Lyapunov Computational Method for Two-Dimensional Boussinesq Equation

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Abstract

A numerical method is developed leading to Lyapunov operators to approximate the solution of two-dimensional Boussinesq equation. It consists of an order reduction method and a finite difference discretization. It is proved to be uniquely solvable and analyzed for local truncation error for consistency. The stability is checked by using Lyapunov criterion and the convergence is ...... Some numerical implementations are provided at the end of the paper to validate the theoretical results.

Key words: Bousinesq equation, Finite-difference scheme, Stability analysis, Consistency, Convergence, Error estimates, Lyapunov operator.

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1 Introduction

The present paper is devoted to the development of a numerical method based on two-dimensional finite difference scheme to approximate the solution of the nonlinear Boussinesq equation in $\mathbb{R}^2$ written on the form

$$u_{tt} = \Delta\, u + u_{xxxx} + (u^2)_{xx}, \quad ((x, y), t) \in \Omega \times (t_0, +\infty) \tag{1}$$

with initial conditions

$$u(x, y, t_0) = u_0(x, y) \quad \text{and} \quad \frac{\partial\, u}{\partial\, t}(x, y, t_0) = \varphi(x, y), \quad (x, y) \in \Omega \tag{2}$$

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and boundary conditions

$$\frac{\partial u}{\partial n}(x,y,t)=0, \quad ((x,y),t)\in\partial\Omega\times(t_0,+\infty) \tag{3}$$

on a rectangular domain $\Omega=]L_0,L_1[\times]L_0,L_1[$ in $\mathbb{R}^2$. $t_0\geq 0$ is a real parameter fixed as the initial time, u_{tt} is the second order partial derivative in time, $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ is the Laplace operator in $\mathbb{R}^2$, u_{xx} and u_{xxxx} are respectively the second order and the fourth order partial derivative according to x. $\dfrac{\partial}{\partial n}$ is the outward normal derivative operator along the boundary $\partial\Omega$. Finally, u, u_0 and φ are real valued functions with u_0 and φ are $\mathcal{C}^2$ on $\overline{\Omega}$.

The Boussinesq equation ....
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Our main idea consists in developing a numerical scheme to approximate the solution of (1)-(3). The time and the space partial derivatives are replaced by finite-difference approximations in order to transform the initial boundary-value problem (1)-(3) into a linear algebraic system. An order reduction method is adapted leading to a system of coupled PDEs which is transformed by the next to a discrete algebraic system. The resulting method is analyzed for local truncation error and stability and the scheme is proved to be uniquely solvable and convergent.

2 Two-Dimensional Finite Difference Scheme

Consider the rectangular domain $\Omega=]L_0,L_1[\times]L_0,L_1[\subset\mathbb{R}^2$ and an integer $J\in\mathbb{N}^*$. Denote $h=\dfrac{L_1-L_0}{J}$ for the space step, $x_j=L_0+jh$ and $y_m=L_0+mh$ for all $(j,m)\in I^2=\{0,1,\ldots,J\}^2$. Let $l=\Delta t$ be the time step and $t_n=t_0+nl$, $n\in\mathbb{N}$ for the discrete time grid. For $(j,m)\in I$ and $n\geq 0$, $u^n_{j,m,k}$ will be the net function $u(x_j,y_m,t_n)$ and $U^n_{j,m}$ the numerical solution. The following discrete approximations will be applied for the different differential operators involved in the problem. For time derivatives, we set

$$u_t \rightsquigarrow \frac{U^{n+1}_{j,m}-U^{n-1}_{j,m}}{2l} \quad \text{and} \quad u_{tt} \rightsquigarrow \frac{U^{n+1}_{j,m}-2U^n_{j,m}+U^{n-1}_{j,m}}{l^2}$$

and for space derivatives, we shall use

$$u_x \rightsquigarrow \frac{U^n_{j+1,m}-U^n_{j-1,m}}{2h} \quad \text{and} \quad u_y \rightsquigarrow \frac{U^n_{j,m+1}-U^n_{j,m-1}}{2h}$$

for first order derivatives and

$$u_{xx} \rightsquigarrow \frac{U^{n,\alpha}_{j+1,m} - 2U^{n,\alpha}_{j,m} + U^{n,\alpha}_{j-1,m}}{h^2}, \quad u_{yy} \rightsquigarrow \frac{U^{n,\alpha}_{j,m+1} - 2U^{n,\alpha}_{j,m} + U^{n,\alpha}_{j,m-1}}{h^2}$$

for second order ones, where for $n \in \mathbb{N}^*$ and $\alpha \in \mathbb{R}$,

$$u^{n,\alpha} = \alpha\, U^{n+1} + (1-2\alpha)U^n + \alpha\, U^{n-1}.$$

Finally, we denote $\sigma = \dfrac{l^2}{h^2}$ and $\delta = \dfrac{1}{h^2}$.

3 Discrete Two-Dimensional Boussinesq Equation

The method is based on an order reduction idea in which we set

$$v = u_{xx} + u^2. \tag{4}$$

For $(j,m) \in \mathring{I}^2$ an interior point of the grid I^2, ($\mathring{I} = \{1, 2, \ldots, J-1\}$), and $n \geq 1$, the following discrete equation is deduced from (1).

$$\begin{aligned}
&U^{n+1}_{j,m} - 2U^n_{j,m} + U^{n-1}_{j,m} \\
= &\sigma\alpha \left(U^{n+1}_{j-1,m} - 4U^{n+1}_{j,m} + U^{n+1}_{j+1,m} + U^{n+1}_{j,m-1} + U^{n+1}_{j,m+1}\right) \\
&+\sigma(1-2\alpha) \left(U^{n}_{j-1,m} - 4U^{n}_{j,m} + U^{n}_{j+1,m} + U^{n}_{j,m-1} + U^{n}_{j,m+1}\right) \\
&+\sigma\alpha \left(U^{n-1}_{j-1,m} - 4U^{n-1}_{j,m} + U^{n-1}_{j+1,m} + U^{n-1}_{j,m-1} + U^{n-1}_{j,m+1}\right) \\
&+\sigma\alpha \left(V^{n+1}_{j-1,m} - 2V^{n+1}_{j,m} + V^{n+1}_{j+1,m}\right) \\
&+\sigma(1-2\alpha) \left(V^{n}_{j-1,m} - 2V^{n}_{j,m} + V^{n}_{j+1,m}\right) \\
&+\sigma\alpha \left(V^{n-1}_{j-1,m} - 2V^{n-1}_{j,m} + V^{n-1}_{j+1,m}\right).
\end{aligned} \tag{5}$$

Similarly, the following discrete equation is obtained from equation (4).

$$\begin{aligned}
V^{n+1}_{j,m} + V^{n-1}_{j,m} = &\, 2\delta\alpha \left(U^{n+1}_{j-1,m} - 2U^{n+1}_{j,m} + U^{n+1}_{j+1,m}\right) \\
&+2\delta(1-2\alpha) \left(U^{n}_{j-1,m} - 2U^{n}_{j,m} + U^{n}_{j+1,m}\right) \\
&+2\delta\alpha \left(U^{n-1}_{j-1,m} - 2U^{n-1}_{j,m} + U^{n-1}_{j+1,m}\right) \\
&+2\widehat{F(U^n_{j,m})}
\end{aligned} \tag{6}$$

where $F(u) = u^2$, $F^n = F(u^n)$ and $\widehat{F^n} = \dfrac{F^{n-1} + F^n}{2}$. The discrete boundary conditions are written for $n \geq 0$ as

$$U^n_{1,m} = U^n_{-1,m} \quad \text{and} \quad U^n_{J-1,m} = U^n_{J+1,m}, \tag{7}$$

$$U^n_{j,1} = U^n_{j,-1} \quad \text{and} \quad U^n_{j,J-1} = U^n_{j,J+1}, \tag{8}$$

Next, as it is motioned in the introduction, the idea consists in applying Lyapunov operators to approximate the solution of the continuous problem (1)-(3) or its discrete equivalent system (5)-(8). Denote

$$a_1 = \frac{1}{2} + 2\alpha\sigma, \quad a_2 = -\alpha\sigma,$$

$$b_1 = 1 - 2(1-2\alpha)\sigma, \quad b_2 = (1-2\alpha)\sigma,$$

$$c_1 = (1-2\alpha)\delta \quad \text{and} \quad c_2 = \alpha\delta.$$

Equation (5) becomes

$$\begin{aligned}
& a_2U^{n+1}_{j-1,m} + a_1U^{n+1}_{j,m} + a_2U^{n+1}_{j+1,m} + a_2U^{n+1}_{j,m-1} + a_1U^{n+1}_{j,m} + a_2U^{n+1}_{j,m+1} \\
& +a_2\left(V^{n+1}_{j-1,m} - 2V^{n+1}_{j,m} + V^{n+1}_{j+1,m}\right) \\
= & \; b_2U^n_{j-1,m} + b_1U^n_{j,m} + b_2U^n_{j+1,m} + b_2U^n_{j,m-1} + b_1U^n_{j,m} + b_2U^n_{j,m+1} \\
& -a_2U^{n-1}_{j-1,m} - a_1U^{n-1}_{j,m} - a_2U^{n-1}_{j+1,m} - a_2U^{n-1}_{j,m-1} - a_1U^{n-1}_{j,m} - a_2U^{n-1}_{j,m+1} \\
& +b_2\left(V^n_{j-1,m} - 2V^n_{j,m} + V^n_{j+1,m}\right) \\
& -a_2\left(V^{n-1}_{j-1,m} - 2V^{n-1}_{j,m} + V^{n-1}_{j+1,m}\right).
\end{aligned} \tag{9}$$

Equation (6) becomes

$$\begin{aligned}
& V^{n+1}_{j,m} - 2c_2\left(U^{n+1}_{j-1,m} - 2U^{n+1}_{j,m} + U^{n+1}_{j+1,m}\right) \\
= & \; 2c_1\left(U^n_{j-1,m} - 2U^n_{j,m} + U^n_{j+1,m}\right) \\
& +2c_2\left(U^{n-1}_{j-1,m} - 2U^{n-1}_{j,m} + U^{n-1}_{j+1,m}\right) \\
& -V^{n-1}_{j,m} + 2\widehat{F(U^n_{j,m})}.
\end{aligned} \tag{10}$$

Denote A, B and R the matrices defined by

$$A = \begin{pmatrix} a_1 & 2a_2 & 0 & \dots & \dots & 0 \\ a_2 & a_1 & a_2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a_2 & a_1 & a_2 \\ 0 & \dots & \dots & 0 & 2a_2 & a_1 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & 2b_2 & 0 & \dots & \dots & 0 \\ b_2 & b_1 & b_2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & b_2 & b_1 & b_2 \\ 0 & \dots & \dots. & 0 & 2b_2 & b_1 \end{pmatrix}$$

and

$$R = \begin{pmatrix} -2 & 2 & 0 & \dots & \dots & 0 \\ 1 & -2 & 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 & -2 & 1 \\ 0 & \dots & \dots & 0 & 2 & -2 \end{pmatrix}.$$

The system (7)-(10) can be written on the matrix form

$$\begin{cases} \mathcal{L}_A(U^{n+1}) + a_2 R V^{n+1} = \mathcal{L}_B(U^n) - \mathcal{L}_A(U^{n-1}) + R(b_2 V^n - a_2 V^{n-1}), \\ \frac{1}{2} V^{n+1} - c_2 R U^{n+1} = R(c_1 U^n + c_2 U^{n-1}) - \frac{1}{2} V^{n-1} + \widehat{F^n} \end{cases} \quad (11)$$

for all $n \geq 1$ where

$$U^n = (U^n_{j,m})_{0 \leq j,m \leq J}, \; V^n = (V^n_{j,m})_{0 \leq j,m \leq J} \text{ and } F^n = (F(U^n_{j,m}))_{0 \leq j,m \leq J}$$

and for a matrix $Q \in \mathcal{M}_{(J+1)^2}(\mathbb{R})$, $\mathcal{L}_Q$ is the Lyapunov operator defined by

$$\mathcal{L}_Q(X) = QX + XQ, \; \forall\, X \in \mathcal{M}_{(J+1)^2}(\mathbb{R}).$$

4 Solvability of the Discrete Problem

We are now ready to set our first main result.

Theorem 1 *The system (11) is uniquely solvable whenever U^0 and U^1 are known.*

Proof. It reposes on the inverse of Lyapunov operators. Consider the endomorphism Φ which associates to any element (X, Y) in $\mathcal{M}_{(J+1)^2}(\mathbb{R}) \times \mathcal{M}_{(J+1)^2}(\mathbb{R})$ its image $\Phi(X, Y) = (AX + XA + a_2 RY, \frac{1}{2} Y - c_2 RX)$. To prove Theorem 1, it suffices to show that $ker \Phi$ is reduced to 0. Indeed,

$$\Phi(X, Y) = 0 \Longleftrightarrow (AX + XA + a_2 RY, \frac{1}{2} Y - c_2 RX) = (0, 0)$$

or equivalently,

$$Y = 2c_2 RX \quad \text{and} \quad (A + 2a_2 c_2 R^2)X + XA = 0.$$

So, the problem is transformed to the resolution of a Lyapunov type equation of the form

$$\mathcal{L}_{W,A}(X) = WX + XA = 0 \tag{12}$$

where W is the matrix given by $W = A + 2a_2c_2R^2$. Denoting

$$\omega = 2a_2c_2,\ \omega_1 = a_1 + 6\omega,\ \overline{\omega}_1 = \omega_1 + \omega \quad \text{and} \quad \omega_2 = a_2 - 4\omega$$

the matrix W is explicitly given by

$$W = \begin{pmatrix} \omega_1 & 2\omega_2 & 2\omega & 0 & \dots & \dots & \dots & 0 \\ \omega_2 & \overline{\omega}_1 & \omega_2 & \omega & \ddots & \ddots & \ddots & \vdots \\ \omega & \omega_2 & \omega_1 & \omega_2 & \omega & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \omega & \omega_2 & \omega_1 & \omega_2 & \omega \\ \vdots & \ddots & \ddots & \ddots & \omega & \omega_2 & \overline{\omega}_1 & \omega_2 \\ 0 & \dots & \dots & \dots & 0 & 2\omega & 2\omega_2 & \omega_1 \end{pmatrix}.$$

Next, we use the following result.

Lemma 4.1 *Let E be a finite dimensional ($\mathbb{R}$ or $\mathbb{C}$) vector space and $(\Phi_n)_n$ be a sequence of endomorphisms converging uniformly to an invertible endomorphism Φ. Then, there exists n_0 such that, for any $n \geq n_0$, the endomorphism Φ_n is invertible.*

Assume now that $l = o(h^{2+s})$, with $s > 0$ which is always possible. Then, the coefficients appearing in A and W will satisfy as $h \longrightarrow 0$ the following.

$$A_{i,i} = \frac{1}{2} + \varepsilon\, h^{2+2s} \longrightarrow \frac{1}{2}.$$

For $1 \leq\ i \leq\ J-1$,

$$A_{i,i-1} = A_{i,i+1} = \frac{A_{0,1}}{2} = \frac{A_{J,J-1}}{2} = -\varepsilon\, h^{2+2s} \longrightarrow 0.$$

For $2 \leq\ i \leq\ J-2$,

$$W_{i,i} = W_{0,0} = W_{J,J} = \frac{1}{2} + 2\alpha\varepsilon\, h^{2+2s} - 12\alpha^2\varepsilon\, h^{2s} \longrightarrow \frac{1}{2}.$$

Similarly,

$$W_{1,1} = W_{J-1,J-1} = \frac{1}{2} + 2\alpha\varepsilon\, h^{2+2s} - 14\alpha^2\varepsilon\, h^{2s} \longrightarrow \frac{1}{2}$$

and

$$W_{i,i-1} = W_{i,i+1} = \frac{W_{0,1}}{2} = \frac{W_{J,J-1}}{2} = -\alpha\varepsilon\, h^{2+2s} + 8\alpha^2\varepsilon\, h^{2s} \longrightarrow 0$$

Finally,

$$W_{i,i-2} = W_{i,i+2} = \frac{W_{0,2}}{2} = \frac{W_{J,J-2}}{2} = -2\alpha^2\varepsilon\, h^{2s} \longrightarrow 0.$$

Next, observing that for all X in the space $\mathcal{M}_{(J+1)^2}(\mathbb{R}) \times \mathcal{M}_{(J+1)^2}(\mathbb{R})$,

$$\begin{aligned} \|(\mathcal{L}_{W,A} - I)(X)\| &= \|(W - \tfrac{1}{2}I)X + X(A - \tfrac{1}{2}I)\| \\ &\le \left[\|W - \tfrac{1}{2}I\| + \|A - \tfrac{1}{2}I\|\right] \|X\|, \end{aligned}$$

it results that

$$\|\mathcal{L}_{W,A} - I\| \le \|W - \frac{1}{2}I\| + \|A - \frac{1}{2}I\| \le C(\alpha)h^{2s}. \tag{13}$$

Consequently, the Lyapunov endomorphism $\mathcal{L}_{W,A}$ converges uniformly to the identity I as h goes towards 0 and $l = o(h^{2+s})$ with $s > 0$. Using Lemma 4.1, the operator $\mathcal{L}_{W,A}$ is invertible for h small enough.

5 Convergence of the Discrete Method

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6 Consistency and Stability of the Discrete Method

The consistency of the proposed method is done by evaluating the the local truncation error arising from the discretization of the system

$$\begin{cases} u_{tt} - \Delta\, u - v_{xx} = 0, \\ v = q u_{xx} + u^2. \end{cases} \tag{14}$$

The principal part of the first equation is

$$\begin{aligned} \mathcal{L}^1_{u,v}(t,x,y) = &\frac{l^2}{12}\frac{\partial^4 u}{\partial\, t^4} - \frac{h^2}{12}\left(\frac{\partial^4 u}{\partial\, x^4} + \frac{\partial^4 u}{\partial\, y^4}\right) - \alpha\, l^2 \frac{\partial^2(\Delta\, u)}{\partial\, t^2} \\ &- \frac{h^2}{12}\frac{\partial^2 v}{\partial\, x^4} - \alpha\, l^2 \frac{\partial^4 v}{\partial\, t^2 \partial\, x^2} + O(l^2 + h^2). \end{aligned}$$

The principal part of the local error truncation due to the second part is

$$\mathcal{L}^2_{u,v}(t,x,y) = \frac{l^2}{2}\frac{\partial^2 v}{\partial t^2} + \frac{l^4}{24}\frac{\partial^4 v}{\partial t^4} - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4} - \alpha\, l^2 \frac{\partial^4 u}{\partial t^2 \partial x^2} + O(l^2 + h^2).$$

It is clear that the two operators $\mathcal{L}^1_{u,v}$ and $\mathcal{L}^2_{u,v}$ tend toward 0 as l and h tend to 0, which ensures the consistency of the method. Furthermore, the method is consistent with an order 2 in time and space.

We now proceed by proving the stability of the method by applying the Lyapunov criterion. A linear system $\mathcal{L}(x_{n+1}, x_n, x_{n-1}, \dots) = 0$ is stable in the sense of Lyapunov if for any bounded initial solution x_0 the solution x_n remains bounded for all $n \geq 0$. Here, we will precisely prove the following result.

Lemma 2 $\mathcal{P}_n$: *The solution (U^n, V^n) is bounded independently of n whenever the initial solution (U^0, V^0) is bounded.*

We will proceed by recurrence on n. Assume firstly that $\|(U^0, V^0)\| \leq \eta$ for some η positive. Using the system (11), we obtain

$$\begin{cases} \mathcal{L}_{W,A}(U^{n+1}) = \mathcal{L}_{\widetilde{B},B}(U^n) + b_2 R V^n - \mathcal{L}_{W,A}(U^{n-1}) - a_2 R(F^{n-1} + F^n), \\ V^{n+1} = 2c_2 R U^{n+1} + 2R(c_1 U^n + c_2 U^{n-1}) - V^{n-1} + 2\widehat{F^n}. \end{cases} \tag{15}$$

where $\widetilde{B} = B - 2a_2 c_1 R^2$. Consequently,

$$\|\mathcal{L}_{W,A}(U^{n+1})\| \leq \|\mathcal{L}_{\widetilde{B},B}\|.\|U^n\| + 2|b_2|.\|V^n\| + \|\mathcal{L}_{W,A}\|.\|U^{n-1}\| + 2|a_2|(\|F^{n-1}\| + \|F^n\|) \tag{16}$$

and

$$\|V^{n+1}\| \leq 4|c_2|.\|U^{n+1}\| + 4(|c_1|.\|U^n\| + |c_2|.\|U^{n-1}\|) + \|V^{n-1}\| + \|F^{n-1}\| + \|F^n\|. \tag{17}$$

Next, recall that, for $l = o(h^{s+2})$ small enough, $s > 0$, we have

$$a_1 = \frac{1}{2} + 2\alpha\, h^{2s+2} \to \frac{1}{2}, \quad a_2 = -\alpha\, h^{2s+2} \to 0,$$

$$b_1 = 1 - 2(1-2\alpha)h^{2s+2} \to 1, \quad b_2 = (1-2\alpha)h^{2s+2} \to 0,$$
$$c_1 = (1-2\alpha)h^{-2} \to \infty \quad \text{and} \quad c_2 = \alpha\, h^{2s+2} \to \infty,$$
$$a_2 c_1 = -\alpha(1-2\alpha)h^{2s} \to 0.$$

As a consequence, for h small enough,

$$\|\mathcal{L}_{\widetilde{B},B}\| \leq 2\|B\| + 2|a_2 c_1|\|R\|^2 \leq 2\max(|b_1|, 2|b_2|) + 4|a_2 c_1| \leq 2 + 4 = 6, \tag{18}$$

and the following lemma deduced from (13).

Lemma 6.1 *For h small enough, it holds for all $X \in \mathcal{M}_{(J+1)^2}(\mathbb{R})$ that*

$$\frac{1}{2}\|X\| \le (1 - C(\alpha)h^{2s})\|X\| \le \|\mathcal{L}_{W,A}(X)\| \le (1 + C(\alpha)h^{2s})\|X\| \le \frac{3}{2}\|X\|.$$

As a result, (16) yields that

$$\frac{1}{2}\|U^{n+1}\| \le 6\|U^n\| + 2\|V^n\| + \frac{3}{2}\|U^{n-1}\| + 2(\|F^{n-1}\| + \|F^n\|). \tag{19}$$

For $n = 0$, this implies that

$$\|U^1\| \le 12\|U^0\| + 4\|V^0\| + 3\|U^{-1}\| + 4(\|F^{-1}\| + \|F^0\|). \tag{20}$$

Using the discrete initial condition

$$U^0 = U^{-1} + l\varphi.$$

Here we identify the function φ to the matrix whom coefficients are $\varphi_{j,m} = \varphi(x_j, y_m)$. We obtain

$$\|U^{-1}\| \le \|U^0\| + l\|\varphi\|. \tag{21}$$

Observing that

$$F^{-1}_{j,m} = F(U^{-1}_{j,m}) = (U^0_{j,m} - l\varphi_{j,m})^2,$$

it results that

$$|F^{-1}_{j,m}| \le |U^0_{j,m}|^2 + 2l|\varphi_{j,m})|.|U^0_{j,m}| + l^2|\varphi_{j,m}|^2$$

and consequently,

$$\|F^{-1}\| \le \|U^0\|^2 + 2l\|\varphi\|.\|U^0\| + l^2\|\varphi\|^2. \tag{22}$$

Hence, equation (20) yields that

$$\|U^1\| \le (15 + 8l\|\varphi\|)\|U^0\| + 4\|V^0\| + 8\|F^0\| + 3l\|\varphi\| + 4l^2\|\varphi\|^2. \tag{23}$$

Now, the Lyapunov criterion for stability states exactly that

$$\forall\, \varepsilon > 0, \exists\, \eta > 0 \;\; s.t; \;\; \|(U^0, V^0)\| \le \eta \;\Rightarrow\; \|(U^n, V^n)\| \le \varepsilon, \; \forall\, n \ge 0. \tag{24}$$

For $n = 1$ and $\|(U^1, V^1)\| \le \varepsilon$, we seek an $\eta > 0$ for which $\|(U^0, V^0)\| \le \eta$. Indeed, using (23), this means that, it suffices to find η such that

$$8\eta^2 + (19 + 8l\|\varphi\|)\eta + 3l\|\varphi\| + 4l^2\|\varphi\|^2 - \varepsilon < 0. \tag{25}$$

Choosing l small enough, we obtain a discriminant

$$\Delta = (19 + 8l\|\varphi\|)^2 - 32(3l\|\varphi\| + 4l^2\|\varphi\|^2 - \varepsilon) > 0$$

and a zero $\eta_1 = \dfrac{\sqrt{\Delta} - (19 + 8l\|\varphi\|)}{16} > 0$. Consequently, it suffices to choose $\eta \in]0, \eta_1[$ to obtain (25). Finally, (23) yields that $\|U^1\| \leq \varepsilon$.
Now, equation (17), for $n = 0$, implies that

$$\|V^1\| \leq A(l, h, \varphi)\|U^0\|^2 + B(l, h, \varphi)\|U^0\| + C(l, h, \varphi) + 16|c_2|\|V^0\|, \quad (26)$$

where

$$A(l, h, \varphi) = 3 + 32|c_2|,$$
$$B(l, h, \varphi) = 4\left(|c_1| + 8|c_2|(2 + l\varphi) + l\varphi + \frac{1}{h^2}\right),$$

and

$$C(l, h, \varphi) = 2(1 + 8|c_2|)l^2\|\varphi\|^2 + 4l(4|c_2| + \frac{1}{h^2})\|\varphi\|.$$

Choosing $\|(U^0, V^0)\| \leq \eta$, it suffices to study the inequality

$$A(l, h, \varphi)\eta^2 + (B(l, h, \varphi) + 16|c_2|)\,\eta + C(l, h, \varphi) - \varepsilon \leq 0. \quad (27)$$

Its discriminant satisfies

$$\Delta \sim \frac{16}{h^4}\left(1 + 20\alpha + |1 - 2\alpha|\right)^2 - \frac{128\alpha}{h^2}\left((4 + 16\alpha)\|\varphi\|h^s - \varepsilon\right) > 0$$

for h small enough. The zero $\eta_1' = \dfrac{\sqrt{\Delta} - (B(l, h, \varphi) + 16|c_2|)}{A(l, h, \varphi)} > 0$. As a consequence, for $\eta \in]0, \eta_1'[$ we obtain $\|V^1\| \leq \varepsilon$. Finally, for $\eta \in]0, \eta_0[$ with $\eta_0 = \min(\eta_1, \eta_1')$, we obtain $\|(U^1, V^1)\| \leq \varepsilon$ whenever $\|(U^0, V^0)\| \leq \eta$. Assume now that the (U^k, V^k) is bounded for $k = 1, 2, \ldots, n$ (by ε_1) whenever (U^0, V^0) is bounded by η and let $\varepsilon > 0$. We shall prove that it is possible to choose η satisfying $\|(U^{n+1}, V^{n+1})\| \leq \varepsilon$. Indeed, from (19), we have

$$\|U^{n+1}\| \leq 19\varepsilon_1 + 8\varepsilon_1^2. \quad (28)$$

So, one seeks, ε_1 for which $8\varepsilon_1^2 + 19\varepsilon_1 - \varepsilon \leq 0$. The discriminant is $\Delta = 361 + 32\varepsilon$, leading to one positive zeros $\varepsilon_1 = \dfrac{\sqrt{361 + 32\varepsilon} - 19}{16}$. Then $\|U^{n+1}\| \leq \varepsilon$ whenever $\|(U^k, V^k)\| \leq \varepsilon_1$, $k = 1, 2, \ldots, n$. Next, using (17) and (28), we have

$$\|V^{n+1}\| \leq (4|c_1| + 80|c_2| + 1)\,\varepsilon_1 + (32|c_2| + 2)\,\varepsilon_1^2. \quad (29)$$

So, it suffices as previously to choose ε_1 such that

$$(32|c_2| + 2)\,\varepsilon_1^2 + (4|c_1| + 80|c_2| + 1)\,\varepsilon_1 - \varepsilon \leq 0.$$

The discriminant is $\Delta = (4|c_1| + 80|c_2| + 1)^2 + 4(32|c_2| + 2)\varepsilon$, leading to one positive zeros $\varepsilon_1' = \dfrac{\sqrt{\Delta} - (4|c_1| + 80|c_2| + 1)}{2(32|c_2| + 2)}$. Then $\|U^{n+1}\| \leq \varepsilon$ whenever $\|(U^k, V^k)\| \leq \varepsilon_1'$, $k = 1, 2, \ldots, n$. Next, it holds from the recurrence hypothesis for $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_1')$, that there exists $\eta > 0$ for which $\|(U^0, V^0)\| \leq \eta$ implies that $\|(U^k, V^k)\| \leq \varepsilon_0$, for $k = 1, 2, \dots, n$, which by the next induces that $\|(U^{n+1}, V^{n+1})\| \leq \varepsilon$.

7 Numerical implementations

The initial data becomes

$$U_{j,m}^0 = u(x_j, y_m, t_0) = u_0(x_j, y_m), \quad (j, m) \in I, \tag{30}$$

$$U_{j,m}^1 = \frac{1}{2}\left(2u_0(x_j, y_m) + l^2\Delta\, u_0(x_j, y_m) + (v_0)_{xx}(x_j, y_m) - 2lg(x_j, y_m)\right). \tag{31}$$

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