A Bochner theorem for Dunkl polynomials

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Abstract

We establish an analogue of the Bochner theorem for first order operators of Dunkl type, that is we classify all such operators having polynomial solutions. Under natural conditions it is seen that the only families of orthogonal polynomials in this category are limits of little and big q-Jacobi polynomials as $q = -1$.

This paper is dedicated to our friend Willard Miller Jr on the occasion of his retirement

1. Introduction

We have introduced recently new families of "classical" polynomials [12], [13] through limits of little and big q-Jacobi polynomials when q goes to -1. The term "classical" is taken to mean that the polynomials $P_n(x)$ are eigenfunctions of some differential or difference operator L

$$LP_n(x) = \lambda_n P_n(x). \tag{1.1}$$

In the case of the novel classes of polynomials that we have identified, the operator L is of first order in the derivative operator ∂_x and contains moreover the reflection operator R defined by $Rf(x) = f(-x)$. In other words the operator L can then be said to be a first order differential-difference operator of Dunkl type [5].

The purpose of the present paper is to examine in general the eigenvalue problem (1.1) for such operators and hence to classify the families of polynomials that are eigenfunctions of first order operators of Dunkl type. As a result, under the natural condition that the operator L is potentially self-adjoint, we show that the big -1 Jacobi polynomials (and their little -1 Jacobi polynomials limit) exhaust the list of orthogonal polynomials with that property.

2. Dunkl-type operators of first order and their polynomial solutions

Consider the most general form of linear differential operators L of first order which contain also the reflection operator R:

$$L = F_0(x) + F_1(x)R + G_0(x)\partial_x + G_1(x)\partial_x R, \tag{2.1}$$

where $F_0(x), F_1(x), G_0(x), G_1(x)$ are arbitrary functions.

We are seeking polynomial eigensolutions of the operator L, i.e. for every n we assume that there exists a monic polynomial $P_n(x) = x^n + O(x^{n-1})$ which is an eigenfunction of the operator L with eigenvalue λ_n (1.1). In what follows we will assume that

$$\lambda_n \neq 0 \quad \text{for } n = 1, 2, \dots, \quad \lambda_n \neq \lambda_m \quad \text{for } n \neq m. \tag{2.2}$$

We first establish the necessary conditions for the existence of such eigensolutions.

Consider the monomials x^n, $n = 0, 1, 2, \dots$. If condition (1.1) holds for all n, the operator L must preserve the space of polynomials of any dimension, i.e. for any polynomial $\pi(x)$ of degree n, we have $L\pi(x) = \tilde{\pi}(x)$, where $\tilde{\pi}(x)$ is a polynomial of the same degree as $\pi(x)$. Indeed, since $\pi(x)$ can be expanded as a finite linear combination of the polynomials $P_i(x)$, $i = 0, 1, \dots, n$, the condition $\lambda_n \neq 0$ guarantees that the polynomial $\tilde{\pi}(x) = L\pi(x)$ and $\pi(x)$ have the same degree.

Hence condition (1.1) is equivalent to the condition

$$Lx^n = Q_n(x), \ n = 0, 1, 2, \dots \tag{2.3}$$

where $Q_n(x)$ are polynomials of degree n. Obviously

$$Q_n(x) = \lambda_n x^n + O(x^{n-1}) \tag{2.4}$$

We shall thus proceed to verify condition (2.3) for the first values of n, i.e. $n = 0, 1, 2, 3$. This will provide necessary conditions for the functions $F_i(x), G_i(x)$ to satisfy.

Consider the case $n = 0$. Without loss of generality we can assume that

$$L\{1\} = 0 \tag{2.5}$$

Indeed, by condition (2.3) we have $L\{1\} = c$ for some constant c. Since this constant can be incorporated additively into the function $F_0(x)$, we can assume that $c = 0$. Then, from (2.3) we find that necessarily $F_1(x) = -F_0(x)$, i.e. that the operator L must have the form

$$L = F(x)(I - R) + G_0(x) + G_1(x)\partial_x R, \tag{2.6}$$

where $F(x) = F_0(x)$ and I is the identity operator. In what follows we will use the form (2.6) for the operator L.

Consider now the case $n = 1$. We have $L\{x\} = Q_1(x)$ with $Q_1(x)$ a linear function. Hence, we obtain the condition

$$2xF(x) + G_0(x) - G_1(x) = Q_1(x). \tag{2.7}$$

Analogously, for $n = 2, 3$ we obtain the conditions

$$2x(G_0(x) + G_1(x)) = Q_2(x) \tag{2.8}$$

and

$$2x^3 F(x) + 3x^2(G_0(x) - G_1(x)) = Q_3(x). \tag{2.9}$$

We thus have 3 equations (2.7), (2.8), (2.9) for the 3 unknown functions $F(x), G_0(x), G_1(x)$. Solving these equations, we can present their general solution in the form

$$G_0(x) = \frac{\mu}{x^2} + \frac{\nu_0}{x} + \rho_0 + \tau_0 x, \quad G_1(x) = -\frac{\mu}{x^2} + \frac{\nu_1}{x} + \rho_1 + \tau_1 x,$$
$$F(x) = -\frac{\mu}{x^3} + \frac{\nu_1 - \nu_0}{2x^2} + \frac{\xi}{x} + \eta, \tag{2.10}$$

where $\mu, \nu_0, \nu_1, \rho_0, \rho_1, \tau_0, \tau_1, \xi, \eta$ are arbitrary constants. In what follows we will assume the restrictions

$$\tau_1 \neq \pm\tau_0, \quad 2\eta + (2N+1)(\tau_0 - \tau_1) \neq 0, \ N = 0, 1, 2, \ldots \tag{2.11}$$

We thus have

Theorem 1 *Let $\mu, \nu_0, \nu_1, \rho_0, \rho_1, \tau_0, \tau_1, \xi, \eta$ be arbitrary complex parameters subjected to conditions (2.11). Construct the operator L using formulas (2.6) and (2.10). Then, for any n, condition (2.3) holds with*

$$\lambda_n = \begin{cases} (\tau_0 + \tau_1)n & \text{if } n \text{ is even} \\ 2\eta + (\tau_0 - \tau_1)n & \text{if } n \text{ is odd} \end{cases} \tag{2.12}$$

and the operator L has a complete set of polynomial eigenfunctions $LP_n(x) = \lambda_n P_n(x)$, $n = 0, 1, 2, \ldots$ with the nondegeneracy conditions (2.2).

The proof of the theorem follows from the simple observation that for any $n = 1, 2, 3, \ldots$ we have

$$Lx^n = \lambda_n x^n + \kappa_n^{(1)} x^{n-1} + \kappa_n^{(2)} x^{n-2} + \kappa_n^{(3)} x^{n-3} = Q_n(x)$$

where the coefficients $\kappa_n^{(i)}$, $i = 1, 2, 3$ are straightforwardly found from the explicit expression of the operator L. Hence the operator L preserves the space of polynomials. The nondegeneracy conditions (2.2) are equivalent to the conditions (2.11).

3. Symmetrizable operators L

In the previous section we identified the most general form of first-order operators L of Dunkl-type that have nondegenerate polynomial eigensolutions $P_n(x)$. Assume now that the polynomials $P_n(x)$ are orthogonal, i.e. that there exists a nondegenerate linear functional σ such that

$$\langle \sigma, P_n(x) P_m(x) \rangle = h_n \delta_{nm} \tag{3.1}$$

with some nonzero normalization constants h_n.

This condition means that the operator L is symmetric with respect to the functional σ:

$$\langle \sigma, L\{V(x)\} W(x) \rangle = \langle \sigma, L\{W(x)\} V(x) \rangle \tag{3.2}$$

where $V(x), W(x)$ are arbitrary polynomials. Property (3.2) trivially follows from (1.1) and the completeness of the system of polynomials $P_n(x)$.

In the case of a positive definite functional, i.e. $h_n > 0$, there is a realization of (3.1) in terms of an integral

$$\langle \sigma, P_n(x) P_m(x) \rangle = \int_a^b P_n(x) P_m(x) d\sigma(x) \tag{3.3}$$

with respect to a measure $d\sigma(x)$, where $\sigma(x)$ is a nondecreasing function of bounded variation [7]. The limits a, b of integration may be finite or infinite.

We restrict ourselves to the case where there is a positive weight function $w(x) = d\sigma(x)/dx > 0$ inside the interval $[a, b]$ (we do not exclude the existence of concentrated masses at the endpoints a, b of the interval).

Under such restrictions we have the following necessary condition for the operator L:

$$(w(x)L)^* = w(x)L, \tag{3.4}$$

where M^* stands for the Lagrange adjoint operator with respect to the operator M. Moreover, in what follows we will assume that all functions $F(x), G_0(x)$ and $G_1(x)$ are real for the real values of the argument x. Recall that if the operator M is a differential operator

$$M = \sum_{k=0}^{N} A_k(x) \partial_x^k,$$

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with real-valued functions $A_k(x)$ then the Lagrange adjoint operator is defined by

$$M^* = \sum_{k=0}^{N} (-1)^k \partial_x^k A_k(x)$$

In the presence of the reflection operator R, i.e. in the case where the operator M has the form

$$M = \sum_{k=0}^{N} A_k(x)\partial_x^k + \sum_{k=0}^{N} B_k(x)\partial_x^k R$$

with some real-valued functions $A_k(x), B_k(x)$, we define the Lagrange adjoint operator by

$$M^* = \sum_{k=0}^{N} (-1)^k \partial_x^k A_k(x) + \sum_{k=0}^{N} (-1)^k R \partial_x^k B_k(x) = \sum_{k=0}^{N} (-1)^k \partial_x^k A_k(x) + \sum_{k=0}^{N} \partial_x^k B_k(-x) R, \tag{3.5}$$

where we used the following formal rules

$$R\partial_x = -\partial_x R, \quad R^* = R, \quad RB(x) = B(-x)R.$$

These formulas assume that the interval of orthogonality is necessarily symmetric, i.e. $b = -a$ (note that this includes possible situations where the interval of orthogonality is the union of several intervals; in such a case, pairs of corresponding intervals should be symmetric with respect to the reflection $x \to -x$).

We thus restrict our considerations to operators L which possess property (3.4) with some positive function $w(x)$ inside the symmetric interval (or intervals) of orthogonality. Such operators are called symmetrizable with respect to the function $w(x)$ [6].

It is easy to see that this symmetrizability property excludes the existence of the term $G_0(x)\partial_x$ in (2.1). Hence we restrict ourselves in the following, to operators of the form

$$L = F(x)(1 - R) + G_1(x)\partial_x R, \tag{3.6}$$

with $G_0(x) = 0$.

This implies $\mu = \nu_0 = \rho_0 = \tau_0 = 0$ and from (2.10), we obtain the following expressions for the functions $G_1(x)$ and $F(x)$:

$$G_1(x) = \frac{\nu_1}{x} + \rho_1 + \tau_1 x, \quad F(x) = \frac{\nu_1}{2x^2} + \frac{\xi}{x} + \eta \tag{3.7}$$

with 5 arbitrary parameters $\nu_1, \rho_1, \tau_1, \xi, \eta$.

Note that we can multiply the operator L by an arbitrary nonzero constant $L \to \kappa_0 L$. A scaling transformation of the independent variable $x \to \kappa_1 x$ with an arbitrary nonzero constant κ_1 can also be performed. Using this freedom, without loss of generality, we can always reduce the number of arbitrary parameters to 3.

Assume that $\nu_1 \tau_1 \neq 0$. We can then rewrite $G_1(x)$ in the form

$$G_1(x) = \frac{g_1(x - d)(x + c)}{x}, \tag{3.8}$$

where g_1, d, c are real constants. Under the condition that $d \neq c$, we can set $g_1 = 2$ and $d = 1$ by an appropriate choice of the parameters κ_0, κ_1 so as to have

$$G_1(x) = \frac{2(x-1)(x+c)}{x} \tag{3.9}$$

with one arbitrary real parameter c. We will assume that $0 < c < 1$. (The cases $c > 1$ or $c < -1$ can be analyzed analogously). For the function $F(x)$ we have

$$F(x) = -\frac{c}{x^2} + \frac{\beta - \alpha c}{x} - \alpha - \beta - 1 \tag{3.10}$$

with two arbitrary real parameters α, β, c.

In [13] it was shown that the operator L with the coefficients $G_1(x), F(x)$ given by (3.8) and (3.10) have big -1 Jacobi polynomials as eigensolutions. In the limiting $c = 0$ the little -1 Jacobi polynomials are obtained.

Hence we have

Theorem 2 *The functions $G_1(x)$ and $F(x)$ given by (3.8) and (3.10) thus provide the most general first order operator of Dunkl-type under our hypotheses. They define the operator L that has the big -1 Jacobi polynomials as eigenfunctions*

Note that we have considered the generic (nondegenerate) choice of real parameters in the functions $G_1(x)$ and $F(x)$. In the next section we will examine the symmetrizability property of the operator L and consider also the degenerate cases with respect to the parameters.

4.   Pearson-type equation for the weight function

It is instructive to derive directly the expression for the weight function $w(x)$ from condition (3.4).

Denote $M = w(x)L = w(x)F(x)(1 - R) + w(x)G_1(x)\partial_x R$. Assuming that all parameters of the functions $F(x), G_1(x), w(x)$ are real, we have

$$M^* = w(x)F(x) - w(-x)F(-x)R - \partial_x R w(x)G_1(x) =$$

$$w(x)F(x) - w(-x)F(-x)R + \frac{d(w(-x)G_1(-x))}{dx}R + w(-x)G_1(-x)\partial_x R.$$

condition $M^* = M$ thus means that the following two equations hold:

$$w(x)G_1(x) = w(-x)G_1(-x) \tag{4.1}$$

and

$$w(-x)F(-x) - w(x)F(x) = \frac{d}{dx}w(-x)G_1(-x) = \frac{d}{dx}w(x)G_1(x) \tag{4.2}$$

With $G_1(x)$ given by (3.9), we have from (4.1)

$$w(x) = \theta(x)(x+1)(x-c)W(x^2), \tag{4.3}$$

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where $\theta(x) = x/|x|$ is the standard sign function and $W(x^2)$ is an arbitrary even function of x. We will suppose that $W(x^2) > 0$ inside the interval $[c, 1]$. Assume that $\alpha > -1$, $\beta > -1$. Then from (4.3) it follows that the function $w(x)$ is positive inside the two symmetric intervals $[-1, -c]$ and $[c, 1]$.

Substituting expression (4.3) into the second condition of (4.2) and using the explicit expression (3.10), we obtain a differential equation for the function $W(y)$:

$$2(y - 1)(y - c^2)W'(y) + \left((2 - \alpha - \beta)y + \beta - 1 + c^2(\alpha - 1)\right)W(y) = 0$$

which has the following general solution

$$W(y) = C(1 - y)^{(\alpha-1)/2}(y - c^2)^{(\beta-1)/2}.$$

with an arbitrary constant C. Thus the function $w(x)$ has the expression

$$w(x) = C\theta(x)(x + 1)(x - c)(1 - x^2)^{(\alpha-1)/2}(x^2 - c^2)^{(\beta-1)/2}. \tag{4.4}$$

Assuming $C > 0$ and $\alpha > -1$, $\beta > -1$, we see that $w(x) > 0$ inside the two symmetric intervals $[-1, -c]$ and $[c, 1]$. This coincides with the weight function $w(x)$ corresponding to the big -1 Jacobi polynomials which was derived in [13] from a completely different approach.

Consider now possible degenerate cases of the function $G_1(x)$.

(i) Assume that $\nu_1 = 0$ but $\rho_1 \tau_1 \neq 0$. Then using scaling transformations, we can always reduce the functions $G_1(x)$ and $F(x)$ to

$$G_1(x) = 2(1 - x), \quad F(x) = \alpha + \beta + 1 - \beta/x.$$

From (4.1) we get $w(x) = (x + 1)W(x^2)$ with some function $W(y)$. Then from (4.3) we obtain a Pearson-type differential equation for the function $W(y)$:

$$W'(y) + ((1 - \alpha - \beta) + \beta)W(y) = 0 \tag{4.5}$$

Its general solution is

$$W(y) = Cy^{\beta/2}(1 - y)^{(\alpha-1)/2}. \tag{4.6}$$

Hence we obtain for the weight function

$$w(x) = C(x + 1)|x|^\beta(1 - x^2)^{(\alpha-1)/2}. \tag{4.7}$$

This weight function corresponds to the one of the little -1 Jacobi polynomials which are orthogonal on the interval $[-1, 1]$ [12]. Equivalently, this case corresponds to setting $c = 0$ in formulas (3.9) and (3.10).

(ii) Assume that $\nu_1 = \rho_1 = 0$ but $\tau_1 \neq 0$. Then we can write without loss of generality

$$G_1(x) = 2x, \quad F(x) = \alpha + \beta + 1 - \beta/x$$

From (4.1) we now have $w(x) = \theta(x)W(x^2)$ and from (4.2) follows Pearson type equation

$$2yW'(y) = (\alpha + \beta)W(y)$$

with solution $W(y) = Cy^{(\alpha+\beta)/2}$. Whence

$$w(x) = C\theta(x)|x|^{(\alpha+\beta)/4}. \tag{4.8}$$

The function $w(x)$ of (4.8) is not positive however, in any symmetric interval $[-a, a]$ (or any union of symmetric intervals). Hence, this case does not lead to orthogonal polynomials with positive measure on the real axis.

(iii) Consider the degenerate case when the numerator in expression (3.8) has coinciding zeros. Using scaling transformations, $G_1(x)$ and $F(x)$ can be cast in the form

$$G_1(x) = \frac{2(x-1)^2}{x}, \quad F(x) = x^{-2} + ax^{-1} + b \tag{4.9}$$

with arbitrary constants a, b.

From condition (4.1) we find

$$w(x) = \theta(x)(x+1)^2 W(x^2) \tag{4.10}$$

with an arbitrary even function $W(x^2)$. From condition (4.2) we then obtain a differential equation for the function $W(y)$:

$$2(y-1)^2 W'(y) + ((b+3)y + 2a + b - 1) W(y) = 0$$

with general solution

$$W(y) = C(y-1)^{-(b+3)/2} \exp\left(\frac{a+b+1}{y-1}\right) \tag{4.11}$$

with an arbitrary constant C. Thus the weight function can be presented in the form

$$w(x) = C\theta(x)(x-1)^2 (x^2-1)^{-(b+3)/2} \exp\left(\frac{a+b+1}{x^2-1}\right). \tag{4.12}$$

It is easily seen that due to presence of the sign function $\theta(x)$ there are no symmetric intervals $[-d, d]$ (or pairs of symmetric intervals) inside which the function $w(x)$ can be positive. Hence in this (degenerate) case the operator L is not symmetrizable.

(iv) Consider the case $\tau_1 = 0$, $\nu_1 \rho_1 \neq 0$. We can here posit $G_1(x) = 2(1 - x^{-1})$ and $F(x) = -x^{-2} - \alpha x^{-1} - \beta - 1$. This leads to $G(x) = \theta(x)(x+1)W(x^2)$ and for the function $W(y)$ we have the equation

$$2(y-1)W'(y) + (\alpha+2)W(y) = 0$$

whence

$$w(x) = C\theta(x)(x+1)(1-x^2)^{(\alpha+\beta)/2}.$$

Again it is impossible to realize the condition $w(x) > 0$ on a symmetric interval.

(v) Finally consider the case when $\tau_1 = \rho_1 = 0$. Then $G_1(x) = -2/x$, $F(x) = -x^{-2} - \alpha x^{-1} - \beta$ This case corresponds to the weight function

$$w(x) = C\theta(x)\exp\left(-\frac{\beta}{2}x^2\right)$$

and here also it is impossible to have a positive weight function on a symmetric interval.

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We see that in all cases, the function $w(x)$ possesses a factorization $w(x) = \pi(x)W(x^2)$, where $\pi(x)$ is a polynomial of first or second degree, and the function $W(y)$ obeys a Pearson-type equation that we have derived. Roughly speaking, the function $W(y)$ resembles the corresponding weight function for the classical orthogonal polynomials [10]. But, in contrast to the classical case, no analogs of Laguerre or Hermite polynomials appear in the case $q = -1$.

5. Conclusions

We showed that the symmetrizability property of the Dunkl operator L with a positive weight function $w(x)$ on a symmetric interval (or a union of symmetric intervals) leads to very strong restrictions upon the associated "classical" orthogonal polynomials. Namely, only big and little -1-Jacobi polynomials admit Dunkl analogs. A number of generalizations offer themselves.

For example it was shown in [2] that the generalized Hermite polynomials $H_n^{(\mu)}(x)$ and the generalized Gegenbauer polynomials $S_n^{(\xi,\eta)}(x)$ (see, e.g. [3], [11], [1] for their definition and properties) are eigenfunctions

$$L_\mu P_n((x) = \lambda_n P_n(x)$$

of an operator L which is quadratic

$$L = \sigma(x)T_\mu^2 + \tau(x)T_\mu \tag{5.1}$$

with respect to the Dunkl operator

$$T_\mu = \partial_x + \mu x^{-1}(I - R).$$

Here $\sigma(x)$ is a polynomial of degree not exceeding 2 and $\deg(\tau(x)) = 1$.

Hence the generalized Hermite and Gegenbauer polynomials can be considered as "classical", but in contradistinction to the general situation considered here, they satisfy an eigenvalue equation of second order in the Dunkl operator. The classification of all second-order Dunkl-type operators having orthogonal polynomials as solutions is a more involved problem.

Note that the operator L defined by (5.1) is a direct generalization of the classical hypergeometric operators

$$L = \sigma(x)\partial_x^2 + \tau(x)\partial_x$$

leading to the classical orthogonal polynomials as eigensolutions [10].

Let us mention also that operators of Dunkl-type have also been used in [8] and [4] to construct some polynomial eigenvalue solutions. In these cases however, the Dunkl operators are not symmetrizable, and the solutions therefore do not belong to the class of ordinary orthogonal polynomials.

Acknowledgments

References

[1] S. Belmehdi, *Generalized Gegenbauer polynomials*, J. Comput. Appl Math. **133** (2001), 195-205.

[2] Y. Ben Cheikh and M.Gaied, *Characterization of the Dunkl-classical symmetric orthogonal polynomials*, Appl. Math. and Comput. **187**, (2007) 105–114.

[3] T. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, NY, 1978.

[4] F.Chouchene,*Harmonic analysis associated with the Jacobi-Dunkl operator on* $[-\frac{\pi}{2}, \frac{\pi}{2}]$, J. Computat. Appl. Math. **178**, (2005), 75–89.

[5] C.F.Dunkl, *Integral kernels with reflection group invariance.* Canadian Journal of Mathematics, **43** (1991) 1213–1227.

[6] W.N. Everitt, K.H. Kwon, L.L. Littlejohn, R.Wellman, *Orthogonal polynomial solutions of linear ordinary differential equations*, J. Comput. Appl. Math. **133** (2001), 85–109.

[7] M.E.H.Ismail, *Classical and Quantum orthogonal polynomials in one variable.* Encyclopedia of Mathematics and its Applications (No. 98), Cambridge, 2005.

[8] T.Koornwinder and F. Bouzeffour, *Nonsymmetric Askey-Wilson polynomials as vector-valued polynomials*, arXiv:1006.1140v1

[9] K.H. Kwon, L.L. Littlejohn and B.H. Yoo, *New characterizations of classical orthogonal polynomials*, Indag. Mathem., N.S., **7** (1996), 199–213.

[10] A.F. Nikiforov, S.K. Suslov, and V.B. Uvarov, *Classical Orthogonal Polynomials of a Discrete Variable*, Springer, Berlin, 1991.

[11] M. Rosenblum, *Generalized Hermite Polynomials and the Bose-like Oscillator Calculus*, in: Oper. Theory Adv. Appl., vol. **73**, Birkhauser, Basel, 1994, pp. 369-396.

[12] L.Vinet and A.Zhedanov, *A "missing" family of classical orthogonal polynomials*, CRM Preprint 3305 (2010).

[13] L.Vinet and A.Zhedanov, *A limit* $q = -1$ *for big q-Jacobi polynomials*, CRM Preprint 3308 (2010).