SCHUR-FINITENESS IN λ-RINGS

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Abstract. We introduce the notion of a Schur-finite element in a λ-ring.

Since the beginning of algebraic K-theory in [G57], the splitting principle has proven invaluable for working with λ-operations. Unfortunately, this principle does not seem to hold in some recent applications, such as the K-theory of motives. The main goal of this paper is to introduce the subring of Schur-finite elements of any λ-ring, and study its main properties, especially in connection with the virtual splitting principle.

A rich source of examples comes from Heinloth's theorem [Hl], that the Grothendieck group $K_0(\mathcal{A})$ of an idempotent-complete $\mathbb{Q}$-linear tensor category $\mathcal{A}$ is a λ-ring. For the category $\mathcal{M}^{\text{eff}}$ of effective Chow motives, we show that $K_0(Var) \to K_0(\mathcal{M}^{\text{eff}})$ is not an injection, answering a question of Grothendieck.

When $\mathcal{A}$ is the derived category of motives $\mathbf{DM}_{gm}$ over a field of characteristic 0, the notion of Schur-finiteness in $K_0(\mathbf{DM}_{gm})$ is compatible with the notion of a Schur-finite object in $\mathbf{DM}_{gm}$, introduced in [Mz].

We begin by briefly recalling the classical splitting principle in Section 1, and answering Grothendieck's question in Section 2. In section 3 we recall the Schur polynomials, the Jacobi-Trudi identities and the Pieri rule from the theory of symmetric functions. Finally, in Section 4, we define Schur-finite elements and show that they form a subring of any λ-ring. We also state the conjecture that every Schur-finite element is a virtual sum of line elements.

Notation. We will use the term λ-ring in the sense of [Ber, 2.4]; we warn the reader that our λ-rings are called *special λ-rings* by Grothendieck, Atiyah and others; see [G57] [AT] [A].

A $\mathbb{Q}$-linear category $\mathcal{A}$ is a category in which each hom-set is uniquely divisible (i.e., a $\mathbb{Q}$-module). By a $\mathbb{Q}$-linear tensor category (or $\mathbb{Q}$TC) we mean a $\mathbb{Q}$-linear category which is also symmetric monoidal and such that the tensor product is $\mathbb{Q}$-linear. We will be interested in $\mathbb{Q}$TC's which are idempotent-complete.

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1. Finite-dimensional λ-rings

Almost all λ-rings of historical interest are finite-dimensional. This includes the complex representation rings $R(G)$ and topological K-theory of compact spaces [AT, 1.5] as well as the algebraic K-theory of algebraic varieties [G57]. In this section we present this theory from the viewpoint we are adopting. Little in this section is new.

Recall that an element x in a λ-ring R is said to be *even* of finite degree n if $\lambda_t(x)$ is a polynomial of degree n, or equivalently that there is a λ-ring homomorphism from the ring Λ_n defined in 1.2 to R, sending a to x. We say that x is a *line element* if it is even of degree 1, i.e., if $\lambda^n(x) = 0$ for all $n > 1$.

We say that x is *odd* of degree n if $\sigma_t(x) = \lambda_{-t}(x)^{-1}$ is a polynomial of finite degree n. Since $\sigma_{-t}(x) = \lambda_t(-x)$, we see that x is odd just in case $-x$ is even. Therefore there is a λ-ring homomorphism from the ring Λ_{-n} defined in 1.2 to R sending b to x.

We say that an element x is *finite-dimensional* if it is the difference of two even elements, or equivalently if x is the sum of an even and an odd element. The subset of even elements in R is closed under addition and multiplication, and the subset of finite-dimensional elements forms a subring of R.

Example 1.1. If R is a binomial λ-ring, then r is even if and only if some $r(r-1)\cdots(r-n) = 0$, and odd if and only if some $r(r+1)\cdots(r+n) = 0$. The binomial rings $\prod_{i=1}^{n}\mathbb{Z}$ are finite dimensional. If R is connected then the subring of finite-dimensional elements is just $\mathbb{Z}$.

There is a well known family of universal finite-dimensional λ-rings $\{\Lambda_n\}$.

Definition 1.2. Following [AT], let Λ_n denote the free λ-ring generated by one element $a = a_1$ of finite degree n (i.e., subject to the relations that $\lambda^k(a) = 0$ for all $k > n$). By [Ber, 4.9], Λ_n is just the polynomial ring $\mathbb{Z}[a_1, ..., a_n]$ with $a_i = \lambda^i(a_1)$.

Similarly, we write Λ_{-n} for the free λ-ring generated by one element $b = b_1$, subject to the relations that $\sigma^k(b) = 0$ for all $k > n$. Using the antipode S, we see that there is a λ-ring isomorphism $\Lambda_{-n} \cong \Lambda_n$ sending b to $-a$, and hence that $\Lambda_{-n} \cong \mathbb{Z}[b_1, ..., b_n]$ with $b_k = \sigma^k(b)$.

Consider finite-dimensional elements in λ-rings R which are the difference of an even element of degree m and an odd element of degree n. The maps $\Lambda_m \to R$ and $\Lambda_{-n} \to R$ induce a λ-ring map from $\Lambda_m \otimes \Lambda_{-n}$ to R.

Lemma 1.3. *If an element x is both even and odd in a λ-ring, then x and all the $\lambda^i(x)$ are nilpotent. Thus $\lambda_t(x)$ is a unit of $R[t]$.*

Proof. If x is even and odd then $\lambda_t(x)$ and $\sigma_{-t}(x)$ are polynomials in $R[t]$ which are inverse to each other. It follows that the coefficients $\lambda^i(x)$ of the t^i are nilpotent for all $i > 0$. □

If R is a graded λ-ring, an element $\sum r_i$ is even (resp., odd, resp., finite-dimensional) if and only if each homogeneous term r_i is even (resp., odd, resp., finite-dimensional). This is because the operations λ^n multiply the degree of an element by n.

The forgetful functor from λ-rings to commutative rings has a right adjoint; see [Kn, pp. 20–21]. It follows that the category of λ-rings has all colimits. In particular, if $B \leftarrow A \rightarrow C$ is a diagram of λ-rings, the tensor product $B \otimes_A C$ has the structure of a λ-ring. Here is a typical, classical application of this construction, originally proven in [AT, 6.1].

Proposition 1.4 (Splitting Principle)**.** *If x is any even element of finite degree n in a λ-ring R, there exists an inclusion $R \subseteq R'$ of λ-rings and line elements $\ell_1, ..., \ell_n$ in R' so that $x = \sum \ell_i$.*

Proof. Let Ω_n denote the tensor product of n copies of the λ-ring $\Lambda_1 = \mathbb{Z}[\ell]$; this is a λ-ring whose underlying ring is the polynomial ring $\mathbb{Z}[\ell_1, ..., \ell_n]$, and the λ-ring Λ_n of Definition 1.2 is the subring of symmetric polynomials in Ω_n; see [AT, §2]. Let R' be the pushout of the diagram $\Omega_n \leftarrow \Lambda_n \rightarrow R$. Since the image of x is $1 \otimes x = a \otimes 1 = (\sum \ell_i) \otimes 1$, it suffices to show that $R \rightarrow R'$ is an injection. This follows from the fact that Ω_n is free as a Λ_n-module. □

Corollary 1.5. *If x is any finite-dimensional element of a λ-ring R, there is an inclusion $R \subseteq R'$ of λ-rings and line elements ℓ_i, ℓ'_j in R' so that*

$$x = (\sum \ell_i) - (\sum \ell'_j).$$

Scholium 1.6. For later use, we record an observation, whose proof is implicit in the proof of Proposition 4.2 of [AT]: $\lambda^m(\lambda^n x) = P_{m,n}(\lambda^1 x, \dots, \lambda^{mn} x)$ is a sum of monomials, each containing a term $\lambda^i x$ for $i \geq n$. For example, $\lambda^2(\lambda^3 x) = \lambda^6 x - x\,\lambda^5 x + \lambda^4 x\,\lambda^2 x$ (see [Kn, p. 11]).

2. K_0 of tensor categories

The Grothendieck group of a $\mathbb{Q}$-linear tensor category provides numerous examples of λ-rings, and forms the original motivation for introducing the notion of Schur-finite elements in a λ-ring.

A $\mathbb{Q}$-linear tensor category is *exact* if it has a distinguished family of sequences, called *short exact sequences* and satisfying the axioms of [Q], and such that each $A \otimes -$ is an exact functor. In many applications $\mathcal{A}$ is *split exact*: the only short exact sequences are those which split. By $K_0(\mathcal{A})$ we mean the Grothendieck group as an exact category, i.e., the quotient of the free abelian group on the objects $[A]$ by the relation that $[B] = [A] + [C]$ for every short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$.

Let $\mathcal{A}$ be an idempotent-complete exact category which is a $\mathbb{Q}$TC for $\otimes$. For any object A in $\mathcal{A}$, the symmetric group Σ_n (and hence the group ring $\mathbb{Q}[\Sigma_n]$) acts on the n-fold tensor product $A^{\otimes n}$. If $\mathcal{A}$ is idempotent-complete, we define $\wedge^n A$ to be the direct summand of $A^{\otimes n}$ corresponding to the alternating idempotent $\sum (-1)^\sigma \sigma / n!$ of $\mathbb{Q}[\Sigma_n]$. Similarly, we can define the symmetric powers $\mathrm{Sym}^n(A)$. It turns out that $\lambda^n(A)$ only depends upon the element $[A]$ in $K_0(\mathcal{A})$, and that λ^n extends to a well defined operation on $K_0(\mathcal{A})$.

The following result was proven by F. Heinloth in [Hl, Lemma 4.1], but the result seems to have been in the air; see [Dav, p. 486], [LL04, 5.1] and [B1, B2]. A special case of this result was proven long ago by Swan in [Sw].

Theorem 2.1. *If $\mathcal{A}$ is any idempotent-complete exact $\mathbb{Q}TC$, $K_0(\mathcal{A})$ has the structure of a λ-ring. If A is any object of $\mathcal{A}$ then $\lambda^n([A]) = [\wedge^n A]$.*

Kimura [Kim] and O'Sullivan have introduced the notion of an object C being finite-dimensional in any $\mathbb{Q}$TC $\mathcal{A}$: C is the direct sum of an even object A (one for which some $\wedge^n A \cong 0$) and an odd object B (one for which some $\mathrm{Sym}^n(B) \cong 0$). It is immediate that $[C]$ is a finite-dimensional element in the λ-ring $K_0(\mathcal{A})$. Thus the two notions of finite dimensionality are related.

Example 2.2. Let $\mathcal{M}^{\mathrm{eff}}$ denote the category of $\mathbb{Q}$-linear pure effective Chow motives with respect to rational equivalence over a field k. Its objects are summands of smooth projective varieties over a field k and morphisms are given by Chow groups. Thus $K_0(\mathcal{M}^{\mathrm{eff}})$ is the group generated by the classes of objects, modulo the relation $[M_1 \oplus M_2] = [M_1] + [M_2]$. Since $\mathcal{M}^{\mathrm{eff}}$ is a $\mathbb{Q}$TC, $K_0(\mathcal{M}^{\mathrm{eff}})$ is a λ-ring.

By adjoining an inverse to the Lefschetz motive to $\mathcal{M}^{\mathrm{eff}}$, we obtain the category $\mathcal{M}$ of Chow motives (with respect to rational equivalence). This is also a $\mathbb{Q}$TC, so $K_0(\mathcal{M})$ is a λ-ring.

The category $\mathcal{M}^{\text{eff}}$ embeds into the triangulated category $\mathbf{DM}^{\text{eff}}_{gm}$ of effective geometric motives; see [MVW, 20.1]. Similarly, The category $\mathcal{M}$ embeds in the triangulated category $\mathbf{DM}_{gm}$ of geometric motives [MVW, 20.2]. Bondarko proved in [Bo, 6.4.3] that $K_0(\mathbf{DM}^{\text{eff}}_{gm}) \cong K_0(\mathcal{M}^{\text{eff}})$ and $K_0(\mathbf{DM}_{gm}) \cong K_0(\mathcal{M})$. Thus we may investigate λ-ring questions in these triangulated settings. As far as we know, it is possible that every element of $K_0(\mathbf{DM}_{gm})$ is finite-dimensional.

Here is an application of these ideas. Recall that any quasiprojective scheme X has a motive with compact supports in $\mathbf{DM}^{\text{eff}}$, $M^c(X)$. If k has characteristic 0, this is an effective geometric motive, and if U is open in X with complement Z there is a triangle $M^c(Z) \to M^c(X) \to M^c(U)$; see [MVW, 16.15]. It follows that $[M^c(X)] = [M^c(U)] + [M^c(Z)]$ in $K_0(\mathcal{M}^{\text{eff}})$. (This was originally proven by Gillet and Soulé in [GS, Thm. 4] before the introduction of $\mathbf{DM}$, but see [GS, 3.2.4].

Definition 2.3. Let $K_0(Var)$ be the Grothendieck ring of varieties obtained by imposing the relation $[U]+[X \setminus U] = [X]$ for any variety X. By the above remarks, there is a well defined ring homomorphism $K_0(Var) \to K_0(\mathcal{M}^{\text{eff}})$.

Grothendieck asked in [G64, p.174] if this morphism was far from being an isomorphism. We can now answer his question.

Theorem 2.4. *The homomorphism $K_0(Var) \to K_0(\mathcal{M}^{eff})$ is not an injection.*

For the proof, we need to introduce Kapranov's zeta-function. If X is any quasiprojective variety, its symmetric power S^nX is the quotient of X^n by the action of the symmetric group. We define $\zeta_t(X) = \sum [S^nX]t^n$ as a power series with coefficients in $K_0(Var)$.

Lemma 2.5. *(*[Gul]*) The following diagram is commutative:*

$$\begin{array}{ccc} K_0(Var) & \xrightarrow{\zeta_t} & 1 + K_0(Var)[[t]] \\ {\scriptstyle M^c}\downarrow & & \downarrow{\scriptstyle M^c} \\ K_0(\mathcal{M}^{eff}) & \xrightarrow{\sigma_t} & 1 + K_0(\mathcal{M}^{eff})[[t]]. \end{array}$$

Proof. It suffices to show that $[M^c(S^nX)] = \text{Sym}^n[M^c(X)]$ in $K_0(\mathcal{M}^{\text{eff}})$ for any X. This is proven by del Baño and Navarro in [dBN, 5.3]. □

Definition 2.6. Following [LL04, 2.2], we say that a power series $f(t) = \sum r_n t^n \in R[[t]]$ is *determinentally rational* over a ring R if there exists an $m > 0$ such that the $m \times m$ symmetric matrices $(r_{n+i+j})^m_{i,j=1}$ have determinant 0 for all large n. The name comes from the classical fact ([1894]) that when R is a field (or a domain) a power series is determinentally rational if and only if it is a rational function.

Clearly, if $f(t)$ is not determinentally rational over R and $R \subset R'$ then $f(t)$ cannot be determinentally rational over R'.

If $x = a + b$ is a finite-dimensional element of a λ-ring R, with a even and b odd, then $\lambda_t(a)$ and $\lambda_t(-b)$ are polynomials so $\lambda_t(x) = \lambda_t(a)\lambda_t(-b)$ and $\sigma_t(x) = \lambda_t(x)^{-1}$ are rational functions, and hence rational functions. This was observed by André in [A05].

Proof of Theorem 2.4. Let X be the product $C \times D$ of two smooth projective curves of genus > 0, so that $p_g(X) > 0$. Larsen and Lunts showed in [LL04, 2.4, 3.9] that $\zeta_t(X)$ is not determinentally rational over $R = K_0(Var)$. On the other hand, Kimura proved in [Kim] that X is a finite-dimensional object in $\mathcal{M}^{\text{eff}}$, so $\sigma_t(X) = \lambda_t(X)^{-1}$ is a determinentally rational function in $R' = K_0(\mathcal{M}^{\text{eff}})$. It follows that $R \to R'$ cannot be an injection. □

3. Symmetric functions

We devote this section to a quick study of the ring Λ of symmetric functions, and especially the Schur polynomials s_π, referring the reader to [Macd] for more information. In the next section, we will use these polynomials to define the notion of Schur-finite elements in a λ-ring.

The ring Λ is defined as the ring of symmetric polynomials in variables x_i; a major role is played by the *elementary symmetric polynomials* $e_i \in \Lambda$ and the *homogeneous power sums* $h_n = \sum x_{i_1} \cdots x_{i_n}$ (where the sum being taken over $i_1 \le \cdots \le i_n$). Their generating functions $E(t) = \sum e_n t^n$ and $H(t) = \sum h_n t^n$ are $\prod(1 + x_i t)$ and $\prod(1 - x_i t)^{-1}$, so that $H(t)E(-t) = 1$. In fact, Λ is a graded polynomial ring in two relevant ways (with e_n and h_n in degree n):

$$\Lambda = \mathbb{Z}[e_1, ..., e_n, ...] = \mathbb{Z}[h_1, ..., h_n, ...].$$

Given a partition $\pi = (n_1, ..., n_r)$ of n (so that $\sum n_i = n$), we let $s_\pi \in \Lambda_n$ denote the Schur polynomial of π. The elements e_n and h_n of Λ are identified with $s_{(1,...,1)}$ and $s_{(n)}$, respectively. The Schur polynomials also form a $\mathbb{Z}$-basis of Λ by [Macd, 3.3]. By abuse, we will say that a partition π *contains* a partition $\lambda = (\lambda_1, ..., \lambda_s)$ if $n_i \ge \lambda_i$ and $r \ge s$, which is the same as saying that the Young diagram for π contains the Young diagram for λ.

Here is another description of Λ, taken from [Kn]: Λ is isomorphic to the direct sum R_* of the representation rings $R(\Sigma_n)$, made into a ring via the outer product $R(\Sigma_m) \otimes R(\Sigma_n) \to R(\Sigma_{m+n})$. Under this identification, $e_n \in \Lambda_n$ is identified with the class of the trivial simple representation V_n of Σ_n. More generally, s_π corresponds to the class $[V_\pi]$ in $R(\Sigma_n)$ of the irreducible respesentation corresponding to π. (See [Kn, III.3].)

Proposition 3.1. *Λ is a graded Hopf algebra, with coproduct Δ and antipode S determined by the formulas*

$$\Delta(e_n) = \sum_{i+j=n} e_i \otimes e_j, \quad S(e_n) = h_n \text{ and } S(h_n) = e_n.$$

Proof. The graded bialgebra structure is well known and due to Burroughs [Bu], who defined the coproduct on R_* as the map induced from the restriction maps $R(\Sigma_{m+n}) \to R(\Sigma_m) \otimes R(\Sigma_n)$, and established the formulas $\Delta(e_n) = \sum_{i+j=n} e_i \otimes e_j$. The fact that there is a ring involution S interchanging e_n and h_n is also well known. The fact that S is an antipode does not seem to be well known, but it is immediate from the formula $\sum(-1)^r e_r h_{n-r}$ of [Macd, (2.6)]. □

Remark 3.2. Atiyah shows in [A, 1.2] that Λ is isomorphic to the graded dual $R^* = \oplus \mathrm{Hom}(R(\Sigma_n), \mathbb{Z})$. That is, if $\{v_\pi\}$ is the dual basis in R^n to the basis $\{[V_\pi]\}$ of simple representations in R_n and the restriction of $[V_\pi]$ is $\sum c_\pi^{\mu\nu} [V_\mu] \otimes [V_\nu]$ then $v_\mu v_\nu = \sum_\pi c_\pi^{\mu\nu} v_\pi$ in R^*. Thus the product studied by Atiyah on the graded dual R^* is exactly the algebra structure dual to the coproduct Δ.

Let π' denote the conjugate partition to π. The *Jacobi-Trudi identities* $s_\pi = \det|h_{\pi_i+j-i}| = \det|e_{\pi'_i+j-i}|$ show that the antipode S interchanges s_π and $s_{\pi'}$. (Jacobi conjectured the identities, and his student Nicoló Trudi verified them in 1864; they were rediscovered by Giovanni Giambelli in 1903 and are sometimes called the *Giambelli identities*).

Let $I_{e,n}$ denote the ideal of Λ generated by the e_i with $i \ge n$. The quotient $\Lambda/I_{e,n}$ is the polynomial ring $\Lambda_{n-1} = \mathbb{Z}[e_1, ..., e_{n-1}]$. Let $I_{h,n}$ denote $S(I_{e,n})$, i.e., the ideal of Λ generated by the h_i with $i \ge n$.

Proposition 3.3. *The Schur polynomials s_π for partitions π containing (1^n) (i.e., with at least n rows) form a $\mathbb{Z}$-basis for the ideal $I_{e,n}$. The Schur polynomials with at most n rows form a $\mathbb{Z}$-basis of Λ_n.*

Similarly, the Schur polynomials s_π for partitions π containing (n) (i.e., with $\pi_1 \geq n$) form a $\mathbb{Z}$-basis for the ideal $I_{h,n}$.

Proof. We prove the assertions about $I_{e,n}$; the assertion about $I_{h,n}$ follows by applying the antipode S. By [Macd, 3.2], the s_π which have fewer than n rows project onto a $\mathbb{Z}$-basis of $\Lambda_{n-1} = \Lambda/I_{e,n}$. Since the s_π form a $\mathbb{Z}$-basis of Λ, it suffices to show that every partition $\pi = (\pi_1, ..., \pi_r)$ with $r > n$ is in $I_{e,n}$. Expansion along the first row of the Jacobi-Trudi identity $s_\pi = \det|e_{\pi'_i+j-i}|$ shows that s_π is in the ideal $I_{e,r}$. □

Corollary 3.4. *The ideal $I_{h,m} \cap I_{e,n}$ of Λ has a $\mathbb{Z}$-basis consisting of the Schur polynomials s_π for partitions π containing the hook $(m, 1^{n-1}) = (m, 1, ..., 1)$.*

Definition 3.5. For any partition $\lambda = (\lambda_1, ..., \lambda_r)$, let I_λ denote the subgroup of Λ generated by the Schur polynomials s_π for which π contains λ, i.e., $\pi_i \geq \lambda_i$ for $i = 1, ..., r$. We have already encountered the special cases $I_{e,n} = I_{(1,...,1)}$ and $I_{h,n} = I_{(n)}$ in Proposition 3.3, and $I_{(m,1,...,1)} = I_{h,m} \cap I_{e,n}$ in Corollary 3.4.

Example 3.6. Consider the partition $\lambda = (2, 1)$. Since $I_\lambda = I_{h,2} \cap I_{e,2}$ by Corollary 3.4, Λ_λ is the pullback of $\mathbb{Z}[a]$ and $\mathbb{Z}[b]$ along the common quotient $\mathbb{Z}[a]/(a^2) = \Lambda/(I_{(1,1)} + I_{(2)})$. The universal element of Λ_λ is $x = (a, b)$ and if we set $y = (0, b^2)$ then $\Lambda_{(2,1)} \cong \mathbb{Z}[x, y]/(y^2 - x^2 y)$. Since $\lambda^n(b) = b^n$ for all n, it is easy to check that $\lambda^{2i}(x) = y^i$ and $\lambda^{2i+1}(x) = xy^i$.

Lemma 3.7. *The I_λ are ideals of Λ, and $\{I_\lambda\}$ is closed under intersection.*

Proof. The Pieri rule writes $h_p s_\pi$ as a sum of s_μ, where μ runs over partitions containing π. Thus I_λ is closed under multiplication by the h_p. As every element of Λ is a polynomial in the h_p, I_λ is an ideal.

If $\mu = (\mu_1, ..., \mu_s)$ is another partition, then s_π is in $I_\lambda \cap I_\mu$ if and only if $\pi_i \geq \max\{\lambda_i, \mu_i\}$ Thus $I_\lambda \cap I_\mu = I_{\lambda \cup \mu}$. □

Remark 3.8. The λ-ideal $I_\lambda + I_\mu$ need not be of the form I_ν for any ν. For example, $I = I_{(2)} + I_{(1,1)}$ contains every Schur polynomial except 1 and $s_1 = e_1$.

We conclude this section by connecting Λ with λ-rings. Recall from [Ber, 4.4], [G57, I.4] or [AT, §2] that the universal λ-ring on one generator $a = a_1$ is the polynomial ring $\mathbb{Z}[a_1, \dots, a_n, \dots]$, with $\lambda^n(a) = a_n$. This ring is naturally isomorphic to the ring of natural operations on the category of λ-rings, with a_n corresponding to the operation λ^n; an operation ϕ corresponds to $\phi(a) \in \Lambda$.

Following [A] and [Kn], we may identify this universal λ-ring with Λ, where the a_i are identified with the $e_i \in \Lambda$. The operation σ^n, defined by $\sigma^n(x) = (-1)^n \lambda^n(-x)$, corresponds to h_n; this may be seen by comparing the generating functions $H(t) = E(-t)^{-1}$ and $\sigma_t(x) = \lambda_{-t}(x)^{-1}$.

Proposition 3.9. *If ϕ is an element of Λ, and $\Delta(\phi) = \sum \phi'_i \otimes \phi''_i$ then the corresponding natural operation on λ-rings satisfies $\phi(x + y) = \sum \phi'_i(x)\phi''_i(y)$.*

Proof. Consider the set Λ' of all operations in Λ satisfying the condition of the proposition. Since Δ is a ring homomorphism, Λ' is a subring of Λ. Since $\Delta(e_n) = \sum e_i \otimes e_{n-i}$ and $\lambda^n(x + y) = \sum \lambda^i(x)\lambda^{n-i}(y)$, Λ' contains the generators e_n of Λ, and hence $\Lambda' = \Lambda$. □

The Littlewood-Richardson rule states that $\Delta([V_\pi])$ is a sum $\sum c_\pi^{\mu\nu}[V_\mu] \otimes [V_\nu]$, where $\mu \subseteq \pi$ and π is obtained from μ by concatenating ν in a certain way. By Proposition 3.9, we then have

Corollary 3.10. $s_\pi(x+y) = \sum c_\pi^{\mu\nu} s_\mu(x) s_\nu(y)$.

4. Schur-finite λ-rings

In this section we introduce the notion of a Schur-finite element in a λ-ring R, and show that these elements form a subring of R containing the subring of finite-dimensional elements. We conjecture that they are the elements for which the virtual splitting principle holds.

Definition 4.1. We say that an element x in a λ-ring R is *Schur-finite* if there exists a partition λ such that $s_\mu(x) = 0$ for every partition μ containing λ. That is, I_λ annihilates x. We call such a λ a *bound* for x.

By Remark 3.8, $x \in R$ may have no unique minimal bound λ. By Example 4.5 below, $s_\lambda(x) = 0$ does not imply that λ is a bound for x.

Proposition 4.2. *Each I_λ is a λ-ideal, and $\Lambda_\lambda = \Lambda/I_\lambda$ is a λ-ring. Thus every Schur-finite $x \in R$ with bound λ determines a λ-ring map $f : \Lambda_\lambda \to R$ with $f(a) = x$.*

Moreover, if λ is a rectangular partition then I_λ is a prime ideal, and Λ_λ is a subring of a polynomial ring in which a becomes finite-dimensional.

Proposition 4.2 verifies Conjecture 3.9 of [KKT].

Proof. Fix a rectangular partition $\beta = ((m+1)^{n+1}) = (m+1, ..., m+1)$, and set $a = \sum_1^m a_i$, $b = \sum_1^n b_j$. Consider the universal λ-ring map

$$f : \Lambda \to \Lambda_m \otimes \Lambda_{-n} \cong \mathbb{Z}[a_1, ..., a_m, b_1, ..., b_n]$$

sending e_1 to the finite-dimensional element $a + b$ (see Definition 1.2). We claim that the kernel of f is I_β. Since $\mathrm{Ker}(f)$ is a λ-ideal, this proves that I_β is a λ-ideal and that Λ/I_β embeds into the polynomial ring $\mathbb{Z}[a_1, ..., a_m, b_1, ..., b_n]$. Since any partition λ can be written as a union of rectangular partitions β_i, Lemma 3.7 implies that $I_\lambda = \cap I_{\beta_i}$ is also a λ-ideal.

By the Littlewood-Richardson rule 3.10, $s_\pi(a+b) = \sum c_\pi^{\mu\nu} s_\mu(a) s_\nu(b)$, where π is obtained from μ by concatenating ν in a certain way. If π contains β then $s_\pi(a+b) = 0$, because either the length of μ is $> m$ or else $\nu_1 > n$; by Proposition 3.3, $s_\mu(a) = 0$ in the first case and $s_\nu(b) = 0$ in the second case. Thus $I_\lambda \subseteq \mathrm{Ker}(f)$.

If π does not contain β then the length of μ is at most m and $\nu_1 \le n$. By Proposition 3.3, $s_\mu(a) \neq 0$ in Λ_m and $s_\nu(b) \neq 0$ in Λ_{-n}. As the $s_\mu(a)$ run over a basis of Λ_m and the $s_\nu(b)$ run over a basis of Λ_{-n}, by Proposition 3.3, we have $f(s_\pi) = s_\pi(a+b) \neq 0$. Thus $I_\lambda = \mathrm{Ker}(f)$, as claimed. □

Corollary 4.3. $\Lambda_{(2,2)}$ *is the subring $\mathbb{Z} + x\mathbb{Z}[a,b]$ of $\mathbb{Z}[a,b]$, where $x = a + b$.*

Proof. By Proposition 4.2, $\Lambda_{(2,2)}$ is the subring of $\mathbb{Z}[a,b]$ generated $x = a + b$ and the $\lambda^n(x)$. Since

$$\lambda^{n+1}(x) = a\lambda^n(b) + \lambda^{n+1}(b) = ab^n + b^{n+1} = xb^n,$$

we have $\Lambda_{(2,2)} = \mathbb{Z}[x, xb, xb^2, \dots, xb^n, \dots] = \mathbb{Z} + x\mathbb{Z}[a,b]$. □

Remark 4.4. The ring $\Lambda_{(2,2)}$ was studied in [KKT, 3.8], where it was shown that $\Lambda_{(2,2)}$ embeds into $\mathbb{Z}[x,y]$ sending e_n to xy^{n-1}. This is the same as the embedding in Corollary 4.3, up to the change of coordinates $(x,y) = (a+b, b)$.

Example 4.5. Let I be the ideal of $\Lambda_{(2,2)}$ generated by the $\lambda^{2i}(x)$ $(i > 0)$ and set $R = \Lambda_{(2,2)}/I$. Then R is a λ-ring and x is a nonzerodivisor such that $\lambda^{2i}(x) = 0$ but $\lambda^{2i+1}(x) \neq 0$. In particular, $\lambda^2(x) = 0$ yet $\lambda^3(x) \neq 0$.

To see this, we use the embedding of Corollary 4.3 to see that I contains $x(xb^{2i-1})$ and $(xb)(xb^{2i-1})$ and hence the ideal J of $\mathbb{Z}[a,b]$ generated by x^2b. In fact, I is additively generated by J and the $\{xb^{2i-1}\}$. It follows that R has basis $\{1, x^n, xb^{2n} | n \geq 1\}$. Since $\lambda^n(\lambda^{2i}(x))$ is equivalent to $\lambda^{2in}(x) = xb^{2in-1}$ modulo J (by 1.6), it lies in I. Hence I is a λ-ideal of $\Lambda_{(2,2)}$.

Lemma 4.6. *If x and y are Schur-finite, so is $x + y$.*

Proof. Given a partition λ, there is a partition π_0 such that whenever π contains π_0, one of the partitions μ and ν appearing in the Littlewood-Richardson rule 3.10 must contain λ. If x and y are both killed by all Schur polynomials indexed by partitions containing λ, we must therefore have $s_\pi(x + y) = 0$. □

Corollary 4.7. *Finite-dimensional elements are Schur-finite.*

Proof. Proposition 3.3 shows that even and odd elements are Schur-finite. □

Example 4.8. If R is a binomial ring containing $\mathbb{Q}$, then every Schur-finite element is finite-dimensional. This follows from Example 1.1 and [Macd, Ex. I.3.4], which says that $s_\pi(r)$ is a rational number times a product of terms $r - c(x)$, where the $c(x)$ are integers.

Example 4.9. The universal element x of $\Lambda_{(2,1)}$ is Schur-finite but not finite-dimensional. To see this, recall from Example 3.6 that $\Lambda_{(2,1)} \cong \mathbb{Z}[x, y]/(y^2 - x^2 y)$. Because $\Lambda_{(2,1)}$ is graded, if x were finite-dimensional it would be the sum of an even and odd element in the degree 1 part $\{nx\}$ of $\Lambda_{(2,1)}$. If $n \in \mathbb{N}$, nx cannot be even because the second coordinate of $\lambda^k(nx)$ is $\binom{-n}{k} b^k$ by 1.2. And nx cannot be odd, because the first coordinate of $\sigma^k(nx)$ is $(-1)^k \binom{-n}{k} a^k$.

Lemma 4.10. *Let $R \subset R'$ be an inclusion of λ-rings. If $x \in R$ is Schur-finite in R', then x is Schur-finite in R. In particular, if x is finite-dimensional in R', then x is Schur-finite in R.*

Proof. Since $s_\pi(x)$ may be computed in either R or R', the set of partitions π for which $s_\pi(x) = 0$ is the same for R and R'. The final assertion follows from Lemma 4.7. □

Lemma 4.11. *If π is a partition on n, $s_{\pi'}(-x) = (-1)^n s_\pi(x)$.*

Proof. Write s_π as a homogeneous polynomial $f(e_1, e_2, ...)$ of degree n. Applying the antipode S in Λ, we have $s_{\pi'} = f(h_1, h_2, ...)$. It follows that $s_{\pi'}(-x) = f(\sigma^1, \sigma^2, ...)(-x)$. Since $\sigma^i(-x) = (-1)^i \lambda^i(x)$, and f is homogeneous, we have

$$s_{\pi'}(-x) = f(-\lambda^1, +\lambda^2, ...)(x) = (-1)^n f(\lambda^1, \lambda^2, ...)(x) = s_\pi(x). \qquad \square$$

Theorem 4.12. *The Schur-finite elements form a subring of any λ-ring, containing the subring of finite-dimensional elements.*

Proof. The Schur-finite elements are closed under addition by Lemma 4.6. Since π contains λ just in case π' contains λ', Lemma 4.11 implies that $-x$ is Schur-finite whenever x is. Hence the Schur-finite elements form a subgroup of R. It suffices to show that if x and y are Schur-finite in R, then xy and all $\lambda^i(x)$ are Schur-finite.

Let x be Schur-finite with rectangular bound μ, so there is a map from the λ-ring Λ_μ to R sending the generator e to x. Embed Λ_μ in $R' = \mathbb{Z}[a_1, \dots, b_1, \dots]$ using Proposition 4.2. Since every element of R' is finite-dimensional, $\lambda^n(e)$ is finite-dimensional in R', and hence Schur-finite in Λ_μ by Lemma 4.10. It follows that the image $\lambda^n(x)$ of $\lambda^n(e)$ in R is also Schur-finite.

Let x and y be Schur-finite with rectangular bounds μ and ν, and let $\Lambda_\mu \to R$ and $\Lambda_\nu \to R$ be the λ-ring maps sending the generators e_μ and e_ν to x and y. Since the induced map $\Lambda_\mu \otimes \Lambda_\nu \to R$ sends $e_\mu \otimes e_\nu$ to xy, we only need to show that $e_\mu \otimes e_\nu$ is Schur-finite. But $\Lambda_\mu \otimes \Lambda_\nu \subset \mathbb{Z}[a_1, \dots, b_1, \dots] \otimes \mathbb{Z}[a_1, \dots, b_1, \dots]$, and in the larger ring every element is finite-dimensional, including the tensor product. By Lemma 4.10, $e_\mu \otimes e_\nu$ is Schur-finite in $\Lambda_\mu \otimes \Lambda_\nu$. $\square$

Conjecture 4.13 (Virtual Splitting principle). Let x be a Schur-finite element of a λ-ring R. Then R is contained in a larger λ-ring R' such that x is finite-dimensional in R', i.e., there are line elements ℓ_i, ℓ'_j in R' so that

$$x = (\sum \ell_i) - (\sum \ell'_j).$$

Example 4.14. The virtual splitting principle holds in the universal case, where $R_0 = \Lambda_\beta$. Indeed, we know that x is $\sum a_i + \sum b_j$ in $R'_0 = \mathbb{Z}[a_1, \dots, b_1, \dots]$. Since $\ell_j = -b_j$ is a line element, x is a difference of sums of line elements in R'_0.

Unfortunately, although the induced map $f : R \to R \otimes_{R_0} R'_0$ sends a Schur-finite element x to a difference of sums of line elements, the map f need not be an injection.

As partial evidence for Conjecture 4.13, we show that the virtual splitting principle holds for elements bounded by the hook $(2, 1)$.

Theorem 4.15. *Let x be a Schur-finite element in a λ-ring R. If x has bound $(2,1)$, then R is contained in a λ-ring R' in which x is a virtual sum $\ell_1 + \ell_2 - a$ of line elements.*

Proof. The polynomial ring $R[a]$ becomes a λ-ring once we declare a to be a line element. Set $y = x + a$, and let I be the ideal of $R[a]$ generated by $\lambda^3(y)$.

For all $n \geq 2$, the equation $s_{n,1}(x) = 0$ yields $\lambda^{n+1}(x) = x\lambda^n(x) = x^{n-1}\lambda^2(x)$ in R, and therefore $\lambda^{n+1}(y) = (a + x)x^{n-2}\lambda^2(x) = x^{n-2}\lambda^3(y)$. It follows from Scholium 1.6 that $\lambda^m(\lambda^3 y) \in I$ for all $m \geq 1$ and hence that

$$\lambda^n(f \cdot \lambda^3 y) = P_n(\lambda^1(f), \dots, \lambda^n(f); \lambda^1(\lambda^3 y), \dots, \lambda^n(\lambda^3 y))$$

is in I for all $f \in R[a]$. Thus I is a λ-ideal of $R[a]$, $A = R[a]/I$ is a λ-ring, and the image of y in A is even of degree 2. By the Splitting Principle 1.4, the image of $x = y - a$ in some λ-ring A' containing A is a virtual sum $\ell_1 + \ell_2 - a$ of line elements.

To conclude, it suffices to show that R injects into $A = R[a]/I$. If $r \in R$ vanishes in A then $r = f\lambda^3(y)$ for some $f = f(a)$ in $R[a]$. We may take f to have minimal degree $d \geq 0$. Writing $f(a) = c\,a^d + g(a)$, with $c \in R$ and $\deg(g) < d$, the coefficient of a^{d+1} in $f\lambda^3(y)$, namely $c\,\lambda^2(x)$, must be zero. But then $c\lambda^3 y = 0$, and $r = g\,\lambda^3 y$, contradicting the minimality of f. $\square$

Remark 4.16. The rank of a Schur-finite object with bound π cannot be well defined unless π is a rectangular partition. This is because any rectangular partition $\mu = (m+1)^{n+1}$ contained in π yields a map $R \to R'$ sending x to an element of rank $m - n$. If π is not rectangular there are different maximal rectangular subpartitions with different values of $m - n$.

Example 4.16.1. *Let x be the element of Theorem 4.15. By Lemma 4.11, $-x$ also has bound $(2,1)$. Applying Theorem 4.15 to $-x$ shows that R is also contained in a λ-ring R'' in which x is a virtual sum $a - \ell_1 - \ell_2$ of line bundles. Therefore x has rank 1 in R', and has rank -1 in R''.*

Let R be a λ-ring and $x \in R$. One central question is to determine when the power series $\lambda_t(x)$ is a rational function. (See [A05], [LL04], [Hl], [Gul], [B1, B2], [KKT] for example.) For concreteness, we consider the question of being determinentally rational (see 2.6). This is connected to Schur-finiteness.

Proposition 4.17. *If x is Schur-finite, then $\lambda_t(x)$ is determinentally rational.*

Conversely, if $\lambda_t(x)$ is determinentally rational, there is an m such that the sequence $s_{(1^m)}(x), \dots, s_{(n^m)}(x), \dots$ is eventually 0.

The first assertion of this proposition was proven in [KKT, 3.10] for λ-rings of the form $K_0(\mathcal{A})$ using categorical methods.

Proof. By definition, $\lambda_t(x)$ is determinentally rational if and only if for some m the determinants of the $m \times m$ matrices $A_n = (\lambda^{n+i+j}(x))$ are 0 for all large n. Reversing the rows in A_{n-m} yields the matrix in the Jacobi-Trudi identity for $s_\pi(x)$, $\pi = (n^m) = (n, n, ..., n)$. Since $\det(A_{m-n}) = \pm s_\pi(x)$, $\lambda_t(x)$ is determinentally rational if and only if for some m the sequence $\{s_{(n^m)}(x)\}$ is eventually 0.

If x is Schur-finite, some bound for x is a rectangular partition (N^m). Then $s_{(n^m)}(x) = 0$ for all $n \geq N$, because the partition (n^m) contains (N^m). □

We conclude by connecting our notion of Schur-finiteness to the notion of a Schur-finite object in a $\mathbb{Q}$-linear tensor category $\mathcal{A}$, given in [Mz]). By definition, an object A is *Schur-finite* if some $S_\lambda(A) \cong 0$ in $\mathcal{A}$. By [Mz, 1.4], this implies that $S_\mu(A) = 0$ for all μ containing λ. It is evident that if A is a Schur-finite object of $\mathcal{A}$ then $[A]$ is a Schur-finite element of $K_0(\mathcal{A})$. However, the converse need not hold. For example, if $\mathcal{A}$ contains infinite direct sums then $K_0(\mathcal{A}) = 0$ by the Eilenberg swindle, so $[A]$ is always Schur-finite.

Here are two examples of Schur-finite objects whose class in $K_0(\mathcal{A})$ is finite-dimensional even though they are not finite-dimensional objects.

Example 4.18. Let $\mathcal{A}$ denote the abelian category of positively graded modules over the graded ring $A = \mathbb{Q}[\varepsilon]/(\varepsilon^2 = 0)$. It is well known that $\mathcal{A}$ is a tensor category under $\otimes_{\mathbb{Q}}$, with the λ-ring $K_0(\mathcal{A}) \cong \Lambda_{-1} = \mathbb{Z}[b]$; $1 = [Q]$ and $b = [\mathbb{Q}[1]]$. The graded object A is Schur-finite but not finite-dimensional in $\mathcal{A}$ by [Mz, 1.12]. However, $[A]$ is a finite-dimensional element in $K_0(\mathcal{A})$ because $[A] = [\mathbb{Q}] + [\mathbb{Q}[1]]$.

Example 4.19 (O'Sullivan). Let X a Kummer surface; then there is an open subvariety U of X, whose complement Z is a finite set of points, such that $M(U)$ is Schur-finite but not finite-dimensional in the Kimura-O'Sullivan sense [Mz, 3.3]. However, it follows from the distinguished triangle

$$M(Z)(2)[3] \to M(U) \to M(X) \to M(Z)(2)[4]$$

that $[M(U)] = [M(Z)(2)[3]] + [M(X)]$ in $K_0(\mathbf{DM}_{gm}$ and hence in $K_0(\mathcal{M})$. Since both $M(X)$ and $M(Z)(2)[3]$ are finite-dimensional, $[M(U)]$ is a finite-dimensional element of $K_0(\mathcal{M})$.

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