EXPANSIONS OF ONE DENSITY VIA POLYNOMIALS ORTHOGONAL WITH RESPECT TO THE OTHER.

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Abstract. We expand Chebyshev polynomials and some of its linear combination in linear combinations of $q-$Hermite, Rogers and Al Salam-Chihara polynomials and vice versa. We use these expansions to obtain expansions of the some densities, including $q-$Normal and some related to it, in infinite series of orthogonal polynomials allowing deeper analysis, discovering new properties. On the way we find an easy proof of expansion of of Poisson-Mehler kernels for $q-$Hermite polynomials and also its inverse. We also formulate simple rule relating one set of orthogonal polynomials to the other given the properties of the ratio of the respective densities of measures orthogonalizing these polynomials sets.

1. Introduction

The aim of this paper is formulate a simple rule of expanding one density in terms of products of the other density times orthogonal polynomial with respect to this density and present some of its consequences and applications. The original aim of such expansions was to use them to find some ' easy to generate', simple densities that bound from above other densities that are given in the form of infinite products. Another words the original aim of such expansions was practical and connected with the idea of generating i.i.d. sequences of observations drawn from distributions given by densities that have some difficult to analyze form (e.g. in the form of infinite product). Later however, it turned out that such expansions are interesting by its own allowing deeper insight into distributions that are defined by the densities involved. In particular 'two lines proofs' are possible of identities that traditionally are proved on half or more pages. Simple reflection leads to conclusion that we deal with this type of situation in case of e.g. Mehler expansion formula, recently obtained (see [18]) expansion of $q-$Normal density in terms products of Wigner density times Chebyshev polynomials. Thus it is the time to generalize it, formulate general rule and obtain some new expansions. It will turn out that following this general rule, the difficulty of obtaining expansion of the type discussed in the paper is shifted to difficulties in obtaining so called "connection coefficients" obtained in expanding one family of orthogonal polynomials with respect to the other.

We will also obtain new expansions of so called $q-$Conditional Normal density in series of Kesten type density times some special combinations of Chebyshev

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polynomials as well as inversions of so to say Mehler, expansion formula. The two above mentioned densities and distributions defined by them appeared recently in works of Bożejko or Bryc at al. They were originally defined in terms of infinite products and thus it is difficult to work with them for someone not familiar with notation, notions and basic results in so called $q-$series theory. The ideas we are presenting here are universal and can be applied to any densities and systems of orthogonal polynomials.

The paper is organized as follows. The next section 2 presents general idea of expansion, the main subject of the paper, as well as simple Proposition presenting relationship between sets of polynomials given the ratio of the densities of measures orthogonalizing the sets of polynomials. The task in a sense inverse to the idea of expansion. This section contains also subsection 2.1 that presents some, believed to be interesting, examples of expansions between simple measures orthogonalizing well known sets of polynomials such as Chebyshev, Hermite and some of their combinations. In the next section 3 we introduce notation used in $q-$series theory and then families of orthogonal polynomials that will be used in the sequel as well as measures that make these sets of polynomials orthogonal. In particular we present here $q-$Hermite, Al-Salam-Chihara and Rogers ($q-$utraspherical) polynomials. The next section 4 is devoted to finding some connection coefficients between some families of polynomials related to above mentioned families. Section 5 presents main results of the paper, that is expansions of one density in the series of the other density times series of polynomials orthogonal to this other measure. We also give some (by no means all possible) immediate consequences that lead to some interesting identities. Finally section 6 contains lengthy proofs of the results of section 4.

2. Idea of expansion

The idea of expansion that we are going to pursue is general, simple and is not new (appeared e.g. in [21], Exercise 2.9). We believe that it is very fruitful, has not been sufficiently exploited and is as follows.

Suppose we have two measures α and β defined on $\mathbb{R}$. Let us define two spaces $L_2(\mathbb{R},\mathcal{B},\alpha)$ and $L_2(\mathbb{R},\mathcal{B},\beta)$, where $\mathcal{B}$ denotes a set of Borel subsets, of real functions defined on $\mathbb{R}$, square integrable with respect to measures α and β respectively. Assume also that $\operatorname{supp}\beta\subseteq\operatorname{supp}\alpha$. Further suppose that we know the sets of orthogonal polynomials $\{a_n(x)\}_{n\geq0}$ and $\{b_n(x)\}_{n\geq0}$ defined on $\mathbb{R}$ that are orthogonal with respect to measures α and β respectively. That is assume that we know, that $\forall m,n\geq0$ $\int_{\mathbb{R}}a_n(x)a_m(x)d\alpha(x)=\delta_{nm}\hat{a}_n$ and $\int_{\mathbb{R}}b_n(x)b_m(x)d\beta(x)=\delta_{mn}\hat{b}_n$. Suppose also that we know connection coefficients between sets $\{a_n(x)\}_{n\geq0}$ and $\{b_n(x)\}_{n\geq0}$ i.e. we know numbers $\gamma_{k,n}$ such that $\forall n\geq1$ $a_n(x)=\sum_{k=0}^{n}b_k(x)\gamma_{k,n}$. Further suppose that measures α & β have densities $A(x)$ and $B(x)$ respectively. Then

$$B(x)=A(x)\sum_{n=0}^{\infty}c_na_n(x), \tag{2.1}$$

where $c_n=\gamma_{0,n}\hat{b}_0/\hat{a}_n$. The sense of (2.1) and the type of convergence in it depends on the properties of functions $B(x)$, $A(x)$ and coefficients $\{c_n\}_{n\geq1}$. If

$\int_{\mathbb{R}}\left(B(x)^2/A^2(x)\right)d\alpha(x)<\infty$ that is if $B(x)/A(x)\in L_2(\mathbb{R},\mathcal{B},\alpha)$, series $\sum_{n=0}^{\infty}c_na_n(x)$ converges in $L_2(\mathbb{R},\mathcal{B},\alpha)$ and depending on coefficients $\{c_n\}_{n\geq 0}$ we can even have almost (with respect. to α) pointwise (if $\sum_{i\geq 1}|c_n|^2\log n<\infty$, see Rademacher-Menshov thm.). However in general $B(x)/A(x)$ is only integrable with respect to measure α. Then one has to refer to distribution theory and $\sum_{n=0}^{\infty}c_na_n(x)$ is in general a distribution of 0 order.

To see that really $c_n=\gamma_{0,n}\hat{b}_0/\hat{a}_n$, $n\geq 0$ let us multiply both sides of (2.1) by $\alpha_m(x)$ let us integrate over $\mathbb{R}$. On the left hand side we will get $\gamma_{0,m}b_0$ since $\int_u^v b_k(x)\,d\beta(x)=0$ for $k\geq 1$. On the right hand side we get $c_m\hat{a}_m$.

Remark 1. *Of course to get expansion (2.1) one needs only to calculate* $\int_{\mathbb{R}}\alpha_m(x)\,d\beta(x)$ $=\gamma_{0,m}$ *on the other hand to get connection coefficients one needs to do some algebra without integration. This sometimes can be simpler.*

The idea of relating sets of polynomials given the relationship between measures that make these sets of polynomials orthogonal is not new (see e.g. [21] Thm.2.7.1 (by Christoffel)), assertion iii). Christoffel's relationship between sets of polynomials given the fact that the ratio between orthogonalizing these polynomials measures is a polynomial is accurate given zeros of this polynomial. If polynomial is of order more than 2 it is hard to find these zeros as functions of coefficients. This is of course limitation of possible applications of Christoffels result. The following simple Proposition can be viewed as simplified modification of Christoffel's Theorem. It contains series of simple remarks concerning relationships between discussed sets of polynomials. They do not give precise relationship but in particular situation, confronted together can give such connection. Besides here the only thing one has to know about the ration of the measures is its expansion with respect to one of these sets.

Proposition 1. *Suppose α and β are two probability measures with densities respectively $A(x)$ and $B(x)$. Suppose further that monic*[1] *polynomials $\{a_i\}_{i\geq 1}$ and $\{b_i\}_{i\geq 1}$ are orthogonal with respect to, measures α and β respectively. Suppose additionally that we know that $B(x)/A(x)=W(x)$, where W can expanded in the series of polynomials $a_i(x)$: $W(x)=1+\sum_{i=1}^{N}w_ia_i(x)/\hat{a}_i$ where $\hat{a}_i=\int a_i^2(x)\,A(x)\,dx$, converging in $L_2(\mathbb{R},\mathcal{B},\alpha)$. Define $w_0=1$. Number N can be finite or infinite. Let us recursively define the sequence of numbers $\{f_n\}_{n\geq 0}$, $f_0=1$ by the sequence of equations for*

$$n\geq 1:\sum_{i=0}^{n}f_{n-i}w_i=0,$$

where set $w_i=0$ for $i\geq N+1$ if N is finite.

i) Then monic polynomials defined by:

$$\phi_n(x)=\sum_{i=0}^{n}f_{n-i}a_i(x)$$

satisfy $\int_{\mathbb{R}}\phi_n(x)\,B(x)\,dx=0$, $n=1,2,\ldots$. Besides we have equality $\forall n\geq 1$:

$$a_n(x)=\sum_{i=0}^{n}w_{n-i}\phi_i(x).$$

[1] Polynomial $p_n(x)$ of order n is called monic if coefficient by x^n is equal to 1.

ii) If N is finite, then $\int a_i(x)\,dB(x) = w_i,\ i = 1,\ldots,N,$ *and* $\int a_i(x)\,dB(x) = 0,\ \forall i \geq N+1$. *Hence in particular:*

$$a_n(x) = \phi_n + \sum_{i=1}^{N} w_i \phi_{n-i}(x),$$

for $n \geq N+1$.

iii) If N is finite then there exist N sequences $\left\{\gamma_{n,j}\right\}_{n\geq 1, 1\leq j\leq N}$ *such that* $\forall n \geq 1$

$$a_n(x) = b_n(x) + \sum_{j=1}^{N} \gamma_{n,j} b_{n-j}(x).$$

Proof. Is moved to section 6 □

2.1. **Simple examples.**

(1) Let us consider $A(x) = \frac{1}{\pi\sqrt{1-x^2}}$ if $x \in (-1,1)$ and 0 otherwise and $a_n(x) = T_n(x)$ (Chebyshev Polynomials of the first kind). Further $B(x) = \frac{2}{\pi}\sqrt{1-x^2}$ if $x \in (-1,1)$ and 0 otherwise and $b_n(x) = U_n(x)$ (Chebyshev Polynomials of the second kind). It is known that

$$\begin{aligned}\int_{-1}^{1} a_n(x)\,a_m(x)\,A(x)\,dx &= \begin{cases} 1 & for \quad n=m=0 \\ \frac{1}{2}\delta_{nm} & for \quad n \neq 0 \text{ or } m \neq 0\end{cases}, \\ \int_{-1}^{1} b_n(x)\,b_m(x)\,B(x)\,dx &= \delta_{nm}.\end{aligned}$$

Polynomials $\{T_n\}$ and $\{U_n\}$ satisfy the same three term recurrence with different however initial condition for $n = 1$. Namely $T_{-1}(x) = U_{-1}(x) = 0$, $T_0(x) = U_0(x) = 1$, $T_1(x) = x$, $U_1(x) = 2x$. Now notice that $(U_1(x) - U_{-1}(x))/2 = x = T_1(x)$. Besides we have $x(U_n(x) - U_{n-2}(x))/2 = (U_{n+1} + U_{n-1}(x) - U_{n-1}(x) + U_{n-3}(x)) = (U_{n+1}(x) - U_{n-1}(x))/2 + (U_{n-1}(x) - U_{n-3}(x))/2$, which is a three term recurrence satisfied by polynomials T_n. Hence:

$$\forall n \geq 1 : T_n(x) = \left(U_n(x) - U_{n-2}(x)\right)/2. \tag{2.2}$$

Hence sequence we have $\gamma_{0,0} = 1$, $\gamma_{k,n} = \begin{cases} 1/2 & if \quad k=n \\ -1/2 & if \quad k=n-2 \\ 0 & if \quad otherwise\end{cases}$ for $n \geq 1$. Thus $\gamma_{0,0} = 1$, $\gamma_{0,1} = 0, \gamma_{0,2} = -1/2$, $\gamma_{0,n} = 0$ for $n \geq 3$. So we have elementary relationship $B(x) = A(x)(1 - T_2(x)) = 2A(x)(1-x^2)$.

Similarly one can deduce that :

$$\forall n \geq 1 : U_n(x) = 2\sum_{i=0}^{\lfloor n/2\rfloor} T_{n-2i}(x) - \left(1 + (-1)^n\right)/2.$$

Hence $\gamma_{0,2i+1} = 0$, $\gamma_{0,2i} = 1$, $i = 0,1,2,\ldots$. Thus we have

$$A(x) = B(x)\sum_{i=0}^{\infty} U_{2i}(x)$$

and we do not have neither pointwise nor even mod β convergence. One can deduce, following definition of distributions that right hand side of the above equality is a distribution t_α for which $\forall n \geq 1\ \ t_\alpha(T_n) = 0$, by (2.2)and orthogonality of $\{U_i\}_{i\geq 1}$ with respect to $B(x)$. However we are not going to continue this topic since our main concern are regular, convergent cases.

(2) Let

$$A(x|y,\rho)=\frac{\left(1-\rho^2\right)\sqrt{4-x^2}}{2\pi((1-\rho^2)^2-\rho xy(1+\rho^2)+\rho^2(x^2+y^2))}$$

if $x\in(-2,2)$ and 0 otherwise and $|y|\leq 2$, $|\rho|<1$ be a particular case of Kesten density considered also in the sequel. It is known (also it follows the fact that it is a particular case of considered in the sequel distribution f_{CN}) that the following polynomials

$$k_n(x|y,\rho)=U_n(x/2)-\rho yU_{n-1}(x/2)+\rho^2U_{n-2}(x/2)$$

when $n\geq 2$, $k_1(x|y,\rho)=x-\rho y$ and $k_0(x|y\rho)=1$ are orthogonal with respect to measure defined by A.

As measure β let us take same measure as in the previous example but re-scaled by 2. More precisely let β have density

$$B(x)=\frac{1}{2\pi}\sqrt{4-x^2}.$$

Hence re-scaled Chebyshev polynomials $U_n(x/2)$ are orthogonal with respect to β. If expansion of B is concerned we have

$$\gamma_{0,n}=\left\{\begin{array}{ccc}0 & if & n>2\\ \rho^2 & if & n=2\\ -\rho y & if & n=1\end{array}\right..$$

Besides it is known (also from (3.9c)) that $\int_{-2}^{2}k_n^2(x|y,\rho,0)A(x|y,\rho)=(1-\rho^2)$. Hence we have:

$$\begin{aligned}B(x)&=A(x|y,\rho)\left(1-\frac{\rho y}{(1-\rho^2)}k_1(x|y,\rho)+\frac{\rho^2}{1-\rho^2}k_2(x|y,\rho)\right)\\&=A(x|y,\rho)\left((1-\rho^2)^2-\rho(1-q)xy(1+\rho^2)+(1-q)\rho^2(x^2+y^2)\right)/(1-\rho^2).\end{aligned}$$

On the other hand one can easily derive (or it follows (4.7) in [19] considered for $q=0$ and noting that $h_n(x|0)=U_n(x)$) that

$$U_n(x/2)=\sum_{j=0}^{n}\rho^{n-j}U_{n-j}(y/2)k_j(x|y,\rho).$$

Thus we have

$$A(x|y,\rho)=B(x)\sum_{i=0}^{\infty}\rho^iU_i(y/2)U_i(x/2),\tag{2.3}$$

which is a particular case of Poisson -Mehler kernel to be discussed in the sequel.

(3) Following well known (see e.g. [1] Ex. 5, p. 339) formula concerning Hermite polynomials H_n orthogonal with respect to measure $d\alpha(x)=\frac{1}{\sqrt{2\pi}}\exp\left(-x^2/2\right)dx\overset{df}{=}A(x)dx$

$$\forall\rho\in(-1,1),\forall n\geq 1:H_n\left(\rho x+y\sqrt{1-\rho^2}\right)=\sum_{i=0}^{n}\binom{n}{i}\rho^i\left(\sqrt{1-\rho^2}\right)^{n-i}H_i(x)H_{n-i}(y),$$

we can rewrite it in the following form :

$$\forall\rho\in(-1,1),\forall n\geq 1: H_n(x)=\sum_{i=0}^{n}\binom{n}{i}\rho^iH_i(y)\left(\sqrt{1-\rho^2}\right)^{n-i}H_{n-i}\left(\frac{(x-\rho y)}{\sqrt{1-\rho^2}}\right),$$

since we have trivially $x=\rho y+\sqrt{1-\rho^2}\frac{(x-\rho y)}{\sqrt{1-\rho^2}}$, and view as 'connection coefficient formula' between sets of polynomials $\{H_n(x)\}_{n\geq 0}$ that are orthogonal with respect measure $d\alpha$ and $\left\{\left(\sqrt{1-\rho^2}\right)^nH_n\left(\frac{(x-\rho y)}{\sqrt{1-\rho^2}}\right)\right\}_{n\geq 0}$ that are orthogonal with respect to measure $d\beta(x)=\frac{1}{\sqrt{2\pi(1-\rho^2)}}\exp\left(-\frac{(x-\rho y)^2}{2(1-\rho^2)}\right)dx\overset{df}{=}B(x)dx$. An easy calculation gives $\gamma_{0,n}=\rho^nH_n(y)$ and $\hat{a}=n!$ and we end up with famous Mehler Hermite Polynomial Formula

$$\frac{1}{\sqrt{2\pi(1-\rho^2)}}\exp\left(-\frac{(x-\rho y)^2}{2(1-\rho^2)}\right)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)\sum_{i=0}^{\infty}\frac{\rho^i}{i!}H_i(x)H_i(y),\tag{2.4}$$

which is better known in a form obtained from the above after dividing both sides by $\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$ and whose proof takes about a page in popular handbooks of special functions.

2.2. **Densities defined by infinite products.** As it follows with those three examples the idea of expanding one density with a help of another, can be fruitful and lead to interesting formulae and consequently to deeper understanding of the expanded distribution. Besides some recently used distributions have densities that are defined with a help of infinite series. Infinite series are in many ways difficult to deal with in particular to calculate many quantities that are interesting for probabilists. That is why we will use this technique of expansion to accustom that is to obtain another, more suitable for further analysis and research, form of three densities that appeared recently and that are defined with through infinite series.

Two of these three distributions appeared in the context of so called Bryc processes or fields (see details [2] and [4]) or $q-$Gaussian processes (for details see [8]). Their densities are as follows:

$$f_N(x|q)=\frac{\sqrt{1-q}}{2\pi\sqrt{4-(1-q)x^2}}\prod_{k=0}^{\infty}\left((1+q^k)^2-(1-q)x^2q^k\right)\prod_{k=0}^{\infty}(1-q^{k+1})\tag{2.5}$$

defined for $|q|<1$ and $|x|<\frac{2}{\sqrt{1-q}}$ that will be sometimes referred to $q-$Normal (briefly $q-$N) distribution and

$$f_{CN}(x|y,\rho,q)=\frac{\sqrt{1-q}}{2\pi\sqrt{4-(1-q)x^2}}\times\tag{2.6a}$$

$$\prod_{k=0}^{\infty}\frac{(1-\rho^2q^k)\left(1-q^{k+1}\right)\left((1+q^k)^2-(1-q)x^2q^k\right)}{(1-\rho^2q^{2k})^2-(1-q)\rho q^k(1+\rho^2q^{2k})xy+(1-q)\rho^2(x^2+y^2)q^{2k}}\tag{2.6b}$$

defined for $|q|<1$, $|\rho|<1$, $|x|,|y|<\frac{2}{\sqrt{1-q}}$ that will be referred to as $(y,\rho,q)-$Conditional Normal, (briefly $(y,\rho,q)-CN$) distribution.

Third one is a so called $q-$utraspherical density (density with respect to which Rogers (called also $q-$utraspherical) polynomials are orthogonal). It is given by
(2.7)

$$f_R\left(x|\beta,q\right)=\frac{\sqrt{1-q}}{2\pi\sqrt{4-(1-q)x^2}}\prod_{k=0}^{\infty}\frac{(1-q^{k+1})\left((1+q^k)^2-(1-q)x^2q^k\right)\left(1-\beta^2q^k\right)}{(1-\beta q^k)(1-\beta q^{k+1})\left((1+\beta q^k)^2-(1-q)\beta x^2q^k\right)}.$$

defined also for $|q|<1$ and $|x|<\frac{2}{\sqrt{1-q}}$ and $|\beta|<1$. This distributions is closely related to distributions $q-$N and $(y,\rho,q)-$CN. Namely we have the following.

Remark 2. *Note that we have*
(2.8)

$$f_R\left(x|\beta,q\right)=f_N\left(x|q\right)\times\prod_{k=0}^{\infty}\frac{\left(1-\beta^2q^k\right)}{(1-\beta q^k)(1-\beta q^{k+1})\left((1+\beta q^k)^2-(1-q)\beta x^2q^k\right)}.$$

Moreover we also have $f_{CN}\left(x|x,\rho,q\right)=f_R\left(x|\rho,q\right)/(1-\rho)$ *since* $(1-\rho^2q^{2k})^2-(1-q)\rho q^k(1+\rho^2q^{2k})x^2+2(1-q)\rho^2x^2q^{2k}=\left(1-\rho q^k\right)^2\left(\left(1+\rho q^k\right)^2-(1-q)\rho x^2q^k\right)$ *and* $\lim_{\beta\rightarrow1^-}f_R\left(x|\beta,q\right)=\frac{\sqrt{1-q}}{2\pi\sqrt{4-(1-q)x^2}}$.

These distributions appeared in the context of special functions in particular in the context of Rogers polynomials . However only recently their importance to both commutative and noncommutative probability became apparent. As mentioned before distributions $f_N\left(x|q\right)$ and $f_{CN}\left(x|y,\rho,q\right)$ reappeared in 1997 in the paper [8] of Bożejko and Speicher in purely noncommutative probability context. These families turn out to be important both for classical and noncommutative probabilists as well as for physicists.

The following section of the paper is devoted to listing known properties of q-Normal, $(y,\rho,q)-CN$, $(\beta,q)-$R distributions (section 3). The other two sections of the paper are devoted to discussion of new properties of these distributions that are obtained from expansion presented in section 5. Final section of the paper is contains longer proofs of the results presented in the paper.

3. Families of orthogonal polynomials. Their properties and relationships

We will use traditional notation of $q-$series theory i.e. $\left[0\right]_q=0;\left[n\right]_q=1+q+\ldots+q^{n-1}=\frac{1-q^n}{1-q}$, $\left[n\right]_q!=\prod_{i=1}^{n}\left[i\right]_q$,with $\left[0\right]_q!=1$, $\left[\begin{smallmatrix}n\\k\end{smallmatrix}\right]_q=\left\{\begin{array}{cc}\frac{\left[n\right]_q!}{\left[n-k\right]_q!\left[k\right]_q!} & when\quad n\geq k\geq0\\ 0 & when\quad otherwise\end{array}\right.$. Sometimes it will be useful to use so called $q-$Pochhammer symbol for $n\geq1$: $\left(a|q\right)_n=\prod_{i=0}^{n-1}\left(1-aq^i\right)$, with $\left(a|q\right)_0=1$, $\left(a_1,a_2,\ldots,a_k|q\right)_n=\prod_{i=1}^{k}\left(a_i|q\right)$. It is easy to notice that $\left(q|q\right)_n=\left(1-q\right)^n\left[n\right]_q!$ and that $\left[\begin{smallmatrix}n\\k\end{smallmatrix}\right]_q=\left\{\begin{array}{cc}\frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k} & when\quad n\geq k\geq0\\ 0 & when\quad otherwise\end{array}\right.$. Notice that $\left(a|0\right)_n=1-a$ for $n\geq1$ and $\left(a|1\right)_n=\left(1-a\right)^n$. If it will not cause misunderstanding Pochhammer symbol $(a|q)_n$ will sometimes abbreviated to $\left(a\right)_n$ if choice of q will be obvious. Let us also denote $I_A\left(x\right)=\left\{\begin{array}{ccc}1 & if & x\in A\\ 0 & if & x\notin A\end{array}\right.$ and $S\left(q\right)$ $=\left[-2/\sqrt{1-q},2/\sqrt{1-q}\right]$. Recall that every family of orthogonal polynomials is defined by 3 term recursive relationship. The 3 families of orthogonal polynomials that we are going to use are defined by the following recursive relationships:

$$H_{n+1}(x|q) = xH_n(x|q) - [n]_q H_{n-1}(x|q), \tag{3.1}$$

$$R_{n+1}(x|\beta,q) = (1-\beta q^n) xR_n(x|\beta,q) \tag{3.2}$$

$$-\left(1-\beta^2 q^{n-1}\right)[n]_q R_{n-1}(x|\beta,q), \tag{3.3}$$

$$P_{n+1}(x|y,\rho,q) = (x-\rho y q^n)P_n(x|y,\rho,q) \tag{3.4}$$

$$-(1-\rho^2 q^{n-1})[n]_q P_{n-1}(x|y,\rho,q), \tag{3.5}$$

with $H_{-1}(x|q) = R_{-1}(x|\beta,q) = P_{-1}(x|y,\rho,q) = 0$, $H_0(x|q) = R_0(x|\beta,q) = P_0(x|y,\rho,q) = 1$. Here parameters β,ρ,y,q have the following bounds: $|q| \leq 1$, $|\beta|,|\rho| < 1$, $y \in \mathbb{R}$. The family (3.1) will be referred to as family of $q-$Hermite polynomials, family (3.2) will be referred to as family of Rogers polynomials. Finally family (3.4) will be referred to as the family of Al-Salam-Chihara polynomials.

In fact in the literature (see e.g. [1]) more popular are these families transformed. Namely as $q-$Hermite polynomials often function polynomials

$$h_n(x|q) = (1-q)^{n/2} H_n\left(\frac{2x}{\sqrt{1-q}}|q\right), n \geq 1 \tag{3.6}$$

sometimes called also continuous $q-$Hermite polynomials as Rogers polynomials function

$$C_n(x|\beta,q) = (q)_n (1-q)^{n/2} R_n\left(\frac{2x}{\sqrt{1-q}}|\beta,q\right), n \geq 1 \tag{3.7}$$

and as Al-Salam-Chihara function polynomials

$$p_n(x|a,b,q) = (1-q)^{n/2} P_n\left(\frac{2x}{\sqrt{1-q}}|\frac{2a}{\sqrt{(1-q)b}},\sqrt{b},q\right), \tag{3.8}$$

for $|\beta| < 1, a^2 > b \geq 0$ (if we want these polynomials to have probabilistic interpretation). For our purposes, closely connected with probability, the families defined by (3.1), (3.2) and (3.4) are more suitable.

Families of polynomials $\{H_n\}_{\geq 1}, \{P_n\}_{n\geq 1}$ and $\{R_n\}_{\geq 1}$ have the following basic properties:

Lemma 1.

$$\forall n \in \mathbb{N}: \int_{S(q)} H_n(x|q) H_m(x|q) f_N(x|q)\, dx = \begin{cases} 0 & when \quad n \neq m \\ [n]_q! & when \quad n = m \end{cases}, \tag{3.9a}$$

$$\int_{S(q)} H_n(x|q) f_{CN}(x|y,\rho,q)\, dx = \rho^n H_n(y|q), \tag{3.9b}$$

(3.9c)

$$\int_{S(q)} P_n(x|y,\rho,q) P_m(x|y,\rho,q) f_{CN}(x|y,\rho,q)\, dx = \begin{cases} 0 & when \quad n \neq m \\ (\rho^2)_n [n]_q! & when \quad n = m \end{cases},$$

(3.9d)

$$\int_{S(q)} R_n(x|\beta,q) R_n(x|\beta,q) f_R(x|\beta,q)\, dx = \begin{cases} 0 & when \quad n \neq m \\ \frac{(1-\beta)(\beta^2)_n [n]_q!}{(1-\beta q^n)} & when \quad n = m \end{cases},$$

(3.9e)

$$\forall |\rho_1|,|\rho_2| < 1: \int_{S(q)} f_{CN}(x|y,\rho_1,q) f_{CN}(y|z,\rho_2,q)\, dy = f_{CN}(x|z,\rho_1\rho_2,q).$$

$$\max_{x\in S(q)}\left|H_n\left(x|q\right)\right|\leq\frac{W_n\left(q\right)}{(1-q)^{n/2}},\ \max_{x\in S(q)}\left|R_n\left(x|\beta,q\right)\right|\leq\frac{V_n\left(q,\beta\right)}{\left(q\right)_n\left(1-q\right)^{n/2}},\tag{3.10}$$

where $W_n\left(q\right)=\sum_{i=0}^{n}\left[\begin{smallmatrix}n\\ i\end{smallmatrix}\right]_q$, $V_n\left(q,\beta\right)=\sum_{i=0}^{n}\frac{\left(\beta|q\right)_i\left(\beta|q\right)_{n-i}}{\left(q|q\right)_i\left(q|q\right)_{n-i}}$

Proof. (3.9a), (3.9b), follow [21] (and also [2], [4]). To prove (3.9c) and (3.9d) we use [1] , standard knowledge on orthogonal polynomials and formulae (3.4) and (3.2). First let us denote $A_n=\int_{S(q)}R_n^2(x|\beta,q)f_R\left(x|\beta,q\right)dx$ and $Q_n=\int_{S(q)}P_n^2\left(x|y,\rho,q\right)f_{CN}\left(x|y,\rho,q\right)dx$. Further we multiply both sides (3.4) and (3.2) once respectively by $P_{n-1}\left(x|y,\rho,q\right)$ and $R_{n-1}(x|\beta,q)$ and then by $P_{n+1}\left(x|y,\rho,q\right)$ and $R_{n+1}\left(x|\beta,q\right)$ and integrate respectively with respect. f_{CN} and f_R over $S\left(q\right)$ obtaining respectively $\int_{S(q)}x\times P_n\left(x|y,\rho,q\right)\times P_{n-1}\left(x|y,\rho,q\right)\times f_{CN}\left(x|y,\rho,q\right)dx=(1-\rho^2q^{n-1})[n]_qQ_{n-1}$ and $(1-\beta q^n)\int_{S(q)}x\times R_n\left(x|\beta,q\right)\times R_{n-1}\left(x|\beta,q\right)\times f_R\left(x|\beta,q\right)dx=(1-\beta^2q^{n-1})\left[n\right]_qA_{n-1}$ and then $Q_{n+1}=\int_{S(q)}xP_{n+1}\left(x|y,\rho,q\right)P_n\left(x|y,\rho,q\right)f_{CN}\left(x|y,\rho,q\right)dx$ and $A_{n+1}=(1-\beta q^n)\int_{S(q)}x\times R_n\left(x|\beta,q\right)\times R_{n+1}\left(x|\beta,q\right)\times f_R\left(x|\beta,q\right)dx$. From these equations we deduce that respectively $Q_n=(1-\rho^2q^{n-1})[n]_qQ_{n-1}$and $A_n=\frac{1-\beta q^{n-1}}{1-\beta q^n}\left(1-\beta^2q^{n-1}\right)\left[n\right]_qA_{n-1}$ from which follows (3.9d) by iteration. (3.9c) we prove in a similar way.

Formula (3.9e) is taken from [2] and [3] can be also found in [8]. Formula (3.10) follows formulae 13.1.10 and 13.2.16 of [21] and (3.6) and (3.7). □

We will also use, mentioned already Chebyshev, polynomials of the first $T_n\left(x\right)$ defined by $T_n\left(\cos\theta\right)=\cos n\theta$ and second kind $U_n\left(x\right)$ defined by $U_n\left(\cos\theta\right)=\frac{\sin(n+1)\theta}{\sin\theta}$ and ordinary (probabilistic) Hermite polynomials $H_n\left(x\right)$ i.e. polynomials orthogonal with respect to $\frac{1}{\sqrt{2\pi}}\exp(-x^2/2)$. They satisfy 3−term recurrence. Recall that Chebyshev polynomials satisfy 3−term recurrence (3.11), below while Polynomials H_n satisfy 3−term recurrence (3.12) below.

$$2xU_n\left(x\right)=U_{n+1}\left(x\right)+U_{n-1}\left(x\right),\tag{3.11}$$

$$xH_n\left(x\right)=H_{n+1}\left(x\right)+nH_{n-1}\tag{3.12}$$

with $T_0\left(x\right)=U_0\left(x\right)=H_0\left(x\right)=1$, $T_1\left(x\right)=H_1\left(x\right)=x$, $U_1\left(x\right)=2x$. Moreover we will be using re-scaled versions of polynomials T_n and U_n that is $\hat{T}_n\left(x|q\right)=T_n\left(x\sqrt{1-q}/2\right)/\left(1-q\right)^{n/2}$ and $\hat{U}_n\left(x|q\right)=U_n\left(x\sqrt{1-q}/2\right)/\left(1-q\right)^{n/2}$. These modified polynomials are orthogonal with respect to modified densities that appear in the context of Chebyshev polynomials. That is we have $\int_{S(q)}\hat{U}_n\left(x|q\right)\hat{U}_m\left(x|q\right)f_U\left(x|q\right)dx=\delta_{mn}$ and $\int_{S(q)}\hat{T}_n\left(x|q\right)\hat{T}_m\left(x|q\right)f_T\left(x|q\right)dx=\delta_{mn}/2$ if $n\vee m\geq1$ and 1 if $n=m=0$, where we denoted

$$f_U\left(x|q\right)=I_{S(q)}\left(x\right)\sqrt{\left(1-q\right)\left(4-\left(1-q\right)x^2\right)}/2\pi,\tag{3.13}$$

$$f_T\left(x|q\right)=I_{S(q)}\left(x\right)/(\sqrt{\left(1-q\right)/\left(4-\left(1-q\right)x^2\right)}\pi).\tag{3.14}$$

Density f_U functions sometimes in the literature as Wigner distribution with radius $2/\sqrt{1-q}$. In the sequel will also appear distribution $f_{CN}\left(x|y,\rho,0\right)$ re-scaled in the following way

(3.15)

$$f_K\left(x|y,\rho,q\right)=\frac{\left(1-\rho^2\right)\sqrt{1-q}\sqrt{4-(1-q)x^2}}{2\pi\left(\left(1-\rho^2\right)^2-\rho(1-q)\left(1+\rho^2\right)xy+(1-q)\rho^2\left(x^2+y^2\right)\right)}I_{S(q)}\left(x\right)$$

that is a particular case of so called Kesten distribution and which is nothing else but re-scaled density $A(x)$ considered above in example 2.

We have Proposition that relates cases defined by special values of parameters to known families of polynomials or distributions:

Proposition 2. 1. $f_{CN}(x|y,0,q) = f_R(x|0,q) = f_N(x|q) = f_U(x|q)(q)_\infty \times \prod_{k=1}^{\infty}\left((1+q^k)^2-(1-q)x^2q^k\right)$,

2. $\forall n \geq 0 : R_n(x|0,q) = H_n(x|q),\ H_n(x|0) = U_n(x/2),\ H_n(x|1) = H_n(x)$, $\lim_{\beta->1^-}\frac{R_n(x|\beta,q)}{(\beta)_n} = 2\frac{T_n\left(x\sqrt{1-q}/2\right)}{(1-q)^{n/2}}$

3. $\forall n \geq 0 : P_n(x|x,\rho,q) = R_n(x|\rho,q),\ P_n(x|y,0,q) = H_n(x|q),\ P_n(x|y,\rho,1) = (1-\rho^2)^{n/2}H_n\left(\frac{x-\rho y}{\sqrt{1-\rho^2}}\right),\ P_n(x|y,\rho,0) = U_n(x/2) - \rho y U_{n-1}(x/2) + \rho^2 U_{n-2}(x/2) \overset{df}{=} k_n(x|y,\rho)$.

4. *relationship (3.9b) reduces for $\rho = 0$ to relationship (3.9a) with $m = 0$,*

5. $f_N(x|0) = \frac{1}{2\pi}\sqrt{4-x^2}I_{<-2,2>}(x),\ f_N(x|1) = \frac{1}{\sqrt{2\pi}}\exp\left(-x^2/2\right),\ f_R(x|1,q) = f_T(x|q)$,

6. $f_{CN}(x|y,\rho,0) = f_K(x|y,\rho),\ f_{CN}(x|y,\rho,1) = \frac{1}{\sqrt{2\pi(1-\rho^2)}}\exp\left(-\frac{(x-\rho y)^2}{2(1-\rho^2)}\right)$.

Proof. 1. is obvious. 2. follows observation that (3.1) simplifies to (3.11) and (3.12) for $q = 0$ and $q = 1$ respectively while (3.2) simplifies to (3.1). To calculate $\lim_{\beta->1^-}\frac{R_n(x|\beta,q)}{(\beta)_n}$ let us first denote $Q_n(x|\beta,q) = \frac{R_n(x|\beta,q)}{(\beta)_n}$ and write $3-$term recurrence for it obtaining $Q_{n+1}(x|\beta,q) = xQ_n(x|\beta,q) - \frac{(1-q^n)(1-\beta^2q^{n-1})}{(1-q)(1-\beta q^n)(1-\beta q^{n-1}}Q_{n-1}(x|\beta,q)$, with $Q_{-1}(x|\beta,q) = 0$, $Q_0(x|\beta,q) = 1$. Let $\beta->1^-$. Then one gets recognizes $3-$term recurrence for polynomials $\frac{T_n\left(x\sqrt{1-q}/2\right)}{(1-q)^{n/2}}$. This form property can also be found in [21] , formula 13.2.15.
3. First three assertions follow either direct observation in the case of $P_n(x|y,\rho,0)$ or comparison of (3.4) and (3.12) considered for substitution $x->(x-\rho y)/\sqrt{1-\rho^2}$ and then multiplication of both sides by $\left(1-\rho^2\right)^{(n+1)/2}$ third assertion follows following observations: $P_{-1}(x|y,\rho,0) = 0,\ P_0(x|y,\rho,0) = 1,\ P_1(x|y,\rho,0) = x-\rho y$, $P_2(x|y,\rho,0) = x(x-\rho y) - \left(1-\rho^2\right), P_{n+1}(x|y,\rho,0) = xP_n(x|y,\rho,0) - P_{n-1}(x|y,\rho,0)$ for $n \geq 2$ which is equation (3.11). 5. and 6. Their first assertions are obvious. Secondly we notice that passing to the limit $q->1^-$ and applying 2. and 3. we obtain well known relationships defining Hermite polynomials. Hence Hermite polynomials are orthogonal with respect to measure defined by $f_N(x|1)$. Thus distributions defined by f_N and f_{CN} tend to normal $N(0,1)$ and $N\left(\rho y, \left(1-\rho^2\right)\right)$ distributions weakly as $q->1^-$. So it is natural to define $f_N(x|1)$ and $f_{CN}(x|y,\rho,q)$ as they are in 5. and 6. □

As suggested in assertion 3. of the Proposition 2 we will be using notation $k_n(x|y,\rho)$ instead $P_n(x|y,\rho,0)$ which seems to be simpler. Besides we have $k_0(x|y,\rho) = 1$, $k_1(x|y,\rho) = x-\rho y$, $k_2(x|y,\rho) = x(x-\rho y) - (1-\rho^2)$ and $k_{n+1}(x|y,\rho) = xk_n(x|y\rho) - k_{n-1}(x|y,\rho)$.

Remark 3. *Since polynomials $\{k_n(x|y,\rho)\}_{n\geq 0}$ are orthogonal to measure with density $A(x)$ of example 2, or more precisely with density $f_K(x|y,\rho,0)$, we deduce (by simple change change of variables in appropriate integral) that polynomials $\left\{k_n\left(x\sqrt{1-q}|y\sqrt{1-q},\rho\right)\right\}_{n\geq 0}$ are orthogonal with respect to $f_K(x|y,\rho,q)$*

Hence in particular f_N is a generalization of $N(0,1)$ density, while f_{CN} is a generalization of $N\left(\rho y, 1-\rho^2\right)$ density. It is also known see e.g. [4] that $f_{CN}(x|y,\rho,q)/f_N(x|q)$ follows Lancaster type expansion. Namely we have:

$$\prod_{k=0}^{\infty}\frac{(1-\rho^2q^k)}{(1-\rho^2q^{2k})^2-(1-q)\rho q^k(1+\rho^2q^{2k})xy+(1-q)\rho^2(x^2+y^2)q^{2k}} \tag{3.16}$$
$$=\sum_{n=0}^{\infty}\frac{\rho^n}{[n]_q!}H_n(x|q)H_n(y|q),$$

converges uniformly and defines the Poisson-Mehler kernel. It is an almost obvious generalization of (2.4) and (2.3). We will prove and generalize it by our expansion idea in the next section.

4. Auxiliary results

In this section we are going either to recall or to calculate connection coefficients of one family of orthogonal polynomials with respect to the others. First we will recall known results, exposing some of the families of connection coefficients. To do this let us introduce one more family of polynomials $\{B_n(x|q)\}_{n\geq 0}$ that are orthogonal but with respect to some complex measure. They play an auxiliary role. They satisfy the following 3−term recursive equation:

$$B_{n+1}(y|q)=-q^nyB_n(y|q)+q^{n-1}[n]_qB_{n-1}(y|q)\,;n\geq 0, \tag{4.1}$$

with $B_{-1}(y|q)=0$, $B_0(y|q)=1$. Formula (16) of [4] allows to express then through $q-$Hermite polynomials.
Namely we have: $B_n(x|q)=\left\{\begin{array}{ll} i^nq^{n(n-2)/2}H_n(i\sqrt{q}\,x|\frac{1}{q}) & \text{for } q>0\\ (-1)^{n(n-1)/2}|q|^{n(n-2)/2}H_n(-\sqrt{|q|}\,x|\frac{1}{q}) & \text{for } q<0\end{array}\right.$,
where $i=\sqrt{-1}$. Obviously we have $B_n(x|0)=0$ for $n>2$ and also we can see that $B_n(x|1)=i^nH_n(iy)$, $n\geq 0$. The properties of families of polynomials $\{H_n\}_{n\geq 0}$, $\{P_n\}_{n\geq 0}$, $\{R_n\}_{n\geq 0}$ including met in the literature 'connection coefficient formulae' are be collected in the following Lemma

Lemma 2. *i)* $\forall n\geq 1: P_n(x|y,\rho,q)=\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}\rho^{n-j}B_{n-j}(y|q)H_j(x|q)$,
ii) $\forall n>0:\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}B_{n-j}(x|q)H_j(x|q)=0$,
iii) $\forall n\geq 0: H_n(x|q)=\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}\rho^{n-j}H_{n-j}(y|q)P_j(x|y,\rho,q)$,
iv) $\forall n\geq 0: U_n\left(x\sqrt{1-q}/2\right)=\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j(1-q)^{n/2-j}q^{j(j+1)/2}\begin{bmatrix}n-j\\j\end{bmatrix}_qH_{n-2j}(x|q)$
and $H_n(y|q)=\sum_{k=0}^{\lfloor n/2\rfloor}(1-q)^{-n/2}q^k\left(\begin{bmatrix}n\\k\end{bmatrix}_q-q^{n-2k+1}\begin{bmatrix}n\\k-1\end{bmatrix}_q\right)U_{n-2k}\left(y\sqrt{1-q}/2\right)$
v) $\forall n\geq 1,|\beta|,|\gamma|<1: R_n(x|\gamma,q)=\sum_{k=0}^{\lfloor n/2\rfloor}\beta^k\frac{(\gamma/\beta)_k(\gamma)_{n-k}(q)_n}{(q)_k(q)_{n-2k}(\beta q)_{n-k}}\frac{1-\beta q^{n-2k}}{1-\beta}(1-q)^{-k}R_{n-2k}(x|\beta,q)$,
in particular: $R_n(x|\gamma,q)=\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\gamma^kq^{k(k-1)/2}\frac{(\gamma)_{n-k}(q)_n}{(q)_k(q)_{n-2k}(1-q)^k}H_{n-2k}(x|q)$
and $H_n(x|q)=\sum_{k=0}^{\lfloor n/2\rfloor}\beta^k\frac{(q)_n\left(1-\beta q^{n-2k}\right)}{(1-\beta)(q)_k(q)_{n-2k}(\beta q)_{n-k}(1-q)^k}R_{n-2k}(x|\beta,q)$

Proof. Formulae given in assertions i) and ii) are given in Remark 1 following Theorem 1 in [4]. iii) We start with formula (4.7) in [19] that gives connection coefficients of h_n with respect to p_n. Then we pass to polynomials H_n &

P_n using formulae $h_n(x|q) = (1-q)^{n/2} H_n\left(\frac{2x}{\sqrt{1-q}}|q\right)$, $n \geq 1$ and $p_n(x|a,b,q) = (1-q)^{n/2} P_n\left(\frac{2x}{\sqrt{1-q}}|\frac{2a}{\sqrt{(1-q)b}}, \sqrt{b}, q\right)$. By the way notice that this formula can be easily derived from assertions i) and ii) by standard change of order of summation. iv) Follows 'change of base' formula in continuous $q-$Hermite polynomials (*i.e.* polynomials h_n) in e.g. [20], [22] or [23] (formula 7.2) that states that

$$h_n(x|p) = \sum_{k=0}^{\lfloor n/2 \rfloor} c_{n,n-2k}(p,q) h_{n-2k}(x|q)$$

where

$$\begin{aligned} c_{n,n-2k}(p,q) &= \sum_{j=0}^{k} (-1)^j p^{k-j} q^{j(j+1)/2} \begin{bmatrix} n-2k+j \\ j \end{bmatrix}_q \times \\ &\quad (\begin{bmatrix} n \\ k-j \end{bmatrix}_p - p^{n-2k+2j+1} \begin{bmatrix} n \\ k-j-1 \end{bmatrix}_p) \end{aligned}$$

again expressed for polynomials h_n, next one observes that $h_n(x|0) = U_n(x)$, $\begin{bmatrix} n \\ k \end{bmatrix}_0 = 1$ for $n \geq 0$, $k = 0, \ldots, n$ hence we have $c_{n,n-2k}(0,q) = (-1)^k q^{k(k+1)/2} \begin{bmatrix} n-k \\ k \end{bmatrix}_q$ and consequently $U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k q^{k(k+1)/2} \begin{bmatrix} n-k \\ k \end{bmatrix}_q h_{n-2k}(x|q)$, similarly we get $c_{n,n-2k}(q,0) = q^k(\begin{bmatrix} n \\ k \end{bmatrix}_q - q^{n-2k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q)$ and consequently $h_n(x|q) = \sum_{k=0}^{\lfloor n/2 \rfloor} q^k(\begin{bmatrix} n \\ k \end{bmatrix}_q - q^{n-2k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q) U_{n-2k}(x)$. Now it remains to return to polynomials H_n . v) It is in fact celebrated connection coefficient formula for Rogers polynomials which was in fact expressed in term of polynomials C_n. Other formulae in this assertions are in fact applications of the first formula with $\beta = 0$ in the first case and $\gamma = 0$ in the second and using the fact that $R_n(x|0,q) = H_n(x|q)$. □

We have an important proposition generalizing assertion ii) of the Lemma above. We will use it in the proof of the Lemma 3 below.

Lemma 3. $\forall n \geq 0:$
i)

$$U_n\left(x\sqrt{1-q}/2\right) = \sum_{k=0}^{n} D_{k,n}(y,\rho,q) P_k(x|y,\rho,q), \tag{4.2}$$

where

$$\begin{aligned} D_{k,n}(y,\rho,q) &= \sum_{j=0}^{\lfloor (n-k)/2 \rfloor} (-1)^j (1-q)^{n/2-j} q^{j(j+1)/2} \begin{bmatrix} n-j \\ n-k-j \end{bmatrix} \times \\ &\quad \begin{bmatrix} n-k-j \\ n-k-2j \end{bmatrix} \rho^{n-k-2j} H_{n-k-2j}(y|q). \end{aligned}$$

ii)

$$k_n\left(x\sqrt{1-q}|y\sqrt{1-q},\rho\right) = \sum_{k=0}^{n} C_{k,n}(y,\rho,q) P_k(x|y,\rho,q), \tag{4.3}$$

where

$$C_{k,n}\left(y,\rho,q\right) = \sum_{j=0}^{\lfloor (n-k)/2\rfloor}\left(-1\right)^{j}\left(1-q\right)^{n/2-j}q^{n-k+j(j-3)/2}\begin{bmatrix}n-1-j\\n-k-2j\end{bmatrix}_q\times$$
$$\left(\begin{bmatrix}j+k\\k\end{bmatrix}_q-\rho^2q^k\begin{bmatrix}j+k-1\\k\end{bmatrix}_q\right)\rho^{n-k-2j}H_{n-k-2j}\left(y|q\right)$$

Remark 4. *Notice that* $D_{k,n}\left(y,\rho,q\right)\left(\rho^2\right)_k[k]_q! = \int_{-2/\sqrt{1-q}}^{2/\sqrt{1-q}}U_n\left(x\sqrt{1-q}/2\right)P_k\left(x|y,\rho,q\right)f_{CN}\left(x|y,\rho,q\right)dx$ *and* $C_{k,n}\left(y,\rho,q\right)\left(\rho^2\right)_k[k]_q! = \int_{-2/\sqrt{1-q}}^{2/\sqrt{1-q}}P_n\left(x\sqrt{1-q}|y\sqrt{1-q},\rho,0\right)P_k\left(x|y,\rho,q\right)f_{CN}\left(x|y,\rho,q\right)dx.$

Let us define the following quantity: $[2k-1]_q!! = \begin{cases}1 & if\quad k=0\\ \prod_{i=1}^{k}[2i-1]_q & if\quad k\geq 1\end{cases}$

We have also some interesting corollaries based on the following easy observations following simple induction applied to formulae (3.11), (3.1), assertion 3. of the Proposition 2, (4.1), and (3.2).

Remark 5. *i)* $U_n\left(0\right) = \begin{cases}0 & if\quad n=2k-1\\ (-1)^k & if\quad n=2k\end{cases}, k=1,2,\ldots$

ii) $U_n\left(1\right) = \left(-1\right)^n U_n\left(-1\right) = \left(n+1\right),$

iii) $U_n\left(\frac{1}{2}\right) = \left(-1\right)^{3\lfloor (n+2)/3\rfloor}\left(\left(n+1-3\left\lfloor (n+2)/3\right\rfloor\right)\right),$

iv) $H_n\left(0|q\right) = \begin{cases}0 & if\quad n=2k-1\\ (-1)^k[2k-1]_q!! & if\quad n=2k\end{cases}, k=1,2\ldots,$

$H_n\left(\frac{2}{\sqrt{1-q}}\right) = \frac{W_n(q)}{(1-q)^{n/2}}$, *where* $W_n\left(q\right)$ *by 3.10 and* $n\geq 1,$

v) $k_n\left(0|y,\rho\right) = \begin{cases}(-1)^k\left(1-\rho^2\right) & if\quad n=2k\\ (-1)^{k-1}\rho y & if\quad n=2k-1\end{cases}, k=1,2,\ldots$

$k_n\left(1|y,\rho\right) = \begin{cases}(-1)^k(1-\rho^2) & if\quad n=3k\\ (-1)^{k-1}(-\rho y+\rho^2) & if\quad n=3k-1\\ (-1)^{k-1}\left(1-\rho y\right) & if\quad n=3k-2\end{cases}, k=1,2,\ldots$

vi) $B_n\left(0|q\right) = \begin{cases}0 & if\quad n=2k-1\\ q^{k(k-1)}[2k-1]_q!! & if\quad n=2k\end{cases}, k=1,2,\ldots$

vii) $R_n\left(0,\rho,q\right) = \begin{cases}0 & if\quad n=2k-1\\ (-1)^k\left(\rho^2|q^2\right)_k[2k-1]_q!! & if\quad n=2k\end{cases}.$

Corollary 1. *i)* $\forall q\in(-1,1); n\geq 1: 1-q^{n(n+1)/2} = \sum_{j=0}^{n-1}(1-q)^{n-j}q^{j(j+1)/2}\begin{bmatrix}2n-j\\j\end{bmatrix}_q[2n-2j-1]_q!!$

ii) $\forall q,\rho\in(-1,1); n\geq 1: P_n\left(0|y,\rho,q\right) = \sum_{j=0}^{\lfloor n/2\rfloor}\begin{bmatrix}n\\2j\end{bmatrix}_q\left(-1\right)^j\rho^{n-2j}B_{n-2j}\left(y|q\right)[2j-1]_q!!$

Proof. i) We put $x=0$ in assertion *iv)* of Lemma 2, use assertion iv) of Remark 5, substitute $n->2n$, perform necessary simplifications, we get including fact that: $(1-q)^k[2k-1]_q!! = (q|q^2)_{k-1}$ and $H_0\left(0|q\right)=1$ which leads to conclusion that the summand for $j=n$ is equal to $q^{n(n+1)/2}$.

ii) We put $x=0$ and apply assertion iii) of Lemma 2 and then use iv) of the Remark 5 □

5. Expansions

In this section we are going to apply general idea of expansion presented in section 2, use results of section 4 and obtain expansions of some presented above densities in terms of the others. Since there will many such expansions to formulate all of them in one theorem would lead to clumsy and unclear statement. Instead we device this section unto many subsections entitled by the names of densities that will expanded in its body.

5.1. **f_N and f_U.** Using assertion of iv) of lemma 2 we deduce that coefficients $\gamma_{0,n}$ in expanding f_N is given by $\gamma_{0,n}=\left\{\begin{array}{ccc} 0 & if & n=2k+1, \\ (-1)^k q^{k(k+1)/2} & if & n=2k, \end{array}\right.$ $k=0,1,\ldots$ and we end up with an expansion

$$f_N(x|q)=f_U(x|q)\sum_{k=0}^{\infty}(-1)^k q^{k(k+1)/2}U_{2k}\left(x\sqrt{1-q}/2\right), \tag{5.1}$$

which was obtained and discussed in [18] with a help of so called "triple product identity".

Using another assertion of iv) of lemma 2 we get the inverse of the above expansion. Namely we have

$$\gamma_{0,n}=\left\{\begin{array}{ccc} 0 & if & n=2k+1, \\ (1-q)^{-k}q^k\left(\left[\begin{smallmatrix}2k\\k\end{smallmatrix}\right]_q-q\left[\begin{smallmatrix}2k\\k-1\end{smallmatrix}\right]_q\right) & if & n=2k, \end{array}\right. .$$

Notice that $(1-q)^{-k}q^k\left(\left[\begin{smallmatrix}2k\\k\end{smallmatrix}\right]_q-q\left[\begin{smallmatrix}2k\\k-1\end{smallmatrix}\right]_q\right)/[2k]_q!=\frac{q^k(1-q)^{k+1}}{(q)_k(q)_{k+1}}$. Since we have also 3.9a, we see that we have:

$$f_U(x|q)=f_N(x|q)\sum_{k=0}^{\infty}\frac{q^k(1-q)^{k+1}}{(q)_k(q)_{k+1}}H_{2k}(x|q). \tag{5.2}$$

As corollaries we get the following useful formulae:

$$(q)_\infty\prod_{k=1}^{\infty}\left((1+q^k)^2-(1-q)x^2q^k\right)=\sum_{k=0}^{\infty}(-1)^k q^{k(k+1)/2}U_{2k}\left(x\sqrt{1-q}/2\right),$$

which reduces (after putting $x=0$) to well known

$$(q)_\infty(-q)_\infty^2=(-q)_\infty\left(q^2|q^2\right)_\infty=\sum_{k=0}^{\infty}q^{k(k+1)/2}$$

which is a particular case of 'triple product identity' or to (after putting $x^2(1-q)=4$) to:

$$(q)_\infty^3=\sum_{k=0}^{\infty}(-1)^k(k+1)q^{k(k+1)}$$

and

$$\prod_{k=1}^{\infty}\left((1+q^k)^2-(1-q)x^2q^k\right)^{-1}=\sum_{k=0}^{\infty}\frac{q^k\left(q^{k+1}\right)_\infty(1-q)^k}{(q^2)_k}H_{2k}(x|q),$$

since $(q)_{\infty}/(q)_k = \left(q^{k+1}\right)_{\infty}$ and $(q)_{k+1} = (1-q)\left(q^2\right)_k$ from which we get for example (by setting $x=0$) identity

$$\frac{1}{(q)_{\infty}(-q)_{\infty}^{2}} = 1+\sum_{k=1}^{\infty}(-1)^{k}\frac{q^{k}(1-q)^{k}}{(q)_{k}(q^{2})_{k}}[2k-1]_{q}!!,$$

or (after inserting $x^2(1-q)=4$ and applying assertion iv) of Remark 5):

$$(q)_{\infty}^{-3} = \sum_{k=0}^{\infty}\frac{q^{k}W_{2k}(q)}{(q)_{k}(q)_{k+1}}.$$

5.2. f_N **and** f_{CN}**.** We use assertion i) of lemma 2 we deduce that coefficients $\gamma_{0,n}$ in expanding f_{CN} is given by $\gamma_{0,n}=\rho^n B_n(y|q)$. Keeping in mind (3.9c) we get

$$f_N(x|q) = f_{CN}(x|y,\rho,q)\sum_{n=0}^{\infty}\frac{\rho^n}{(\rho^2)_n[n]_q!}B_n(y|q)P_n(x|y,\rho,q). \tag{5.3}$$

We use assertion iii) of lemma 2 we deduce that coefficients $\gamma_{0,n}$ in expanding f_{CN} is given by $\gamma_{0,n}=\rho^n H_n(y|q)$. Keeping in mind (3.9a) we get:

$$f_{CN}(x|y,\rho,q) = f_N(x|q)\sum_{n=0}^{\infty}\frac{\rho^n}{[n]_q!}H_n(y|q)H_n(x|q). \tag{5.4}$$

Notice that (5.4) is in fact famous Poisson-Mehler kernel of $q-$ Hermite polynomials, while (5.3) is its inverse. Compare [5] for another proof of (5.4). Notice that for every fixed m, $\sum_{n=0}^{m}\frac{\rho^n}{(\rho^2)_n[n]_q!}B_n(y|q)P_n(x|y,\rho,q)$ is not a symmetric function of x and y, while when $m=\infty$ it is!

As a corollary (after putting $y=x$ and then using Remark 2) we get the following interesting expansion

$$\frac{\left(\rho^2\right)_{\infty}}{(\rho)_{\infty}^{2}\prod_{k=0}^{\infty}\left(\left(1+\rho q^k\right)^2-(1-q)\rho x^2 q^k\right)} \tag{5.5a}$$

$$= \sum_{n=0}^{\infty}\frac{\rho^n}{[n]_q!}H_n^2(x|q) \tag{5.5b}$$

, which reduces to well known formula (see [21], Exercise 12.2(b))

$$\frac{\left(\rho^2\right)_{\infty}}{(\rho)_{\infty}^{4}} = \sum_{n=0}^{\infty}\frac{\rho^n}{(q)_n}W_n^2(q),$$

after inserting $x=2/\sqrt{1-q}$ and applying 3.10 or after inserting $x=0$, to

$$\prod_{k=0}^{\infty}\frac{\left(1-\rho^2q^{2k+1}\right)}{\left(1-\rho^2q^{2k}\right)} = 1+\sum_{k=1}^{\infty}\rho^{2k}\prod_{j=1}^{k}\frac{\left(1-q^{2j-1}\right)}{\left(1-q^{2j}\right)},$$

since as it can be easily noticed $\frac{\left([2k-1]_q!!\right)^2}{[2k]_q!} = \prod_{j=1}^{k}\frac{\left(1-q^{2j-1}\right)}{\left(1-q^{2j}\right)}$ and $\frac{\left(\rho^2\right)_{\infty}}{(\rho)_{\infty}^{2}(-\rho^2)_{\infty}^{2}} = \prod_{k=0}^{\infty}\frac{\left(1-\rho^2q^{2k+1}\right)}{\left(1-\rho^2q^{2k}\right)}$.

As far as convergence of series (5.3) and (5.4) is concerned then we see that for $|\rho|,|q|<1$ and $x,y\in S_q$ function $g(x|y,\rho,q) = f_{CN}(x|y,\rho,q)/f_N(x|q) = (\rho_2)_{\infty}\prod_{k=0}^{\infty}\frac{1}{(1-\rho^2q^{2k})^2-(1-q)\rho q^k(1+\rho^2q^{2k})xy+(1-q)\rho^2(x^2+y^2)q^{2k}}$ both bounded and 'cut away from zero' hence its square as well as inverse of this square are integrable on compact interval S_q.

Remark 6. *Dividing both sides of (5.3) and (5.4) by $f_N(x|q)$, passing with $q->1^-$ and keeping in mind that $B_n(x|1) = i^n H_n(ix)$ and that $P_n(x|y,\rho,1) = \left(\sqrt{1-\rho^2}\right)^n H_n\left(\frac{(x-\rho y)}{\sqrt{1-\rho^2}}\right)$ we get:*

$$1/\sum_{n=0}^{\infty}\frac{\rho^n}{n!}H_n(x)H_n(y) = \sum_{n=0}^{\infty}\frac{\rho^n i^n}{n!(1-\rho^2)^{n/2}}H_n(ix)H_n\left(\frac{(x-\rho y)}{\sqrt{1-\rho^2}}\right) \tag{5.6}$$

Here however situation is different. Series $\sum_{n=0}^{\infty}\frac{\rho^n}{n!}H_n(x)H_n(y)$, as it is known, is convergent for all $x,y\in\mathbb{R}$ and $|\rho|<1$, while series (5.6) only for $x,y\in\mathbb{R}$ and $\rho^2<1/2$ since then function $f_N^2(x|q)/f_{CN}(x|y,\rho,q) = \exp(-\frac{(x-\rho y)^2}{2(1-\rho^2)}+x^2)$ is integrable with respect x over whole $\mathbb{R}$.

5.3. **f_N and f_R.** We use last two statements of assertion v) of lemma 2 we deduce that coefficients $\gamma_{0,n}$ in expanding f_R is given by

$$\gamma_{0,n} = \begin{cases} 0 & if \quad n=2k+1, \\ \frac{\beta^k (q)_{2k}}{(q)_k(\beta q)_k(1-q)^k} & if \quad n=2k, \end{cases},$$

$k=0,1,\ldots$. Keeping in mid (3.9a) we get:

$$f_R(x|\beta,q) = f_N(x|q)\sum_{k=0}^{\infty}\frac{\beta^k}{[k]_q!(\beta q)_k}H_{2k}(x|q). \tag{5.7}$$

As a corollary let us take $\beta=\rho$ and use (2.8) and compare it with (5.5). We will get then for $|q|,|\rho|<1, x^2(1-q)\le 2$:

$$(1-\rho)\sum_{n=0}^{\infty}\frac{\rho^n}{[n]_q!}H_n^2(x) = \sum_{n=0}^{\infty}\frac{\rho^n}{[n]_q!(\rho q)_n}H_{2n}(x|q).$$

Next we use second assertion of v) of lemma 2 and deduce that coefficient $\gamma_{0,n}$ in expanding f_N is by

$$\gamma_{0,n} = \begin{cases} 0 & if \quad n=2k+1, \\ (-\gamma)^k q^{k(k-1)/2}\frac{(\gamma)_k(q)_{2k}}{(q)_k(1-q)^k} & if \quad n=2k, \end{cases},$$

$k=0,1,\ldots$. We use also (3.9d) and get

$$f_N(x|q) = f_R(x|\gamma,q)\sum_{k=0}^{\infty}(-\gamma)^k q^{k(k-1)/2}\frac{(\gamma)_k\left(1-\gamma q^{2k}\right)}{(1-\gamma)[k]_q!(\gamma^2)_{2k}}R_{2k}(x|\gamma,q). \tag{5.8}$$

Again we can deduce that one of the series is the inverse of the other.

5.4. **f_K and f_{CN}..** We will be using assertion ii) of Lemma 3, Remark 3, and the fact that for $n\ge 1$:

$$\begin{aligned}&\int_{-2/\sqrt{1-q}}^{2/\sqrt{1-q}} f_K(\xi|y,\rho,q)\,k_n^2\left(\xi\sqrt{1-q}|y\sqrt{1-q},\rho\right)d\xi \\ =& \frac{1}{\sqrt{1-q}}\int_{-2}^{2} f_K\left(x/\sqrt{1-q}|y/\sqrt{1-q},\rho,0\right)k_n^2(x|y,\rho)\,dx = \frac{(1-\rho^2)}{\sqrt{1-q}}.\end{aligned}$$

Beside notice that $C_{0,1}(y,\rho,q)=1$. Hence $\beta_1(y,\rho,q)=0$. Consequently we get $\forall x\in<\frac{-2}{\sqrt{1-q}},\frac{2}{\sqrt{1-q}}>;y\in<\frac{-2}{\sqrt{1-q}},\frac{2}{\sqrt{1-q}}>;0<|\rho|<1;q\in(-1,1)$

$$f_{CN}(x|y,\rho,q)=f_K(x|\rho,q)(1+\sum_{n=2}^{\infty}\beta_n(y,\rho,q)k_n(x\sqrt{1-q}|y\sqrt{1-q},\rho))),$$

where $\beta_k(y,\rho,q)=\sum_{j=1}^{\lfloor k/2\rfloor}(-1)^j(1-q)^{k/2-j}q^{k+j(j-3)/2}\left[\begin{smallmatrix}k-1-j\\k-2j\end{smallmatrix}\right]\rho^{k-2j}H_{k-2j}(y|q)$.

5.5. f_U **and** f_{CN}. Using assertion i) of Lemma 3 and calculating in the similar way we get: $\forall x\in<\frac{-2}{\sqrt{1-q}},\frac{2}{\sqrt{1-q}}>;y\in<\frac{-2}{\sqrt{1-q}},\frac{2}{\sqrt{1-q}}>;0<|\rho|<1;q\in(-1,1)$,

$$f_{CN}(x|y,\rho,q)=f_U(x|q)(1+\sum_{k=1}^{\infty}\gamma_k(y,\rho,q)U_k\left(x\sqrt{1-q}/2\right),\tag{5.9}$$

$\gamma_k(y,\rho,q)=\sum_{j=0}^{\lfloor k/2\rfloor}(-1)^j(1-q)^{k/2-j}\times q^{j(j+1)/2}\left[\begin{smallmatrix}k-j\\k-2j\end{smallmatrix}\right]_q\rho^{k-2j}H_{k-2j}(y|q)$.

Corollary 2.

$$\begin{aligned}&\left(q^3|q^3\right)_\infty\sum_{k=0}^{\infty}\frac{(1-q)^{k/2}\rho^k}{(q)_k}H_k(y|q)\eta_k(q)\\=&\frac{\left(\rho^2\right)_\infty\left(q^3|q^3\right)_\infty}{\prod_{k=0}^{\infty}\left(1+\rho^2q^{2k}+\rho^4q^{4k}-\sqrt{1-q}\rho yq^k(1+\rho^2q^{2k})+(1-q)\rho^2y^2q^{2k}\right)}\\=&1+\sum_{k=1}^{\infty}(-1)^{3k}\left(\gamma_{3k}(y,\rho,q)+\gamma_{3k+1}(y,\rho,q)\right),\end{aligned}$$

where $\{\eta_k(q)\}_{k\geq-1}$ *are given recursively* $\eta_{-1}(q)=0,\eta_0(q)=1$, $\eta_{k+1}(q)=\eta_k(q)-\left(1-q^k\right)\eta_{k-1}(q)$, $k\geq0$.

Proof. We insert $x=1/\sqrt{1-q}$ in (5.9) and use assertion iii of Remark 5 which simplifies to simple rule $U_{3m+2}(1/2)=0,U_{3m}(1/2)=U_{3m+1}(1/2)(-1)^{3m}$. Then we insert $x=1/\sqrt{1-q}$ in (2.6) and (3.13) and use the fact that $(1-\rho^2q^{2k})^2+\rho^2q^{2k}=1+\rho^2q^{2k}+\rho^4q^{4k}$. On the way we also use (3.16), identity $(q)_\infty\prod_{k=1}^{\infty}\left(1+q^k+q^{2k}\right)=\prod_{k=1}^{\infty}\left(1-q^{3k}\right)=\left(q^3|q^3\right)_\infty$, the fact $(1-q)^{k/2}H_k\left(\frac{1}{\sqrt{1-q}}\right)=h_k(1/2)$ and the fact the $q-$Hermite polynomials $h_n(x|q)$ satisfy relationship: $h_{n+1}(x|q)=2xh_n(x|q)-(1-q^n)h_{n-1}(x|q)$. □

6. Proofs

Proof of the Proposition 1. i) Notice that ϕ_n is monic polynomial of degree n, for $n\geq1$. Now let us calculate $\int_{\mathbb{R}}\phi_n(x)B(x)dx$. We have :

$$\begin{aligned}\int_{\mathbb{R}}\phi_n(x)B(x)dx&=\sum_{i=0}^{n}\sum_{j=0}^{N}f_{n-i}\frac{w_j}{\hat{a}_j}\int_{\mathbb{R}}a_i(x)a_j(x)A(x)dx\\&=\sum_{i=0}^{n\wedge m}f_{n-i}w_i=0\end{aligned}$$

for $n\geq1$. Conversely, lest us consider polynomial defined by $p_n(x)=\sum_{i=0}^{n}w_{n-i}\phi_i(x)$. We have $p_n(x)=\sum_{i=0}^{n}w_{n-i}\sum_{j=0}^{i}f_{i-j}a_j(x)=\sum_{j=0}^{n}a_j(x)\sum_{i=j}^{n}w_{n-i}f_{i-j}=\sum_{j=0}^{n}a_j(x)\sum_{k=0}^{n-j}w_{n-j-k}f_k=\sum_{j=0}^{n}a_j(x)\sum_{s=0}^{n-j}w_sf_{n-j-s}=a_n(x)$.

ii) Let us define coefficients $c_{n,i}$ by the following expansion:

$$a_n(x) = \sum_{i=0}^{n} c_{n,i} b_i(x),$$

The fact that $\{a_n\}$ and $\{b_n\}$ are monic implies that $\forall n \geq 0 : c_{n,n} = 1$. ii) implies that $c_{i,0} = w_i$, $i \leq n; c_{n,0} = 0$ for $n \geq N+1$. Besides we have the following relationships between coefficients $c_{n,i}$ that is implied by $3-$terms recurrences satisfied by families $\{a_i\}$ and $\{b_i\}$. On one hand we have $xa_n(x) = a_{n+1} + \alpha_n a_n + \hat{\alpha}_n a_{n-1} = b_{n+1} + (\alpha_n + c_{n+1,n})b_n + \sum_{i=0}^{n-1}(c_{n+1,i} + \alpha_n c_{n,i} + \hat{\alpha}_n c_{n-1,i}) b_i$ on the other $xa_n(x)$ $= \sum_{i=0}^{n} c_{n,i}(b_{i+1} + \beta_i b_i + \hat{\beta}_i b_{i-1}) = b_{n+1} + (c_{n,n-1} + \beta_n)b_n + \sum_{i=1}^{n-1}(c_{n,i-1} + \beta_i c_{n,i}$ $+ \hat{\beta}_i c_{n,i+1}) + \beta_0 c_{n,0} + \hat{\beta}_1 c_{n,1}$. Equating these two sides we get:

$$\alpha_n + c_{n+1,n} = c_{n,n-1} + \beta_n,$$
$$\forall 1 \leq i \leq n-1 : c_{n+1,i} + \alpha_n c_{n,i} + \hat{\alpha}_n c_{n-1,i} = c_{n,i-1} + \beta_i c_{n,i} + \hat{\beta}_i c_{n,i+1},$$
$$c_{n+1,0} + \alpha_n c_{n,0} + \hat{\alpha}_n c_{n-1,0} = \beta_0 c_{n,0} + \hat{\beta}_1 c_{n,1}.$$

From the last of these equations we deduce that $c_{n,1} = 0$ for $n \geq N+2$. Similarly by considering equation

$$c_{n+1,1} + \alpha_n c_{n,1} + \hat{\alpha}_n c_{n-1,1} = c_{n,0} + \beta_i c_{n,1} + \hat{\beta}_i c_{n,2}$$

we deduce that $c_{n,2} = 0$ for $n \geq N+3$ and so on. We see that then $c_{n,i} = 0$ for $n \geq N+i+1$. In particular it means that $c_{n,n-j} = 0$ for $j \geq N+1$. □

Proof. of Lemma 3. i) We will argue straightforwardly using formulae from assertions *i*) and *ii*) of Lemma 2 and then comparing it with assertion iv) of the same lemma.

We have
$\sum_{k=0}^{n} D_{k,n}(y,\rho,q) P_k(x|y,\rho,q) = \sum_{k=0}^{n} D_{k,n}(y,\rho,q) \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q \rho^{k-i} B_{k-i}(y|q) H_i(x|q)$
$=$
$\sum_{i=0}^{n} H_i(x|q) \sum_{k=i}^{n} \begin{bmatrix} k \\ i \end{bmatrix}_q D_{k,n}(y,\rho,q) B_{k-i}(y|q)$. Let us denote
$G_{i,n}(y,\rho,q) = \sum_{k=i}^{n} \begin{bmatrix} k \\ i \end{bmatrix}_q D_{k,n}(y,\rho,q) B_{k-i}(y|q)$. We have using formula $D_{k,n}(y,\rho,q)$.

$$\begin{aligned} G_{i,n}(y,\rho,q) &= \sum_{k=i}^{n} \begin{bmatrix} k \\ i \end{bmatrix}_q \rho^{k-i} B_{k-i}(y|q) \times D_{k,n}(y,\rho,q) \\ &= \sum_{k=i}^{n} \begin{bmatrix} k \\ i \end{bmatrix}_q \rho^{k-i} B_{k-i}(y|q) \sum_{j=0}^{\lfloor (n-k)/2 \rfloor} (-1)^j (1-q)^{n/2-j} q^{j(j+1)/2} \begin{bmatrix} n-j \\ n-k-j \end{bmatrix} \\ &\quad \times \begin{bmatrix} n-k-j \\ n-k-2j \end{bmatrix} \rho^{n-k-2j} H_{n-k-2j}(y|q) \\ &= \sum_{j=0}^{\lfloor (n-i)/2 \rfloor} (-1)^j (1-q)^{n/2-j} q^{j(j+1)/2} \rho^{n-i-2j} \begin{bmatrix} n-j \\ j \end{bmatrix}_q \begin{bmatrix} n-2j \\ i \end{bmatrix}_q \times \\ &\quad \sum_{k=i}^{n-2j} \begin{bmatrix} n-i-2j \\ k-i \end{bmatrix}_q B_{k-i}(y|q) H_{n-k-2j}(y|q). \end{aligned}$$

Now $\sum_{k=i}^{n-2j}\begin{bmatrix}n-i-2j\\k-i\end{bmatrix}_qB_{k-i}\left(y|q\right)H_{n-k-2j}\left(y|q\right)=$ $\sum_{s=0}^{n-i-2j}\begin{bmatrix}n-i-2j\\s\end{bmatrix}_qB_s\left(y|q\right)H_{n-i-2j-s}\left(y|q\right)=\left\{\begin{array}{ccc}1 & if & n-i=2j\\0 & if & n-i>2j\end{array}\right.$ by assertion ii) of Lemma 2.

Hence $G_{i,n}\left(y,\rho,q\right)=\left\{\begin{array}{cc}0 & if\quad n-i \text{ is odd}\\(-1)^m(1-q)^{n/2-m}q^{m(m+1)/2}\begin{bmatrix}n-m\\m\end{bmatrix}_q & if\quad n-i=2m\end{array}\right.$.

So $\sum_{k=0}^{n}D_{k,n}\left(y,\rho,q\right)P_k\left(x|y,\rho,q\right)=\sum_{i=0}^{n}H_i\left(x|q\right)G_{i,n}\left(y,\rho,q\right)=\sum_{m=0}^{\lfloor n/2\rfloor}(-1)^m(1-q)^{n/2-m}q^{m(m+1)/2}\begin{bmatrix}n-m\\m\end{bmatrix}_qH_{n-2m}\left(x|q\right)=U\left(x\sqrt{1-q}/2\right)$ by assertion iv) of Lemma 2.

ii) Notice that $C_{0,0}\left(y,\rho,q\right)=1$, $C_{n,n}\left(y,\rho,q\right)=(1-q)^{n/2}$, $C_{0,n}\left(y,\rho,q\right)=(1-\rho^2)\times\sum_{j=1}^{\lfloor n/2\rfloor}(-1)^j(1-q)^{n/2-j}q^{n+j(j-3)/2}\times\begin{bmatrix}n-1-j\\j\end{bmatrix}_q\rho^{n-2j}H_{n-2j}\left(y|q\right)$, $C_{n-1,n}\left(y,\rho,q\right)=\left(1-q\right)^{n/2}q\rho y\left[n-1\right]_q$. Hence $C_{0,1}(y,\rho,q)=0$ and $C_{1,1}(y,\rho,q)=(1-q)^{1/2}$, $C_{0,2}\left(y,\rho,q\right)=-(1-\rho^2)$, $C_{1,2}\left(y,\rho,q\right)=(1-q)q\rho y$. Thus equation (4.3) is satisfied for $n=0,1,2$. For larger n formula will be proved straightforwardly. Let us consider expression $W_n\left(x|y,\rho,q\right)=\sum_{k=0}^{n}C_{k,n}\left(y,\rho,q\right)P_k\left(x|y,\rho,q\right)$. We have

$$\begin{aligned}W_n\left(x|y,\rho,q\right) &= \sum_{k=0}^{n}P_k\left(x|y,\rho,q\right)\sum_{j=0}^{\lfloor(n-k)/2\rfloor}\left(-1\right)^j\left(1-q\right)^{n/2-j}q^{n-k+j(j-3)/2}\begin{bmatrix}n-1-j\\n-k-2j\end{bmatrix}_q\\ &\times\left(\begin{bmatrix}j+k\\k\end{bmatrix}_q-\rho^2q^k\begin{bmatrix}j+k-1\\k\end{bmatrix}_q\right)\rho^{n-k-2j}H_{n-k-2j}\left(y|q\right)\\ &= \sum_{j=0}^{\lfloor n/2\rfloor}\left(-1\right)^j\left(1-q\right)^{n/2-j}q^{j(j+1)/2}\sum_{k=0}^{n-2j}\begin{bmatrix}n-1-j\\n-k-2j\end{bmatrix}_q\\ &\times\left(\begin{bmatrix}j+k\\k\end{bmatrix}_q-\rho^2q^k\begin{bmatrix}j+k-1\\k\end{bmatrix}_q\right)\rho^{n-k-2j}H_{n-k-2j}\left(y|q\right)P_k\left(x|y,\rho,q\right)\end{aligned}$$

Now $n-k+j\left(j-3\right)/2=j\left(j+1\right)/2+n-k-2j$, $\begin{bmatrix}n-1-j\\n-k-2j\end{bmatrix}_q\begin{bmatrix}j+k\\k\end{bmatrix}_q=\frac{[n-1-j]_q![j+k]_q}{[n-k-2j]_q![k]_q![j]_q!}$ $=\frac{[j+k]_q}{[n-j]_q}\begin{bmatrix}n-j\\j\end{bmatrix}_q\begin{bmatrix}n-2j\\k\end{bmatrix}_q$ and $\begin{bmatrix}n-1-j\\n-k-2j\end{bmatrix}_q\begin{bmatrix}j+k-1\\k\end{bmatrix}_q=\frac{[n-1-j]_q!}{[n-k-2j]_q![k]_q![j-1]_q!}=\begin{bmatrix}n-1-j\\j-1\end{bmatrix}_q\begin{bmatrix}n-2j\\k\end{bmatrix}_q$, hence

$$\begin{aligned}W_n\left(x|y,\rho,q\right) &= \sum_{j=0}^{\lfloor n/2\rfloor}\left(-1\right)^j\left(1-q\right)^{n/2-j}q^{j(j+1)/2}\frac{1}{\left[n-j\right]_q}\begin{bmatrix}n-j\\j\end{bmatrix}_q\\ &\times\sum_{k=0}^{n-2j}\begin{bmatrix}n-2j\\k\end{bmatrix}_q q^{n-k-2j}\rho^{n-k-2j}H_{n-k-2j}\left(y|q\right)P_k\left(x|y,\rho,q\right)\\ &-\rho^2\sum_{j=1}^{\lfloor n/2\rfloor}\left(-1\right)^j\left(1-q\right)^{n/2-j}q^{j(j+1)/2}q^{n-2j}\begin{bmatrix}n-1-j\\j-1\end{bmatrix}_q\\ &\times\sum_{k=0}^{n-2j}\begin{bmatrix}n-2j\\k\end{bmatrix}_q\rho^{n-k-2j}H_{n-k-2j}\left(y|q\right)P_k\left(x|y,\rho,q\right).\end{aligned}$$

Now we apply assertion iii) of Lemma 2 and also the simple fact that $q^{n-k-2j}[k+j]_q = [n-j]_q - [n-k-2j]_q$ and get after applying assertion iv) of Lemma 2

$$\begin{aligned} W_n(x|y,\rho,q) &= U_n\left(x\sqrt{1-q}/2\right) - \sum_{j=0}^{\lfloor n/2\rfloor} (-1)^j (1-q)^{n/2-j} q^{j(j+1)/2} \frac{[n-2j]_q}{[n-j]_q} \begin{bmatrix} n-j \\ j \end{bmatrix}_q \\ &\times \sum_{k=0}^{n-2j-1} \begin{bmatrix} n-2j-1 \\ k \end{bmatrix}_q \rho^{n-2j-k} H_{n-k-2j}(y|q) P_k(x|y,\rho,q) \\ &-\rho^2 \sum_{j=1}^{\lfloor n/2\rfloor} (-1)^j (1-q)^{n/2-j} q^{j(j+1)/2} q^{n-2j} \begin{bmatrix} n-1-j \\ j-1 \end{bmatrix}_q H_{n-2j}(x|q). \end{aligned}$$

Now we apply formula $H_{n-k-2j}(y|q) = yH_{n-1-k-2j}(y|q) - [n-1-2j-k]_q H_{n-2-2j-k}(y|q)$ and split first sum into two. Since $\frac{[n-2j]_q}{[n-j]_q}\begin{bmatrix} n-j \\ j \end{bmatrix}_q = \begin{bmatrix} n-1-j \\ j \end{bmatrix}_q$ we see that the first of these two sums is equal to $\rho\sqrt{1-q}yU_{n-1}\left(x\sqrt{1-q}/2\right)$. Hence

$$\begin{aligned} W_n(x|y,\rho,q) &= U_n\left(x\sqrt{1-q}/2\right) - \rho\sqrt{1-q}yU_{n-1}\left(x\sqrt{1-q}/2\right) \\ &+ \sum_{j=0}^{\lfloor n/2\rfloor} (-1)^j (1-q)^{n/2-j} q^{j(j+1)/2} \frac{[n-2j]_q}{[n-j]_q} \begin{bmatrix} n-j \\ j \end{bmatrix}_q \\ &\times \sum_{k=0}^{n-2j-1} \begin{bmatrix} n-2j-1 \\ k \end{bmatrix}_q [n-1-k-2j]_q \rho^{n-2j-k} H_{n-2-k-2j}(y|q) P_k(x|y,\rho,q) \\ &+\rho^2 \sum_{j=0}^{\lfloor n/2-1\rfloor} (-1)^j (1-q)^{n/2-1-j} q^{j(j+1)/2} q^{n-j-1} \begin{bmatrix} n-2-j \\ j \end{bmatrix}_q H_{n-2-2j}(x|q). \end{aligned}$$

Notice that $\sum_{k=0}^{n-2j-1}\begin{bmatrix} n-2j-1 \\ k \end{bmatrix}_q [n-1-k-2j]_q \rho^{n-2j-k} H_{n-2-k-2j}(y|q) P_k(x|y,\rho,q) = [n-1-2j]_q \rho^2 H_{n-2-2j}(x|q)$ by assertion iii) of Lemma 2. Besides $\frac{[n-2j]_q}{[n-j]_q}\begin{bmatrix} n-j \\ j \end{bmatrix}_q = \begin{bmatrix} n-1-j \\ j \end{bmatrix}_q$. Thus sum of the last two summands is equal to
$\rho^2(1-q)\sum_{j=0}^{\lfloor n/2\rfloor -1} (-1)^j (1-q)^{n/2-1-j} q^{j(j+1)/2} \begin{bmatrix} n-1-j \\ j \end{bmatrix}_q [n-1-2j]_q H_{n-2-2j}(x|q) + \rho^2 \sum_{j=0}^{\lfloor n/2-1\rfloor} (-1)^j (1-q)^{n/2-1-j} q^{j(j+1)/2} q^{n-j-1} \begin{bmatrix} n-2-j \\ j \end{bmatrix}_q H_{n-2-2j}(x|q)$. Now $\begin{bmatrix} n-1-j \\ j \end{bmatrix}_q [n-1-2j]_q = [n-1-j]_q \begin{bmatrix} n-2-2j \\ j \end{bmatrix}_q$ and $(1-q)[n-1-j] = 1-q^{n-1-j}$, hence the sum of last two summands is equal to
$\rho^2 \sum_{j=0}^{\lfloor n/2-1\rfloor} (-1)^j (1-q)^{n/2-1-j} q^{j(j+1)/2} \begin{bmatrix} n-2-j \\ j \end{bmatrix}_q H_{n-2-2j}(x|q) = \rho^2 U_{n-2}\left(x\sqrt{1-q}/2\right)$ by assertion iv) of Lemma 2. □

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