

# CPHD filter derivation for extended targets

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In this document, an attempt to derive a cardinalized probability hypothesis density filter (CPHD) (corrector formula only) for extended targets is given.

## 1 Derivation

We first give some quantities related to the derivation.

- The extended target measurements are distributed according to an iid cluster process. The corresponding likelihood is given as

$$f(Z|x) = n!P_z(n|x) \prod_{z \in Z} p_z(z|x) \quad (1)$$

- The false alarms are distributed according to an iid cluster process also.

$$f(Z_{FA}) = n!P_{FA}(n) \prod_{z \in Z_{FA}} p_{FA}(z) \quad (2)$$

- The multitarget prior  $f(X_k|Z_{0:k-1})$  is assumed to be an iid cluster process.

$$f(X_k|Z_{0:k-1}) = n!P_{k+1|k}(n) \prod_{x_k \in X_k} p_{k+1|k}(x_k) \quad (3)$$

where

$$p_{k+1|k}(x_k) \triangleq N_{k|k-1}^{-1} D_{k|k-1}(x_k) \quad (4)$$

with  $N_{k|k-1} \triangleq \int D_{k|k-1}(x_k) dx_k$ .

The p.g.fl corresponding to the updated multitarget density  $f(X_k|Z_k)$  is then given as

$$G_{k|k}[h] = \frac{\frac{\delta}{\delta Z_k} F[0, h]}{\frac{\delta}{\delta Z_k} F[0, 1]} \quad (5)$$

where

$$F[g, h] \triangleq \int h^X G[g|X] f(X|Z_{k-1}) \delta X \quad (6)$$

$$G[g|X] \triangleq \int g^Z f(Z|X) \delta Z \quad (7)$$

- Calculation of  $G[g|X]$ : Suppose given the target states  $X$  the measurement sets corresponding to different targets are independent. Assume that targets are detected with probabilities  $p_D(x)$ . Then the p.g.fl for a single targets measurements become

$$G[g|x] = 1 - p_D(x) + p_D(x)G_Z[g|x] \quad (8)$$

Then p.g.fl for the measurements belonging to all targets become

$$G[g|X] = (1 - p_D(x) + p_D(x)G_Z[g|x])^X \quad (9)$$

With the addition of false alarms we have

$$G[g|X] = G_{FA}[g](1 - p_D(x) + p_D(x)G_Z[g|x])^X \quad (10)$$

- Calculation of  $F[g, h]$ : substituting  $G[g|X]$  above into the definition of  $F[g, h]$  we get

$$F[g, h] = \int h^X G_{FA}[g](1 - p_D(x) + p_D(x)G_Z[g|x])^X f(X|Z_{k-1})\delta X \quad (11)$$

$$= G_{FA}[g] \int (h(1 - p_D(x) + p_D(x)G_Z[g|x]))^X f(X|Z_{k-1})\delta X \quad (12)$$

$$= G_{FA}[g]G_{k|k-1}[h(1 - p_D(x) + p_D(x)G_Z[g|x])] \quad (13)$$

$$= G_{FA}[g]G_{k|k-1}[h(1 - p_D + p_D G_Z[g])] \quad (14)$$

where we omitted the arguments  $x$  of the functions.

For the iid cluster processes, we know the following identities

$$G_{FA}[g] = G_{FA}(p_{FA}[g]) \quad (15)$$

$$G_Z[g|x] = G_Z(p_z[g]|x) \quad (16)$$

where the functions  $G_{FA}(\cdot)$  and  $G_Z(\cdot|x)$  are the probability generating functions for the cardinality distributions  $P_{FA}(\cdot)$  and  $P_z(\cdot|x)$  respectively. We also know the following derivative expressions.

$$\frac{\delta}{\delta Z} G[g] = G^{(n)}(p[g]) \prod_{z \in Z} p(z) \quad (17)$$

Now taking the derivatives of  $F[g, h]$  with respect to  $Z$ .

$$\begin{aligned} \frac{\delta}{\delta z_1} F[g, h] &= G_{FA}^{(1)}(p_{FA}[g])p_{FA}(z_1)G_{k|k-1}\left(p_{k|k-1}[h(1 - p_D + p_D G_Z(p_z[g]))]\right) \\ &\quad + G_{FA}(p_{FA}[g])G_{k|k-1}^{(1)}\left(p_{k|k-1}[h(1 - p_D + p_D G_Z(p_z[g]))]\right)p_{k|k-1}[hp_D G_Z^{(1)}(p_z[g])p_z(z_1)] \\ &= F[g, h]p_{FA}(z_1)\left(\frac{G_{FA}^{(1)}(p_{FA}[g])}{G_{FA}(p_{FA}[g])}\right) \\ &\quad + \frac{G_{k|k-1}^{(1)}\left(p_{k|k-1}[h(1 - p_D + p_D G_Z(p_z[g]))]\right)}{G_{k|k-1}\left(p_{k|k-1}[h(1 - p_D + p_D G_Z(p_z[g]))]\right)}p_{k|k-1}\left[hp_D G_Z^{(1)}(p_z[g])\frac{p_z(z_1)}{p_{FA}(z_1)}\right] \\ &= F[g, h]p_{FA}(z_1)\left(\zeta_{FA}^{(1)}(p_{FA}[g])\right) \\ &\quad + \zeta_{k|k-1}^{(1)}\left(p_{k|k-1}[h(1 - p_D + p_D G_Z(p_z[g]))]\right)p_{k|k-1}\left[hp_D G_Z^{(1)}(p_z[g])\frac{p_z(z_1)}{p_{FA}(z_1)}\right] \end{aligned} \quad (18)$$

where we use the functions  $\zeta_{FA}^{(i)}(\cdot)$  and  $\zeta_{k|k-1}^{(i)}(\cdot)$  defined as

$$\zeta_{FA}^{(i)}(x) \triangleq \left(\frac{G_{FA}^{(1)}(x)}{G_{FA}(x)}\right)^{(i-1)} = \log^{(i)} G_{FA}(x) \quad (19)$$

$$\zeta_{k|k-1}^{(i)}(x) \triangleq \left(\frac{G_{k|k-1}^{(1)}(x)}{G_{k|k-1}(x)}\right)^{(i-1)} = \log^{(i)} G_{k|k-1}(x) \quad (20)$$

The second order derivative is given as

$$\begin{aligned}
\frac{\delta}{\delta z_1 \delta z_2} F[g, h] = & F[g, h] \left( \prod_{z'=z_1, z_2} p_{FA}(z') \right) \left( \prod_{z'=z_1, z_2} \left( \zeta_{FA}^{(1)}(p_{FA}[g]) \right. \right. \\
& + \zeta_{k|k-1}^{(1)} \left( p_{k|k-1} [h(1 - p_D + p_D G_Z(p_z[g]))] \right) p_{k|k-1} \left[ h p_D G_Z^{(1)}(p_z[g]) \frac{p_z(z')}{p_{FA}(z')} \right] \\
& + \zeta_{FA}^{(2)}(p_{FA}[g]) + \zeta_{k|k-1}^{(2)} \left( p_{k|k-1} [h(1 - p_D + p_D G_Z(p_z[g]))] \right) \\
& \times \prod_{z'=z_1, z_2} p_{k|k-1} \left[ h p_D G_Z^{(1)}(p_z[g]) \frac{p_z(z')}{p_{FA}(z')} \right] \\
& \left. \left. + \zeta_{k|k-1}^{(1)} \left( p_{k|k-1} [h(1 - p_D + p_D G_Z(p_z[g]))] \right) p_{k|k-1} \left[ h p_D G_Z^{(2)}(p_z[g]) \prod_{z'=z_1, z_2} \frac{p_z(z')}{p_{FA}(z')} \right] \right) \right) \quad (21)
\end{aligned}$$

Then we can write the general formula as

$$\begin{aligned}
\frac{\delta}{\delta Z} F[g, h] = & F[g, h] \left( \prod_{z' \in Z} p_{FA}(z') \right) \sum_{\mathcal{P} \triangleleft Z} \prod_{W \in \mathcal{P}} \left( \zeta_{FA}^{(|W|)}(p_{FA}[g]) \right. \\
& \left. + \sum_{\mathcal{Q} \triangleleft W} \zeta_{k|k-1}^{(|\mathcal{Q}|)} \left( p_{k|k-1} [h(1 - p_D + p_D G_Z(p_z[g]))] \right) \prod_{V \in \mathcal{Q}} \eta_V[g, h] \right) \quad (22)
\end{aligned}$$

where

$$\eta_V[g, h] \triangleq p_{k|k-1} \left[ h p_D G_Z^{(|V|)}(p_z[g]) \prod_{z' \in V} \frac{p_z(z')}{p_{FA}(z')} \right] \quad (23)$$

A proof of (22) can be found in Section 6. Setting  $g = 0$  and taking derivative with respect to  $x$ .

$$\begin{aligned}
\frac{\delta}{\delta x} \frac{\delta}{\delta Z} F[0, h] = & \frac{\delta}{\delta Z} F[0, h] \zeta_{k|k-1}^{(1)} \left( p_{k|k-1} [h(1 - p_D + p_D G_Z(0))] \right) \\
& \times (1 - p_D(x) + p_D(x) G_Z(0)) p_{k|k-1}(x) \\
& + F[0, h] \left( \prod_{z' \in Z} p_{FA}(z') \right) \sum_{\mathcal{P} \triangleleft Z} \left( \prod_{W \in \mathcal{P}} \left( \zeta_{FA}^{(|W|)}(0) \right. \right. \\
& \left. \left. + \sum_{\mathcal{Q} \triangleleft W} \zeta_{k|k-1}^{(|\mathcal{Q}|)} \left( p_{k|k-1} [h(1 - p_D + p_D G_Z(0))] \right) \prod_{V \in \mathcal{Q}} \eta_V[0, h] \right) \right) \\
& \times \sum_{W \in \mathcal{P}} \frac{\frac{\delta}{\delta x} \sum_{\mathcal{Q} \triangleleft W} \zeta_{k|k-1}^{(|\mathcal{Q}|)} \left( p_{k|k-1} [h(1 - p_D + p_D G_Z(0))] \right) \prod_{V \in \mathcal{Q}} \eta_V[0, h]}{\left( \zeta_{FA}^{(|W|)}(0) + \sum_{\mathcal{Q} \triangleleft W} \zeta_{k|k-1}^{(|\mathcal{Q}|)} \left( p_{k|k-1} [h(1 - p_D + p_D G_Z(0))] \right) \prod_{V \in \mathcal{Q}} \eta_V[0, h] \right)} \quad (24) \\
= & \frac{\delta}{\delta Z} F[0, h] \zeta_{k|k-1}^{(1)} \left( p_{k|k-1} [h(1 - p_D + p_D G_Z(0))] \right) \\
& \times (1 - p_D(x) + p_D(x) G_Z(0)) p_{k|k-1}(x) \\
& + F[0, h] \left( \prod_{z' \in Z} p_{FA}(z') \right) \sum_{\mathcal{P} \triangleleft Z} \left( \prod_{W \in \mathcal{P}} \left( \zeta_{FA}^{(|W|)}(0) \right. \right. \\
& \left. \left. + \sum_{\mathcal{Q} \triangleleft W} \zeta_{k|k-1}^{(|\mathcal{Q}|)} \left( p_{k|k-1} [h(1 - p_D + p_D G_Z(0))] \right) \prod_{V \in \mathcal{Q}} \eta_V[0, h] \right) \right)
\end{aligned}$$

$$\times \sum_{W \in \mathcal{P}} \frac{\left( \sum_{Q \subseteq W} \left( \prod_{V \in Q} \eta_V[0, h] \right) \left( \zeta_{k|k-1}^{(|Q|+1)} \left( p_{k|k-1} [h(1-p_D + p_D G_Z(0))] \right) \right) \right.}{\left( \zeta_{FA}^{(|W|)}(0) + \sum_{Q \subseteq W} \zeta_{k|k-1}^{(|Q|)} \left( p_{k|k-1} [h(1-p_D + p_D G_Z(0))] \right) \prod_{V \in Q} \eta_V[0, h] \right)} \left( \begin{aligned} & \times (1-p_D(x) + p_D(x)G_Z(0))p_{k|k-1}(x) \\ & + \zeta_{k|k-1}^{(|Q|)} \left( p_{k|k-1} [h(1-p_D + p_D G_Z(0))] \right) \\ & \times \sum_{V \in Q} \frac{p_D(x)G_Z^{(|W|)}(0) \prod_{z' \in V} \frac{p_z(z'|x)}{p_{FA}(z')} p_{k|k-1}(x)}{\eta_V[0, h]} \end{aligned} \right) \quad (25)$$

Evaluating at  $h = 1$ , we get

$$\begin{aligned} \frac{\delta}{\delta x} \frac{\delta}{\delta Z} F[0, 1] &= \frac{\delta}{\delta Z} F[0, 1] \zeta_{k|k-1}^{(1)} \left( p_{k|k-1} [1-p_D + p_D G_Z(0)] \right) (1-p_D(x) + p_D(x)G_Z(0)) p_{k|k-1}(x) \\ &+ F[0, 1] \left( \prod_{z' \in Z} p_{FA}(z') \right) \sum_{\mathcal{P} \subseteq Z} \left( \prod_{W \in \mathcal{P}} \left( \zeta_{FA}^{(|W|)}(0) \right. \right. \\ &+ \left. \left. \sum_{Q \subseteq W} \zeta_{k|k-1}^{(|Q|)} \left( p_{k|k-1} [1-p_D + p_D G_Z(0)] \right) \prod_{V \in Q} \eta_V[0, 1] \right) \right) \\ &\times \sum_{W \in \mathcal{P}} \frac{\left( \sum_{Q \subseteq W} \left( \prod_{V \in Q} \eta_V[0, 1] \right) \left( \zeta_{k|k-1}^{(|Q|+1)} \left( p_{k|k-1} [1-p_D + p_D G_Z(0)] \right) \right) \right.}{\left( \zeta_{FA}^{(|W|)}(0) + \sum_{Q \subseteq W} \zeta_{k|k-1}^{(|Q|)} \left( p_{k|k-1} [1-p_D + p_D G_Z(0)] \right) \prod_{V \in Q} \eta_V[0, 1] \right)} \left( \begin{aligned} & \times (1-p_D(x) + p_D(x)G_Z(0))p_{k|k-1}(x) \\ & + \zeta_{k|k-1}^{(|Q|)} \left( p_{k|k-1} [1-p_D + p_D G_Z(0)] \right) \\ & \times \sum_{V \in Q} \frac{p_D(x)G_Z^{(|W|)}(0) \prod_{z' \in V} \frac{p_z(z'|x)}{p_{FA}(z')} p_{k|k-1}(x)}{\eta_V[0, 1]} \end{aligned} \right) \end{aligned} \quad (26)$$

Defining additional quantities

$$\alpha_Q \triangleq \prod_{V \in Q} \eta_V[0, 1] \quad (27)$$

$$\beta_W \triangleq \zeta_{FA}^{(|W|)}(0) + \sum_{Q \subseteq W} \zeta_{k|k-1}^{(|Q|)} \left( p_{k|k-1} [1-p_D + p_D G_Z(0)] \right) \alpha_Q \quad (28)$$

we get

$$\begin{aligned} \frac{\delta}{\delta x} \frac{\delta}{\delta Z} F[0, 1] &= \frac{\delta}{\delta Z} F[0, 1] \zeta_{k|k-1}^{(1)} \left( p_{k|k-1} [1-p_D + p_D G_Z(0)] \right) (1-p_D(x) + p_D(x)G_Z(0)) p_{k|k-1}(x) \\ &+ F[0, 1] \left( \prod_{z' \in Z} p_{FA}(z') \right) \sum_{\mathcal{P} \subseteq Z} \left( \prod_{W \in \mathcal{P}} \beta_W \right) \\ &\times \sum_{W \in \mathcal{P}} \frac{1}{\beta_W} \left( \begin{aligned} & \sum_{Q \subseteq W} \alpha_Q \left( \zeta_{k|k-1}^{(|Q|+1)} \left( p_{k|k-1} [1-p_D + p_D G_Z(0)] \right) \right) \\ & \times (1-p_D(x) + p_D(x)G_Z(0))p_{k|k-1}(x) \\ & + \zeta_{k|k-1}^{(|Q|)} \left( p_{k|k-1} [1-p_D + p_D G_Z(0)] \right) \\ & \times \sum_{V \in Q} \frac{p_D(x)G_Z^{(|W|)}(0) \prod_{z' \in V} \frac{p_z(z'|x)}{p_{FA}(z')} p_{k|k-1}(x)}{\eta_V[0, 1]} \end{aligned} \right) \end{aligned} \quad (29)$$

Now dividing by  $\frac{\delta}{\delta Z}F[0,1]$  to obtain  $D_{k|k}(x)$ , we get

$$D_{k|k}(x) = (1 - p_D(x) + p_D(x)G_Z(0))\zeta_{k|k-1}^{(1)}\left(p_{k|k-1}[1 - p_D + p_D G_Z(0)]\right)p_{k|k-1}(x) \\ + \frac{\sum_{\mathcal{P} \setminus \mathcal{L} \setminus \mathcal{Z}} \left( \prod_{W \in \mathcal{P}} \beta_W \right) \sum_{W \in \mathcal{P}} \frac{1}{\beta_W} \left( \begin{array}{l} \sum_{\mathcal{Q} \setminus \mathcal{L} \setminus \mathcal{W}} \alpha_{\mathcal{Q}} \left( \zeta_{k|k-1}^{(|\mathcal{Q}|+1)} \left( p_{k|k-1}[1 - p_D + p_D G_Z(0)] \right) \right) \\ \times (1 - p_D(x) + p_D(x)G_Z(0)) p_{k|k-1}(x) \\ + \zeta_{k|k-1}^{(|\mathcal{Q}|)} \left( p_{k|k-1}[1 - p_D + p_D G_Z(0)] \right) \\ \times \sum_{V \in \mathcal{Q}} \frac{p_D(x)G_Z^{(|W|)}(0) \prod_{z' \in V} \frac{p_z(z'|x)}{p_{FA}(z')}}{\eta_V[0,1]} p_{k|k-1}(x) \end{array} \right)}{\sum_{\mathcal{P} \setminus \mathcal{L} \setminus \mathcal{Z}} \prod_{W \in \mathcal{P}} \beta_W} \quad (30)$$

Defining coefficients  $\omega_{\mathcal{P}}$  as

$$\omega_{\mathcal{P}} \triangleq \frac{\prod_{W \in \mathcal{P}} \beta_W}{\sum_{\mathcal{P} \setminus \mathcal{L} \setminus \mathcal{Z}} \prod_{W \in \mathcal{P}} \beta_W} \quad (31)$$

we get

$$D_{k|k}(x) = (1 - p_D(x) + p_D(x)G_Z(0))\zeta_{k|k-1}^{(1)}\left(p_{k|k-1}[1 - p_D + p_D G_Z(0)]\right)p_{k|k-1}(x) \\ + \sum_{\mathcal{P} \setminus \mathcal{L} \setminus \mathcal{Z}} \omega_{\mathcal{P}} \sum_{W \in \mathcal{P}} \frac{1}{\beta_W} \left( \begin{array}{l} \sum_{\mathcal{Q} \setminus \mathcal{L} \setminus \mathcal{W}} \alpha_{\mathcal{Q}} \left( \zeta_{k|k-1}^{(|\mathcal{Q}|+1)} \left( p_{k|k-1}[1 - p_D + p_D G_Z(0)] \right) \right) \\ \times (1 - p_D(x) + p_D(x)G_Z(0)) p_{k|k-1}(x) \\ + \zeta_{k|k-1}^{(|\mathcal{Q}|)} \left( p_{k|k-1}[1 - p_D + p_D G_Z(0)] \right) \\ \times \sum_{V \in \mathcal{Q}} \frac{p_D(x)G_Z^{(|W|)}(0) \prod_{z' \in V} \frac{p_z(z'|x)}{p_{FA}(z')}}{\eta_V[0,1]} p_{k|k-1}(x) \end{array} \right) \quad (32) \\ = (1 - p_D(x) + p_D(x)G_Z(0))\zeta_{k|k-1}^{(1)}\left(p_{k|k-1}[1 - p_D + p_D G_Z(0)]\right)p_{k|k-1}(x) \\ + \sum_{\mathcal{P} \setminus \mathcal{L} \setminus \mathcal{Z}} \omega_{\mathcal{P}} \sum_{W \in \mathcal{P}} \frac{1}{\beta_W} \left( \begin{array}{l} \sum_{\mathcal{Q} \setminus \mathcal{L} \setminus \mathcal{W}} \alpha_{\mathcal{Q}} \left( \zeta_{k|k-1}^{(|\mathcal{Q}|+1)} \left( p_{k|k-1}[1 - p_D + p_D G_Z(0)] \right) \right) \\ \times (1 - p_D(x) + p_D(x)G_Z(0)) \\ + \zeta_{k|k-1}^{(|\mathcal{Q}|)} \left( p_{k|k-1}[1 - p_D + p_D G_Z(0)] \right) \\ \times \sum_{V \in \mathcal{Q}} \frac{p_D(x)G_Z^{(|W|)}(0) \prod_{z' \in V} \frac{p_z(z'|x)}{p_{FA}(z')}}{\eta_V[0,1]} \end{array} \right) p_{k|k-1}(x) \quad (33)$$

If now we define a constant

$$\kappa \triangleq \sum_{\mathcal{P} \setminus \mathcal{L} \setminus \mathcal{Z}} \omega_{\mathcal{P}} \sum_{W \in \mathcal{P}} \frac{1}{\beta_W} \sum_{\mathcal{Q} \setminus \mathcal{L} \setminus \mathcal{W}} \alpha_{\mathcal{Q}} \zeta_{k|k-1}^{(|\mathcal{Q}|+1)} \left( p_{k|k-1}[1 - p_D + p_D G_Z(0)] \right) \quad (34)$$

we get

$$D_{k|k}(x) = (\zeta_{k|k-1}^{(1)} + \kappa)(1 - p_D(x) + p_D(x)G_Z(0))p_{k|k-1}(x) \\ + \sum_{\mathcal{P} \setminus \mathcal{L} \setminus \mathcal{Z}} \omega_{\mathcal{P}} \sum_{W \in \mathcal{P}} \frac{1}{\beta_W} \sum_{\mathcal{Q} \setminus \mathcal{L} \setminus \mathcal{W}} \alpha_{\mathcal{Q}} \zeta_{k|k-1}^{(|\mathcal{Q}|)} \sum_{V \in \mathcal{Q}} \frac{p_D(x)G_Z^{(|W|)}(0)}{\eta_V[0,1]} \prod_{z' \in V} \frac{p_z(z'|x)}{p_{FA}(z')} p_{k|k-1}(x) \quad (35)$$

where we have dropped the argument of the terms  $\zeta_{k|k-1}^{(\cdot)}\left(p_{k|k-1}[1 - p_D + p_D G_Z(0)]\right)$  for simplicity.

We have  $G_{k|k}[h]$  as

$$G_{k|k}[h] = \frac{\left( \begin{array}{c} F[0, h] \sum_{\mathcal{P} \setminus Z} \prod_{W \in \mathcal{P}} \left( \zeta_{FA}^{(|W|)}(0) \right. \\ \left. + \sum_{\mathcal{Q} \setminus W} \zeta_{k|k-1}^{(|\mathcal{Q}|)} \left( p_{k|k-1} [h(1 - p_D + p_D G_Z(0))] \right) \prod_{V \in \mathcal{Q}} \eta_V[0, h] \right) \end{array} \right)}{\left( \begin{array}{c} F[0, 1] \sum_{\mathcal{P} \setminus Z} \prod_{W \in \mathcal{P}} \left( \zeta_{FA}^{(|W|)}(0) \right. \\ \left. + \sum_{\mathcal{Q} \setminus W} \zeta_{k|k-1}^{(|\mathcal{Q}|)} \left( p_{k|k-1} [1 - p_D + p_D G_Z(0)] \right) \prod_{V \in \mathcal{Q}} \eta_V[0, 1] \right) \end{array} \right)} \quad (36)$$

$$\begin{aligned} &= \frac{G_{k|k-1} \left( p_{k|k-1} [h(1 - p_D + p_D G_Z(0))] \right)}{G_{k|k-1} \left( p_{k|k-1} [1 - p_D + p_D G_Z(0)] \right)} \\ &\times \frac{\sum_{\mathcal{P} \setminus Z} \prod_{W \in \mathcal{P}} \left( \zeta_{FA}^{(|W|)}(0) + \sum_{\mathcal{Q} \setminus W} \zeta_{k|k-1}^{(|\mathcal{Q}|)} \left( p_{k|k-1} [h(1 - p_D + p_D G_Z(0))] \right) \prod_{V \in \mathcal{Q}} \eta_V[0, h] \right)}{\sum_{\mathcal{P} \setminus Z} \prod_{W \in \mathcal{P}} \beta_W} \end{aligned} \quad (37)$$

Then,

$$\begin{aligned} G_{k|k}(x) &= \frac{G_{k|k-1} \left( x p_{k|k-1} [1 - p_D + p_D G_Z(0)] \right)}{G_{k|k-1} \left( p_{k|k-1} [1 - p_D + p_D G_Z(0)] \right)} \\ &\times \frac{\sum_{\mathcal{P} \setminus Z} \prod_{W \in \mathcal{P}} \left( \zeta_{FA}^{(|W|)}(0) + \sum_{\mathcal{Q} \setminus W} \alpha_{\mathcal{Q}} x^{|\mathcal{Q}|} \zeta_{k|k-1}^{(|\mathcal{Q}|)} \left( x p_{k|k-1} [1 - p_D + p_D G_Z(0)] \right) \right)}{\sum_{\mathcal{P} \setminus Z} \prod_{W \in \mathcal{P}} \beta_W} \end{aligned} \quad (38)$$

$$\begin{aligned} &= \frac{G_{k|k-1} \left( x p_{k|k-1} [\cdot] \right)}{G_{k|k-1} \left( p_{k|k-1} [\cdot] \right)} \frac{\sum_{\mathcal{P} \setminus Z} \prod_{W \in \mathcal{P}} \left( \zeta_{FA}^{(|W|)}(0) + \sum_{\mathcal{Q} \setminus W} \alpha_{\mathcal{Q}} x^{|\mathcal{Q}|} \zeta_{k|k-1}^{(|\mathcal{Q}|)} \left( x p_{k|k-1} [\cdot] \right) \right)}{\sum_{\mathcal{P} \setminus Z} \prod_{W \in \mathcal{P}} \beta_W} \end{aligned} \quad (39)$$

where we skipped the arguments of  $p_{k|k-1} [1 - p_D + p_D G_Z(0)]$  for the sake of clarity.

## 2 Summary

Calculate the quantities  $D_{k|k}(x)$  and  $G_{k|k}(x)$  from the given quantities  $D_{k|k-1}(x)$  and  $G_{k|k-1}(x)$  as follows.

$$D_{k|k}(x) = (\zeta_{k|k-1}^{(1)} + \kappa)(1 - p_D(x) + p_D(x)G_Z(0))p_{k|k-1}(x) + \sum_{\mathcal{P} \subset \mathcal{Z}} \omega_{\mathcal{P}} \sum_{W \in \mathcal{P}} \frac{1}{\beta_W} \sum_{\mathcal{Q} \subset W} \alpha_{\mathcal{Q}} \zeta_{k|k-1}^{(|\mathcal{Q}|)} \sum_{V \in \mathcal{Q}} \frac{p_D(x)G_Z^{(|W|)}(0)}{\eta_V[0, 1]} \prod_{z' \in V} \frac{p_z(z'|x)}{p_{FA}(z')} p_{k|k-1}(x) \quad (40)$$

$$G_{k|k}(x) = \frac{G_{k|k-1}(xp_{k|k-1}[\cdot]) \sum_{\mathcal{P} \subset \mathcal{Z}} \prod_{W \in \mathcal{P}} \left( \zeta_{FA}^{(|W|)}(0) + \sum_{\mathcal{Q} \subset W} \alpha_{\mathcal{Q}} x^{|\mathcal{Q}|} \zeta_{k|k-1}^{(|\mathcal{Q}|)}(xp_{k|k-1}[\cdot]) \right)}{G_{k|k-1}(p_{k|k-1}[\cdot]) \sum_{\mathcal{P} \subset \mathcal{Z}} \prod_{W \in \mathcal{P}} \beta_W} \quad (41)$$

where we skipped the arguments of  $p_{k|k-1}[1 - p_D + p_D G_Z(0)]$  for the sake of clarity and

$$p_{k|k-1}(x) = \frac{D_{k|k-1}(x)}{\int D_{k|k-1}(x) dx} \quad (42)$$

$$\zeta_{FA}^{(i)}(x) \triangleq \left( \frac{G_{FA}^{(1)}(x)}{G_{FA}(x)} \right)^{(i-1)} \quad (43)$$

$$\zeta_{k|k-1}^{(i)}(x) \triangleq \left( \frac{G_{k|k-1}^{(1)}(x)}{G_{k|k-1}(x)} \right)^{(i-1)} \quad (44)$$

$$\kappa \triangleq \sum_{\mathcal{P} \subset \mathcal{Z}} \omega_{\mathcal{P}} \sum_{W \in \mathcal{P}} \frac{1}{\beta_W} \sum_{\mathcal{Q} \subset W} \alpha_{\mathcal{Q}} \zeta_{k|k-1}^{(|\mathcal{Q}|+1)}(p_{k|k-1}[1 - p_D + p_D G_Z(0)]) \quad (45)$$

$$\omega_{\mathcal{P}} \triangleq \frac{\prod_{W \in \mathcal{P}} \beta_W}{\sum_{\mathcal{P} \subset \mathcal{Z}} \prod_{W \in \mathcal{P}} \beta_W} \quad (46)$$

$$\beta_W \triangleq \zeta_{FA}^{(|W|)}(0) + \sum_{\mathcal{Q} \subset W} \zeta_{k|k-1}^{(|\mathcal{Q}|)}(p_{k|k-1}[1 - p_D + p_D G_Z(0)]) \alpha_{\mathcal{Q}} \quad (47)$$

$$\alpha_{\mathcal{Q}} \triangleq \prod_{V \in \mathcal{Q}} \eta_V[0, 1] \quad (48)$$

$$\eta_V[g, h] \triangleq p_{k|k-1} \left[ h p_D G_Z^{(|V|)}(p_z[g]) \prod_{z' \in V} \frac{p_z(z')}{p_{FA}(z')} \right] \quad (49)$$



### 3 Calculating Posterior Cardinality Distribution $P_{k|k}(n)$

In this section, we are going to calculate the posterior cardinality distribution  $P_{k|k}(n)$  based on the posterior cardinality probability generating function (p.g.f.)  $G_{k|k}(x)$ . For any cardinality distribution, p.g.f. pair given as  $P(n)$ – $G(x)$  we know the relationship

$$P(n) = \frac{1}{n!} G^{(n)}(0) \quad (50)$$

which we are going to use in this calculation. In the previous sections, we calculated the posterior cardinality p.g.f.  $G_{k|k}(x)$  as

$$G_{k|k}(x) = \frac{G_{k|k-1}(xp_{k|k-1}[\cdot]) \sum_{\mathcal{P} \setminus \mathcal{L} \setminus \mathcal{Z}} \prod_{W \in \mathcal{P}} \left( \zeta_{FA}^{(|W|)}(0) + \sum_{\mathcal{Q} \setminus \mathcal{L} \setminus \mathcal{W}} \alpha_{\mathcal{Q}} x^{|\mathcal{Q}|} \zeta_{k|k-1}^{(|\mathcal{Q}|)}(xp_{k|k-1}[\cdot]) \right)}{G_{k|k-1}(p_{k|k-1}[\cdot]) \sum_{\mathcal{P} \setminus \mathcal{L} \setminus \mathcal{Z}} \prod_{W \in \mathcal{P}} \beta_W} \quad (51)$$

Here, the denominators on the right hand side are constant with respect to  $x$  and for this reason, we are going to take the derivative of only the numerator terms.

For the derivative of the multiplication of two functions, we have

$$(f(x)g(x))^{(n)} = \sum_{i=1}^n C_{n,i} f^{(n-i)}(x) g^{(i)}(x) \quad (52)$$

where  $C_{n|i}$  is the binomial coefficient given as

$$C_{n|i} = \frac{n!}{i!(n-i)!}. \quad (53)$$

We have the generalization of this derivative rule as

$$\left( \prod_{j=1}^M f_j(x) \right)^{(n)} = \sum_{\substack{0 \leq i_1, i_2, \dots, i_M \leq M \\ i_1 + i_2 + \dots + i_M = n}} C_{n|i_1, \dots, i_M} \prod_{j=1}^M f_j^{(i_j)}(x) \quad (54)$$

where  $C_{n|i_1, \dots, i_M}$  is the multinomial coefficient given as

$$C_{n|i_1, \dots, i_M} = \frac{n!}{i_1! i_2! \dots i_M!}. \quad (55)$$

We are going to use the following lemmas in the calculation.

**Lemma 1** Derivatives of the term  $G_{k|k-1}(xp_{k|k-1}[\cdot])$  are given as

$$G_{k|k-1}^{(n)}(xp_{k|k-1}[\cdot]) \Big|_{x=0} = (p_{k|k-1}[\cdot])^n G_{k|k-1}^{(n)}(0) \quad (56)$$

$$= n! (p_{k|k-1}[\cdot])^n P_{k|k-1}(n) \quad (57)$$

where  $P_{k|k-1}(\cdot)$  is the predicted cardinality distribution.  $\square$

**Lemma 2** Derivatives of the term  $\left( \zeta_{FA}^{(|W|)}(0) + \sum_{\mathcal{Q} \setminus \mathcal{L} \setminus \mathcal{W}} \alpha_{\mathcal{Q}} x^{|\mathcal{Q}|} \zeta_{k|k-1}^{(|\mathcal{Q}|)}(xp_{k|k-1}[\cdot]) \right)$  are given as

$$\left( \zeta_{FA}^{(|W|)}(0) + \sum_{\mathcal{Q} \setminus \mathcal{L} \setminus \mathcal{W}} \alpha_{\mathcal{Q}} x^{|\mathcal{Q}|} \zeta_{k|k-1}^{(|\mathcal{Q}|)}(xp_{k|k-1}[\cdot]) \right)^{(n)} \Big|_{x=0} = \zeta_{FA}^{(|W|)}(0) \delta_{n,0} + n! \sum_{\substack{\mathcal{Q} \setminus \mathcal{L} \setminus \mathcal{W} \\ |\mathcal{Q}|=n}} \alpha_{\mathcal{Q}} \zeta_{k|k-1}^{(|\mathcal{Q}|)}(0) \quad (58)$$

Notice that the derivative is equal to 0 if  $n > |W|$ .  $\square$

**Lemma 3** *The  $n$ th derivative of the summation*

$$\sum_{\mathcal{P} \subset \mathcal{Z}} \prod_{W \in \mathcal{P}} \left( \zeta_{FA}^{(|W|)}(0) + \sum_{\mathcal{Q} \subset W} \alpha_{\mathcal{Q}} x^{|\mathcal{Q}|} \zeta_{k|k-1}^{(|\mathcal{Q}|)}(xp_{k|k-1}[\cdot]) \right) \quad (59)$$

is given as

$$\sum_{\substack{\mathcal{P} \subset \mathcal{Z} \\ \mathcal{P} \triangleq \{W_j\}_{j=1}^{|\mathcal{P}|}}} \sum_{\substack{0 \leq i_1, i_2, \dots, i_{|\mathcal{P}|} \leq |\mathcal{P}| \\ i_1 + i_2 + \dots + i_{|\mathcal{P}|} = n \\ i_j \leq |W_j| \quad \forall j}} C_{n|i_1, \dots, i_{|\mathcal{P}|}} \prod_{j=1}^{|\mathcal{P}|} \left( \zeta_{FA}^{(|W_j|)}(0) \delta_{i_j, 0} + i_j! \sum_{\substack{\mathcal{Q} \subset W_j \\ |\mathcal{Q}| = i_j}} \alpha_{\mathcal{Q}} \zeta_{k|k-1}^{(|\mathcal{Q}|)}(0) \right) \quad (60)$$

$$= n! \sum_{\substack{\mathcal{P} \subset \mathcal{Z} \\ \mathcal{P} \triangleq \{W_j\}_{j=1}^{|\mathcal{P}|}}} \sum_{\substack{0 \leq i_1, i_2, \dots, i_{|\mathcal{P}|} \leq |\mathcal{P}| \\ i_1 + i_2 + \dots + i_{|\mathcal{P}|} = n \\ i_j \leq |W_j| \quad \forall j}} \prod_{j=1}^{|\mathcal{P}|} \left( \zeta_{FA}^{(|W_j|)}(0) \delta_{i_j, 0} + \sum_{\substack{\mathcal{Q} \subset W_j \\ |\mathcal{Q}| = i_j}} \alpha_{\mathcal{Q}} \zeta_{k|k-1}^{(|\mathcal{Q}|)}(0) \right) \quad (61)$$

where we used (54) and the condition  $i_j < |W_j| \quad \forall j$  of the second summation is added just to reduce the number of possibilities.  $\square$

Now we are ready for the final result using the derivative of the multiplication rule (52) and the three lemmas above as follows.

$$P_{k|k}(n) \triangleq \frac{1}{G_{k|k-1}(p_{k|k-1}[\cdot]) \sum_{\mathcal{P} \subset \mathcal{Z}} \prod_{W \in \mathcal{P}} \beta_W} \sum_{i=1}^n (p_{k|k-1}[\cdot])^{n-i} P_{k|k-1}(n-i) \\ \times \sum_{\substack{\mathcal{P} \subset \mathcal{Z} \\ \mathcal{P} \triangleq \{W_j\}_{j=1}^{|\mathcal{P}|}}} \sum_{\substack{0 \leq i_1, i_2, \dots, i_{|\mathcal{P}|} \leq |\mathcal{P}| \\ i_1 + i_2 + \dots + i_{|\mathcal{P}|} = i \\ i_j \leq |W_j| \quad \forall j}} \prod_{j=1}^{|\mathcal{P}|} \left( \zeta_{FA}^{(|W_j|)}(0) \delta_{i_j, 0} + \sum_{\substack{\mathcal{Q} \subset W_j \\ |\mathcal{Q}| = i_j}} \alpha_{\mathcal{Q}} \zeta_{k|k-1}^{(|\mathcal{Q}|)}(0) \right) \quad (62)$$

## 4 Reducing CPHD for extended targets to PHD for extended targets

Now we check whether the CPHD equations given above reduce to PHD when the cluster processes are replaced by Poisson processes. This is done also for checking the formulas given above. First we give the related quantities.

- The extended target measurements are distributed according to a Poisson process. The corresponding likelihood is given as

$$f(Z|x) = n! P_z(n|x) \prod_{z \in Z} p_z(z|x) \quad (63)$$

where

$$P_z(n|x) = \frac{e^{-\gamma(x)} \gamma^n(x)}{n!} \quad (64)$$

- The false alarms are distributed according to a Poisson process also.

$$f(Z_{FA}) = n! P_{FA}(n) \prod_{z \in Z_{FA}} p_{FA}(z) \quad (65)$$

where

$$P_{FA}(n) = \frac{e^{-\lambda} \lambda^n}{n!} \quad (66)$$

- The multitarget prior  $f(X_k|Z_{0:k-1})$  is assumed to be a Poisson process.

$$f(X_k|Z_{0:k-1}) = n! P_{k+1|k}(n) \prod_{x_k \in X_k} p_{k+1|k}(x_k) \quad (67)$$

where

$$p_{k+1|k}(x_k) \triangleq N_{k|k-1}^{-1} D_{k|k-1}(x_k) \quad (68)$$

with  $N_{k|k-1} \triangleq \int D_{k|k-1}(x_k) dx_k$ . We have the cardinality distribution  $P_{k+1|k}(n)$  given as

$$P_{k+1|k}(n) = \frac{e^{-N_{k|k-1}} N_{k|k-1}^n}{n!} \quad (69)$$

We here must note that the probability generating function and functionals corresponding to a Poisson process with parameter  $\lambda$  and density  $p(\cdot)$  corresponds to

$$G(x) = \exp(\lambda x - \lambda) \quad \text{and} \quad G[h] = \exp(\lambda p[h] - \lambda) \quad (70)$$

We have

$$G^{(n)}(x) = \lambda^n G(x) = \lambda^n \exp(\lambda x - \lambda) \quad (71)$$

Also the following expression holds.

$$G^{(n)}(0) = \lambda^n e^{-\lambda} \quad (72)$$

The functions  $\zeta_{FA}$  and  $\zeta_{k|k-1}$  are then given as

$$\zeta_{FA}^{(i)}(x) \triangleq \left( \frac{G_{FA}^{(1)}(x)}{G_{FA}(x)} \right)^{(i-1)} = (\lambda)^{(i-1)} = \begin{cases} 0 & i > 1 \\ \lambda & i = 1 \end{cases} = \lambda \delta_{i,1} \quad (73)$$

$$\zeta_{k|k-1}^{(i)}(x) \triangleq \left( \frac{G_{k|k-1}^{(1)}(x)}{G_{k|k-1}(x)} \right)^{(i-1)} = (N_{k|k-1})^{(i-1)} = \begin{cases} 0 & i > 1 \\ N_{k|k-1} & i = 1 \end{cases} = N_{k|k-1} \delta_{i,1} \quad (74)$$

Then in the formula

$$\begin{aligned} \frac{\delta}{\delta Z} F[g, h] = & F[g, h] \left( \prod_{z' \in Z} p_{FA}(z') \right) \sum_{\mathcal{P} \subset Z} \prod_{W \in \mathcal{P}} \left( \zeta_{FA}^{(|W|)}(p_{FA}[g]) \right. \\ & \left. + \sum_{\mathcal{Q} \subset W} \zeta_{k|k-1}^{(|\mathcal{Q}|)} \left( p_{k|k-1}[h(1 - p_D + p_D G_Z(p_z[g]))] \right) \prod_{V \in \mathcal{Q}} \eta_V[g, h] \right), \end{aligned} \quad (75)$$

all terms in the summation  $\sum_{\mathcal{Q} \subset W}$  becomes zero except  $\mathcal{Q} = \{W\}$  which means that there is only one set in the partition  $\mathcal{Q}$  and it is  $V = W$ . Also from above, we have  $\zeta_{FA}^{(|W|)}(x) = \lambda \delta_{|W|,1}$ . Substituting these into the above equation

$$\frac{\delta}{\delta Z} F[g, h] = F[g, h] \left( \prod_{z' \in Z} p_{FA}(z') \right) \sum_{\mathcal{P} \subset Z} \prod_{W \in \mathcal{P}} \left( \lambda \delta_{|W|,1} + N_{k|k-1} \eta_W[g, h] \right) \quad (76)$$

This equation is the same as the equation Mahler derived for  $\frac{\delta}{\delta Z} F[g, h]$  [1, Eq. (28)]. In the CPHD equation

$$\begin{aligned} D_{k|k}(x) = & (\zeta_{k|k-1}^{(1)} + \kappa)(1 - p_D(x) + p_D(x) G_Z(0)) p_{k|k-1}(x) \\ & + \sum_{\mathcal{P} \subset Z} \omega_{\mathcal{P}} \sum_{W \in \mathcal{P}} \frac{1}{\beta_W} \sum_{\mathcal{Q} \subset W} \alpha_{\mathcal{Q}} \zeta_{k|k-1}^{(|\mathcal{Q}|)} \sum_{V \in \mathcal{Q}} \frac{p_D(x) G_Z^{(|W|)}(0)}{\eta_V[0, 1]} \prod_{z' \in V} \frac{p_z(z'|x)}{p_{FA}(z')} p_{k|k-1}(x) \end{aligned} \quad (77)$$

By above facts, it is easy to see that  $\kappa = 0$ . Substituting above facts into this equation we have

$$D_{k|k}(x) = (1 - p_D(x) + p_D(x)G_Z(0))D_{k|k-1}(x) + \sum_{\mathcal{P} \setminus \mathcal{Z}} \omega_{\mathcal{P}} \sum_{W \in \mathcal{P}} \frac{\alpha_{\{W\}} p_D(x) G_Z^{(|W|)}(0)}{\beta_W \eta_W[0, 1]} \prod_{z' \in W} \frac{p_z(z'|x)}{p_{FA}(z')} D_{k|k-1}(x) \quad (78)$$

Seeing that  $\alpha_{\{W\}} = \eta_W[0, 1]$  we get

$$D_{k|k}(x) = (1 - p_D(x) + p_D(x)G_Z(0))D_{k|k-1}(x) + \sum_{\mathcal{P} \setminus \mathcal{Z}} \omega_{\mathcal{P}} \sum_{W \in \mathcal{P}} \frac{p_D(x) G_Z^{(|W|)}(0)}{\beta_W} \prod_{z' \in W} \frac{p_z(z'|x)}{p_{FA}(z')} D_{k|k-1}(x) \quad (79)$$

We also see that

$$\beta_W \triangleq \zeta_{FA}^{(|W|)}(0) + \sum_{\mathcal{Q} \setminus W} \zeta_k^{(|\mathcal{Q}|)} \left( p_{k|k-1} [1 - p_D + p_D G_Z(0)] \right) \alpha_{\mathcal{Q}} \quad (80)$$

$$= \lambda \delta_{|W|, 1} + N_{k|k-1} \alpha_{\{W\}} \quad (81)$$

$$= \lambda \delta_{|W|, 1} + N_{k|k-1} \eta_W[0, 1] \quad (82)$$

$$= \lambda \delta_{|W|, 1} + N_{k|k-1} p_{k|k-1} \left[ p_D G_Z^{(|W|)}(0) \prod_{z' \in W} \frac{p_z(z')}{p_{FA}(z')} \right] \quad (83)$$

$$= \lambda \delta_{|W|, 1} + D_{k|k-1} \left[ p_D G_Z^{(|W|)}(0) \prod_{z' \in W} \frac{p_z(z')}{p_{FA}(z')} \right] \quad (84)$$

$$(85)$$

Dividing and multiplying all quantities in the last summation by  $\lambda^{|W|}$  and distributing, we obtain

$$D_{k|k}(x) = (1 - p_D(x) + p_D(x)G_Z(0))D_{k|k-1}(x) + \sum_{\mathcal{P} \setminus \mathcal{Z}} \omega_{\mathcal{P}} \sum_{W \in \mathcal{P}} \frac{p_D(x) G_Z^{(|W|)}(0)}{\lambda^{-|W|} \beta_W} \prod_{z' \in W} \frac{p_z(z'|x)}{\lambda p_{FA}(z')} D_{k|k-1}(x) \quad (86)$$

Defining the new coefficient

$$d_W = \lambda^{-|W|} \beta_W \quad (87)$$

$$= \frac{\lambda}{\lambda^{-|W|}} \delta_{|W|, 1} + D_{k|k-1} \left[ p_D G_Z^{(|W|)}(0) \prod_{z' \in W} \frac{p_z(z')}{\lambda p_{FA}(z')} \right] \quad (88)$$

$$= \delta_{|W|, 1} + D_{k|k-1} \left[ p_D G_Z^{(|W|)}(0) \prod_{z' \in W} \frac{p_z(z')}{\lambda p_{FA}(z')} \right] \quad (89)$$

which are the same coefficients defined in [1]. Also it is easy to see that

$$\omega_{\mathcal{P}} \triangleq \frac{\prod_{W \in \mathcal{P}} \beta_W}{\sum_{\mathcal{P} \setminus \mathcal{Z}} \prod_{W \in \mathcal{P}} \beta_W} \quad (90)$$

$$= \frac{\prod_{W \in \mathcal{P}} \lambda^{-|W|} \beta_W}{\sum_{\mathcal{P} \setminus \mathcal{Z}} \prod_{W \in \mathcal{P}} \lambda^{-|W|} \beta_W} \quad (91)$$

$$= \frac{\prod_{W \in \mathcal{P}} d_W}{\sum_{\mathcal{P} \setminus \mathcal{Z}} \prod_{W \in \mathcal{P}} d_W} \quad (92)$$

which are the same  $\omega_{\mathcal{P}}$  coefficients in [1]. Knowing that  $G_Z^{|W|}(0) = \gamma^{|W|}(x)e^{-\gamma(x)}$  we get

$$D_{k|k}(x) = (1 - p_D(x) + p_D(x)e^{-\gamma(x)})D_{k|k-1}(x) + \sum_{\mathcal{P} \setminus Z} \omega_{\mathcal{P}} \sum_{W \in \mathcal{P}} \frac{p_D(x)\gamma^{|W|}(x)e^{-\gamma(x)}}{d_W} \prod_{z' \in W} \frac{p_z(z'|x)}{\lambda p_{FA}(z')} D_{k|k-1}(x) \quad (93)$$

which is the same formula in [1, Eq. (5)].

## 5 Reducing CPHD for extended targets to CPHD for standard targets

For standard targets,  $P_z(n|x) = 0$  except for  $n = 1$  where it is unity i.e.,  $P_z(1|x) = 1$  which makes  $G_Z(x) = x$ . Then

$$\eta_V[0, h] \triangleq p_{k|k-1} \left[ h p_D G_Z^{(|V|)}(0) \prod_{z' \in V} \frac{p_z(z')}{p_{FA}(z')} \right] \quad (94)$$

$$= \begin{cases} p_{k|k-1} \left[ h p_D \frac{p_z(z')}{p_{FA}(z')} \right] & |V| = 1, V = \{z'\} \\ 0 & \text{otherwise} \end{cases} \quad (95)$$

$$= p_{k|k-1} \left[ h p_D \frac{p_z(z')}{p_{FA}(z')} \right] \delta_{|V|,1} \quad (96)$$

This gives

$$\alpha_{\mathcal{Q}} \triangleq \prod_{V \in \mathcal{Q}} \eta_V[0, 1] = \prod_{z \in W} p_{k|k-1} \left[ p_D \frac{p_z(z')}{p_{FA}(z')} \right] \delta_{|\mathcal{Q}|, |W|} \quad (97)$$

where  $\mathcal{Q}$  is a partition of the set  $W$ .

$$\beta_W \triangleq \zeta_{FA}^{(|W|)}(0) + \sum_{\mathcal{Q} \setminus W} \zeta_{k|k-1}^{(|\mathcal{Q}|)} \left( p_{k|k-1} [1 - p_D + p_D G_Z(0)] \right) \alpha_{\mathcal{Q}} \quad (98)$$

$$= \zeta_{FA}^{(|W|)}(0) + \zeta_{k|k-1}^{(|W|)} \left( p_{k|k-1} [1 - p_D] \right) \prod_{z' \in W} p_{k|k-1} \left[ p_D \frac{p_z(z')}{p_{FA}(z')} \right] \quad (99)$$

Substituting these into  $G_{k|k}[h]$  we get

$$G_{k|k}[h] \triangleq \frac{G_{k|k-1} \left( p_{k|k-1} [h(1 - p_D + p_D G_Z(0))] \right)}{G_{k|k-1} \left( p_{k|k-1} [1 - p_D + p_D G_Z(0)] \right)} \times \frac{\sum_{\mathcal{P} \setminus Z} \prod_{W \in \mathcal{P}} \left( \zeta_{FA}^{(|W|)}(0) + \sum_{\mathcal{Q} \setminus W} \zeta_{k|k-1}^{(|\mathcal{Q}|)} \left( p_{k|k-1} [h(1 - p_D + p_D G_Z(0))] \right) \right) \prod_{V \in \mathcal{Q}} \eta_V[0, h]}{\sum_{\mathcal{P} \setminus Z} \prod_{W \in \mathcal{P}} \beta_W} \quad (100)$$

$$= \frac{G_{k|k-1} \left( p_{k|k-1} [h(1 - p_D)] \right)}{G_{k|k-1} \left( p_{k|k-1} [1 - p_D] \right)} \times \frac{\sum_{\mathcal{P} \setminus Z} \prod_{W \in \mathcal{P}} \left( \zeta_{FA}^{(|W|)}(0) + \zeta_{k|k-1}^{(|W|)} \left( p_{k|k-1} [h(1 - p_D)] \right) \right) \prod_{z' \in W} p_{k|k-1} \left[ h p_D \frac{p_z(z')}{p_{FA}(z')} \right]}{\sum_{\mathcal{P} \setminus Z} \prod_{W \in \mathcal{P}} \left( \zeta_{FA}^{(|W|)}(0) + \zeta_{k|k-1}^{(|W|)} \left( p_{k|k-1} [1 - p_D] \right) \right) \prod_{z' \in W} p_{k|k-1} \left[ p_D \frac{p_z(z')}{p_{FA}(z')} \right]} \quad (101)$$

$$\begin{aligned}
&= \frac{G_{k|k-1}(p_{k|k-1}[h(1-p_D)])}{G_{k|k-1}(p_{k|k-1}[1-p_D])} \\
&\times \frac{\sum_{\mathcal{P}\mathcal{L}\mathcal{Z}} \prod_{W \in \mathcal{P}} \frac{\delta}{\delta W} \left( \log G_{FA}(p_{FA}[g]) + \log G_{k|k-1}(p_{k|k-1}[h(1-p_D + p_D p_z[g])]) \right) \Big|_{g=0}}{\sum_{\mathcal{P}\mathcal{L}\mathcal{Z}} \prod_{W \in \mathcal{P}} \frac{\delta}{\delta W} \left( \log G_{FA}(p_{FA}[g]) + \log G_{k|k-1}(p_{k|k-1}[1-p_D + p_D p_z[g]) \right) \Big|_{g=0}}
\end{aligned} \tag{102}$$

$$\begin{aligned}
&= \frac{G_{k|k-1}(p_{k|k-1}[h(1-p_D)])}{G_{k|k-1}(p_{k|k-1}[1-p_D])} \\
&\times \frac{\sum_{\mathcal{P}\mathcal{L}\mathcal{Z}} \prod_{W \in \mathcal{P}} \frac{\delta}{\delta W} \log \left( G_{FA}(p_{FA}[g]) G_{k|k-1}(p_{k|k-1}[h(1-p_D + p_D p_z[g])]) \right) \Big|_{g=0}}{\sum_{\mathcal{P}\mathcal{L}\mathcal{Z}} \prod_{W \in \mathcal{P}} \frac{\delta}{\delta W} \log \left( G_{FA}(p_{FA}[g]) G_{k|k-1}(p_{k|k-1}[1-p_D + p_D p_z[g]) \right) \Big|_{g=0}}
\end{aligned} \tag{103}$$

where we used the facts that  $\int \zeta_{FA}^{(1)} = \log G_{FA}$  and  $\int \zeta_{k|k-1}^{(1)} = \log G_{k|k-1}$  to obtain (102). We can see that for any p.g.fl.,  $G[g]$

$$\sum_{\mathcal{P}\mathcal{L}\mathcal{Z}} \prod_{W \in \mathcal{P}} \left( G^{|W|}[g] + \frac{\delta G}{\delta W}[g] \right) = \frac{\delta G^{|Z|}}{\delta Z}[g] \tag{104}$$

from the product rule for functional derivatives (See [2, 8th row of Table 11.2]). Since  $G[g]$  is arbitrary, we can replace it with  $G[g] - G[0]$  which gives

$$\sum_{\mathcal{P}\mathcal{L}\mathcal{Z}} \prod_{W \in \mathcal{P}} \left( (G[g] - G[0])^{|W|} + \frac{\delta G}{\delta W}[g] \right) = \frac{\delta (G[g] - G[0])^{|Z|}}{\delta Z} \tag{105}$$

Now, evaluating both sides at  $g = 0$ , we get

$$\sum_{\mathcal{P}\mathcal{L}\mathcal{Z}} \prod_{W \in \mathcal{P}} \frac{\delta G}{\delta W}[0] = \frac{\delta (G[g] - G[0])^{|Z|}}{\delta Z} \Big|_{g=0} \tag{106}$$

Now suppose we have two functionals and we need to evaluate the following ration.

$$\frac{\sum_{\mathcal{P}\mathcal{L}\mathcal{Z}} \prod_{W \in \mathcal{P}} \frac{\delta G_1}{\delta W}[0]}{\sum_{\mathcal{P}\mathcal{L}\mathcal{Z}} \prod_{W \in \mathcal{P}} \frac{\delta G_2}{\delta W}[0]} = \frac{\frac{\delta (G_1[g] - G_1[0])^{|Z|}}{\delta Z} \Big|_{g=0}}{\frac{\delta (G_2[g] - G_2[0])^{|Z|}}{\delta Z} \Big|_{g=0}} \tag{107}$$

This division gives  $\frac{0}{0}$  indeterminate form. One must use l'Hopital's rule  $|Z| - 1$  times (take derivatives of the numerator and denominator  $|Z| - 1$  times with respect to  $g$ ), to get

$$\frac{\sum_{\mathcal{P}\mathcal{L}\mathcal{Z}} \prod_{W \in \mathcal{P}} \frac{\delta G_1}{\delta W}[0]}{\sum_{\mathcal{P}\mathcal{L}\mathcal{Z}} \prod_{W \in \mathcal{P}} \frac{\delta G_2}{\delta W}[0]} = \frac{\frac{\delta G_1[g]}{\delta Z} \Big|_{g=0}}{\frac{\delta G_2[g]}{\delta Z} \Big|_{g=0}} \tag{108}$$

Using this formula for evaluating (103), we obtain

$$G_{k|k}[h] = \frac{G_{k|k-1}(p_{k|k-1}[h(1-p_D)])}{G_{k|k-1}(p_{k|k-1}[1-p_D])} \times \frac{\frac{\delta}{\delta Z} \log \left( G_{FA}(p_{FA}[g]) G_{k|k-1}(p_{k|k-1}[h(1-p_D+p_DPz[g])]) \right) \Big|_{g=0}}{\frac{\delta}{\delta Z} \log \left( G_{FA}(p_{FA}[g]) G_{k|k-1}(p_{k|k-1}[1-p_D+p_DPz[g])]) \right) \Big|_{g=0}} \quad (109)$$

$$= \frac{\frac{\delta}{\delta Z} (G_{FA}[g] G_{k|k-1}[h(1-p_D+p_DPz[g])]) \Big|_{g=0}}{\frac{\delta}{\delta Z} (G_{FA}[g] G_{k|k-1}[1-p_D+p_DPz[g]) \Big|_{g=0}} \quad (110)$$

which is the formula for CPHD for standard targets (See [3, Equations (113) and (114)]).

## 6 Proof of the Main Equation (22)

The proof is by induction as in [1]. The formula can be seen to be satisfied for first and second order derivatives in (18) and (21) respectively. We assume that for  $|Z| = m$  the formula is satisfied as below.

$$\frac{\delta}{\delta Z} F[g, h] = F[g, h] \left( \prod_{z' \in Z} p_{FA}(z') \right) \sum_{\mathcal{P} \setminus Z} \prod_{W \in \mathcal{P}} \left( \zeta_{FA}^{(|W|)}(p_{FA}[g]) + \sum_{Q \setminus W} \zeta_{k|k-1}^{(|Q|)} \prod_{V \in Q} \eta_V[g, h] \right) \quad (111)$$

Now we need to show that the formula is satisfied for  $Z \cup z_*$ .

$$\begin{aligned} \frac{\delta}{\delta z_*} \frac{\delta}{\delta Z} F[g, h] &= \frac{\delta}{\delta z_*} \left( F[g, h] \left( \prod_{z' \in Z} p_{FA}(z') \right) \sum_{\mathcal{P} \setminus Z} \prod_{W \in \mathcal{P}} \left( \zeta_{FA}^{(|W|)}(p_{FA}[g]) + \sum_{Q \setminus W} \zeta_{k|k-1}^{(|Q|)} \prod_{V \in Q} \eta_V[g, h] \right) \right) \quad (112) \\ &= F[g, h] \left( \prod_{z' \in Z \cup z_*} p_{FA}(z') \right) \left( \zeta_{FA}^{(1)}(p_{FA}[g]) \right. \\ &\quad \left. + \zeta_{k|k-1}^{(1)} \eta_{\{z_*\}}[g, h] \right) \\ &\quad \times \sum_{\mathcal{P} \setminus Z} \prod_{W \in \mathcal{P}} \left( \zeta_{FA}^{(|W|)}(p_{FA}[g]) + \sum_{Q \setminus W} \zeta_{k|k-1}^{(|Q|)} \prod_{V \in Q} \eta_V[g, h] \right) \\ &\quad + F[g, h] \left( \prod_{z' \in Z \cup z_*} p_{FA}(z') \right) \sum_{\mathcal{P} \setminus Z} \left( \prod_{W \in \mathcal{P}} \left( \zeta_{FA}^{(|W|)}(p_{FA}[g]) + \sum_{Q \setminus W} \zeta_{k|k-1}^{(|Q|)} \prod_{V \in Q} \eta_V[g, h] \right) \right) \\ &\quad \times \sum_{W \in \mathcal{P}} \frac{\left( \zeta_{FA}^{(|W|+1)}(p_{FA}[g]) + \sum_{Q \setminus W} \left( \prod_{V \in Q} \eta_V[g, h] \right) \left( \zeta_{k|k-1}^{(|Q|+1)} \eta_{\{z_*\}}[g, h] + \sum_{V \in Q} \frac{\eta_{V \cup z_*}[g, h]}{\eta_V[g, h]} \right) \right)}{\zeta_{FA}^{(|W|)}(p_{FA}[g]) + \sum_{Q \setminus W} \zeta_{k|k-1}^{(|Q|)} \prod_{V \in Q} \eta_V[g, h]} \quad (113) \\ &= F[g, h] \left( \prod_{z' \in Z \cup z_*} p_{FA}(z') \right) \sum_{\mathcal{P} \setminus Z} \prod_{W \in \mathcal{P} \cup \{z_*\}} \left( \zeta_{FA}^{(|W|)}(p_{FA}[g]) + \sum_{Q \setminus W} \zeta_{k|k-1}^{(|Q|)} \prod_{V \in Q} \eta_V[g, h] \right) \\ &\quad + F[g, h] \left( \prod_{z' \in Z \cup z_*} p_{FA}(z') \right) \sum_{\mathcal{P} \setminus Z} \left( \prod_{W \in \mathcal{P}} \left( \zeta_{FA}^{(|W|)}(p_{FA}[g]) + \sum_{Q \setminus W} \zeta_{k|k-1}^{(|Q|)} \prod_{V \in Q} \eta_V[g, h] \right) \right) \end{aligned}$$

$$\begin{aligned}
& \times \sum_{W \in \mathcal{P}} \frac{\left( \zeta_{FA}^{(|W|+1)}(p_{FA}[g]) + \sum_{Q \subseteq W} \zeta_{k|k-1}^{(|Q|+1)} \prod_{V \in Q \cup \{z_\star\}} \eta_V[g, h] \right)}{\zeta_{FA}^{(|W|)}(p_{FA}[g]) + \sum_{Q \subseteq W} \zeta_{k|k-1}^{(|Q|)} \prod_{V \in Q} \eta_V[g, h]} \quad (114) \\
& = F[g, h] \left( \prod_{z' \in Z \cup z_\star} p_{FA}(z') \right) \sum_{\mathcal{P} \subseteq Z} \prod_{W \in \mathcal{P} \cup \{z_\star\}} \left( \zeta_{FA}^{(|W|)}(p_{FA}[g]) + \sum_{Q \subseteq W} \zeta_{k|k-1}^{(|Q|)} \prod_{V \in Q} \eta_V[g, h] \right) \\
& \quad + F[g, h] \left( \prod_{z' \in Z \cup z_\star} p_{FA}(z') \right) \sum_{\mathcal{P} \subseteq Z} \left( \prod_{W \in \mathcal{P}} \left( \zeta_{FA}^{(|W|)}(p_{FA}[g]) + \sum_{Q \subseteq W} \zeta_{k|k-1}^{(|Q|)} \prod_{V \in Q} \eta_V[g, h] \right) \right) \\
& \quad \times \sum_{W \in \mathcal{P}} \frac{\zeta_{FA}^{(|W|+1)}(p_{FA}[g]) + \sum_{Q \subseteq W \cup \{z_\star\}} \zeta_{k|k-1}^{(|Q|)} \prod_{V \in Q} \eta_V[g, h]}{\zeta_{FA}^{(|W|)}(p_{FA}[g]) + \sum_{Q \subseteq W} \zeta_{k|k-1}^{(|Q|)} \prod_{V \in Q} \eta_V[g, h]} \quad (115) \\
& = F[g, h] \left( \prod_{z' \in Z \cup z_\star} p_{FA}(z') \right) \sum_{\mathcal{P} \subseteq Z \cup \{z_\star\}} \prod_{W \in \mathcal{P}} \left( \zeta_{FA}^{(|W|)}(p_{FA}[g]) + \sum_{Q \subseteq W} \zeta_{k|k-1}^{(|Q|)} \prod_{V \in Q} \eta_V[g, h] \right) \quad (116)
\end{aligned}$$

which completes the proof.

## References

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