A limit $q = -1$ for big q-Jacobi polynomials

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Abstract

We study a new family of "classical" orthogonal polynomials which satisfy (apart from a 3-term recurrence relation) an eigenvalue problem with differential operators of Dunkl-type. These polynomials can be obtained from the big q-Jacobi polynomials in the limit $q \to -1$. An explicit expression of these polynomials in terms of Gauss' hypergeometric functions is found. We also show that these polynomials provide a nontrivial realization of the Askey-Wilson algebra for $q \to -1$.

Keywords: classical orthogonal polynomials, Jacobi polynomials, big q-Jacobi polynomials.

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1. Introduction

We constructed in [13] a system of "classical" orthogonal polynomials $P_n(x)$ containing two real parameters α, β and corresponding to the limit $q \to -1$ of the little q-Jacobi polynomials. By "classical" we mean that these polynomials satisfy (apart from a 3-term recurrence relation) a nontrivial eigenvalue equation of the form

$$LP_n(x) = \lambda_n P_n(x). \tag{1.1}$$

The novelty lies in the fact that L is a differential-difference operator of special type. Namely, L is a linear operator which is of first order in the derivative operator ∂_x and contains also the reflection operator R which acts as $Rf(x) = f(-x)$. Roughly speaking, one can say that L belongs to the class of Dunkl operators [4] which contain both the operators ∂_x and R. Nevertheless, the operator L differs from the standard Dunkl operators in a fundamental way. Indeed, L preserves the linear space of polynomials of any given maximal degree. This basic property allows to construct a complete system of polynomials $P_n(x)$, $n = 0, 1, 2, \dots$ as eigenfunctions of the operator L.

Guided by the $q \to -1$ limit of the little q-Jacobi polynomials, we derived in [13] an explicit expression of the polynomials $P_n(x)$ in terms of Gauss' hypergeometric functions. We also found explicitly the recurrence coefficients and showed that the polynomaials $P_n(x)$ are orthogonal on the interval $[-1, 1]$ with a weight function related to the weight function of the generalized Jacobi polynomials [3]. We also proved that they admit the Dunkl classical property [1] and further demonstrated that the operator L together with the multiplication operator x form a special case of the Askey-Wilson algebra $AW(3)$ [14] corresponding to the parameter $q = -1$.

In this paper we construct similarly, a new family of "classical" orthogonal polynomials which are obtained as a nontrivial limit of the big q-Jacobi polynomials when $q \to -1$. We will call them "big -1 Jacobi polynomials"

In contast to the little -1 Jacobi polynomials, the big -1 Jacobi polynomials contain 3 real parameters α, β, c. This leads to more complicated formulas for the recurrence coefficients as well as for the explicit expression in terms of the Gauss hypergeometric function. Moreover, in contrast to the little -1 Jacobi polynomials the big -1 Jacobi polynomials are orthogonal on the union of the *two intervals* $[-1, -c]$ and $[c, 1]$ (it is assumed that $0 < c < 1$) When $c = 0$ these intervals connect into one interval $[-1, 1]$. This corresponds to the degeneration of the big -1 Jacobi polynomials into the little -1 Jacobi polynomials

The fundamental "classical" property (1.1) holds for the big -1 Jacobi polynomials as well. The operator L is again a first order differential operator of Dunkl type which preserves the space of polynomials. This means that both little and big -1 Jacobi polynomials provide two "missing" families of classical orthogonal polynomials which should be included into the Askey table as special cases.

We also show that the big -1 Jacobi polynomials provide a convenient realization of the $AW(3)$ algebra for $q = -1$.

2. Big q-Jacobi polynomials in the limit $q = -1$

The big q-Jacobi polynomials $P_n(x; a, b, c)$ depend on 3 parameters and are defined by the following 3-term recurrence relation (for brevity, we will sometimes omit the dependence on the parameters a, b, c):

$$P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = x P_n(x), \tag{2.1}$$

where

$$u_n = A_{n-1} C_n, \quad b_n = 1 - A_n - C_n$$

with

$$A_n = \frac{(1 - aq^{n+1})(1 - abq^{n+1})(1 - cq^{n+1})}{(1 - abq^{2n+1})(1 - abq^{2n+2})}, \quad C_n = -acq^{n+1}\, \frac{(1 - q^n)(1 - abc^{-1}q^n)(1 - bq^n)}{(1 - abq^{2n+1})(1 - abq^{2n})} \tag{2.2}$$

In terms of basic hypergeometric functions [8], [9] they are given by

$$P_n(x; a, b, c) = \kappa_n {}_3\varphi_2 \left(\begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, cq \end{matrix} \Big| q; q \right) \tag{2.3}$$

where the coefficient κ_n ensures that $P_n(x)$ is monic: $P_n(x) = x^n + O(x^{n-1})$. We shall not need the explicit expression of $< kappa_n$ in the following.

The big q-Jacobi polynomials satisfy the eigenvalue equation [8], [9]

$$L P_n(x) = \lambda_n P_n(x), \quad \lambda_n = (q^{-n} - 1)(1 - abq^{n+1}) \tag{2.4}$$

where the operator L is

$$L f(x) = B(x)(f(xq) - f(x)) + D(x)(f(xq^{-1}) - f(x)) \tag{2.5}$$

with

$$B(x) = \frac{aq(x-1)(bx-c)}{x^2}, \quad D(x) = \frac{(x - aq)(x - cq)}{x^2} \tag{2.6}$$

The orthogonality relation is

$$\int_{cq}^{aq} w(x) P_n(x) P_m(x) d_q x = h_n \delta_{nm}, \quad h_n = u_1 u_2 \dots u_n, \tag{2.7}$$

with the q-integral defined as [8], [9]

$$\int_{cq}^{aq} f(x) d_q x = aq(1-q) \sum_{s=0}^{\infty} f(aq^{s+1}) q^s - cq(1-q) \sum_{s=0}^{\infty} f(cq^{s+1}) q^s$$

and the weight function

$$w(x) = g \frac{(a^{-1}x; q)_\infty (c^{-1}x; q)_\infty}{(x; q)_\infty (bc^{-1}x; q)_\infty}, \tag{2.8}$$

where

$$(a; q)_s = (1 - a)(1 - aq) \dots (1 - aq^{s-1})$$

is the shifted q-factorial [9] and $(a; q)_\infty = \lim_{s\to\infty} (a; q)_s$ (In (2.8), g is a normalization factor which is not essential for our considerations).

Consider the operator $(q+1)^{-1}L$, where the operator L is defined by (2.5). Put

$$q = -\exp(\epsilon),\ a = -\exp(\epsilon\alpha),\ b = -\exp(\epsilon\beta) \tag{2.9}$$

and take the limit $\epsilon \to 0$ which corresponds to the limit $q \to -1$. It is not difficult to verify that the limit does exist and that we have

$$L_0 = \lim_{q\to -1}(q+1)^{-1}L = g_0(x)(R-I) + g_1(x)\partial_x R, \tag{2.10}$$

where

$$g_0(x) = \alpha+\beta+1+(c\alpha-\beta)x^{-1}+cx^{-2}, \quad g_1(x) = \frac{2(x-1)(x+c)}{x}. \tag{2.11}$$

The operator I is the identity operator and R is the reflection operator $Rf(x) = f(-x)$.

Equivalently, the operator L_0 can be presented through its action on $f(x)$:

$$L_0 f(x) = g_0(x)(f(-x)-f(x)) - g_1(x)f'(-x) \tag{2.12}$$

On monomials x^n the operator L_0 acts as follows.

For n even,

$$L_0 x^n = 4n(x-1)(x+c)x^{n-2}. \tag{2.13}$$

For n odd

$$L_0 x^n = -2(\alpha+\beta+n+1)x^n + 2(\beta - c\alpha + n - c)x^{n-1} + 2(n-1)cx^{n-2} \tag{2.14}$$

In any case, the operator L_0 is lower triangular, with 3 diagonals, in the basis x^n:

$$Lx^n = \xi_n x^n + \eta_n x^{n-1} + \zeta_n x^{n-2} \tag{2.15}$$

with the coefficients ξ_n, η_n, ζ_n straightforwardly obtained from (2.13), (2.14). It is easily seen that the operator L_0 preserves the linear space of polynomials of any fixed dimension. Hence for every $n = 0, 1, 2, \dots$ there are monic polynomials eigenfunctions $P_n^{(-1)}(x) = x^n + O(x^{n-1})$ of the operator L_0

This eigenvalue equation is obtained as the $q \to -1$ limit of the eigenvalue equation (2.4):

$$L_0 P_n^{(-1)}(x) = \lambda_n\, P_n^{(-1)}(x), \tag{2.16}$$

where

$$\lambda_n = \begin{cases} 2n, & n \quad \text{even} \\ -2(\alpha+\beta+n+1), & n \quad \text{odd} \end{cases} \tag{2.17}$$

Consider the limit $q \to -1$ for the recurrence coefficients. Assuming (2.9), we have

$$A_n^{(-1)} = \lim_{\epsilon\to 0} A_n = \begin{cases} \frac{(c+1)(\alpha+n+1)}{\alpha+\beta+2n+2}, & n \quad \text{even} \\ \frac{(1-c)(\alpha+\beta+n+1)}{\alpha+\beta+2n+2}, & n \quad \text{odd} \end{cases} \tag{2.18}$$

and

$$C_n^{(-1)} = \lim_{\epsilon\to 0} C_n = \begin{cases} \frac{(1-c)n}{\alpha+\beta+2n}, & n \quad \text{even} \\ \frac{(1+c)(\beta+n)}{\alpha+\beta+2n}, & n \quad \text{odd} \end{cases} \tag{2.19}$$

Hence for the recurrence coefficients we have

$$u_n^{(-1)} = \lim_{\epsilon\to 0} A_{n-1}C_n = \begin{cases} \frac{(1-c)^2 n(\alpha+\beta+n)}{(\alpha+\beta+2n)^2}, & n \quad \text{even} \\ \frac{(1+c)^2(\alpha+n)(\beta+n)}{(\alpha+\beta+2n)^2}, & n \quad \text{odd} \end{cases} \tag{2.20}$$

and

$$b_n^{(-1)} = \lim_{\epsilon\to 0} 1 - A_n - C_n = \begin{cases} -c + \frac{(c-1)n}{\alpha+\beta+2n} + \frac{(1+c)(\beta+n+1)}{\alpha+\beta+2n+2}, & n \quad \text{even} \\ c + \frac{(1-c)(n+1)}{\alpha+\beta+2n+2} - \frac{(c+1)(\beta+n)}{\alpha+\beta+2n}, & n \quad \text{odd} \end{cases} \tag{2.21}$$

The polynomials $P_n^{(-1)}(x)$ satisfy the 3-term recurrence relation

$$P_{n+1}^{(-1)}(x) + b_n^{(-1)} P_n^{(-1)}(x) + u_n^{(-1)} P_{n-1}^{(-1)}(x) = x P_n^{(-1)}(x). \tag{2.22}$$

For any real $c \neq 1$ and real α, β satisfying the restriction $\alpha > -1,\ \beta > -1$, the recurrence coefficients $b_n^{(-1)}$ are real and the recurrence coeficients u_n are positive. This means that the polynomials $P_n^{(-1)}(x)$ are positive definite orthogonal polynomials.

Let us consider expression (2.3) in details,

$$P_n(x) = \kappa_n \sum_{s=0}^{n} \frac{(q^{-n};q)_s (abq^{n+1};q)_s (x;q)_s}{(q;q)_s (aq;q)_s (cq;q)_s} q^s, \tag{2.23}$$

In the limit $q \to -1$ it is easy to obtain that

$$\frac{(x;q)_s}{(cq;q)_s} = \begin{cases} \left(\frac{1-x^2}{1-c^2}\right)^{s/2}, & s \quad \text{even} \\ \frac{1-x}{1+c}\left(\frac{1-x^2}{1-c^2}\right)^{(s-1)/2}, & s \quad \text{odd} \end{cases}$$

Hence, in the limit $q \to -1$ the sum (2.23) is divided in two parts. The first part is an even polynomial with respect to x, i.e. $p(x^2)$, where $p(x)$ is a polynomial. The second part will have the form $(1-x)q(x^2)$ with another polynomial $q(x)$. Simple calculations lead to the following formulas.

If n is even

$$P_n^{(-1)}(x) = \kappa_n \left[{}_2F_1 \left(\left. \begin{matrix} -\frac{n}{2}, \frac{n+\alpha+\beta+2}{2} \\ \frac{\alpha+1}{2} \end{matrix} \right| \frac{1-x^2}{1-c^2} \right) + \frac{n(1-x)}{(1+c)(\alpha+1)} {}_2F_1 \left(\left. \begin{matrix} 1-\frac{n}{2}, \frac{n+\alpha+\beta+2}{2} \\ \frac{\alpha+3}{2} \end{matrix} \right| \frac{1-x^2}{1-c^2} \right) \right] \tag{2.24}$$

If n is odd

$$P_n^{(-1)}(x) = \kappa_n \left[{}_2F_1 \left(\left. \begin{matrix} -\frac{n-1}{2}, \frac{n+\alpha+\beta+1}{2} \\ \frac{\alpha+1}{2} \end{matrix} \right| \frac{1-x^2}{1-c^2} \right) - \frac{(\alpha+\beta+n+1)(1-x)}{(1+c)(\alpha+1)} {}_2F_1 \left(\left. \begin{matrix} -\frac{n-1}{2}, \frac{n+\alpha+\beta+3}{2} \\ \frac{\alpha+3}{2} \end{matrix} \right| \frac{1-x^2}{1-c^2} \right) \right] \tag{2.25}$$

The normalization coefficient is given by

$$\kappa_n = \begin{cases} \frac{(1-c^2)^{n/2}((\alpha+1)/2)_{n/2}}{((n+\alpha+\beta+2)/2)_{n/2}}, & n \quad \text{even} \\ (1+c)\frac{(1-c^2)^{(n-1)/2}((\alpha+1)/2)_{(n+1)/2}}{((n+\alpha+\beta+1)/2)_{(n+1)/2}}, & n \quad \text{odd} \end{cases} \tag{2.26}$$

The remaining problem is to find the othogonality relation and the corresponding weight function $w(x)$ for the big -1 Jacobi polynomials. Of course, this could be done directly from the known orthogonality relation for the big q-Jacobi polynomials by taking the limit $q \to -1$. However it is more instructive to derive the weight function using the method of polynomial mappings [6], [10]. This method will allow to find nontrivial relations between the big -1 Jacobi polynomials and the ordinary Jacobi polynomials. This will explain the origin of the rather "strange" expressions (2.24) and (2.25).

3. Polynomial systems and the Christoffel transform

In this section we consider a scheme allowing to obtain a new family of orthogonal polynomial starting from two sets of orthogonal polynomials related by the Christoffel transform. This scheme is a simple generalization of the well known Chihara method for constructing symmetric orthogonal polynomials from a pair of orthogonal polynomials and their kernel partner [2]. It is also very close to the scheme proposed by Marcellán and Petronilho in [10].

Let $P_n(x),\ n = 0, 1, 2, \ldots$ be a set of monic orthogonal polynomials satisfying the recurrence relation

$$P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = x P_n(x). \tag{3.1}$$

Consider a partner family of orthogonal polynomials $Q_n(x)$ related to $P_n(x)$ by the Christoffel transform [11]

$$Q_n(x) = \frac{P_{n+1}(x) - A_n P_n(x)}{x - \nu^2}, \tag{3.2}$$

where ν is a real parameter and $A_n = P_{n+1}(\nu^2)/P_n(\nu^2)$.

If the polynomials $P_n(x)$ are monic orthogonal with respect to the linear functional σ

$$\langle \sigma, P_n(x) P_m(x) \rangle = 0, \quad n \neq m$$

then the polynomials $Q_n(x)$ are monic orthogonal with respect to the functional $\tilde{\sigma} = (x - \nu^2)\sigma$, i.e. [11]

$$\langle \sigma, (x - \nu^2) Q_n(x) Q_m(x) \rangle = 0, \quad n \neq m$$

The polynomials $P_n(x)$ are expressed in terms of the polynomials $Q_n(x)$ via the Geronimus transform [15]

$$P_n(x) = Q_n(x) - B_n Q_{n-1}(x), \tag{3.3}$$

where the coefficients B_n are related to A_n and the recurrence coefficients by the formulas

$$u_n = B_n A_{n-1}, \quad b_n = -A_n - B_n + \nu^2. \tag{3.4}$$

Now, starting from a pair of polynomials $P_n(x), Q_n(x)$ we can construct another family of orthogonal polynomial $R_n(x)$ by proceeding as follows.

For even numbers n, let the polynomials $R_n(x)$ be defined according to

$$R_{2n}(x) = P_n(x^2) \tag{3.5}$$

and for odd numbers n, let

$$R_{2n+1}(x) = (x - \nu)Q_n(x^2) \tag{3.6}$$

It is obvious that for all $n = 0, 1, 2, \ldots$ the polynomials $R_n(x)$ are monic polynomials in x of degree n.

What is more important is that the polynomials $R_n(x)$ are *orthogonal*, since they satisfy the 3-term recurrence relation

$$R_{n+1}(x) + (-1)^n \nu R_n(x) + v_n R_{n-1}(x) = x R_n(x), \tag{3.7}$$

where

$$v_{2n} = -B_n, \ v_{2n+1} = -A_n \tag{3.8}$$

This construction can also be carried out in the reverse.

Assume that the polynomials $R_n(x)$ satisfy the recurrence relation (3.7) with some real parameter ν and positive coefficients v_n, it can easily be shown by induction that

$$R_{2n}(x) = P_n(x^2), \quad R_{2n+1}(x) = (x - \nu)Q_n(x^2),$$

where $P_n(x), Q_n(x)$ are monic polynomials of degree n.

The polynomials $R_n(x)$ are orthogonal with respect to a positive definite linear functional ρ:

$$\langle \rho, R_n(x) R_m(x) \rangle = 0, \quad n \neq m \tag{3.9}$$

Let

$$r_n = \langle \rho, x^n \rangle$$

be the corresponding moments. We use the standard normalization condition $r_0 = 1$. It can then be proven, again by induction, that

$$r_{2n+1} = \nu r_{2n}, \quad n = 0, 1, 2, \ldots \tag{3.10}$$

and that the even moment r_{2n} is an even monic polynomial of degree $2n$ in the argument ν, i.e.

$$r_{2n} = \nu^{2n} + n v_1 \nu^{2n-2} + \frac{n(n-1)}{2} v_1 (v_1 + v_2) \nu^{2n-4} + O(\nu^{2n-6}).$$

It is directly verified that the polynomials $P_n(x)$ and $Q_n(x)$ are orthogonal as they satisfy the recurrence relations

$$P_{n+1}(x) + (v_{2n} + v_{2n+1} + \nu^2) P_n(x) + v_{2n} v_{2n-1} P_{n-1}(x) = x P_n(x)$$

and

$$Q_{n+1}(x) + (v_{2n+2} + v_{2n+1} + \nu^2) Q_n(x) + v_{2n} v_{2n+1} Q_{n-1}(x) = x Q_n(x)$$

Moreover, the polynomials $Q_n(x)$ are Christoffel transforms of the polynomials $P_n(x)$:

$$Q_n(x) = \frac{P_{n+1}(x) + v_{2n+1} P_n(x)}{x - \nu^2}.$$

while the polynomials $P_n(x)$ are Geronimus transforms of $Q_n(x)$:

$$P_n(x) = Q_n(x) + v_{2n} Q_{n-1}(x)$$

Assume that the polynomials $P_n(x)$ have moments c_n. Then one has a simple relation between the moments

$$r_{2n} = c_n, \; r_{2n+1} = \nu c_n, \quad n = 0, 1, 2, \dots \tag{3.11}$$

The moments $\tilde{c}_n$ corresponding to the polynomials $Q_n(x)$ are given by

$$\tilde{c}_n = \frac{c_{n+1} - \nu^2 c_n}{c_1 - \nu^2} \tag{3.12}$$

Expression (3.12) follows easily from the definition of the Christoffel transform [15].

Note that in the special case $\nu = 0$ we recover the well known scheme relating symmetric and non-symmetric polynomials that has been described in details by Chihara [2]. In this case the polynomials $R_n(x)$ are symmetric $R_n(-x) = (-1)^n R_n(x)$ and their odd moments are zero $r_{2n+1} = 0$. All the above formulas remain valid if one puts $\nu = 0$. We have thus provided a generalization of the Chihara scheme with an additional parameter ν. Note that the resulting polynomials $R_n(x)$ are no longer symmetric; they satisfy however the simple recurrence relation (3.7) and have properties very close to those of symmetric orthogonal polynomials.

In [10] a more general problem was studied with the orthogonal polynomials $R_n(x)$ defined as $R_{2n}(x) = P_n(\phi(x))$, where $\phi(x)$ is a polynomial of second degree and $P_n(x)$ is a given system of orthogonal polynomials. Our approach corresponds to the special case $\phi(x) = x^2$. Note that the general case of polynomial mapping has the form $R_{Nn}(x) = P_n(\pi_N(x))$, where $\pi_N(x)$ is a polynomial of degree N. Again it is assumed that both $P_n(x)$ and $R_n(x)$ are nondegenerate orthogonal polynomials. The theory of such mappings was considered in [6].

Consider now the following concrete example connected with Jacobi polynomials. This example will allow to establish the weight function of the big -1-Jacobi polynomials.

Let

$$P_n^{(\xi,\eta)}(x) = G_n \; {}_2F_1 \left({-n, n+\xi+\eta+1 \atop \xi+1} ; x \right)$$

be Jacobi polynomials with orthogonality relation

$$\int_0^1 x^{\xi} (1-x)^{\eta} P_n^{(\xi,\eta)}(x) P_m^{(\xi,\eta)}(x) dx = h_n m \, \delta_{nm}$$

on the interval $[0,1]$.

The normalization coefficient

$$G_n = (-1)^n \, \frac{(\xi+1)_n}{(n+\xi+\eta+1)_n}$$

ensures that $P_n(x)$ is monic $P_n^{(\xi,\eta)}(x) = x^n + O(x^{n-1})$.

Perform first an affine transformation of the argument and consider the new monic orthogonal polynomials

$$P_n(x) = (c^2-1)^n \, P_n^{(\xi,\eta)} \left(\frac{1-x}{1-c^2} \right),$$

where c is a real parameter with the restriction $0 < c < 1$.

In terms of hypergeometric functions

$$P_n(x) = (1-c^2)^n \, \frac{(\xi+1)_n}{(n+\xi+\eta+1)_n} \, {}_2F_1\left({-n, n+\xi+\eta+1 \atop \xi+1} ; \frac{1-x}{1-c^2} \right) \tag{3.13}$$

Clearly, these polynomials are orthogonal on the interval $[c^2, 1]$

$$\int_{c^2}^1 (1-x)^\xi (x-c^2)^\eta P_n(x) P_m(x) dx = 0, \quad n \neq m$$

Introduce also the companion polynomials $Q_n(x)$ through the Christoffel transform

$$Q_n(x) = \frac{P_{n+1}(x) - A_n P_n(x)}{x-1}, \quad A_n = \frac{P_{n+1}(1)}{P_n(1)} \tag{3.14}$$

It is easily seen that the polynomials $Q_n(x)$ are again expressible in terms of Jacobi polynomials with $\xi \to \xi+1$:

$$Q_n(x) = (c^2-1)^n \; P_n^{(\xi+1,\eta)}\left(\frac{1-x}{1-c^2}\right),$$

or, in terms of hypergeometric functions

$$Q_n(x) = (1-c^2)^n \, \frac{(\xi+2)_n}{(n+\xi+\eta+2)_n} \, {}_2F_1\left({-n, n+\xi+\eta+2 \atop \xi+2} ; \frac{1-x}{1-c^2} \right). \tag{3.15}$$

The polynomials $P_n(x)$ and $Q_n(x)$ are connected by the relations (3.2) and (3.3) with $\nu = 1$. The coefficients A_n and B_n can be found from the following observation. Putting $x = 1$, we find from (3.13) and (3.15)

$$P_n(1) = (1-c^2)^n \, \frac{(\xi+1)_n}{(n+\xi+\eta+1)_n}, \quad Q_n(1) = (1-c^2)^n \, \frac{(\xi+2)_n}{(n+\xi+\eta+2)_n}.$$

From these formulas we immediately get

$$A_n = \frac{P_{n+1}(1)}{P_n(1)} = (1-c^2) \, \frac{(\xi+n+1)(\xi+\eta+n+1)}{(2n+\xi+\eta+1)(2n+\xi+\eta+2)} \tag{3.16}$$

and

$$B_n = \frac{Q_n(1) - P_n(1)}{Q_{n-1}(1)} = (1-c^2) \, \frac{n(\eta+n)}{(2n+\xi+\eta)(2n+\xi+\eta+1)}. \tag{3.17}$$

Note that $A_n > 0$, $B_n > 0$ for $n = 1, 2, \dots$ due to the restriction $0 < c < 1$.

Consider now the new monic orthogonal polynomials $R_n(x)$ defined by the relations

$$R_{2n}(x) = P_n(x^2), \quad R_{2n+1}(x) = (x-1) Q_n(x^2) \tag{3.18}$$

According to the general theory of polynomial mappings [6], [10], it is not difficult to show that the polynomials $R_n(x)$ are orthogonal on a domain formed by the union of *two* intervals $[-1, -c]$, $[c, 1]$ of the real axis:

$$\int_{-1}^{-c} R_n(x) R_m(x) W(x) dx + \int_c^1 R_n(x) R_m(x) W(x) dx = 0, \quad n \neq m \tag{3.19}$$

where the (non-normalized) weight function is:

$$W(x) = \theta(x)(1+x)(1-x^2)^\xi (x^2-c^2)^\eta \tag{3.20}$$

and $\theta(x) = x/|x|$ is the sign function. Note that the weight function $W(x)$ is not positive on the interval $[-1, -c]$.

In terms of Gauss' hypergeometric functions we have the expressions

$$R_{2n}(x) = P_n(x^2) = (1-c^2)^n \, \frac{(\xi+1)_n}{(n+\xi+\eta+1)_n} \, {}_2F_1 \left({-n, n+\xi+\eta+1 \atop \xi+1} ; \frac{1-x^2}{1-c^2} \right) \tag{3.21}$$

and

$$R_{2n+1}(x) = (x-1)Q_n(x^2) = (1-c^2)^n \, \frac{(\xi+2)_n}{(n+\xi+\eta+2)_n} \, (x-1) {}_2F_1 \left({-n, n+\xi+\eta+2 \atop \xi+2} ; \frac{1-x^2}{1-c^2} \right) \tag{3.22}$$

The polynomials $R_n(x)$ satisfy the 3-term recurrence relation

$$R_{n+1}(x) + (-1)^n R_n(x) + v_n R_{n-1}(x) = x R_n(x), \tag{3.23}$$

where

$$v_{2n} = -B_n = (c^2-1) \frac{n(\eta+n)}{(2n+\xi+\eta)(2n+\xi+\eta+1)}, \quad v_{2n+1} = -A_n = (c^2-1) \frac{(\xi+n+1)(\xi+\eta+n+1)}{(2n+\xi+\eta+1)(2n+\xi+\eta+2)} \tag{3.24}$$

Note that all the coefficients v_n are negative, $v_n < 0$, which corresponds to the non-positivity of the weight function $W(x)$.

4. The weight function and the orthogonality of the big -1 Jacobi polynomials

In order to determine the weight function and the orthogonality region for the big -1 Jacobi polynomials, we notice that formulas (2.24) and (2.25) can be presented in the following equivalent form:

$$P_n^{(-1)}(x) = R_n(x) - G_n R_{n-1}(x), \tag{4.1}$$

where $R_n(x)$ are the polynomials defined by (3.21), (3.22) that satisfy the recurrence relation (3.23). In these formulas we should put $\xi = (\alpha-1)/2, \ \eta = (\beta+1)/2$. The coefficients G_n have the expression

$$G_n = \begin{cases} \frac{(1-c)n}{2n+\alpha+\beta}, & n \quad \text{even} \\ -\frac{(1+c)(n+\alpha)}{2n+\alpha+\beta}, & n \quad \text{odd} \end{cases} \tag{4.2}$$

It is well known that if two families of orthogonal polynomials are related by a formula such as (4.1), then necessarily the polynomials $P_n^{(-1)}(x)$ are obtained from the polynomials $R_n(x)$ by the Geronimus transform [15]. This is equivalent to the statement that the weight function $w^{(-1)}(x)$ of the polynomials $P_n^{(-1)}(x)$ is obtained from the weight function $W(x)$ of the polynomials $R_n(x)$ as follows:

$$w^{(-1)}(x) = \frac{W(x)}{x-\mu} + M\delta(x-\mu), \tag{4.3}$$

with two additional parameters μ and M. Formula (4.3) means that apart from the division of the weight function $W(x)$ by the linear factor $x-\mu$ there is an additional concentrated mass M that is inserted at the point $x=\mu$.

The parameter μ can be found from the recurrence relation for the coefficients G_n [15]

$$G_{n+1} + (-1)^n + \frac{v_n}{G_n} = \mu \tag{4.4}$$

with the recurrence coefficients v_n given by (3.24).

Substituting (4.2) into (4.4) we obtain $\mu = -c$.

Thus the orthogonality relation for polynomials $P_n^{(-1)}(x)$ takes the form

$$\int_\Gamma P_n^{(-1)}(x)P_m^{(-1)}(x)W(x)(x+c)^{-1}dx + MP_n^{(-1)}(-c)P_m^{(-1)}(-c) = 0, \quad n \neq m \tag{4.5}$$

where the contour Γ is the union of the two intervals $[-1,-c]$ and $[c,1]$ of the real axis.

In order to find the value M of the concentrated mass it is sufficient to consider a special case of the orthogonality relation (4.5) for $n=1, m=0$

$$\int_\Gamma P_1^{(-1)}(x)W(x)(x+c)^{-1}dx + MP_1(-c) = 0 \tag{4.6}$$

Now, $P_1^{(-1)}(x)$ is given by

$$P_1^{(-1)}(x) = x + \zeta$$

where $\zeta = \frac{c(\alpha+1)-\beta-1}{2+\alpha+\beta}$. Substituting this expression into (4.6) and calculating the integral (through an elementary reduction to the Euler beta-integral) we find that $M=0$.

Thus, the orthogonality relation for polynomials $P_n^{(-1)}(x)$ reads

$$\int_\Gamma P_n^{(-1)}(x)P_m^{(-1)}(x)w^{(-1)}(x)dx = 0, \quad n \neq m, \tag{4.7}$$

where the weight function $w^{(-1)}(x)$ can be presented in the form

$$w^{(-1)}(x) = \theta(x)(x+1)(x+c)^{-1}(1-x^2)^{(\alpha-1)/2}(x^2-c^2)^{(\beta+1)/2} \tag{4.8}$$

or, equivalently

$$w^{(-1)}(x) = \theta(x)(x+1)(x-c)(1-x^2)^{(\alpha-1)/2}(x^2-c^2)^{(\beta-1)/2}. \tag{4.9}$$

Note that under the restrictions $\alpha > -1, \beta > -1$, the weight function is positive on the two intervals of Γ and all the moments

$$m_n = \int_\Gamma w^{(-1)}(x)x^n dx$$

are finite for $n = 0, 1, 2, \ldots$.

When $c \to 0$, the big q-Jacobi polynomials reduce to the little q-Jacobi polynomials [8]. In the limit case $q \to -1$ we see that when $c \to 0$, the set of two intervals coalesces to the single interval $[-1,1]$ and the weight function becomes

$$w(x)\,|_{c=0} = (1+x)|x|^\beta(1-x^2)^{(\alpha-1)/2}$$

which corresponds to the weight function of the little -1 Jacobi polynomials [13].

So far, we considered the case $0 < c < 0$. The case $c > 1$ can be treated in an analogous way. This leads to the following orthogonality relation

$$\int_{-c}^{-1} P_n^{(-1)}(x) P_m^{(-1)}(x) w^{(-1)}(x) dx + \int_1^c P_n^{(-1)}(x) P_m^{(-1)}(x) w^{(-1)}(x) dx = 0, \quad n \neq m, \tag{4.10}$$

where the weight function is almost the same as in (4.9) with obvious modifications:

$$w^{(-1)}(x) = \theta(x)(x+1)(c-x)(x^2-1)^{(\alpha-1)/2}(c^2-x^2)^{(\beta-1)/2} \tag{4.11}$$

and where again we have the restrictions $\alpha > -1$, $\beta > -1$.

The case $c = 1$ is degenerate: the recurrence coefficients u_n for even n become zero, $u_{2n} = 0$, which means that orthogonal polynomials $P_n^{(1)}(x)$ are no more positive definite. The two intervals of orthogonality shrink into two points $x = \pm 1$.

5. Anticommutator algebra describing big -1 Jacobi polynomials

The Askey-Wilson polynomials are described by the AW(3)-algebra [14], [12]. Among the different equivalent forms of this algebra, we choose the following one:

$$XY - qYX = \mu_3 Z + \omega_3, \quad YZ - qZY = \mu_1 X + \omega_1, \quad ZX - qXZ = \mu_2 Y + \omega_2, \tag{5.1}$$

which possesses an obvious symmetry with respect to all 3 operators (see, e.g. [7]).

Here q is a fixed parameter corresponding to the "base" parameter in the q-hypergeometric functions defining the Askey-Wilson polynomials [8]. The pairs of operators (X,Y), (Y,Z) and (Z,X) play the role of "Leonard pairs" (see [12], [7]).

The Casimir operator

$$Q = (q^2-1)XYZ + \mu_1 X^2 + \mu_2 q^2 Y^2 + \mu_3 Z^2 + (q+1)(\omega_1 X + \omega_2 qY + \omega_3 Z) \tag{5.2}$$

commutes with all operators X, Y, Z.

The constants ω_i, $i = 1, 2, 2$ (together with the value of the Casimir operator Q) define representations of the $AW(3)$ algebra (see [14] for details).

Consider now the case of the big -1 Jacobi polynomials and choose the following operators

$$X = L_0 + \alpha + \beta + 1, \quad Y = x, \quad Z = -\frac{2}{x}\left(c + (x-1)(x+c)R\right), \tag{5.3}$$

where L_0 is the operator given by (2.10).

It is then easy to verify that these operators satisfy the linear anticommutation relations

$$XY + YX = Z + \omega_3, \quad YZ + ZY = \omega_1, \quad ZX + XZ = 4Y + \omega_2, \tag{5.4}$$

where

$$\omega_1 = -4c,\ \omega_2 = 4(\alpha - \beta c),\ \omega_3 = 2(\beta - \alpha c).$$

The Casimir operator of the algebra defined by (5.4) is

$$Q = Z^2 + 4Y^2$$

In the realization (5.3) the Casimir operator takes the constant value $Q = 4(c^2+1)$.

In this realization the operator X (up to an additive constant) is the operator of which the polynomials $P_n^{(-1)}(x)$ are the eigenfunctions. The operator Y here corresponds to multiplication by x.

The "dual" realization of the algebra (5.4) is obtained if one takes an infinite discrete basis $e_n,\ n = 0, 1, 2, \dots$ on which the operators X, Y act as

$$Xe_n = (\lambda_n + \alpha + \beta + 1)e_n, \quad Ye_n = u_{n+1}^{(-1)} e_{n+1} + b_n^{(-1)} e_n + e_{n-1}, \tag{5.5}$$

where λ_n is the eigenvalue (2.17) and where the recurrence coefficients $u_n^{(-1)}, b_n^{(-1)}$ are given by (2.20), (2.21). Thus in this representation the operator Y is a Jacobi (i.e. tri-diagonal) matrix and the eigenvalue equation

$$Y\vec{P} = x\vec{P}$$

is equivalent to the recurrence relation (2.22) for the big -1 Jacobi polynomials. Indeed, we can present the vector $\vec{P}$ in terms of its expansion coefficients over the basis e_n:

$$\vec{P} = \sum_{n=0}^{\infty} C_n e_n$$

Without loss of generality we can choose $C_0 = 1$. The coefficients C_n in this expansion are then found to satisfy the recurrence relation (2.22) and it is seen moreover that these C_n are monic polynomials in x of degree n. Hence $C_n = P_n^{(-1)}(x)$.

6. Two-diadonal basis for the operator L_0 and a generalization of Gauss' hypergeometric functions

We already showed that the Dunkl-type operator L_0 is tri-diagonal in the ordinary monomial basis x^n (see formula (2.15)). There exists, however, another polynomial basis in which the operator L_0 is two-diagonal. This basis can be constructed as follows

$$\phi_0 = 1,\ \phi_1(x) = x - 1, \phi_2(x) = (x^2-1), \dots, \phi_{2n}(x) = (x^2-1)^n,\ \phi_{2n+1}(x) = (x-1)(x^2-1)^n \tag{6.1}$$

It is easily verified that

$$L_0 \phi_n(x) = \lambda_n \phi_n(x) + \eta_n \phi_{n-1}(x), \tag{6.2}$$

where λ_n is the eigenvalue given by by (2.17), and

$$\eta_n = \begin{cases} 2n(c-1), & \text{if} \quad n \quad \text{even} \\ -2(c+1)(\alpha+n) & \text{if} \quad n \quad \text{odd} \end{cases}$$

Consider now the eigenvalue equation

$$L_0 P_n(x) = \lambda_n P_n(x) \tag{6.3}$$

and expand the polynomials $P_n(x)$ over the basis $\phi_n(x)$:

$$P_n(x) = \sum_{s=0}^{n} A_{ns}\phi_s(x).$$

For the expansion coefficients A_{ns} we have from (6.3):

$$A_{n,s+1} = \frac{A_{ns}(\lambda_n - \lambda_s)}{\eta_{s+1}} \tag{6.4}$$

From (6.4), the coefficients A_{ns} can be found explicitly in terms of A_{n0}:

$$A_{ns} = A_{n0}\, \frac{(\lambda_n - \lambda_0)(\lambda_n - \lambda_1)\dots(\lambda_n - \lambda_{s-1})}{\eta_1\eta_2\dots\eta_s} \tag{6.5}$$

or in terms of the coefficient A_{nn}:

$$A_{ns} = A_{nn}\, \frac{\eta_n\eta_{n-1}\dots\eta_{s+1}}{(\lambda_n - \lambda_{n-1})(\lambda_n - \lambda_{n-2})\dots(\lambda_n - \lambda_s)} \tag{6.6}$$

We thus have the following explicit formula for the polynomials $P_n(x)$

$$P_n(x) = A_{n0} \sum_{s=0}^{n} \frac{(\lambda_n - \lambda_0)(\lambda_n - \lambda_1)\dots(\lambda_n - \lambda_{s-1})}{\eta_1\eta_2\dots\eta_s}\phi_s(x) \tag{6.7}$$

Expression (6.7) resembles Gauss' hypergeometric function and can be considered as a nontrivial generalization of it. Indeed, products in the numerator and denominator of (6.7) can easily be presented in terms of ordinary Pochhammer symbols and we thus recover the explicit formulas (2.24) and (2.25). Note nevertheless, that the form (6.7) looks much simpler.

Moreover, note also that in the basis $\phi_n(x)$ the operators X and Y of the algebra defined by (5.3) and (5.4), become lower and upper triangular:

$$X\phi_n(x) = (L_0 + \alpha + \beta + 1)\phi_n(x) = (\lambda_n + \alpha + \beta + 1)\phi_n(x) + \eta_n\phi_{n-1}(x)$$

and

$$Y\phi_n(x) = x\phi_n(x) = \phi_{n+1}(x) + (-1)^n\phi_n(x)$$

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