bar{\mu}}(x, \varphi_{n+1}(t))$ where $\mu_j = j\,\nu$, for $j = 1,2,\dots,n$ is the density law of a ν-stable subordinator $\{\tilde{\tau}^{(\nu)}_t,\, t>0\}$ with $\nu = 1/(n+1)$, $n \in \mathbb{N}$. Thus, we have*

$$h_\nu(x,t) = e^{\star n}_{\bar{\mu}}(x, \varphi_{n+1}(t)), \quad x,t>0.$$

We recall that $e_\mu = g^{-1}_\mu$ is the 1-dimensional law of the reciprocal gamma process E_μ.

Lemma 5. *[11, Lemma 3] The Mellin convolution $g^{(n+1),\star n}_{\bar{\mu}}(x, \psi_{n+1}(t))$ where $\mu_j = j\,\nu$, $j = 1,2,\dots,n$ and $\nu = 1/(n+1)$, $n \in \mathbb{N}$, is the density law of a ν-inverse process $\{L^{(\nu)}_t,\, t>0\}$. Thus, we have*

$$l_\nu(x,t) = g^{(n+1),\star n}_{\bar{\mu}}(x, \psi_{n+1}(t)), \quad x,t>0.$$

In light of the previous Lemmas and the convolution (3.7) we are able to write the distributions h_ν and l_ν as integral of modified Bessel functions (K_α). Here, the vector $\bar{\mu}$ is given by $\bar{\mu} = \nu\cdot(1,2,\dots,n)$ where $\nu = 1/(n+1)$, $n \in \mathbb{N}$. For instance, from Lemma 4 and equation (3.7), we have that

$$h_{1/3}(x,t) = e^{\star 2}_{(1/3,2/3)}(x, \varphi_3(t)) = e_{1/3} \star e_{2/3}(x, (t/3)^3) = \frac{1}{3\pi}\frac{t^{3/2}}{x^{3/2}} K_{\frac{1}{3}}\left(\frac{2}{3^{3/2}}\frac{t^{3/2}}{\sqrt{x}}\right)$$

whereas, from Lemma 5 and equation (3.7), we obtain

$$l_{1/3}(x,t) = g^{3,\star 2}_{(1/3,2/3)}(x, \psi_3(t)) = g^3_{1/3} \star g^3_{2/3}(x, 3t^{1/3}) = \frac{1}{\pi}\sqrt{\frac{x}{t}} K_{\frac{1}{3}}\left(\frac{2}{3^{3/2}}\frac{x^{3/2}}{\sqrt{t}}\right).$$

If $n \in 2\mathbb{N}$, we obtain n convolutions involving the Bessel function K_α by applying n times the convolution (3.7). Now we focus on the density law l_ν of the inverse process L^ν_t, $t > 0$, where $\nu = 1/(2n+1)$, $n \in \mathbb{N}$. From Lemma 5, Lemma 3 and by making use of the formula (3.7), we can write l_ν as integrals of the functions K_α. Let us consider $\nu = 1/4, 1/5$. For $\nu = 1/4$, we have that

$$l_{1/4}(x,t) = g^{4,\star 3}_{\bar{\mu}}(x, 4t^{1/4}), \quad \bar{\mu} = (1/4, 2/4, 3/4). \tag{3.33}$$

From the Proposition 2 and the Lemma 3, we have that

$$\begin{aligned}
&g^{4,\star 3}_{(1/4,2/4,3/4)} = \tilde{g}^4_{1/4} \circ g^{1,\star 2}_{(3/4,2/4)} \\
=&\tilde{g}^4_{1/4} \circ \left[(g^1_{1/4} * g^1_{1/4} * g^1_{1/4}) \star (g^1_{1/4} * g^1_{1/4}) \right] \\
=&\tilde{g}^4_{1/4} \circ \left[g^{1,\star 2}_{(2/4,1/4)} * g^{1,\star 2}_{(2/4,1/4)} * g^{1,\star 2}_{(2/4,1/4)} \right] \\
=&\tilde{g}^4_{1/4} \circ g^{1,\star 2}_{(6/4,1/4)}(x, 4^4 t) = g^{4,\star 3}_{(6/4,1/4,1/4)}
\end{aligned}$$

and, in explicit form

$$\begin{aligned}
g^{4,\star 3}_{(6/4,1/4,1/4)}(x,t) =& g^{4,\star 2}_{(1/4,1/4)} \star g^4_{3/2}(x,t) \\
&= \int_0^\infty \left[\frac{8}{\Gamma^2(1/4)} \frac{1}{z} K_0\left(\frac{2x^2}{z^2} \right) \right] g^4_{3/2}(z,t)\, dz.
\end{aligned}$$

The latter leads to an alternative form of $l_{1/4}$. Indeed, we can also write $l_{1/4} = l_{1/2} \circ l_{1/2}$ as can be ascertained from (3.17). For $\nu = 1/5$, we have that

$$l_{1/5}(x,t) = g^{5,\star 4}_{\bar{\mu}}(x, 5t^{1/5}), \quad \bar{\mu} = (1/5, 2/5, 3/5, 4/5) \tag{3.34}$$

and, from the Proposition 2 and the Lemma 3,

$$\begin{aligned}
&g^{5,\star 4}_{(1/5,2/5,3/5,4/5)} = \tilde{g}^5_{1/5} \circ g^{1,\star 3}_{(4/5,3/5,2/5)} \\
=&\tilde{g}^5_{1/5} \circ \left[(g^1_{1/5} * g^1_{1/5} * g^1_{1/5} * g^1_{1/5}) \star (g^1_{1/5} * g^1_{1/5} * g^1_{1/5}) \star g^1_{2/5} \right] \\
=&\tilde{g}^5_{1/5} \circ \left[(g^1_{1/5} * g^1_{1/5} * g^1_{1/5} * g^1_{1/5}) \star (g^{1,\star 2}_{(2/5,1/5)} * g^{1,\star 2}_{(2/5,1/5)} * g^{1,\star 2}_{(2/5,1/5)}) \right] \\
=&\tilde{g}^5_{1/5} \circ \left[\divideontimes_{k=1}^{12} \left(g^{1,\star 3}_{(2/5,1/5,1/5)} \right)_k \right] \\
=&\tilde{g}^5_{1/5} \circ g^{1,\star 3}_{24/5,1/5,1/5}(x, 5^5 t) = g^{5,\star 4}_{(24/5,1/5,1/5,1/5)}.
\end{aligned}$$

Finally, we obtain

$$g^{5,\star 4}_{(24/5,1/5,1/5,1/5)}(x,t) = g^{5,\star 2}_{(24/5,1/5)} \star g^{5,\star 2}_{(1/5,1/5)}(x,t).$$

From (3.7) the corresponding integral representation emerges.

We collect all the results given in this Section by stating the following representation Theorem.

Theorem 3. *Let the previous setting prevail. For $\nu = 1/(n+1)$, $n \in \mathbb{N}$, we have that*

$$h_\nu(x,t) = g_{\bar{\mu}}^{-1,\star n}(x, \varphi_{n+1}(t)), \quad \forall\, \bar{\mu} \in \mathscr{P}_{n+1}^{n}(n!) \tag{3.35}$$

and

$$l_\nu(x,t) = g_{\bar{\mu}}^{(n+1),\star n}(x, \psi_{n+1}(t)), \quad \forall\, \bar{\mu} \in \mathscr{P}_{n+1}^{n}(n!) \tag{3.36}$$

where $\varphi_m(s) = (t/m)^m$ and $\psi_m(s) = mt^{1/m}$, $s > 0$, $m \in \mathbb{N}$ and $\varphi_m = \psi_m^{-1}$. Furthermore, for $\gamma \neq 0$, $\nu = 1/(2n+1)$, $n \in \mathbb{N}$,

$$g_{\bar{\mu}}^{\gamma,\star 2n}(x,t) = \frac{2^n|\gamma|^n}{\prod_{j=1}^{2n}\Gamma(\mu_j)}\, \mathcal{E}_{\delta(\bar{\mu})}^{\gamma,\star n}(x,t), \quad \forall\, \bar{\mu} \in \mathscr{P}_{2n+1}^{2n}\left((2n)!\right) \tag{3.37}$$

where $\delta(\bar{\mu}) = (\delta_1(\bar{\mu}), \delta_2(\bar{\mu}), \dots, \delta_n(\bar{\mu})) \in \mathbb{R}_+^n$ and

$$\mathcal{E}_{\delta_j(\bar{\mu})}^{\gamma}(x,t) = \frac{1}{x}\left(\frac{x}{t}\right)^{\frac{\gamma}{2}\sigma_j(\bar{\mu})} K_{\delta_j(\bar{\mu})}\left(2\sqrt{x^\gamma/t^\gamma}\right)$$

with

$$\sigma_j(\bar{\mu}) = \mu_{j+1} + \mu_j, \qquad \delta_j(\bar{\mu}) = \mu_{j+1} - \mu_j, \qquad j = 1, 2, \dots, n.$$

Proof. From Lemma 4, Lemma 5 and Theorem 2 we obtain (3.35) and (3.36). Formula (3.37) comes by applying (3.7). □

Corollary 1. *For $\nu = 1/(n+1)$, $n \in \mathbb{N}$, the explicit solutions to the equation (3.10), with $\theta = 0$, are given by*

$$s_\nu(x,t) = \int_0^\infty \frac{e_{\bar{\mu}}(s, \psi_{n+1}(t))}{\pi(x^2+s^2)}\, s\, ds, \quad x \in \mathbb{R},\ t > 0,\ \bar{\mu} \in \mathscr{P}_{n+1}^{n}(n!)$$

and

$$s_{2\nu}(x,t) = \int_0^\infty \frac{e^{-\frac{x^2}{2s}}}{\sqrt{2\pi s}} e_{\bar{\mu}}(s, \psi_{n+1}(t))\, ds, \quad x \in \mathbb{R},\ t > 0,\ \bar{\mu} \in \mathscr{P}_{n+1}^{n}(n!).$$

Proof. The stochastic solution to the symmetric Riesz operator (3.10), (with $\theta = 0$) is given by the Bochner subordination formula, that is $B(\tilde{\tau}_t^\nu) \overset{law}{=} \tau_t^{2\nu}$. From this and the Proposition 1, the result follows. Indeed, the law s_ν is the density of $C(\tilde{\tau}_t^\nu)$, $t > 0$. □

Corollary 2. *Let us consider the process $E_\mu(t)$, $t > 0$, satisfying the SDE (2.37) and a stable subordinator $\tilde{\tau}_t^\nu$, $t > 0$. For $\nu = 1/(n+1)$, $n \in \mathbb{N}$ and $\bar{\mu} = (\mu_1, \dots, \mu_n) \in \mathbb{R}_+^n$ the following equivalence in law holds*

$$\tilde{\tau}_t^\nu \overset{law}{=} E_{\mu_1}(E_{\mu_2}(\dots E_{\mu_n}((\nu t)^{1/\nu})\dots)), \quad t > 0, \quad \bar{\mu} \in \mathscr{P}_{n+1}^{n}(n!). \tag{3.38}$$

Proof. We recall that $e_\mu = g^{-1}$ is the 1-dimensional law of the process E_μ^1 satisfying the SDE (2.37). For $\nu = 1/(n+1)$, $n \in \mathbb{N}$, by considering that $h_\nu(x,t) = e_\mu^{\star n}(x, (\nu t)^{1/\nu})$, $\bar{\mu} \in \mathscr{P}_{n+1}^{n}(n!)$ (as shown in Theorem 3), the statement immediately follows. □

Examples We list below some possible configurations for $\bar{\mu}$.
For $m = 4$, $\kappa = 5$ we have

$$\mathscr{P}_5^4(4!) = \frac{1}{5}\{(24,1,1,1),(4,3,2,1),(8,6,1,1),(12,2,1,1),\dots,(3,2,2,2)\}$$

and, by choosing $\bar{\mu} = (1,2,3,4)/5$, we obtain

$$g_{\bar{\mu}}^{5,\star 4}(x,t) = g_{(1,2)/5}^{5,\star 2} \star g_{(3,4)/5}^{5,\star 2}(x,t) = \frac{10^2}{\prod_{j=1}^4 \Gamma(j/5)} \mathcal{E}_{\delta(\bar{\mu})}^{5,\star 2}(x,t)$$

with $\delta_1(\bar{\mu}) = \delta_2(\bar{\mu}) = 1/5$ and $\sigma_1(\bar{\mu}) = 3/5$, $\sigma_2(\bar{\mu}) = 7/5$. By taking into account the formula

$$\prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right)\Gamma\left(1-\frac{k}{n}\right) = \frac{(2\pi)^{n-1}}{n}$$

(see Gradshteyn and Ryzhik [19, formula 3.335]), we have that $\prod_{j=1}^4 \Gamma(j/5) = (2\pi)^2/\sqrt{5}$ and thus

$$g_{(1,2,3,4)/5}^{5,\star 4}(x,t) = \frac{5^{5/2}}{\pi^2}\frac{x^{1/2}}{t^{7/2}}\int_0^\infty s\, K_{\frac{1}{5}}\left(2\frac{x^{5/2}}{s^{5/2}}\right) K_{\frac{1}{5}}\left(2\frac{s^{5/2}}{t^{5/2}}\right) ds$$

By choosing $\bar{\mu} = (4,3,2,1)/5$ we have that $\sigma_1(\bar{\mu}) = 7/5$ and $\sigma_2(\bar{\mu}) = 3/5$. Thus,

$$g_{\bar{\mu}}^{5,\star 4}(x,t) = \frac{5^{5/2}}{\pi^2}\frac{x^{5/2}}{t^{3/2}}\int_0^\infty s^{-3}\, K_{\frac{1}{5}}\left(2\frac{x^{5/2}}{s^{5/2}}\right) K_{\frac{1}{5}}\left(2\frac{s^{5/2}}{t^{5/2}}\right) ds.$$

If we choose $\bar{\mu} = (3,2,2,2)/5$ we obtain

$$g_{\bar{\mu}}^{5,\star 4}(x,t) = \frac{2\sqrt{5}\Gamma\left(\frac{1}{5}\right)\Gamma\left(\frac{4}{5}\right)}{\left[\pi\Gamma\left(\frac{2}{5}\right)\right]^2}\frac{x^{3/2}}{t^2}\int_0^\infty s^{\frac{1}{5}-1}K_{\frac{1}{5}}\left(2\frac{x^{5/2}}{s}\right) K_0\left(2\frac{s}{t^{5/2}}\right) ds \tag{3.39}$$

For $m = 6$, $\kappa = 7$, we have that

$$\mathscr{P}_7^6(6!) = \frac{1}{7}\{(720,1,1,1,1,1),(9,5,2,2,2,2),\dots,(5,4,3,3,2,2)\}$$

and, for $\bar{\mu} = (8,6,5,3,1,1)/7$ we obtain

$$g_{\bar{\mu}}^{7,\star 6}(x,t) = g_{(8/7,6/7)}^{7,\star 2} \star g_{(5/7,3/7)}^{7,\star 2} \star g_{(1/7,1/7)}^{7,\star 2}(x,t)$$

whereas, for $\bar{\mu} = (5,4,3,3,2,2)/7$ we have that

$$g_{\bar{\mu}}^{7,\star 6}(x,t) = g_{(5/7,4/7)}^{7,\star 2} \star g_{(3/7,3/7)}^{7,\star 2} \star g_{(2/7,2/7)}^{7,\star 2}(x,t).$$

The latter gives us

$$g_{\bar{\mu}}^{7,\star 6}(x,t) = \frac{7^{7/2}\Gamma\left(\frac{1}{7}\right)\Gamma\left(\frac{6}{7}\right)}{\pi^3\Gamma\left(\frac{2}{7}\right)\Gamma\left(\frac{3}{7}\right)}\frac{x^{7/2}}{t^2}\int_0^\infty s^{-1/2}K_{\frac{1}{7}}\left(2\frac{x^{7/2}}{s^{7/2}}\right) \mathcal{K}_0^{7,\circ 2}(s,t)\, ds$$

where

$$\mathcal{K}_0^{7,\circ 2}(s,t) = \int_0^\infty \frac{1}{s^2} K_0\left(2\frac{s^{7/2}}{z^{7/2}}\right) \frac{1}{z^2} K_0\left(2\frac{z^{7/2}}{t^{7/2}}\right) dz,$$

by considering that $\delta_1(\bar{\mu}) = 1/7$, $\delta_2(\bar{\mu}) = \delta_3(\bar{\mu}) = 0$, $\sigma_1(\bar{\mu}) = 9/7$, $\sigma_2(\bar{\mu}) = 6/7$ and $\sigma_3(\bar{\mu}) = 4/7$. We can also write down

$$g_{\bar{\mu}}^{7,\star 6}(x,t) = 2\frac{7^{3/2}\Gamma\left(\frac{1}{7}\right)\Gamma\left(\frac{6}{7}\right)}{\pi^3\Gamma\left(\frac{2}{7}\right)\Gamma\left(\frac{3}{7}\right)} \frac{x^{7/2}}{t^2} \int_0^\infty s^{\frac{1}{7}-1} K_{\frac{1}{7}}\left(2\frac{x^{7/2}}{s}\right) \mathcal{K}_0^{7,\circ 2}(s^{2/7}, t)\, ds. \tag{3.40}$$

From the formulae (3.39) and (3.40) we obtain

$$l_{1/5}(x,t) = g_{\bar{\mu}}^{5,\star 4}(x, 5t^{1/5}), \quad \text{and} \quad l_{1/7}(x,t) = g_{\bar{\mu}}^{7,\star 6}(x, 7t^{1/7})$$

We observe that in the formulas (3.39) and (3.40) we have used a special configuration of $\bar{\mu}$. In particular we have chosen $\hat{\bar{\mu}} \in \mathscr{P}_\kappa^m$ such that

$$\hat{\sigma}_{\hat{\bar{\mu}}} = \min_{\bar{\mu} \in \mathscr{P}_\kappa^m} \sigma_{\bar{\mu}} \tag{3.41}$$

where $\sigma_{\bar{\mu}} = \sum_{\mu_j \in \bar{\mu}} \mu_j$. Those configurations of $\bar{\mu}$ are the unique factorizations in prime numbers and thus, the representations (3.39) and (3.40) (where the Bessel function K_0 emerges) are unique. Indeed, from the fundamental theorem of arithmetic, we know that every positive integer larger than 1 can be written as a product of the elements of a unique sequence of prime numbers.

4 Fractional and Anomalous diffusions

We recall that a standard diffusion has the mean squared displacement (or time-dependent variance) which is linear in time. Anomalous diffusion is usually met in disordered or fractal media (see e.g. Giona and Roman [16]) and represents a phenomenon for which the mean squared displacement is no longer linear but proportional to a power α of time with $\alpha \neq 1$. Thus we have superdiffusion ($\alpha > 1$) or subdiffusion ($\alpha < 1$) in which diffusion occurs faster or slower than normal diffusion (see e.g. Uchaĭkin [47]). In general, anomalous diffusions are driven by diffusion equations with fractional derivatives in space and/or in time. Physicists derive anomalous diffusions from continuous-time random walks, see e.g. Meerschaert and Scheffler [34]; Metzler and Klafter [37]; Roman and Alemany [43].

In this Section we study anomalous diffusions whose governing equations involve the operator $\mathcal{G}^*$ or its fractional counterpart. First, we briefly discuss on the fractional diffusion equation

$$\frac{\partial^\nu v_\nu}{\partial t^\nu}(x,t) = \frac{\partial^2 v_\nu}{\partial x^2} + \delta(x)\frac{t^{-\nu}}{\Gamma(1-\nu)}, \quad x \in \mathbb{R},\ t > 0. \tag{4.1}$$

Orsingher and Beghin [40] found the solutions to

$$\frac{\partial^\nu u_\nu}{\partial t^\nu}(x,t) = \lambda^2 \frac{\partial^2 u_\nu}{\partial x^2}(x,t), \quad x \in \mathbb{R},\ t > 0,\ \nu \in (0,2] \tag{4.2}$$

which can be written, for $\nu \in (0,2)$, as

$$u_\nu(x,t) = \frac{1}{\nu\pi|x|}\int_0^\infty e^{-w} e^{-\frac{|x|w^{\nu/2}}{\lambda t^{\nu/2}}\cos\frac{\nu\pi}{2}} \sin\left(\frac{|x|w^{\nu/2}}{\lambda t^{\nu/2}}\sin\left(\frac{\nu\pi}{2}\right)\right) dw \tag{4.3}$$

(see [40, Theorem 5.4]) and, for $\nu \in (0,1]$, as

$$u_\nu(x,t) = \frac{\lambda^{2/\nu}\, t}{\nu|x|^{2/\nu+1}} s_{\nu/2}^{\nu/2}\left(\frac{\lambda^{2/\nu}\, t}{|x|^{2/\nu}}, 1\right)$$

(see [40, Theorem 5.5]) whereas, for $\nu = 1/3, 2/3, 4/3$, they presented the explicit (and closed) representations in terms of Airy functions (see [40, Section 4]). We are able to represent the solutions to (4.1), for $\nu = 1/(n+1)$, $n \in \mathbb{N}$, by observing that the stochastic solution to (4.1) is given by the process $B(L_t^\nu)$, $t > 0$ (see Baeumer and Meerschaert [1] and the references therein). Thus, from Theorem 3, we have that

$$v_\nu(x,t) = \int_0^\infty \frac{e^{-\frac{x^2}{2z}}}{\sqrt{2\pi z}}\, g_{\bar{\mu}}^{(n+1),\star n}(z, t^\nu/\nu)\, dz, \quad \bar{\mu} \in \mathscr{P}_{n+1}^n(n!). \tag{4.4}$$

We notice that the equation (4.2) differs from (4.1) because of the forcing term $\delta(x)t^{-\nu}/\Gamma(1-\nu)$ which is introducing a singularity at zero. Because of this, the process $B(L_t^\nu)$, $t > 0$ can not be interpreted as the stochastic solution of (4.2) as the formula (4.3) entails.

Now, we recall that $(L_t^\nu < x) = (\tilde{\tau}_x^\nu > t)$. If we consider the subordinated process $B(\tilde{\tau}_t^\nu)$, $t > 0$, then we have the Bochner's subordination rule from which we can infer (this is the case here) that the subordination leads to the fractional operator acting on space (or the Riesz operator (3.10)). This can be viewed as a special case of the Proposition 1 (the case $\beta = 1$) or, Corollary 2 when the Brownian motion is the guiding process. In the sequel we will generalize such a result (see Theorem 7) by considering the process $S_\mu(\tilde{\tau}_t^\nu)$, $t > 0$ and the governing equation which involve a fractional version of the operator $\mathcal{G}^*$. This result seems to be in line with the fact that compositions involving the inverse to a stable subordinator L^ν (as a random time) are leading to equations with fractional operator acting on time (the time-fractional derivative).

Let us write $D_{0\pm,z}^\alpha f(z)$ in place of $(D_{0\pm}^\alpha f(\cdot))(z)$ in order to streamline the notation as much as possible. Moreover, we recall that $D_{0+,x}^\alpha = (-1)^\alpha D_{0-,x}^\alpha = \frac{\partial^n}{\partial x^n}$ for $\alpha = n$ (see [22, p. 80]). This turns out to be useful later on.

4.1 Regular Sturm-Liouville problem: fractional diffusion on a bounded domain

We focus on time-fractional diffusions in a finite domain. Meerschaert et al. [35] studied the fractional Cauchy problem on bounded domain D involving the uniformly elliptic operator

$$Lu = \sum_{i,j=1}^d \frac{\partial(a_{ij}(x)(\partial u/\partial x_i))}{\partial x_j}$$

acting on the Hilbert space $L^2(D)$. On a finite domain $\Omega_1 = (0,1)$, we study the solution to

$$\begin{cases} D_{0+,t}^\nu m_\nu^{\gamma,\mu} = \mathcal{G}^*\, m_\nu^{\gamma,\mu}, & x \in \Omega_1,\ t > 0, \\ m_\nu^{\gamma,\mu}(x,0) = f(x), & f \in C(\Omega_1) \\ m_\nu^{\gamma,\mu}(x,t) = 0, & x \in \partial\Omega_1,\ t > 0, \end{cases} \tag{4.5}$$

with $\nu \in (0,1]$, $\gamma \neq 0$, $\mu > 0$. These results can be easily extended to the case of $\Omega_a = (0, a)$, $a > 0$. We present the following result

Theorem 4. *The solution to (4.5) can be written as follows*

$$m_\nu^{\gamma,\mu}(x,t) = \mathfrak{w}(x) \sum_n c_n \, E_\nu \left(-(\kappa_n/2)^2 \, t^\nu \right) \frac{\bar{\psi}_{\kappa_n}(x)}{\|\bar{\psi}_{\kappa_n}\|^2_{\mathfrak{w}}} \tag{4.6}$$

where $\mathfrak{w}(x)$ is the weight function,

$$c_n = \int_{\Omega_1} f(x) \bar{\psi}_{\kappa_n}(x)\, dx, \tag{4.7}$$

and $E_\nu(z) = E_{\nu,1}(z)$ is the Mittag-Leffler function (3.19).

Proof. From the fact that $\mathcal{G}^* \, m_\nu^{\gamma,\mu}(x,t) = \mathfrak{w}(x) \mathcal{G} \, \bar{m}_\nu^{\gamma,\mu}(x,t)$ (see (2.33)), the problem (4.5) reduces to

$$D^\nu_{0+,t} \, \bar{m}_\nu^{\gamma,\mu} = \mathcal{G} \, \bar{m}_\nu^{\gamma,\mu}, \quad m_\nu^{\gamma,\mu}(\partial\Omega_1, t) = 0, \quad \bar{m}_\nu^{\gamma,\mu}(x, 0) = f(x)/\mathfrak{w}(x). \tag{4.8}$$

Furthermore, from Lemma 1, we have that $\mathcal{G} \, \bar{\psi}_{\kappa_i} = -(\kappa_i/2)^2 \bar{\psi}_{\kappa_i}$ where

$$\bar{\psi}_{\kappa_i}(x) = x^{\frac{\gamma}{2}(1-\mu)} J_{\mu-1} \left(\kappa_i x^{\gamma/2} \right) \tag{4.9}$$

and κ_i, $i \in \mathbb{N}$ are the zeros of J_α. The orthogonality relation $\langle \bar{\psi}_{\kappa_i}, \, \bar{\psi}_{\kappa_j} \rangle_{\mathfrak{w}} = 0$ if $i \neq j$ where

$$\langle f_1, \, f_2 \rangle_{\mathfrak{w}} = \int_0^1 f_1(x) \, f_2(x) \, \mathfrak{w}(x) \, dx \tag{4.10}$$

leads to the orthonormal system $\left\{ \bar{\psi}_{\kappa_i}(x)/\|\bar{\psi}_{\kappa_i}\|^2_w ; \, i \in \mathbb{N} \right\}$ with respect to the weight function $\mathfrak{w}(x)$. As usual, $\|f\|^2_{\mathfrak{w}} = \langle f, \, f \rangle_{\mathfrak{w}}$. We obtain that

$$\bar{m}^{\gamma,\mu}(x,t) = \sum_{n=1}^\infty c_n \, e^{-t\,(\kappa_n/2)^2} \, \frac{\bar{\psi}_{\kappa_n}(x)}{\|\bar{\psi}_{\kappa_n}\|^2_{\mathfrak{w}}} \tag{4.11}$$

where

$$c_n = \langle f/\mathfrak{w}, \, \bar{\psi}_{\kappa_n} \rangle_{\mathfrak{w}} = \int_{\Omega_1} f(x) \bar{\psi}_{\kappa_n}(x) \, dx$$

and $\|\bar{\psi}_{\kappa_n}\|_{\mathfrak{w}} = J'_{\mu-1}(\kappa_n)/\sqrt{\gamma}$ (see, e.g. [26, p. 130]). Formula (4.11) solves (4.5) in the special case $\nu = 1$. The Riemann-Liouville fractional derivative becomes the usual time-derivative $D^1_{0+,t} = \frac{\partial}{\partial t}$. For $\nu \in (0,1)$ we consider the composition involving the inverse stable subordinator L^ν_t, $t > 0$, and obtain

$$\bar{m}_\nu^{\gamma,\mu}(x,t) = \langle \bar{m}^{\gamma,\mu}(x,\cdot), \, l_\nu(\cdot,t) \rangle = \sum_n c_n \, E_\nu \left(-(\kappa_n/2)^2 \, t^\nu \right) \frac{\bar{\psi}_n(x)}{\|\bar{\psi}_{\kappa_n}\|^2_{\mathfrak{w}}}. \tag{4.12}$$

We have to observe that $m_\nu^{\gamma,\mu}(x,t) = \mathfrak{w}(x) \bar{m}_\nu^{\gamma,\mu}(x,t)$ for the proof to be completed. □

In the previous Theorem we have used the Mittag-Leffler function E_ν which appears very often in the literature. We present a new integral representation of this function.

Lemma 6. *For $\alpha = 1/(n+1)$, $n \in \mathbb{N}$, we can write*

$$E_\alpha(-\mathfrak{q}z) = \int_0^\infty e^{-\mathfrak{q}x} g_{\bar{\mu}}^{(n+1),\star n}(x, z/\alpha)\, dx, \quad \bar{\mu} \in \mathscr{P}^n_{n+1}(n!),\ \mathfrak{q} > 0,\ z > 0.$$

Furthermore, for $n \in 2\mathbb{N}$, the formula (3.37) holds true.

Proof. We have that

$$E_\nu(-\lambda t^\nu) = \int_0^\infty e^{-\lambda x} l_\nu(x,t)\, dx$$

(see (3.17)) and thus, from Theorem 3, the assertion follows. □

From the previous results we have that

$$m_\nu^{1,\mu}(x,t) = E^x \left[f(S_\mu(L_t^\nu))\, \mathbf{1}(L_t^\nu < T_{\Omega_a}(S_\mu)) \right] \tag{4.13}$$

and

$$m_\nu^{-1,\mu}(x,t) = E^x \left[f(E_\mu(L_t^\nu))\, \mathbf{1}(L_t^\nu < T_{\Omega_a}(E_\mu)) \right] \tag{4.14}$$

where $T_D(X) = \inf\{t \geq 0 : X_t \notin D\}$ (see e.g. Bass [2]) and $E^x \phi(X_t) = \phi * f_{X_t}(x)$. We observe that $\mathbf{1}(L_t^\nu < T_{\Omega_a}(X)) = \mathbf{1}(t < T_{\Omega_a}(X(L^\nu)))$, see e.g. Meerschaert et al. [35, Corollary 3.2]. Furthermore, such solutions admit the representation achieved by exploiting the Mellin convolution as pointed out in Lemma 6.

4.2 Time-fractional diffusion in one-dimensional half space

Let us consider the density law

$$\tilde{u}_\nu^{\gamma,\mu}(x,t) = \langle \tilde{g}_\mu^\gamma(x,\cdot),\, l_\nu(\cdot,t) \rangle, \quad x > 0,\, t > 0 \tag{4.15}$$

with

$$\int_0^\infty x^k\, \tilde{u}_\nu^{\gamma,\mu}(x,t) dx = \frac{\Gamma(\mu + k/\gamma)\Gamma(k/\gamma)}{\nu\, \Gamma(\mu)\Gamma(\nu k/\gamma)} t^{\nu k/\gamma}, \quad k \in \mathbb{N}. \tag{4.16}$$

For $k = 2$, we immediately see that formula (4.15) represents a subdiffusion for $2\nu/\gamma < 1$ or a superdiffusion for $2\nu/\gamma > 1$.

Theorem 5. *The density law (4.15) with $\gamma > 0$, $\mu > 0$, $\nu \in (0,1]$, solves the fractional p.d.e. on Ω_∞*

$$D^\nu_{0+,t} \tilde{u}_\nu^{\gamma,\mu} = \mathcal{G}^*\, \tilde{u}_\nu^{\gamma,\mu} \tag{4.17}$$

where $D^\nu_{0+,t}$ stands for the Riemann-Liouville fractional derivative (3.11).

Proof. First we consider $\gamma > 0$. For $\nu = 1$, formula (4.16) becomes the Mellin transform $\mathcal{M}[\tilde{g}_\mu^\gamma(\cdot,t)](k+1)$ and thus we are allowed to consider $\tilde{u}_1^{\gamma,\mu} \equiv \tilde{g}_\mu^\gamma$, see (3.2). The Laplace transform of $\tilde{g}_\mu^\gamma(x,t)$, $x, t > 0$, can be evaluated by recalling that

$$\int_0^\infty x^{\nu-1} \exp\left\{ -\beta x^p - \gamma x^{-p} \right\} dx = \frac{2}{p} \left(\frac{\gamma}{\beta} \right)^{\frac{\nu}{2p}} K_{\frac{\nu}{p}} \left(2\sqrt{\gamma\beta} \right) \tag{4.18}$$

where $p, \gamma, \beta, \nu > 0$ and K_ν is the modified Bessel function (see [19, formula 3.478]). Thus, we can write

$$\mathcal{L}[\tilde{g}^\gamma_\mu(x,\cdot)](\lambda) = 2\frac{x^{\frac{\gamma}{2}(\mu+1)-1}}{\Gamma(\mu)\lambda^{\frac{1-\mu}{2}}} K_{1-\mu}\left(2\lambda^{1/2}x^{\gamma/2}\right) = 2\frac{\mathfrak{w}(x)}{\Gamma(\mu)}\, f(\lambda)\, \psi(x; 2\lambda^{1/2})$$

where $f(\lambda) = \lambda^{(\mu-1)/2}$ and $\psi_\kappa(x) = \psi(x;\kappa)$ is that of the Lemma 1. By considering that $\tilde{g}^\gamma_\mu(x,t) = \mathfrak{w}(x)\tilde{k}^\gamma_\mu(x,t)$ and $\mathcal{G}^*\, \mathfrak{w}(x)\tilde{k}^\gamma_\mu = \mathfrak{w}(x)\, \mathcal{G}\, \tilde{k}^\gamma_\mu$ we obtain

$$\mathcal{L}[\mathcal{G}^*\, \tilde{g}^\gamma_\mu(x,\cdot)](\lambda) = 2\frac{\mathfrak{w}(x)}{\Gamma(\mu)}\, f(\lambda)\, \mathcal{G}\, \psi(x; 2\lambda^{1/2}) = \lambda\, \mathcal{L}[\tilde{g}^\gamma_\mu(x,\cdot)](\lambda)$$

where in the last formula we have used the result (2.17). From the fact that

$$\mathcal{L}\left[\frac{\partial}{\partial t}\tilde{g}^\gamma_\mu(x,\cdot)\right](\lambda) = \lambda\mathcal{L}[\tilde{g}^\gamma_\mu(x,\cdot)](\lambda), \quad x > 0$$

we obtain the claimed result for $\nu = 1$. Now, we consider $\nu \in (0,1)$. From the Laplace transform $\mathcal{L}[l_\nu(x,\cdot)](\lambda) = \lambda^{\nu-1}\exp(-x\lambda^\nu)$ (see formula (3.18)) we obtain that the Laplace transform of $\tilde{u}^{\gamma,\mu}_\nu(x,t) = \langle \tilde{g}^\gamma_\mu(x,\cdot),\, l_\nu(\cdot,t)\rangle$ writes

$$\mathcal{L}[\tilde{u}^{\gamma,\mu}_\nu(x,\cdot)](\lambda) = 2\frac{\mathfrak{w}(x)}{\Gamma(\mu)}\frac{\lambda^{\nu-1}}{\lambda^{\frac{\nu}{2}(1-\mu)}} x^{\frac{\gamma}{2}(1-\mu)} K_{1-\mu}\left(2x^{\gamma/2}\lambda^{\nu/2}\right) = 2\frac{\mathfrak{w}(x)}{\Gamma(\mu)} f(\lambda)\, \psi_\kappa(x)$$

where $\psi_\kappa(x) = \psi(x;\kappa)$ is that in (2.20) with $\kappa = 2\lambda^{\nu/2}$ and $f(\lambda) = \lambda^{\nu(\mu+1)/2-1}$. Thus, in the right-hand side of (4.17) we obtain

$$\mathcal{L}[\mathcal{G}^*\, \tilde{u}^{\gamma,\mu}_\nu(x,\cdot)](\lambda) = \frac{2}{\Gamma(\mu)} f(\lambda)\, \mathcal{G}^*\, \mathfrak{w}(x)\, \psi(x; 2\lambda^{\nu/2}) = 2\frac{\mathfrak{w}(x)}{\Gamma(\mu)} f(\lambda)\, \mathcal{G}\, \psi(x; 2\lambda^{\nu/2})$$

where we have used the fact that $\mathcal{G}^*\, \mathfrak{w}\, f = \mathfrak{w}\, \mathcal{G}\, f$. Finally, from (2.17), we obtain

$$\mathcal{L}[\mathcal{G}^*\, \tilde{u}^{\gamma,\mu}_\nu(x,\cdot)](\lambda) = \lambda^\nu\, \mathcal{L}[\tilde{u}^{\gamma,\mu}_\nu(x,\cdot)](\lambda). \tag{4.19}$$

We note that $|\tilde{u}^{\gamma,\mu}_\nu(\cdot,t)| \leq Be^{-q_0 t}$ for some $B, q_0 > 0$ as a function of t and thus,

$$\mathcal{L}[D^\nu_{0+,t}\tilde{u}^{\gamma,\mu}_\nu(x,\cdot)](\lambda) = \lambda^\nu\mathcal{L}[\tilde{u}^{\gamma,\mu}_\nu(x,\cdot)](\lambda), \tag{4.20}$$

see [22, Lemma 2.14]. By comparing (4.19) with (4.20) the result follows. □

As a direct consequence of the previous Theorem we can state the following result about the kernel $k^{\gamma,\mu}_\nu = k^{\gamma,\mu}_\nu(x,t)$.

Corollary 3. *The function $\tilde{v}^{\gamma,\mu}_\nu(x,t) = \langle \tilde{k}^\gamma_\mu(x,\cdot),\, l_\nu(\cdot,t)\rangle$ solves the following fractional p.d.e. on Ω_∞*

$$D^\nu_{0+,t}\, \tilde{v}^{\gamma,\mu}_\nu = \mathcal{G}\, \tilde{v}^{\gamma,\mu}_\nu \tag{4.21}$$

where $\tilde{v}^{\gamma,\mu}_\nu = \tilde{v}^{\gamma,\mu}_\nu(x,t)$.

Proof. The proof follows easily by observing that

$$\mathfrak{w}(x)\mathcal{L}[\tilde{v}_\nu^{\gamma,\mu}(x,\cdot)](\lambda) = \mathcal{L}[\tilde{u}_\nu^{\gamma,\mu}(x,\cdot)](\lambda)$$

From Lemma 1 and the fact that $\mathcal{G}^* \, \mathfrak{w}\, f = \mathfrak{w}\, \mathcal{G}\, f$, we can argue that

$$\mathfrak{w}(x)\, \mathcal{L}[\mathcal{G}\, \tilde{v}_\nu^{\gamma,\mu}(x,\cdot)](\lambda) = \mathfrak{w}(x)\, \lambda^\nu\, \mathcal{L}[\tilde{v}_\nu^{\gamma,\mu}(x,\cdot)](\lambda).$$

The proof is completed. □

Let us consider the processes $S_\mu(t)$, $t>0$ and $R_{2\mu} = S_{2\mu}^{1/2}$ where S_μ satisfies the stochastic equation (2.36). Furthermore, we consider the process L_t^ν, $t>0$ which is an inverse to a ν-stable subordinator. For $x \in \Omega_\infty$, $t>0$ and $\nu \in (0,1]$, the stochastic solution to

$$D_{0+,t}^\nu \tilde{u}_\nu^{1,\mu}(x,t) = \left(x\frac{\partial^2}{\partial x^2} - (\mu-2)\frac{\partial}{\partial x} \right) \tilde{u}_\nu^{1,\mu}(x,t)$$

is given by $S_\mu(L_t^\nu)$, $t>0$ whereas, the process $R_{2\mu}(L_t^\nu)$, $t>0$ is driven by

$$D_{0+,t}^\nu \tilde{u}_\nu^{2,\mu}(x,t) = \frac{1}{2^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x}\frac{2\mu-1}{x} \right) \tilde{u}_\nu^{2,\mu}(x,t).$$

The solutions to such equations obey radial anomalous diffusion equations.

Corollary 4. *For $\nu = 1/(n+1)$, $n\in\mathbb{N}$, the explicit solution to the fractional p.d.e.* (4.17) *writes*

$$\tilde{u}_\nu^{\gamma,\mu}(x,t) = \int_0^\infty \tilde{g}_\mu^\gamma(x,s) g_{\bar{\mu}}^{(n+1),\star n}(s,t)ds, \quad x>0,\, t>0,\ \bar{\mu}\in\mathscr{P}_{n+1}^n(n!).$$

Let us give an example for $\nu = 1/5$, $\gamma=2$ and $\mu = 3/2$. For $\bar{\mu} = (3,2,2,2)/5$, from the equation (3.39), we have that

$$\tilde{u}_{1/5}^{2,3/2}(x,t) = C\,\frac{x^{2\mu-1}}{t^{2/5}}\int_0^\infty\int_0^\infty \frac{e^{-x^2/z}}{s^{2/5}} K_{\frac{1}{5}}\left(2\frac{z^{5/2}}{s}\right) K_0\left(\frac{2}{5^{3/2}}\frac{s}{t^{1/2}}\right) ds\, dz$$

where

$$C = \frac{2\,\Gamma\left(\frac{1}{5}\right)\Gamma\left(\frac{4}{5}\right)}{5^{3/2}\left[\pi\Gamma\left(\frac{2}{5}\right)\right]^2\Gamma\left(\frac{3}{2}\right)}.$$

This represents an anomalous diffusion of order $\nu = 1/5$ on a spherically symmetric porous media.

4.3 Time- and Space-fractional diffusion in one-dimensional half space

We first introduce the process $\tilde{\tau}_{L_t^\beta}^\nu$, $t>0$ with law

$$\mathfrak{f}_{\nu,\beta}(x,t) = \langle h_\nu(x,\cdot),\, l_\beta(\cdot,t)\rangle, \quad x\in\Omega_\infty\cup\{0\},\ t>0,\ \nu,\beta\in(0,1) \tag{4.22}$$

which has been thoroughly studied by several authors, see e.g. [11; 21; 27; 29] and the references therein. If $\nu = \beta$, then the law (4.22) takes the form $\mathfrak{f}_{\nu,\nu}(x,t) = t^{-1} f_\nu(t^{-1}x)$ where

$$f_\nu(x) = \frac{1}{\pi} \frac{x^{\nu-1} \sin \pi\nu}{1 + 2x^\nu \cos \pi\nu + x^{2\nu}}, \quad x \in \Omega_\infty \cup \{0\},\ t > 0, \quad \nu \in (0,1)$$

and $\tilde{\tau}^\nu_{L^\nu_t} \stackrel{law}{=} t \times {}_1\tilde{\tau}^\nu_t / {}_2\tilde{\tau}^\nu_t$, $t > 0$, ($\tilde{\tau}^\nu_{L^\nu_t} \in \mathbb{P}_1$) where ${}_j\tilde{\tau}^\nu_t$, $j = 1, 2$ are independent stable subordinators and the ratio ${}_1\tilde{\tau}^\nu_t / {}_2\tilde{\tau}^\nu_t$ is independent of t (see [8; 11; 25; 50]). This density law arises in many important contexts, we refer to the paper by James [21] and the references therein for details.

Lemma 7. *The governing equation of the density* (4.22) *writes*

$$(D^\beta_{0+,t} + D^\nu_{0+,x})\mathfrak{f}_{\nu,\beta}(x,t) = \delta(x) \frac{t^{-\beta}}{\Gamma(1-\beta)}, \quad x \in \Omega_\infty \cup \{0\},\ t > 0 \tag{4.23}$$

with $\mathfrak{f}_{\nu,\beta}(\partial\Omega_\infty, t) = 0$ *and* $\mathfrak{f}_{\nu,\beta}(x,0) = \delta(x)$ *or, by considering* (3.16),

$$\left(\frac{\partial^\beta}{\partial t^\beta} + D^\nu_{0+,x} \right) \mathfrak{f}_{\nu,\beta}(x,t) = 0, \quad x \in \Omega_\infty \cup \{0\},\ t > 0. \tag{4.24}$$

Proof. From the Laplace transforms (3.17) and (3.18) we have that

$$\Psi(\xi,\lambda) = \int_0^\infty \int_0^\infty e^{-\lambda t - \xi x} \mathfrak{f}_{\nu,\beta}(x,t)\, dx\, dt = \frac{\lambda^{\beta-1}}{\lambda^\beta + \xi^\nu}.$$

Let us consider the equation (4.24). From the formula (3.16) and the fact that

$$\mathcal{L}\left[\frac{\partial^\alpha f}{\partial z^\alpha}(\cdot) \right](\lambda) = \lambda^\alpha \mathcal{L}[f(\cdot)](\lambda) - \frac{\lambda^{\alpha-1}}{\Gamma(1-\alpha)} f(0^+), \quad \alpha \in (0,1)$$

we obtain

$$\lambda^\beta \Psi(\xi,\lambda) - \lambda^{\beta-1} + \xi^\nu \Psi(\xi,\lambda) = 0$$

which concludes the proof. □

We state the following result.

Theorem 6. *Let us consider the process* $S_\mu(t)$, $t > 0$, *satisfying the stochastic equation* (2.36). *For* $\mu > 0$, $\forall\, \nu \in (0,1]$, *the* $1-$*dimensional density law* g^1_μ *of the process* S_μ *solves the fractional p.d.e. on* Ω_∞

$$D^\nu_{0+,t}\, g^1_\mu(x,t) = D^\nu_{0-,x}\Big(x^{\mu-1+\nu} D^\nu_{0-,x}\Big(x^{1-\mu}\, g^1_\mu(x,t) \Big) \Big). \tag{4.25}$$

Proof. We begin recalling that $g^1_\mu \in \mathbb{M}^\infty_{1-\mu}$ and the formula (3.14) holds. For $\nu = 1$, Lemma 2 holds. Indeed $D^\nu_{0+,t} \equiv \frac{\partial}{\partial t}$ (see [22]) and equation (4.25) becomes (2.31). Thus, we restrict ourselves to the case $\nu \in (0,1)$. From the formula (3.12), we obtain

$$\int_0^\infty x^{\eta-1} D^\nu_{0-,x}\Big(x^{\mu-1+\nu} D^\nu_{0-,x}\Big(x^{1-\mu}\, g^1_\mu(x,t) \Big) \Big)\, dx$$

$$
\begin{aligned}
=&\frac{\Gamma(\eta)}{\Gamma(\eta-\nu)}\int_0^\infty x^{\eta-\nu-1}\Big(x^{\mu-1+\nu}D^\nu_{0-,x}\Big(x^{1-\mu}\,g^1_\mu(x,t)\Big)\Big)\,dx\\
=&\frac{\Gamma(\eta)}{\Gamma(\eta-\nu)}\int_0^\infty x^{(\eta+\mu-1)-1}D^\nu_{0-,x}\Big(x^{1-\mu}\,g^1_\mu(x,t)\Big)\,dx\\
=&\frac{\Gamma(\eta)}{\Gamma(\eta-\nu)}\frac{\Gamma(\eta+\mu-1)}{\Gamma(\eta+\mu-1-\nu)}\int_0^\infty x^{(\eta+\mu-1-\nu)-1}\Big(x^{1-\mu}\,g^1_\mu(x,t)\Big)\,dx\\
=&\frac{\Gamma(\eta)}{\Gamma(\eta-\nu)}\frac{\Gamma(\eta+\mu-1)}{\Gamma(\eta+\mu-1-\nu)}\int_0^\infty x^{(\eta-\nu)-1}g^1_\mu(x,t)\,dx\\
=&\frac{\Gamma(\eta)}{\Gamma(\eta-\nu)}\frac{\Gamma(\eta+\mu-1)}{\Gamma(\eta+\mu-1-\nu)}\mathcal{M}[g^1_\mu(\cdot,t)](\eta-\nu).
\end{aligned}
$$

The x-Mellin transform of both members of (4.25) writes

$$D^\nu_{0+,t}\,\mathcal{M}[g^1_\mu(\cdot,t)](\eta)=\frac{\Gamma(\eta)}{\Gamma(\eta-\nu)}\frac{\Gamma(\eta+\mu-1)}{\Gamma(\eta+\mu-1-\nu)}\mathcal{M}[g^1_\mu(\cdot,t)](\eta-\nu)$$

where $\mathcal{M}[g^1_\mu(\cdot,t)](\eta)=t^{\eta-1}\Gamma(\eta+\mu-1)/\Gamma(\mu)$, $\eta\in\mathbb{H}^\infty_{1-\mu}$. Thus, we have that

$$D^\nu_{0+,t}\,\mathcal{M}[g^1_\mu(\cdot,t)](\eta)=\frac{\Gamma(\eta)}{\Gamma(\eta-\nu)}\frac{\Gamma(\eta+\mu-1)}{\Gamma(\mu)}t^{\eta-1-\nu}$$

because of the fact that $D^\alpha_{0+,t}t^{\beta-1}=\Gamma(\beta)/\Gamma(\beta-\alpha)t^{\beta-\alpha-1}$ (see e.g. [44, Property 2.5]) which concludes the proof. □

We observe that the equation (4.25) leads to the Laguerre polynomials.

Theorem 7. *For $x\in\Omega_\infty$, $t>0$, $\mu>0$ and $\beta,\nu\in(0,1]$ we have that*

i) the density law $\mathfrak{g}^\nu_\mu(x,t)=g^1_\mu\circ h_\nu(x,t)$ solves the fractional p.d.e.

$$-\frac{\partial}{\partial t}\mathfrak{g}^\nu_\mu(x,t)=D^\nu_{0+,x}\Big(x^{\mu-1+\nu}D^\nu_{0-,x}\Big(x^{1-\mu}\,\mathfrak{g}^\nu_\mu(x,t)\Big)\Big),\tag{4.26}$$

ii) the density law $\mathbf{g}^{\nu,\beta}_\mu(x,t)=g^1_\mu\circ\mathfrak{f}_{\nu,\beta}(x,t)$ solves the fractional p.d.e.

$$-D^\beta_{0+,t}\mathbf{g}^{\nu,\beta}_\mu(x,t)=D^\nu_{0+,x}\Big(x^{\mu-1+\nu}D^\nu_{0-,x}\Big(x^{1-\mu}\,\mathbf{g}^{\nu,\beta}_\mu(x,t)\Big)\Big).\tag{4.27}$$

Proof. We proceed as follows: first of all we find out the Mellin transform of the fractional operator acting on space

$$\mathcal{A}f(x,t)=D^\nu_{0+,x}\Big(x^{\mu-1+\nu}D^\nu_{0-,x}\Big(x^{1-\mu}\,f(x,t)\Big)\Big),\quad \nu\in(0,1)\tag{4.28}$$

for a well-defined function $f\in\mathbb{M}^\infty_a$, $a\in\mathbb{R}$ (and for which (3.14) holds, that is $(\mathcal{T}I^{1-\alpha}_{0\pm}f)(\eta)=0$) and second of all we prove *ii)* by exploiting the Mellin technique and then *i)* as a particular case of *ii)*. We also consider $f\in D(\mathcal{G}^*)$. Let us write

$$\Phi_\nu(\eta)=\frac{\Gamma(1-\eta+\nu)}{\Gamma(1-\eta)}\quad\text{and}\quad\Psi_\nu(\eta)=\frac{\Gamma(\eta+\mu-1)}{\Gamma(\eta+\mu-1-\nu)}.$$

From (3.13) we have that

$$\begin{aligned}\int_0^\infty x^{\eta-1}\mathcal{A}f(x,t)dx =&\Phi_\nu(\eta)\int_0^\infty x^{\eta-\nu-1}x^{\mu-1+\nu}D^\nu_{0-,x}\Big(x^{1-\mu}f(x,t)\Big)dx\\ =&\Phi_\nu(\eta)\int_0^\infty x^{\eta+\mu-1}D^\nu_{0-,x}\Big(x^{1-\mu}f(x,t)\Big)dx.\end{aligned}$$

From (3.12) we obtain

$$\begin{aligned}\int_0^\infty x^{\eta-1}\mathcal{A}f(x,t)dx =&\Phi_\nu(\eta)\Psi_\nu(\eta)\int_0^\infty x^{\eta+\mu-\nu-1}x^{1-\mu}f(x,t)dx\\ =&\Phi_\nu(\eta)\Psi_\nu(\eta)\mathcal{M}[f(\cdot,t)](\eta-\nu).\end{aligned}$$

Thus, by collecting all pieces together we have that

$$\mathcal{M}[\mathcal{A}f(\cdot,t)](\eta)=\frac{\Gamma(1-\eta+\nu)}{\Gamma(1-\eta)}\frac{\Gamma(\eta+\mu-1)}{\Gamma(\eta+\mu-1-\nu)}\mathcal{M}[f(\cdot,t)](\eta-\nu). \tag{4.29}$$

Now, we consider the x-Mellin transform

$$\begin{aligned}\mathcal{M}[\mathbf{g}^{\nu,\beta}_\mu(\cdot,t)](\eta) =&\mathcal{M}[g^1_\mu(\cdot,1)](\eta)\times\mathcal{M}[\mathfrak{f}_{\nu,\beta}(\cdot,t)](\eta)\\ =&\frac{\Gamma(\eta+\mu-1)}{\Gamma(\mu)}\mathcal{M}[\mathfrak{f}_{\nu,\beta}(\cdot,t)](\eta)\end{aligned}$$

where the fact that $h_\nu\in\mathbb{F}_\nu$ and $l_\beta\in\mathbb{F}_{1/\beta}$ leads to

$$\mathcal{M}[\mathfrak{f}_{\nu,\beta}(\cdot,t)](\eta)=\mathcal{M}[h_\nu(\cdot,1)](\eta)\times\mathcal{M}[l_\beta(\cdot,t)]\left(\frac{\eta-1}{\nu}+1\right).$$

and, from the formulae (3.20) we obtain

$$\mathcal{M}[\mathbf{g}^{\nu,\beta}_\mu(\cdot,t)](\eta)=\frac{\Gamma(\eta+\mu-1)}{\Gamma(\mu)}\frac{\Gamma\left(\frac{1-\eta}{\nu}\right)}{\nu\,\Gamma(1-\eta)}\frac{\Gamma\left(\frac{\eta-1}{\nu}+1\right)}{\Gamma\left(\frac{\eta-1}{\nu}\beta+1\right)}t^{\frac{\eta-1}{\nu}\beta},\quad \eta\in\mathbb{H}^1_a \tag{4.30}$$

where $a=\max\{0,1-\mu\}$, $\mu>0$. Now, we show that

$$-D^\beta_{0+,t}\mathcal{M}[\mathbf{g}^{\nu,\beta}_\mu(\cdot,t)](\eta)=\mathcal{M}[\mathcal{A}\,\mathbf{g}^{\nu,\beta}_\mu(\cdot,t)](\eta) \tag{4.31}$$

by taking into account the formula (4.29). The right-hand side of the formula (4.31) can be written as

$$\begin{aligned}\mathcal{M}[\mathcal{A}\,\mathbf{g}^{\nu,\beta}_\mu(\cdot,t)](\eta) =&\frac{\Gamma(1-\eta+\nu)}{\Gamma(1-\eta)}\frac{\Gamma(\eta+\mu-1)}{\Gamma(\eta+\mu-\nu-1)}\mathcal{M}[\mathbf{g}^{\nu,\beta}_\mu(\cdot,t)](\eta-\nu)\\ =&\frac{\Gamma(\eta+\mu-1)}{\Gamma(\mu)}\frac{\left(\frac{1-\eta}{\nu}\right)\Gamma\left(\frac{1-\eta}{\nu}\right)}{\nu\,\Gamma(1-\eta)}\frac{\Gamma\left(\frac{\eta-1}{\nu}\right)}{\Gamma\left(\frac{\eta-1}{\nu}\beta-\beta+1\right)}t^{\frac{\eta-1}{\nu}\beta-\beta}\\ =&-\frac{\Gamma(\eta+\mu-1)}{\Gamma(\mu)}\frac{\Gamma\left(\frac{1-\eta}{\nu}\right)}{\nu\,\Gamma(1-\eta)}\frac{\Gamma\left(\frac{\eta-1}{\nu}+1\right)}{\Gamma\left(\frac{\eta-1}{\nu}\beta-\beta+1\right)}t^{\frac{\eta-1}{\nu}\beta-\beta}\\ =&-\mathcal{M}[\mathbf{g}^{\nu,\beta}_\mu(\cdot,1)](\eta)\frac{\Gamma\left(\frac{\eta-1}{\nu}\beta+1\right)}{\Gamma\left(\frac{\eta-1}{\nu}\beta-\beta+1\right)}t^{\frac{\eta-1}{\nu}\beta-\beta}.\end{aligned}$$

From the fact that

$$D^{\beta}_{0+,t} t^{\frac{\eta-1}{\nu}\beta} = \frac{\Gamma\left(\frac{\eta-1}{\nu}\beta+1\right)}{\Gamma\left(\frac{\eta-1}{\nu}\beta-\beta+1\right)} t^{\frac{\eta-1}{\nu}\beta-\beta}$$

(see [44, Property 2.5]) formula (4.31) immediately follows and this prove ii).

For $\beta = 1$, the formula (4.30) takes the form

$$\mathcal{M}[\mathfrak{g}^{\nu}_{\mu}(\cdot,t)](\eta) = \frac{\Gamma(\eta+\mu-1)}{\nu\,\Gamma(\mu)\,\Gamma(1-\eta)} \Gamma\left(\frac{1-\eta}{\nu}\right) t^{\frac{\eta-1}{\nu}}, \quad \eta \in \mathbb{H}^1_a$$

where $a = \max\{0, 1-\mu\}$. This is (for $\beta = 1$) because of the fact that $\tilde{\tau}^{\nu}_{L^1_t} \stackrel{a.s.}{=} \tilde{\tau}^{\nu}_t$, $t > 0$, being $L^1_t \stackrel{a.s.}{=} t$ the elementary subordinator, see e.g. Bertoin [6]. Thus, form (4.29), the Mellin transform of both members of (4.26) becomes

$$-\frac{\partial}{\partial t}\mathcal{M}[\mathfrak{g}^{\nu}_{\mu}(\cdot,t)](\eta) = \frac{\Gamma(1-\eta+\nu)}{\Gamma(1-\eta)} \frac{\Gamma(\eta+\mu-1)}{\Gamma(\eta+\mu-1-\nu)} \mathcal{M}[\mathfrak{g}^{\nu}_{\mu}(\cdot,t)](\eta-\nu)$$

where

$$\begin{aligned}\mathcal{M}[\mathfrak{g}^{\nu}_{\mu}(\cdot,t)](\eta-\nu) =& \frac{\Gamma(\eta+\mu-1-\nu)}{\pi\nu\,\Gamma(\mu)\,\Gamma(1-\eta+\nu)} \Gamma\left(\frac{1-\eta+\nu}{\nu}\right) t^{\frac{\eta-\nu-1}{\nu}}\\ =& -\left(\frac{\eta-1}{\nu}\right) \frac{\Gamma(\eta+\mu-1-\nu)}{\pi\nu\,\Gamma(\mu)\,\Gamma(1-\eta+\nu)} \Gamma\left(\frac{1-\eta}{\nu}\right) t^{\frac{\eta-1}{\nu}-1}\\ =& -\frac{\Gamma(\eta+\mu-1-\nu)}{\pi\nu\,\Gamma(\mu)\,\Gamma(1-\eta+\nu)} \Gamma\left(\frac{1-\eta}{\nu}\right) \frac{\partial}{\partial t} t^{\frac{\eta-1}{\nu}}.\end{aligned}$$

By collecting all pieces together we obtain the result claimed in i). The proof is completed. □

We point out that the density law $\mathfrak{g}^{\nu}_{\mu}(x,t)$, $x, t > 0$ can be viewed as the 1-dimensional law of $S_{\mu}(\tilde{\tau}^{\nu}_t)$, $t > 0$ where S_{μ} is the μ-dimensional squared Bessel process and $\tilde{\tau}^{\nu}$ is the stable subordinator with law h_{ν}, $\nu \in (0,1)$. The infinitesimal generator of S_{μ} is that in (2.15) with $\gamma = 1$. For $\nu = 1/(n+1)$, $n \in \mathbb{N}$, we have that

$$\mathfrak{g}^{\nu}_{\mu}(x,t) = g^1_{\mu} \circ e^{\star n}_{\bar{\mu}}(x, (\nu t)^{1/\nu}), \quad \bar{\mu} \in \mathscr{P}^n_{n+1}(n!)$$

where $g^1_{\mu}, e^{\star n}_{\bar{\mu}} \in \mathbb{F}_1$. Thus, we obtain

$$\begin{aligned}\mathfrak{g}^{\nu}_{\mu}(x,t) =& g^1_{\mu} \star e^{\star n}_{\bar{\mu}}(x, (\nu t)^{1/\nu})\\ =& g^1_{\mu} \star g^{-1}_{\nu} \star e^{\star(n-1)}_{\bar{\mu}}(x, (\nu t)^{1/\nu}), \quad \bar{\mu} \in \mathscr{P}^{n-1}_{n+1}(n!).\end{aligned}$$

where $g^1_{\mu} \star g^{-1}_{\nu}$ is given by (3.6). For $\mu > 0$ and $\bar{\mu} \in \mathscr{P}^{n-1}_{n+1}(n!)$ we have that

$$\mathfrak{g}^{\nu}_{\mu}(x,t) = \frac{\Gamma(\mu+\nu)}{\Gamma(\mu)\Gamma(\nu)} x^{\mu-1} \int_0^{\infty} \frac{z^{\nu}}{(x+z)^{\mu+\nu}}\, e^{\star(n-1)}_{\bar{\mu}}(z, (\nu t)^{1/\nu})\, dz.$$

We give an example for $\nu = 1/2$. The law of $S_{\mu}(\tilde{\tau}^{1/2}_t)$, $t > 0$ writes

$$\mathfrak{g}^{1/2}_{\mu}(x,t) = \frac{\Gamma(\mu+1/2)}{2\sqrt{\pi}\,\Gamma(\mu)} \frac{x^{\mu-1}\, t}{(x+(t/2)^2)^{\mu+1/2}}, \quad x, t > 0,\ \mu > 0$$

and solves the space-fractional equation

$$-\frac{\partial \mathfrak{g}_\mu^{1/2}}{\partial t}(x,t) = D_{0+,x}^{1/2}\left(x^{\mu-1/2} D_{0-,x}^{1/2}\left(x^{1-\mu}\mathfrak{g}_\mu^{1/2}(x,t)\right)\right), \quad x,t>0.$$

We recall that $\tilde{\tau}_t^{1/2} = \inf\{s \geq 0 \,:\, B(s) = t/\sqrt{2}\}$ where B is a standard Brownian motion is the stochastic solution to the equation $\left(\frac{\partial}{\partial t} + D_{0+,x}^{1/2}\right) h_{1/2}(x,t) = 0$.

Remark 1. In the papers by DeBlassie [10]; D'Ovidio and Orsingher [12, 13] and many others, the authors shown the connection between stable processes and higher-order equations. In particular (see [13]) the law of $S_\mu(\tilde{\tau}_t^{1/n})$ solves the equation

$$-\frac{\partial^n}{\partial t^n}\mathfrak{g}_\mu^{1/n}(x,t) = \frac{\partial}{\partial x}\left(x^\mu \frac{\partial}{\partial x}\left(x^{1-\mu}\,\mathfrak{g}_\mu^{1/n}(x,t)\right)\right).$$

The higher-order equation governing the process $R_{2\mu}(\tilde{\tau}_t^{1/2^n})$, $t>0$ writes

$$-\frac{\partial^{2^n}}{\partial t^{2^n}}\mathfrak{g}_\mu^{2,1/2^n}(x,t) = \frac{2^{2^n}}{2^3}\frac{\partial}{\partial x}\left(x^{2\mu-1}\frac{\partial}{\partial x}x^{1-2\mu}\mathfrak{g}_\mu^{2,1/2^n}(x,t)\right) \tag{4.32}$$

where $\mathfrak{g}_\mu^{2,1/2^n} = g_\mu^2 \circ h_{1/2^n}$, (see [13]).

The following Theorem relates the previous results with the representations in terms of Fox's H-functions.

Theorem 8. *Let $\mathcal{A}$ be the operator (4.28), for $\nu,\beta \in (0,1]$, $\mu>0$, the solutions to*

$$D_{0+,t}^{\beta}\mathbf{g}_\mu^{\nu,\beta}(x,t) = \mathcal{A}\,\mathbf{g}_\mu^{\nu,\beta}(x,t), \quad x,t>0 \tag{4.33}$$

can be written in terms of H functions as

$$\mathbf{g}_\mu^{\nu,\beta}(x,t) = \frac{1}{t^{\beta/\nu}}\mathbf{G}_\mu^{\nu,\beta}\left(\frac{x}{t^{\beta/\nu}}\right) \tag{4.34}$$

where

$$\mathbf{G}_\mu^{\nu,\beta}(x) = \frac{1}{x}H_{3,3}^{2,1}\left[x\left|\begin{matrix}(1,\frac{1}{\nu}); & (1,\frac{\beta}{\nu}); & (\mu,0)\\ (\mu,1); & (1,\frac{1}{\nu}); & (1,1)\end{matrix}\right.\right], \quad x>0$$

Proof. From (4.30) and by direct inspection of (2.8) we arrive at

$$\mathcal{M}[\mathbf{G}_\mu^{\nu,\beta}(\cdot)](\eta) = \mathcal{M}_{3,3}^{2,1}\left[\eta\left|\begin{matrix}(1-\frac{1}{\nu},\frac{1}{\nu}); & (1-\frac{\beta}{\nu},\frac{\beta}{\nu}); & (\mu,0)\\ (\mu-1,1); & (1-\frac{1}{\nu},\frac{1}{\nu}); & (0,1)\end{matrix}\right.\right]$$

where $\mathbf{G}_\mu^{\nu,\beta}(x) = \mathbf{g}_\mu^{\nu,\beta}(x,1)$. Thus,

$$\begin{aligned}\mathbf{G}_\mu^{\nu,\beta}(x) =& H_{3,3}^{2,1}\left[x\left|\begin{matrix}(1-\frac{1}{\nu},\frac{1}{\nu}); & (1-\frac{\beta}{\nu},\frac{\beta}{\nu}); & (\mu,0)\\ (\mu-1,1); & (1-\frac{1}{\nu},\frac{1}{\nu}); & (0,1)\end{matrix}\right.\right]\\ =&\frac{1}{x}H_{3,3}^{2,1}\left[x\left|\begin{matrix}(1,\frac{1}{\nu}); & (1,\frac{\beta}{\nu}); & (\mu,0)\\ (\mu,1); & (1,\frac{1}{\nu}); & (1,1)\end{matrix}\right.\right]\end{aligned}$$

thank to the formula (3.24) with $c=1$. From (2.2), by observing that

$$\mathcal{M}\left[\frac{1}{t^{\beta/\nu}}\mathbf{G}_\mu^{\nu,\beta}\left(\frac{\cdot}{t^{\beta/\nu}}\right)\right](\eta) = \mathcal{M}[\mathbf{G}_\mu^{\nu,\beta}(\cdot)](\eta)t^{\frac{\eta-1}{\nu}\beta}$$

we obtain the claimed result. □

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