GEODESICS AND DISTANCE ON THE RIEMANNIAN MANIFOLD OF RIEMANNIAN METRICS

Abstract. Given a fixed closed manifold M, we exhibit an explicit formula for the distance function of the canonical L^2 Riemannian metric on the manifold of all smooth Riemannian metrics on M. Additionally, we examine the completion of the manifold of metrics with respect to the L^2 metric and show that there exists a unique minimal path between any two points. This path is also given explicitly.

1. Introduction

In this paper, we give explicit formulas for the distance between Riemannian metrics, as measured by the canonical L^2 Riemannian metric on the manifold of all metrics over a given closed manifold. We also show that, at least on the metric completion of the manifold of metrics, there exists a unique geodesic connecting any two given points.

Fix any closed manifold M of dimension n, and consider the space $\mathcal{M}$ of all C^∞-smooth Riemannian metrics on M. This space carries a canonical weak Riemannian metric known as the L^2 metric (defined in Sect. 2.2). The L^2 metric itself has many interesting local and infinitesimal properties: for example, Freed–Groisser [FG89] and Gil-Medrano–Michor [GMM91] have shown that its sectional curvatures are nonpositive, and the geodesic equation on $\mathcal{M}$ is explicitly solvable. The L^2 metric has also found numerous applications, for example in the study of moduli spaces. Ebin [Ebi70] used it to construct a slice for the action of the diffeomorphism group on $\mathcal{M}$ (which thus serves a local model for the moduli space of Riemannian metrics). Fischer and Tromba [Tro92] have used the L^2 metric to study Teichmüller spaces of Riemann surfaces, where it naturally gives rise to the well-known Weil-Petersson metric.

In our own work [Cla10, Claa, Clab], we have focused on the global geometry of the L^2 metric, studying the distance it induces between Riemannian metrics on M. This approach, aside from its intrinsic interest, is perhaps most suited to questions related to the convergence of Riemannian manifolds. In fact, Anderson [And92] has used the L^2 metric to study spaces of Einstein metrics. (He refers to the distance function of the L^2 metric on $\mathcal{M}$ as the extrinsic L^2 metric, because he considers this instead of the intrinsic Riemannian distance obtained from restricting the L^2 metric to the submanifold of Einstein metrics.) Studying the global geometry of $\mathcal{M}$ with the L^2 metric is made significantly more difficult by the fact that this is a weak Riemannian metric on an infinite-dimensional manifold (cf. Sect. 2.1). In this setting, many essential results of finite-dimensional Riemannian geometry fail to hold. The Hopf-Rinow Theorem, for example, does not hold in general. Additionally, given any point $g_0 \in \mathcal{M}$, there exist other points at arbitrarily close distance to g_0 that are not in the image of the exponential mapping at g_0. (This last point is directly implied by, for example, [Claa, Lemma 5.11], though it is also easy to see from the work of Gil-Medrano–Michor [GMM91, Rmk. 3.5].)

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As far as convergence of Riemannian manifolds is concerned, one appeal of the L^2 metric is that it provides a very weak notion of convergence. Another appeal is that we have previously shown that convergence in the L^2 metric implies a kind of uniform convergence of the volume forms—which hints that it could be suited for studying convergence of metric measure spaces. Unfortunately, when considered on the full space $\mathcal{M}$, convergence in the L^2 metric is perhaps *too* weak—it does not imply any more synthetic-geometric notion of convergence, such as Gromov–Hausdorff convergence. (For proofs and a more detailed discussion of these facts, we refer to [Clab, Sect. 4.3].) However, as Anderson's work showed, restricted to spaces of Einstein metrics, convergence in the L^2 metric in fact does imply Gromov–Hausdorff convergence (or stronger). An open question is what other subspaces of $\mathcal{M}$ might have this desirable property.

We hope that the formulas given here may serve to provide further applications of the L^2 metric in the area of convergence of Riemannian manifolds.

The paper is organized as follows. In Section 2, we set up the necessary preliminaries, both on L^2 metrics on spaces of sections in general, as well as on the L^2 metric on $\mathcal{M}$ in particular. In Section 3, we find a simplified description for the L^2 distance between metrics, which transforms the problem from finding the infimum of lengths of paths in the infinite-dimensional space $\mathcal{M}$ into a solvable finite-dimensional problem. In Section 4, we show that there exists a unique geodesic connecting any two given metrics in the completion of $\mathcal{M}$. We also write down an explicit formula for this geodesic, which in turn allows us to make the formula for the L^2 distance between metrics explicitly computable. Finally, in Section 5 we outline some open problems regarding the L^2 metric that we find to be of interest.

We have attempted to make the paper as self-contained as possible for the convenience of the reader; however, for conciseness we rely on certain key results from previous works.

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2. Preliminaries

2.1. L^2 **metrics.** In this subsection dealing with general L^2 metrics on spaces of sections, we follow the definitions and results of Freed–Groisser [FG89, Appendix].

Let M be a smooth, closed manifold. Let $\pi : E \to M$ be a smooth fiber bundle, and denote the fiber over $x \in M$ by E_x. Suppose we are given

1. a smooth volume form μ on M with $\mathrm{Vol}(M, \mu) = 1$, and
2. a smooth Riemannian metric g defined on vectors in the vertical tangent bundle $T^v E$.

Let $\mathcal{E}$ denote a space of sections of E, where we allow the possibilities

1. $\mathcal{E} = \Gamma^s(E)$, the space of Sobolev sections $M \to E$ with L^2-integrable weak derivatives up to order s. Here we require $s > n/2$ if $E \to M$ is not a vector bundle.
2. $\mathcal{E} = \Gamma(E)$, the space of smooth sections $M \to E$.

By standard results on mapping spaces, $\mathcal{E}$ is a manifold in either of these cases [Pal68], [Ham82, Example 4.1.2]. (In case 1, it is a separable Hilbert manifold, and in case 2, it is a Fréchet manifold.) With this data, we can define an L^2-type Riemannian metric on $\mathcal{E}$ as follows. The tangent space at $\sigma \in \mathcal{E}$ is identified with the space of vertical vector fields "along σ", that is, with the space of sections of the pulled-back bundle $\sigma^* T^v E$. Now, for

$X, Y \in T_\sigma \mathcal{E}$, define the L^2 metric by

$$(X, Y)_\sigma := \int_M g(\sigma(x))(X(x), Y(x))\, d\mu(x).$$

We denote by d the distance function induced by $(\cdot, \cdot)$ on $\mathcal{E}$, and by d_x the distance function induced by g on E_x. Then d is a pseudometric and d_x is a metric (in the sense of metric spaces). Note that $(\cdot, \cdot)$ is in general a weak Riemannian metric on $\mathcal{E}$, that is, for any σ, the topology induced by $(\cdot, \cdot)_\sigma$ on $T_\sigma \mathcal{E}$ is weaker than the manifold topology. In this case, it is in principle possible that d is not a metric in that it could fail to separate points. There are known examples of this due to the work of Michor–Mumford [MM06, MM05], where weak Riemannian metrics are constructed for which the induced distance between *any* two points is always zero. However, Theorem 1 below will show that L^2 metrics as we have defined them do not suffer from this pathology.

Thinking of E as a bundle of metric spaces $\cup_{x \in M}(E_x, d_x)$ over M, we can define an L^p metric Ω_p on $\mathcal{E}$ by

$$\Omega_p(\sigma, \tau) := \left(\int_M d_x(\sigma(x), \tau(x))^p \, d\mu(x) \right)^{1/p}. \tag{1}$$

Note that Ω_p is indeed a metric (in the sense of metric spaces) on $\mathcal{E}$. All the required properties are easily implied from those of d_x. Only the triangle inequality is not immediately obvious—but this inequality follows, as in the case of an L^p norm, from the triangle inequality for d_x and Hölder's inequality. We can think of the L^p completion of $\mathcal{E}$ in terms of this metric—it consists of all measurable sections of the bundle $\cup_{x \in M}\overline{(E_x, d_x)}$ (where $\overline{(E_x, d_x)}$ of course denotes the completion) that are at finite Ω_p-distance from any fixed section $\sigma \in \mathcal{E}$.

The following theorem gives a positive lower bound for the distance, with respect to d, between distinct elements of $\mathcal{E}$. In the proof, and throughout the rest of the paper, a prime will denote the partial derivative in the variable t.

Theorem 1. *The following inequality holds for any path σ_t, $t \in [0, 1]$, in $\mathcal{E}$:*

$$L_{(\cdot,\cdot)}(\sigma_t)^2 \geq \int_M L_g(\sigma_t(x))^2 \, d\mu, \tag{2}$$

where on the left-hand side, we measure the length in $\mathcal{E}$ with respect to $(\cdot, \cdot)$, while on the right-hand side, we measure the length in E_x with respect to g. In particular, for any $\sigma, \tau \in \mathcal{E}$, we have $d(\sigma, \tau) \geq \Omega_2(\sigma, \tau)$, and so d is a metric on $\mathcal{E}$.

Proof. Without loss of generality, suppose that σ_t is parametrized proportionally to $(\cdot, \cdot)$-arc length. In this case, by a well-known application of Hölder's inequality, we have $L_{(\cdot,\cdot)}(\sigma_t)^2 = E_{(\cdot,\cdot)}(\sigma_t)$, where $E_{(\cdot,\cdot)}$ denotes the energy of the path. On the other hand, using Fubini's theorem followed by another application of Hölder's inequality (this time to the metric g), we have

$$\begin{aligned} E_{(\cdot,\cdot)}(\sigma_t) &= \int_0^1 \int_M g(\sigma_t(x))(\sigma_t'(x), \sigma_t'(x))\, d\mu\, dt = \int_M \int_0^1 g(\sigma_t(x))(\sigma_t'(x), \sigma_t'(x))\, dt\, d\mu \\ &= \int_M E_g(\sigma_t(x))\, dx \geq \int_M L_g(\sigma_t(x))^2 \, d\mu. \end{aligned}$$

This proves (2), from which $d(\sigma, \tau) \geq \Omega_2(\sigma, \tau)$ follows directly. ∎

Now that we have set up the situation for a general L^2 metric, we turn to the main focus of this paper, when $\mathcal{E}$ is the space of smooth Riemannian metrics.

2.2. **Preliminaries on the manifold of metrics.** For any point x in our closed base manifold M, let $\mathcal{S}_x := S^2T_x^*M$ denote the vector space of symmetric $(0,2)$-tensors based at x, and let $\mathcal{S} := \Gamma(S^2T^*M)$ denote the space of smooth, symmetric $(0,2)$-tensor fields. Similarly, denote by $\mathcal{M}_x := S_+^2T_x^*M$ the vector space of positive-definite, symmetric $(0,2)$-tensors at x, and by $\mathcal{M} := \Gamma(S_+^2T^*M)$ the space of smooth sections of this bundle. Thus, $\mathcal{M}$ is the space of smooth Riemannian metrics on M. In the notation of the previous section, we have $E = S_+^2T^*M$, $E_x = \mathcal{M}_x$, and $\mathcal{E} = \mathcal{M}$. Thus we see that $\mathcal{M}$ is a Fréchet manifold, and since $\mathcal{M}$ is an open subset of $\mathcal{S}$, we have a canonical identification of the tangent space $T_g\mathcal{M}$ with $\mathcal{S}$ for any $g \in \mathcal{M}$. (Similarly, the tangent space to $\mathcal{M}_x$ at any $a \in \mathcal{M}_x$ is identified with $\mathcal{S}_x$; thus we have $T_a^v(S_+^2T^*M) \cong \mathcal{S}_x$.)

Any element $\tilde{g} \in \mathcal{M}$ gives rise to a natural scalar product on $T_{\tilde{g}}\mathcal{M} \cong \mathcal{S}$ as follows. For $h, k \in \mathcal{S}$, the canonical scalar product that $\tilde{g}$ induces on $(0,2)$-tensors is

$$\operatorname{tr}_{\tilde{g}}(hk) = \operatorname{tr}(\tilde{g}^{-1}h\tilde{g}^{-1}k) = \tilde{g}^{ij}h_{il}\tilde{g}^{lm}k_{jm},$$

where by expressions like $\tilde{g}^{-1}h$ we of course mean the $(1,1)$-tensor obtained by raising an index of h using $\tilde{g}$. Then $\operatorname{tr}_{\tilde{g}}(hk)$ is a function on M, and by integrating it with respect to the volume form $\mu_{\tilde{g}}$ of $\tilde{g}$, we get a scalar product

$$(h,k)_{\tilde{g}} := \int_M \operatorname{tr}_{\tilde{g}}(hk)\, d\mu_{\tilde{g}}. \tag{3}$$

This L^2 scalar product fits into the framework of the last subsection as follows. For the rest of the paper, we fix some arbitrary reference metric $g \in \mathcal{M}$ that has total volume $\operatorname{Vol}(M,g) = 1$. Given a tensor field $h \in \mathcal{S}$ or a tensor $b \in \mathcal{S}_x$, denote by the capital letter the $(1,1)$-tensor obtained by raising an index using g, i.e., $H = g^{-1}h$ and $B = g(x)^{-1}b$. For each $x \in M$ and $a \in \mathcal{M}_x$, define a scalar product on $T_a\mathcal{M}_x$ (vertical vectors) by

$$\langle b, c\rangle_a := \operatorname{tr}_a(bc)\sqrt{\det A},$$

where $b, c \in T_a\mathcal{M}_x$. Thus, $\langle\cdot,\cdot\rangle$ gives a Riemannian metric on $\mathcal{M}_x$. For the remainder of the paper, we denote by $\mu := \mu_g$ the volume form of g. Then the scalar product (3) is given by the L^2 metric (in the sense of the last section)

$$(h,k)_{\tilde{g}} = \int_M \langle h(x), k(x)\rangle_{\tilde{g}(x)}\, d\mu.$$

As in the last subsection, we denote by d and d_x the distance functions of $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$, respectively. By Theorem 1, it is immediate that d is a metric on $\mathcal{M}$, a fact that we already proved in a less elegant way in [Cla10, Thm. 18].

For $\tilde{g} \in \mathcal{M}$ and $a \in \mathcal{M}_x$, we will denote the norms associated with $(\cdot,\cdot)_{\tilde{g}}$ and $\langle\cdot,\cdot\rangle_a$ by $\|\cdot\|_{\tilde{g}}$ and $|\cdot|_a$, respectively, throughout the remainder of the paper.

In [Claa], we determined the completion of $(\mathcal{M}, d)$, which we will denote in the following by $\overline{\mathcal{M}}$. We will summarize the relevant details of this here.

Let $\tilde{g} : M \to S^2T^*M$ be any measurable section that induces a positive *semi*definite scalar product on each tangent space of M. We call such a section a *measurable semimetric.* A measurable semimetric induces a measurable volume form (and hence a measure) on M using the usual formula $\mu_{\tilde{g}} := \sqrt{\det \tilde{g}}\, dx^1 \wedge \cdots \wedge dx^n$ in local coordinates. We denote by $\mathcal{M}_f$ the set of all measurable semimetrics on M that have finite volume, i.e., with $\int_M d\mu_{\tilde{g}} < \infty$. We also introduce an equivalence relation on $\mathcal{M}_f$ by saying $g_0 \sim g_1$ if and only if the following statement holds almost surely (with respect to the measure μ) on M: $g_0(x)$ fails to be positive semidefinite if and only if $g_1(x)$ fails to be positive semidefinite. We then have the following theorem.

Theorem 2 ([Claa, Thm. 5.17])**.** *There is a natural identification of $\overline{\mathcal{M}}$ with $\widehat{\mathcal{M}_f} := \mathcal{M}_f/\sim$.*

It will not be important in the following what the precise form of this identification is. In fact, we will not use Theorem 2 directly in this paper at all, but will instead use some consequences of it that we have worked out in previous papers. However, having this identification in mind will serve to help keep things conceptually clear. For the reader who is interested in the details of the construction of the identification in Theorem 2, we refer to [Claa].

We do point out, however, that we retain the notation d for the metric induced on the completion $\overline{\mathcal{M}}$ from $(\mathcal{M}, d)$. It will also be convenient to use the above identification to see d as a metric on $\widehat{\mathcal{M}_f}$, and as a *pseudo*metric on $\mathcal{M}_f$. Of course, for $g_0, g_1 \in \mathcal{M}_f$, we have $g_0 \sim g_1$ if and only if $d(g_0, g_1) = 0$.

In what follows, we will also be concerned with special subsets of $\mathcal{M}$ that have convenient properties. They are essentially subsets that are, in a pointwise sense, uniformly bounded away from infinity and the boundary of $\mathcal{M}$.

Definition 3. For $\tilde{g} \in \mathcal{M}$ and $x \in M$, let $\lambda^{\tilde{G}}_{\min}(x)$ denote the minimal eigenvalue of $\tilde{G}(x) = g(x)^{-1}\tilde{g}(x)$. A subset $\mathcal{U} \subset \mathcal{M}$ is called *amenable* if it is of the form

$$\mathcal{U} = \{\tilde{g} \in \mathcal{M} \mid \lambda^{\tilde{G}}_{\min}(x) \geq \zeta \text{ and } |\tilde{g}(x)|_{g(x)} \leq C \text{ for all } \tilde{g} \in \mathcal{U} \text{ and } x \in M\} \tag{4}$$

for some constants $C, \zeta > 0$.

We denote the closure of $\mathcal{U}$ in the L^2 *norm* $\|\cdot\|_g$ by $\mathcal{U}^0$; it consists of all measurable, symmetric $(0,2)$-tensors $\tilde{g}$ satisfying the bounds of (4) a.e.

Remark 4.

(1) Note that the preceding definition differs from that in our previous works (cf. [Claa, Def. 3.1], [Clab, Def. 2.11]). The above definition is coordinate-independent and therefore more satisfying. Additionally, the results we need from those previous works are valid for the definition here because of the following equivalence: If $\mathcal{U} \subset \mathcal{M}$ is amenable in this new sense, then there exist $\mathcal{U}', \mathcal{U}'' \subset \mathcal{M}$ that are amenable in the old sense, and such that $\mathcal{U}' \subset \mathcal{U} \subset \mathcal{U}''$.

(2) If $\mathcal{U} \subset \mathcal{M}$ is amenable, then $\mathcal{U}^0$ is *pointwise convex*, by which we mean the following. Let $g_0, g_1 \in \mathcal{U}^0$, and let ρ be any measurable function on M taking values between 0 and 1. Then $\rho g_0 + (1-\rho) g_1 \in \mathcal{U}^0$. This is straightforward to see by the concavity of the function mapping a matrix to its minimal eigenvalue, and the convexity of the norm $|\cdot|_{g(x)}$.

The following lemma was originally proved in [Claa, Lem. 3.3] for amenable subsets, but the same proof (which is more or less self-evident) works for L^2 closures of amenable subsets.

Lemma 5. *Let $\mathcal{U}$ be an amenable subset. Then there exists a constant $K > 0$ such that for all $\tilde{g} \in \mathcal{U}^0$,*

$$\frac{1}{K} \leq \left(\frac{\mu_{\tilde{g}}}{\mu}\right) \leq K, \tag{5}$$

where by $(\mu_{\tilde{g}}/\mu)$ *we denote the unique measurable function on* M *such that* $\mu_{\tilde{g}} = (\mu_{\tilde{g}}/\mu)\mu$.

To end this subsection, we have a somewhat unexpected and extremely useful result that bounds the distance between two semimetrics *uniformly* based on the intrinsic volume of the subset on which they differ.

Proposition 6 ([Clab, Prop. 2.20])**.** *Let* $g_0, g_1 \in \mathcal{M}_f$ *and* $A := \operatorname{carr}(g_1 - g_0)$*. Then*

$$d(g_0, g_1) \le C(n) \left(\sqrt{\operatorname{Vol}(A, g_0)} + \sqrt{\operatorname{Vol}(A, g_1)} \right),$$

where $C(n)$ *is a constant depending only on* $n = \dim M$*.*

3. $d = \Omega_2$ ON $\mathcal{M}$

In this section, we show that the distance function of the L^2 Riemannian metric is exactly given by the L^2-type metric Ω_2 that we defined in (1).

3.1. **Paths of degenerate metrics and Riemannian distances.** If $g_0, g_1 \in \mathcal{M}$ and g_t is a piecewise differentiable path in $\mathcal{M}$ between them, then $d(g_0, g_1) \le L(g_t)$. The goal of this subsection is to prove a similar inequality for certain paths of semimetrics in $\mathcal{M}_f$.

We first have to be precise about what $L(g_t)$ should mean if $g_t \in \mathcal{M}_f$. We denote by $\mathcal{M}_c \subset \mathcal{M}_f$ the set of all continuous Riemannian metrics on M. By $\mathcal{S}_c$, we denote the closure of $\mathcal{S}$ in the C^0 norm. For $\tilde{g} \in \mathcal{M}_f$, denote by $\mathcal{S}^0_{\tilde{g}}$ the set of measurable $(0,2)$-tensor fields h such that $h(x) = 0$ whenever $\tilde{g}(x)$ is not positive definite, and such that the quantity

$$\|h\|_{\tilde{g}} := \left(\int_{M \setminus X_{\tilde{g}}} \operatorname{tr}_{\tilde{g}}(h^2) \sqrt{\det \tilde{G}}\, d\mu \right)^{1/2}$$

is finite, where in the above $X_{\tilde{g}} \subseteq M$ denotes the set on which $\tilde{g}$ is not positive definite.

We will consider paths of (semi-)metrics g_t, $t \in [0,1]$, in both $\mathcal{M}_c$ and $\mathcal{M}_f$. We will call such a path g_t differentiable in $\mathcal{M}_c$ (resp. $\mathcal{M}_f$) if, for each $x \in M$, $g_t(x)$ is a differentiable path in $\mathcal{M}_x$ and, additionally, g'_t is contained in $\mathcal{S}_c$ (resp. $\mathcal{S}^0_{g_t}$) for all $t \in [0,1]$.

Definition 7. For $E \subseteq M$, we call a path g_t, $t \in [0,1]$, in $\mathcal{M}_f$ *continuous on* E if $g_t(x)|_E$ is continuous in x for all t. If $E = M$, we call g_t simply *continuous.*

To avoid confusion, we emphasize that a continuous path is one that is continuous in x for each t, and a differentiable path is one that is differentiable in t for each x.

Let g_t, $t \in [0,1]$, be a path in $\mathcal{M}_f$ or $\mathcal{M}_c$ that is piecewise differentiable. We denote by $L(g_t)$ the length of g_t as measured in the naive "Riemannian" way:

$$L(g_t) = \int_0^1 \|g'_t\|_{g_t}\, dt.$$

When we refer to the length $L(a_t)$ of a path a_t in $\mathcal{M}_x$, we implicitly mean the length with respect to $\langle \cdot, \cdot \rangle$.

It is easy to see (cf. also the proof of [Cla09, Cor. 3.16]) that the C^0 topology on $\mathcal{M}_c$ is stronger than the Riemannian L^2 topology. Let g_t be a piecewise differentiable path in $\mathcal{M}_c$ connecting two continuous metrics g_0 and g_1. It is intuitive, but perhaps not immediately clear, that using smooth approximations, one could show $d(g_0, g_1) \le L(g_t)$ as in the case of smooth metrics. We formalize this in the following lemma. The proof is straightforward, but we include some details for those readers unfamiliar with regularization of tensors on manifolds.

Lemma 8. *Let $g_0, g_1 \in \mathcal{M}_c$, and suppose that g_t, $t \in [0,1]$, is a piecewise differentiable path in $\mathcal{M}_c$ connecting them. Then $d(g_0, g_1) \leq L(g_t)$.*

Proof. Let $\{U_\alpha, \varphi_\alpha\}$ be a finite atlas of charts $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ for M. Choose a partition of unity p_α subordinate to this atlas. We denote the push-forward of g_t via φ_α by g_t^α and, by an abuse of notation, denote the locally-defined tensor obtained from restricting g_t to U_α by the same. We will regularize these metrics by convolution in local coordinates, letting ϕ be any function on $\mathbb{R}^n$ that has norm 1 in $L^1(\mathbb{R}^n)$ and that vanishes outside the unit ball. Defining $\phi_\epsilon(x) := \epsilon^{-n}\phi(x/\epsilon)$, we have that for all i, j, and α, the convolutions $(g_t^{\alpha,\epsilon})_{ij} := \phi_\epsilon * (g_t^\alpha)_{ij}$ and $(g_t^{\alpha,\epsilon})'_{ij} := (\phi_\epsilon * g_t^\alpha)'_{ij} = \phi_\epsilon * (g_t^\alpha)'_{ij}$ (the prime, as usual, denotes the partial derivative w.r.t. t) are smooth functions converging in the C^0 norm to $(g_t^\alpha)_{ij}$ and $(g_t^\alpha)'_{ij}$, respectively, as $\epsilon \to 0$. Furthermore, since we are dealing with a finite number of indices, for any given t we can choose $\epsilon > 0$ small enough that $(g_t^{\alpha,\epsilon})_{ij}$ and $(g_t^{\alpha,\epsilon})'_{ij}$ are uniformly C^0-close to g_t^α and $(g_t^\alpha)'$, respectively.

One easily sees that for each $\epsilon > 0$, $\|\phi_\epsilon\|_{L^1(\mathbb{R}^n)} = 1$, and also that this implies that convolution with ϕ_ϵ has operator norm 1 when viewed as a linear operator on $C^0(\mathbb{R}^n)$. From this, and the compactness of the time interval on which g_t is defined, one can straightforwardly conclude that $\epsilon > 0$ can be chosen small enough that $(g_t^{\alpha,\epsilon})_{ij}$ and $(g_t^{\alpha,\epsilon})'_{ij}$ are uniformly C^0-close to g_t^α and $(g_t^\alpha)'$, respectively, independently not just of i, j, and α, but also t.

Finally, using our partition of unity, we define $g_t^\epsilon := \sum_\alpha p_\alpha g_t^{\alpha,\epsilon}$ to get a path of Riemannian metrics connecting g_0^ϵ and g_1^ϵ. We have $(g_t^\epsilon)' = \sum_\alpha p_\alpha (g_t^{\alpha,\epsilon})'$, and so one sees that with ϵ small enough, $L(g_t^\epsilon)$ is arbitrarily close to $L(g_t)$. By the above-mentioned fact that the C^0 topology on $\mathcal{M}$ is stronger than the Riemannian L^2 topology, we also have that $d(g_0, g_0^\epsilon)$ and $d(g_1, g_1^\epsilon)$ can be made arbitrarily small, from which the desired result follows. □

Using the above result on paths of continuous metrics, we can prove what we need about paths in $\mathcal{M}_f$. We first briefly set up some notation, and then state the result in a lemma.

Definition 9. Let $E \subseteq M$ be any subset. We denote by $\chi(E)$ the characteristic (or indicator) function of E. The characteristic function of its complement is denoted by $\overline{\chi}(E) := \chi(M \setminus E)$.

Lemma 10. *Let $g_0, g_1 \in \mathcal{M}_c$, and let g_t, $t \in [0,1]$, be any smooth path in $\mathcal{M}_c$ from g_0 to g_1. Furthermore, let $E \subseteq M$ be any measurable subset.*

We define $\tilde{g}_t := \overline{\chi}(E) g_0 + \chi(E) g_t$; in particular $\tilde{g}_1 = \overline{\chi}(E) g_0 + \chi(E) g_1$. Then

$$d(g_0, \tilde{g}_1) \leq L(\tilde{g}_t).$$

Proof. For each $k \in \mathbb{N}$, choose an open set U_k and a closed set Z_k such that $Z_k \subseteq E \subseteq U_k$, and such that $\mu(U_k \setminus Z_k) < \frac{1}{k}$. (This is possible because the Lebesgue measure is regular.) We also choose continuous functions f_k with the properties that

(1) $0 \leq f_k \leq 1$,
(2) if $x \notin U_k$, then $f_k(x) = 0$, and
(3) if $x \in Z_k$, then $f_k(x) = 1$.

For each $t \in [0,1]$, let $\tilde{g}_t^k := f_k g_t + (1 - f_k) g_0$. Thus, we have that in particular, $\tilde{g}_1^k$ coincides with g_1 on Z_k and with g_0 on $M \setminus U_k$. Our goal is to show that $\lim_{k\to\infty} L(\tilde{g}_t^k) \leq L(\tilde{g}_t)$ and $d(\tilde{g}_1^k, \tilde{g}_1) \to 0$ as $k \to \infty$, as we can then conclude from the triangle inequality and Lemma 8 that

$$d(g_0, \tilde{g}_1) \leq d(g_0, \tilde{g}_1^k) + d(\tilde{g}_1^k, \tilde{g}_1) \leq L(\tilde{g}_t^k) + d(\tilde{g}_1^k, \tilde{g}_1).$$

The statement of the lemma then follows by passing to the limit on the right.

We begin with the claim that $\lim_{k\to\infty} L(\tilde{g}_t^k) \le L(\tilde{g}_t)$. Since M and $[0,1]$ are compact, we have that

$$N := \max_{x\in M, t\in[0,1]} |g_t'(x)|^2_{g_t(x)} < \infty.$$

(Recall that by the definition of a smooth path in $\mathcal{M}_c$, we have $g_t' \in \mathcal{S}_c$ for all t.) Therefore, noting that $(\tilde{g}_t^k)' = g_t'$ on Z_k and $(\tilde{g}_t^k)' \equiv 0$ on $M \setminus U_k$, we may estimate

$$\begin{aligned}
\left\|(\tilde{g}_t^k)'\right\|^2_{\tilde{g}_t^k} &= \int_{Z_k} |g_t'|^2_{g_t}\, d\mu + \int_{U_k\setminus Z_k} |f_k g_t'|^2_{f_k g_t}\, d\mu \\
&= \|\chi(Z_k) g_t'\|^2_{g_t} + \int_{U_k\setminus Z_k} \operatorname{tr}_{f_k g_t}((f_k g_t')^2)\sqrt{\det(f_k G_t)}\, d\mu \\
&= \|\chi(Z_k) g_t'\|^2_{g_t} + \int_{U_k\setminus Z_k} f_k^{n/2} \operatorname{tr}_{g_t}((g_t')^2)\sqrt{\det G_t}\, d\mu \\
&\le \|\chi(E) g_t'\|^2_{g_t} + \int_{U_k\setminus Z_k} N\, d\mu.
\end{aligned}$$

The first term in the last line is just $\|\tilde{g}_t'\|^2_{\tilde{g}_t}$, since $\tilde{g}_t = g_t$ on E. The second term is just $N \cdot \mu(U_k \setminus Z_k) < N/k$, which converges to zero as $k \to \infty$. Thus, we have that $\left\|\tilde{g}_t^k\right\|_{\tilde{g}_t^k} \le \|\tilde{g}_t\|_{\tilde{g}_t} + N/k$ for each $t \in [0,1]$, which implies the claim that $\lim_{k\to\infty} L(\tilde{g}_t^k) \le L(\tilde{g}_t)$.

We now move on to the claim that $\lim_{k\to\infty} d(\tilde{g}_1^k, \tilde{g}_1) = 0$. Since g_0 and g_1 are continuous metrics, it is clear that we can find an amenable subset $\mathcal{U}$ such that $g_0, g_1 \in \mathcal{U}^0$. But we also know that at each point, $\tilde{g}_1^k$ and $\tilde{g}_1$ are linear combinations of g_0 and g_1 with coefficients between zero and one. Hence, by the pointwise convexity of L^2 closures of amenable subsets (cf. Remark 4(2)), $\tilde{g}_1^k, \tilde{g}_1 \in \mathcal{U}^0$ for all $k \in \mathbb{N}$. Thus, by Lemma 5, there exists a constant K such that

$$\left(\frac{\mu_{\tilde{g}_1^k}}{\mu}\right) \le K \text{ for all } k \in \mathbb{N} \text{ and } \left(\frac{\mu_{\tilde{g}_1}}{\mu}\right) \le K. \tag{6}$$

Using this, Proposition 6, and the fact that $\tilde{g}_1^k$ and $\tilde{g}_1$ differ only on $U_k \setminus Z_k$, we can conclude

$$d(\tilde{g}_1^k, \tilde{g}_1) \le C(n)\left(\sqrt{\operatorname{Vol}(U_k \setminus Z_k, \tilde{g}_1^k)} + \sqrt{\operatorname{Vol}(U_k \setminus Z_k, \tilde{g}_1)}\right) \le 2C(n)\sqrt{\frac{K}{k}}.$$

This proves the second claim and so, as noted above, the statement of the lemma follows. $\square$

In what follows, we will have to deal with reparametrizations of paths. Given a path in $\mathcal{M}_f$ (or in any space of sections), one can reparametrize globally, in that one replaces g_t, $t \in [0,1]$, with $g_{\varphi(t)}$, for some appropriate $\varphi : [0,1] \to [0,1]$. One can also "reparametrize" pointwise, in that one uses $g_{\varphi_x(t)}$, where for each $x \in M$, $\varphi_x : [0,1] \to [0,1]$ is a function with the appropriate properties. Of course, the latter changes the image of the path in $\mathcal{M}_f$, but for our purposes it can do so in advantageous ways. The next definition deals with the specific reparametrizations we will need.

Definition 11. Let g_t, $t \in [0,1]$, be a path in $\mathcal{M}_f$. By the *pointwise reparametrization of g_t proportional to arc length*, we mean the path in $\tilde{g}_t$ in $\mathcal{M}_f$, $t \in [0,1]$, where for each $x \in M$, $\tilde{g}_t(x)$ is the path obtained from $g_t(x)$ by reparametrization proportional to $\langle\cdot,\cdot\rangle$-arc length.

Given this definition, the following lemma is essentially self-evident.

Lemma 12. *Let $g_0, g_1 \in \mathcal{M}_f$, and let g_t be a piecewise differentiable path in $\mathcal{M}_f$ connecting g_0 and g_1. Suppose g_t fails to have a two-sided t-derivative at times $0 = t_0 < t_1 < \cdots < t_k = 1$. If g_t is continuous on $E \subseteq M$ for all $t \in [0,1]$, and $L(g_t(x)|_{[t_i,t_{i+1}]})$ is continuous as a function of x on E for all $i = 0, \ldots, k-1$, then the path obtained from g_t via pointwise reparametrization proportional to arc length is continuous on E.*

3.2. **Proof that** $d = \Omega_2$**.** We now get into the heavy lifting of this section. We will need two rather technical lemmas to get from the restricted situation of Lemma 10 to the desired general result. In the following, we will always denote by $B_\delta(x)$ the closed geodesic ball around $x \in M$ with radius δ (w.r.t. the fixed reference metric g).

Lemma 13. *Let any $g_0, g_1 \in \mathcal{M}$ and $\epsilon > 0$ be given. Then there exists a $\delta = \delta(\epsilon, g_0, g_1) > 0$ with the property that given any $x_0 \in M$, we can find a path $g_{x_0,t}$ in $\mathcal{M}_f$, for $t \in [-\epsilon, 1+\epsilon]$, from g_0 to $\overline{\chi}(B_\delta(x_0))g_0 + \chi(B_\delta(x_0))g_1$ such that for each $x \in B_\delta(x_0)$, we have*

$$\left|g'_{x_0,t}(x)\right|_{g_{x_0,t}} < \begin{cases} 1, & t \in [-\epsilon, 0) \cup [1, 1+\epsilon], \\ d_x(g_0(x), g_1(x)) + 3\epsilon, & t \in [0,1). \end{cases} \tag{7}$$

Furthermore, for each t, $g_{x_0,t}$ is constant on $M \setminus B_\delta(x_0)$ and is continuous on $B_\delta(x_0)$.

Proof. For a given $x_0 \in M$, we may choose a smooth path $a_{x_0,t}$, $t \in [0,1]$, in $\mathcal{M}_{x_0}$ connecting $g_0(x_0)$ and $g_1(x_0)$ that has length

$$L(a_{x_0,t}) < d_{x_0}(g_0(x_0), g_1(x_0)) + \epsilon. \tag{8}$$

Furthermore, this path can be chosen in such a way that there exist constants $\zeta, \tau > 0$, depending on g_0, g_1, and ϵ but not on x_0, such that

$$a_{x_0,t} \in \mathcal{M}_{x_0}^{\zeta,\tau} := \{a \in \mathcal{M}_{x_0} \mid \det A \geq \zeta \text{ and } |a|_{g(x_0)} \leq \tau \text{ for all } 1 \leq i,j \leq n\}$$

for all $x_0 \in M$ and $t \in [0,1]$. (This relies on the fact that g_0 and g_1 are smooth metrics, and so are contained in a common compact subset of $S^2_+ T^*M$.)

For the rest of the proof, when we refer to geodesics and the Levi-Civita connection ∇ on M, we mean those belonging to our fixed reference metric g. Now, let δ be small enough that for any $x_0 \in M$ and any $x \in B_\delta(x_0)$, there exists a unique minimal geodesic (up to reparametrization) from x_0 to x.

The Levi-Civita connection ∇ can be extended to all tensor fields, and in particular to $T^*M \otimes T^*M$. A brief calculation shows that if $h \in S^2T^*M$ and X is a vector field on M, then $\nabla_X h \in S^2T^*M$. (That is, symmetry is preserved.) Therefore ∇ induces a connection $\overline{\nabla}$ on the vector bundle S^2T^*M.

For each $x_0 \in M$ and $x \in B_\delta(x_0)$, we denote by $P_{x_0,x}$ the parallel transport with respect to $\overline{\nabla}$ along the minimal geodesic from x_0 to x. In local vector bundle coordinates for S^2T^*M, the parallel transport of an element of $\mathcal{S}_{x_0} = S^2T^*_{x_0}M$ is the solution of a first-order linear ODE with coefficients depending smoothly upon x_0 and x. We know that $P_{x_0,x}$ is a linear isometry (w.r.t. the scalar product induced by g) between $\mathcal{S}_{x_0}$ and $\mathcal{S}_x$, so $P_{x_0,x}(a)$ depends smoothly on $a \in s_{x_0}$. Furthermore, since solutions of ODEs behave continuously under perturbations of the coefficients, the mapping $(x_0, x) \mapsto P_{x_0,x}$ is continuous [CL55, Ch. 1, Thm. 7.4].

We next let $\tilde{a}_{x_0,t}(x)$ be the path in $S^2T^*_xM$ obtained from $a_{x_0,t}$ by parallel transport, i.e., $\tilde{a}_{x_0,t}(x) = P_{x_0,x}(a_{x_0,t})$. By the discussion above on the smoothness/continuity of parallel transport, $\tilde{a}_{x_0,t}(x)$ is smooth in the t variable and continuous in the x variable.

Now, since $(x_0, x, a) \mapsto P_{x_0,x}(a)$ is continuous, this mapping is uniformly continuous when restricted to the compact space

$$\bigcup_{x_0 \in M} \left(\{x_0\} \times \bigcup_{x \in B_\delta(x_0)} \mathcal{M}_x^{\zeta,\tau} \right). \tag{9}$$

Thus we may (by shrinking δ if necessary) assume that $P_{x_0,x}(\mathcal{M}_{x_0}^{\zeta,\tau}) \subset \mathcal{M}_x$ for each $x_0 \in M$ and $x \in B_\delta(x_0)$.

Since we have assumed that each path $a_{x_0,t}$ is contained in $\mathcal{M}_{x_0}^{\zeta,\tau}$, this implies that $\tilde{a}_{x_0,t}(x)$ is a path in $\mathcal{M}_x$ running from $P_{x_0,x}(g_0(x_0))$ to $P_{x_0,x}(g_1(x_0))$. Again by continuity of parallel transport, by shrinking δ we may assume that

$$d_x(g_0(x), P_{x_0,x}(g_0(x_0))) \leq \eta \quad \text{and} \quad d_x(g_1(x), P_{x_0,x}(g_1(x_0))) \leq \eta \tag{10}$$

for any choice of $\eta > 0$, uniformly in x_0 and x. Furthermore, since the differential of a linear transformation is again the transformation itself, we have $\tilde{a}'_{x_0,t}(x) = P_{x_0,x}(a'_{x_0,t})$. Thus, by the above-mentioned continuity of $P_{x_0,x}(a)$, one easily sees that we can just as well shrink δ to get the following bound, uniform in x_0 and x:

$$L(\tilde{a}_{x_0,t}(x)) < L(a_{x_0,t}) + \epsilon < d_{x_0}(g_0(x_0), g_1(x_0)) + 2\epsilon. \tag{11}$$

Finally, we note that $d_x(g_0(x), P_{x_0,x}(g_0(x_0)))$, $d_x(g_1(x), P_{x_0,x}(g_1(x_0)))$, and $L(\tilde{a}_{x_0,t}(x))$ are all continuous in x, since all of the quantities involved in their computation are continuous.

For any $x_0 \in M$, $x \in B_\delta(x_0)$ and $\alpha \in \{0, 1\}$, we let $\sigma^\alpha_{x_0,x,t}$, $t \in [0, 1]$, be the geodesic in $\mathcal{M}_x$ connecting $g_\alpha(x)$ and $P_{x_0,x}(g_\alpha(x_0))$. We assume that this geodesic is parametrized proportionally to arc length; note that in this case, $\sigma^\alpha_{x_0,x,t}$ varies continuously in x on $B_\delta(x_0)$ for fixed α, x_0, and t. Referring to (10), we see that for given x and α, if $\eta > 0$ is small enough, then this geodesic indeed exists and is unique. In fact, such a positive η can be found independently of x and α since $g_0(x)$ and $g_1(x)$ lie in the compact region $\cup_x \mathcal{M}_x^{\zeta,\tau} \subset S^2_+ T^* M$. We may shrink δ if necessary to insure that (10) is satisfied for this η.

Define metrics $\hat{g}_0$ and $\hat{g}_1$ by

$$\hat{g}_0(x) := \begin{cases} g_0(x) & \text{if } x \notin B_\delta(x_0), \\ P_{x_0,x}(g_0(x_0)) & \text{if } x \in B_\delta(x_0), \end{cases} \quad \text{and} \quad \hat{g}_1(x) := \begin{cases} g_0(x) & \text{if } x \notin B_\delta(x_0), \\ P_{x_0,x}(g_1(x_0)) & \text{if } x \in B_\delta(x_0). \end{cases}$$

(Note that both metrics equal g_0 on $M \setminus B_\delta(x_0)$.) We consider the paths

$$g^0_{x_0,t}(x) := \begin{cases} g_0(x) & \text{if } x \notin B_\delta(x_0), \\ \sigma^0_{x_0,x,t} & \text{if } x \in B_\delta(x_0), \end{cases} \quad \text{and} \quad g^1_{x_0,t}(x) := \begin{cases} g_0(x) & \text{if } x \notin B_\delta(x_0), \\ \sigma^1_{x_0,x,t} & \text{if } x \in B_\delta(x_0). \end{cases}$$

Then these are smooth paths in $\mathcal{M}_f$ that are continuous on $B_\delta(x_0)$. The path $g^0_{x_0,t}$ connects g_0 and $\hat{g}_0$, while $g^1_{x_0,t}$ connects $\overline{\chi}(B_\delta(x_0))g_0 + \chi(B_\delta(x_0))g_1$ and $\hat{g}_1$. Furthermore, by shrinking δ to obtain $\eta < \epsilon$, we have by (10) that for each $x_0 \in M$ and $x \in B_\delta(x_0)$, we have

$$L(g^\alpha_{x_0,t}(x)) = d_x(g_\alpha(x), P_{x_0,x}(g_\alpha(x_0))) < \epsilon \tag{12}$$

for $\alpha = 0, 1$. We also have that on $B_\delta(x_0)$, $L(g^\alpha_{x_0,t}(x)) = d_x(g_\alpha(x), P_{x_0,x}(g_\alpha(x_0)))$ is continuous in x, as noted after (11).

We define a path $\hat{g}_{x_0,t}$ by

$$\hat{g}_{x_0,t}(x) := \begin{cases} g_0(x) & \text{if } x \notin B_\delta(x_0), \\ \tilde{a}_{x_0,t}(x) & \text{if } x \in B_\delta(x_0). \end{cases} \tag{13}$$

Then $\hat{g}_{x_0,t}$ is a path from $\hat{g}_0$ to $\hat{g}_1$. As noted above, both $\tilde{a}_{0,t}(x)$ and $L(\tilde{a}_{x_0,t}(x))$ vary continuously with x on $B_\delta(x_0)$.

Thus, by the following concatenation,

$$\bar{g}_{x_0,t} := g^0_{x_0,t} * \hat{g}_{x_0,t} * (g^1_{x_0,t})^{-1}, \tag{14}$$

we get a piecewise smooth path in $\mathcal{M}_f$ from g_0 to $\overline{\chi}(B_\delta(x_0))g_0 + \chi(B_\delta(x_0))g_1$ that in continuous on $B_\delta(x_0)$. (Here, $(g^1_{x_0,t})^{-1}$ denotes the path $g^1_{x_0,t}$ run through in reverse.) Let us assume that $\bar{g}_{x_0,t}$ is parametrized such that it runs through $g^0_{x_0,t}$ for $t \in [-\epsilon, 0]$, then $\hat{g}_{x_0,t}$ for $t \in [0,1]$, and finally $(g^1_{x_0,t})$ for $t \in [1, 1+\epsilon]$.

Denote by $g_{x_0,t}$ be the path obtained from $\bar{g}_{x_0,t}$ by pointwise reparametrizing each portion of the concatenation (14) proportionally to arc length. Then by Lemma 12 and the statements following (11), (12), and (13), $g_{x_0,t}$ is a piecewise smooth path in $\mathcal{M}_f$ that is continuous when restricted to $B_\delta(x_0)$, and by construction $g_{x_0,t}(x) = g_0(x)$ for all $t \in [-\epsilon, 1+\epsilon]$ if $x \notin B_\delta(x_0)$. For $x \in B_\delta(x_0)$, the estimates (11) and (12) give

$$\left| g'_{x_0,t}(x) \right|_{g_{x_0,t}} < \begin{cases} 1, & t \in [-\epsilon, 0) \cup [1, 1+\epsilon], \\ d_{x_0}(g_0(x_0), g_1(x_0)) + 2\epsilon, & t \in [0,1). \end{cases}$$

Finally, since g_0 and g_1 are smooth metrics, the function $x \mapsto d_x(g_0(x), g_1(x))$ is continuous. Therefore, we may assume that δ is small enough that $d_{x_0}(g_0(x_0), g_1(x_0)) < d_x(g_0(x), g_1(x)) + \epsilon$ for all $x \in B_\delta(x_0)$. This and the above inequality show that $g_{x_0,t}$ has all the desired properties. ☐

Lemma 14. *Let any $g_0, g_1 \in \mathcal{M}$ and $\epsilon > 0$ be given, and let $\delta = \delta(\epsilon, g_0, g_1) > 0$ be as in Lemma 13. Consider a finite collection of closed subsets $\{F_i \mid i = 1, \dots, N\}$ with the property that for each i, there exists $x_i \in F_i$ such that $F_i \subseteq B_{\delta'}(x_i)$ for some $0 < \delta' < \delta$, and such that $F_i \cap F_j = \emptyset$ for all $i \neq j$. We denote*

$$F := \bigcup_{i=1,\dots,n} F_i.$$

Then there exists a path $\tilde{g}_t$, for $t \in [-\epsilon, 1+\epsilon]$, from g_0 to $\tilde{g}_1 := \overline{\chi}(F)g_0 + \chi(F)g_1$ satisfying

$$L(\tilde{g}_t)^2 < (1+2\epsilon)\left[\Omega_2(g_0,g_1)^2 + 6\epsilon\Omega_1(g_0,g_1) + 9\epsilon^2 + 2\epsilon\right]. \tag{15}$$

Furthermore, $\tilde{g}_t$ satisfies the assumptions of Lemma 10, and so also

$$d(g_0, \tilde{g}_1)^2 < (1+2\epsilon)\left[\Omega_2(g_0,g_1)^2 + 6\epsilon\Omega_1(g_0,g_1) + 9\epsilon^2 + 2\epsilon\right]. \tag{16}$$

Proof. For each $i \in \mathbb{N}$, let $g_{i,t} := g_{x_i,t}$ be the path from g_0 to $\overline{\chi}(B_\delta(x_i))g_0 + \chi(B_\delta(x_i))g_1$ guaranteed by Lemma 13. Then for each $x \in B_\delta(x_i)$, we have

$$\left| g'_{i,t}(x) \right|_{g_{x_0,t}} < \begin{cases} 1, & t \in [-\epsilon, 0) \cup [1, 1+\epsilon], \\ d_x(g_0(x), g_1(x)) + 3\epsilon, & t \in [0,1). \end{cases} \tag{17}$$

Additionally, $g_{i,t}(x)$ is constant in t for $x \notin B_\delta(x_i)$.

Since the sets F_i are pairwise disjoint and closed, we can find $\eta > 0$ such that the closed subsets

$$B_\eta(F_i) = \{x \in M \mid \mathrm{dist}_g(x, F_i) \leq \eta\}$$

are still pairwise disjoint. (Here, dist_g denotes the distance function of the Riemannian metric g on M.) Since $F_i \subseteq B_{\delta'}(x_i)$ for some $0 < \delta' < \delta$, we may also choose η small enough that $B_\eta(F_i) \subseteq B_\delta(x_i)$ for all i.

Now, for each i, we define a continuous function for $x \in B_\eta(F_i)$ by

$$s_i(x,t) := \left(\frac{\eta - \mathrm{dist}_g(x, F_i)}{\eta} \right)(t+\epsilon) - \epsilon,$$

so that $s_i(x, -\epsilon) \equiv -\epsilon$. Furthermore, $s_i(x,t) = t$ for all $x \in F_i$ and $t \in [-\epsilon, 1+\epsilon]$, and $s_i(x,t) = -\epsilon$ for all t if $x \in \partial B_\eta(F_i)$. We define a smooth path in $\mathcal{M}_c$ as follows:

$$\bar{g}_t(x) := \begin{cases} g_{i,t}(x), & x \in F_i, \\ g_{i,s_i(x,t)}(x) & x \in B_\eta(F_i), \\ g_0(x), & x \notin \cup_i B_\eta(F_i), \end{cases} \qquad \text{for } t \in [-\epsilon, 1+\epsilon].$$

With this definition, we can see that the path $\tilde{g}_t := \overline{\chi}(F) g_0 + \chi(F)\bar{g}_t$ satisfies the assumptions of Lemma 10, and hence $d(g_0, \tilde{g}_1) \leq L(\tilde{g}_t)$. We claim that (15) and hence (16) hold as well.

To see this, note that $|\tilde{g}_t'(x)|_{\tilde{g}_t(x)} = \left|g_{i,t}'(x)\right|_{g_{i,t}}$ for all $x \in F$. Therefore, using (17), we can estimate

(18)

$$\begin{aligned} L(\tilde{g}_t)^2 &\leq (1+2\epsilon)E(\tilde{g}_t) = (1+2\epsilon)\sum_{i=1}^N \int_{-\epsilon}^{1+\epsilon} \int_{F_i} |\tilde{g}_t'(x)|^2_{\tilde{g}_t(x)}\, d\mu\, dt \\ &< (1+2\epsilon)\left[\sum_{i=1}^N \int_0^1 \int_{F_i} (d_x(g_0(x), g_1(x)) + 3\epsilon)^2\, d\mu\, dt + \sum_{i=1}^N \int_{[-\epsilon,0)\cup[1,1+\epsilon]} \int_{F_i} d\mu\, dt \right] \\ &= (1+2\epsilon)\left[\sum_{i=1}^N \int_{F_i} d_x(g_0(x), g_1(x))^2\, d\mu \right. \\ &\qquad\qquad \left. + 6\epsilon \cdot \sum_{i=1}^N \int_{F_i} d_x(g_0(x), g_1(x))\, d\mu + (9\epsilon^2 + 2\epsilon) \sum_{i=1}^N \int_{F_i} d\mu \right] \\ &\leq (1+2\epsilon)\left[\Omega_2(g_0, \tilde{g}_1)^2 + 6\epsilon\Omega_1(g_0, \tilde{g}_1) + 9\epsilon^2 + 2\epsilon \right]. \end{aligned}$$

The last line follows by the formulas for Ω_1 and Ω_2, as well as the fact that $\mathrm{Vol}(M, \mu) = 1$.

Finally, we note that $\Omega_1(g_0, \tilde{g}_1) \leq \Omega_1(g_0, g_1)$ and $\Omega_2(g_0, \tilde{g}_1) \leq \Omega_2(g_0, g_1)$ since $\tilde{g}_1$ equals g_1 on F and g_0 everywhere else. Thus (18) in fact implies (15). □

We now have all the pieces necessary to prove the main result of this section.

Theorem 15. *$d(g_0, g_1) = \Omega_2(g_0, g_1)$ for all $g_0, g_1 \in \mathcal{M}$.*

Proof. We have already shown in Theorem 1 that $d(g_0, g_1) \geq \Omega_2(g_0, g_1)$, so it only remains to show the reverse inequality.

Let any $\epsilon > 0$ be given, and let $\delta = \delta(\epsilon, g_0, g_1)$ be the number guaranteed by Lemma 13.

Choose $0 < \delta' < \delta$ and a finite number of points $x_i \in M$, $i = 1, \ldots, N$, such that $B_i := \mathrm{int}(B_{\delta'}(x_i))$ cover M. (Here, int denotes the interior of a set.) For each i, we choose $0 < \delta_i < \delta'$ such that

$$\max\left\{ \mathrm{Vol}(B_i \setminus B_{\delta_i}(x_i), g_0), \mathrm{Vol}(B_i \setminus B_{\delta_i}(x_i), g_1) \right\} < \frac{\epsilon}{2^N - 1}. \tag{19}$$

We then let $F_1 := B_{\delta_1}(x_1)$. For each $i = 2, \dots, N$, define

$$F_i := B_{\delta_i}(x_i) \setminus \bigcup_{j<i} B_j.$$

We wish to see that the sets F_i cover M up to a set of measure ϵ, intrinsically with respect to both g_0 and g_1. By (19), for $\alpha = 0, 1$,

$$\mathrm{Vol}(F_1, g_\alpha) \geq \mathrm{Vol}(B_1, g_\alpha) - \frac{\epsilon}{2^N - 1}.$$

To estimate $\mathrm{Vol}(F_1 \cup F_2, g_\alpha)$, note that

$$F_1 \cup F_2 = B_{\delta_1}(x_1) \cup (B_{\delta_2}(x_2) \setminus B_1) = B_{\delta_1}(x_1) \cup (B_{\delta_2}(x_2) \setminus (B_1 \setminus B_{\delta_1}(x_1))).$$

The first set in the union on the right-hand side is completely contained in B_1, and the second set is completely contained in B_2. Furthermore, they are disjoint. Therefore, again using (19),

$$\begin{aligned} \mathrm{Vol}(F_1 \cup F_2, g_\alpha) &= \mathrm{Vol}(B_{\delta_1}(x_1), g_\alpha) + \mathrm{Vol}(B_{\delta_2}(x_2) \setminus (B_1 \setminus B_{\delta_1}(x_1))) \\ &\geq \left(\mathrm{Vol}(B_1, g_\alpha) - \frac{\epsilon}{2^N - 1}\right) + (\mathrm{Vol}(B_{\delta_2}(x_2)) - \mathrm{Vol}(B_1 \setminus B_{\delta_1}(x_1), g_\alpha)) \\ &\geq \left(\mathrm{Vol}(B_1, g_\alpha) - \frac{\epsilon}{2^N - 1}\right) + \left(\mathrm{Vol}(B_{\delta_2}(x_2)) - \frac{\epsilon}{2^N - 1}\right) \\ &\geq \left(\mathrm{Vol}(B_1, g_\alpha) - \frac{\epsilon}{2^N - 1}\right) + \left(\mathrm{Vol}(B_2, g_\alpha) - 2 \cdot \frac{\epsilon}{2^N - 1}\right) \\ &\geq \mathrm{Vol}(B_1 \cup B_2, g_\alpha) - (1+2)\frac{\epsilon}{2^N - 1}. \end{aligned} \tag{20}$$

If we continue in this way, we find that for $F := \cup_i F_i$,

$$\mathrm{Vol}(F, g_\alpha) \geq \mathrm{Vol}(M, g_\alpha) - \left(\sum_{j=0}^{N-1} 2^j\right) \frac{\epsilon}{2^N - 1} = \mathrm{Vol}(M, g_\alpha) - \epsilon, \tag{21}$$

where we recall that $\cup_i B_i = M$.

Now, note that as defined, the sets F_i satisfy the assumptions of Lemma 14. Let $\tilde{g}_t$ and $\tilde{g}_1$ be as in the lemma. Then we have that

$$d(g_0, \tilde{g}_1)^2 < (1 + 2\epsilon) \left[\Omega_2(g_0, g_1)^2 + 6\epsilon\Omega_1(g_0, g_1) + 9\epsilon^2 + 2\epsilon\right]. \tag{22}$$

On the other hand, $\tilde{g}_1$ and g_1 differ only on $M \setminus F$, where $\tilde{g}_1 = g_0$. Thus, by (21) and Proposition 6, we have

$$d(\tilde{g}_1, g_1) \leq C(n) \left(\sqrt{\mathrm{Vol}(M \setminus F, g_0)} + \sqrt{\mathrm{Vol}(M \setminus F, g_1)}\right) < 2C(n)\sqrt{\epsilon}. \tag{23}$$

Applying the triangle inequality to (22) and (23), we obtain

$$d(g_0, g_1) < \sqrt{(1 + 2\epsilon)\left[\Omega_2(g_0, \tilde{g}_1)^2 + 6\epsilon\Omega_1(g_0, \tilde{g}_1) + 9\epsilon^2 + 2\epsilon\right]} + 2C(n)\sqrt{\epsilon}.$$

Sending $\epsilon \to 0$ then gives the desired result, $d(g_0, g_1) \leq \Omega_2(g_0, g_1)$. $\square$

Since the completion of $(\mathcal{M}, d)$ is already known, the previous theorem implies that the L^2 completion of $\mathcal{M}$ (in the sense discussed following (1)) is given by $\widehat{\mathcal{M}_f}$.

4. Geodesics: Existence, uniqueness, explicit formulas

In this section, we use the formula $d = \Omega_2$, together with an analysis of the geometry of $(\mathcal{M}_x, \langle \cdot, \cdot \rangle)$, to obtain results about geodesics (that is, minimal paths) in $\overline{\mathcal{M}}$. Of course, as the completion of the path metric space $(\mathcal{M}, d)$, $\overline{\mathcal{M}}$ is itself a complete path metric space. (By *path metric space*, we mean that the distance between points is equal to the infimum of the lengths of paths between those points. Some authors refer to this as an inner or intrinsic metric space.) However, since $\overline{\mathcal{M}}$ is not locally compact, completeness is no guarantee that minimal paths exists between arbitrary points—even in the Riemannian case, this does not hold, as an example of McAlpin shows [McA65, Sect. I.E] (see also [Lan95, Sect. VIII.6]).

As the title of the section implies, we will nevertheless show that a unique geodesic exists between any two points, and we can give an explicit and easily computable formula for this geodesic. Our analysis of the geometry of $\mathcal{M}_x$ builds upon the foundation set up by Freed–Groisser [FG89] and Gil-Medrano–Michor [GMM91].

4.1. **The metric d_x on $\mathcal{M}_x$.** In this subsection, we investigate the properties of the metric d_x as a preparation for studying $\overline{\mathcal{M}}$.

Given a tensor $a \in \mathcal{S}_x$, we have as before $A = g(x)^{-1} a$. We will denote by $\sqrt{A}$ the square root of the determinant of A, as far as this is well-defined. Similarly, $\sqrt[4]{A}$ simply denotes $\sqrt[4]{\det A}$. Note that these quantities are coordinate-independent.

Our first result bounds d_x from below based on the determinants of two given elements, and is the pointwise analog of [Cla10, Lemma 12]. It will come in useful when showing that given paths are minimal in $\mathcal{M}_x$.

Lemma 16. *Let $a_0, a_1 \in \mathcal{M}_x$. Then*

$$d_x(a_0, a_1) \geq \frac{4}{\sqrt{n}} \left| \sqrt[4]{A_1} - \sqrt[4]{A_0} \right|.$$

Proof. The proof is essentially the same as [Cla10, Lemma 12], but for completeness we include it here.

First, let $a \in \mathcal{M}_x$, and suppose that $b \in T_a \mathcal{M}_x \cong \mathcal{S}_x$. Let b_1 be the pure-trace part of b ($b_1 = \frac{1}{n} \operatorname{tr}_a b$) and b_0 be the trace-free part ($b_0 = b - b_1$). It is easy to see that $\operatorname{tr}_a(b_0 b_1) = 0$, and therefore

$$\operatorname{tr}_a(b^2) = \operatorname{tr}_a(b_0^2) + \operatorname{tr}_a(b_1^2) = \operatorname{tr}_a(b_0^2) + \frac{1}{n} (\operatorname{tr}_a b)^2.$$

Since $\operatorname{tr}_a(b_0^2) \geq 0$, we can conclude that $(\operatorname{tr}_a b)^2 \leq n \operatorname{tr}_a(b^2)$.

Let a_t, $t \in [0, 1]$, be any path connecting a_0 and a_1. We can estimate

$$\begin{aligned} \partial_t \sqrt[4]{A_t} &= \frac{1}{4} (\det A_t)^{-3/4} \operatorname{tr}_{a_t}(a_t') (\det A_t) = \frac{1}{4} \left(\operatorname{tr}_{a_t}(a_t')^2 \sqrt{A_t} \right)^{1/2} \\ &\leq \frac{1}{4} \left(n \operatorname{tr}_{a_t}((a_t')^2) \sqrt{A_t} \right)^{1/2} = \frac{\sqrt{n}}{4} \left\| a_t' \right\|_{a_t}, \end{aligned}$$

where we have used the inequality of the last paragraph. Integrating this last estimate gives

$$\sqrt[4]{A_1} - \sqrt[4]{A_0} = \int_0^1 \partial_t \sqrt[4]{A_t} \, dt \leq \frac{\sqrt{n}}{4} \int_0^1 \left\| a_t' \right\|_{a_t} \, dt = \frac{\sqrt{n}}{4} L(a_t).$$

Since this holds for any path a_t between a_0 and a_1, and we can just as easily exchange a_0 and a_1, the statement of the lemma is proved. $\square$

The next result is qualitatively the reverse of the previous, bounding d_x from above based on determinants. It is the pointwise analog of Proposition 6 and will help us in determining the completion of $(\mathcal{M}_x, d_x)$.

Lemma 17. *Let $\tilde{a}, \hat{a} \in \mathcal{M}_x$. Then*

$$d_x(\tilde{a}, \hat{a}) \leq C'(n) \left| \sqrt[4]{\tilde{A}} + \sqrt[4]{\hat{A}} \right|.$$

Proof. This proof is very similar to, but simpler than, the proof of [Claa, Prop. 4.1]. However, for completeness, we include it here.

First, define paths $\tilde{a}^s_t$ and $\hat{a}^s_t$, for $0 < s \leq 1$ and $t \in [s, 1]$, by

$$\tilde{a}^s_t := t\tilde{a} \quad \text{and} \quad \hat{a}^s_t := t\hat{a}.$$

We consider these as a family of paths in the time variable t with domain depending on the family parameter s.

Second, define a family $\bar{a}^s_t$ of paths in t depending on the family parameter s by

$$\bar{a}^s_t := s\left((1-t)\tilde{a} + t\hat{a}\right),$$

where again $0 < s \leq 1$ but this time $t \in [0, 1]$.

Then the concatenation $a^s_t := (\tilde{a}^s_t)^{-1} * \bar{a}^s_t * \hat{a}^s_t$ (here, $(\tilde{a}^s_t)^{-1}$ means we run through that path backwards) is, for each s, a path from $\tilde{a}$ to $\hat{a}$. We will prove that

$$\lim_{s \to 0} L(a^s_t) \leq C'(n) \left(\sqrt[4]{\det \tilde{A}} + \sqrt[4]{\det \hat{A}} \right),$$

which will imply the result immediately.

First, note that $L(\tilde{a}^s_t) \leq \lim_{s \to 0} L(\tilde{a}^s_t)$ for all s. To estimate the right-hand side, we first compute

$$\langle (\tilde{a}^s_t)', (\tilde{a}^s_t)' \rangle_{\tilde{a}^s_t} = \operatorname{tr}_{t\tilde{a}}(\tilde{a}^2) \sqrt{\det(t\tilde{A})} = n t^{\frac{n}{2}-2} \sqrt{\tilde{A}}.$$

Therefore,

$$L(\tilde{a}^s_t) \leq \lim_{s \to 0} L(\tilde{a}^s_t) = \sqrt{n} \cdot \sqrt[4]{\tilde{A}} \int_0^1 t^{\frac{n}{4}-1}\, dt.$$

Since $\frac{n}{4} - 1 > -1$, the above integral is finite, with a value depending only on n. Hence we have

$$L(\tilde{a}^s_t) \leq C'(n) \sqrt[4]{\tilde{A}}.$$

In exactly the same way, we can show

$$L(\hat{a}^s_t) \leq C'(n) \sqrt[4]{\hat{A}},$$

even using the same constant.

Now, if we can show that $\lim_{s \to 0} L(\bar{a}^s_t) = 0$, we will be finished. So we compute

$$\begin{aligned}
\langle (\bar{a}^s_t)', (\bar{a}^s_t)' \rangle_{\bar{a}^s_t} &= \operatorname{tr}_{s((1-t)\hat{a}+t\tilde{a})}\left(s^2(\tilde{a}-\hat{a})^2\right) \sqrt{\det\left(s\left((1-t)\,\tilde{A} + t\hat{A}\right)\right)} \\
&= s^{n/2} \operatorname{tr}_{(1-t)\tilde{a}+t\hat{a}}\left((\hat{a}-\tilde{a})^2\right) \sqrt{\det\left((1-t)\,\tilde{A} + t\hat{A}\right)} \\
&= s^{n/2} \langle (a^1_t)', (a^1_t)' \rangle_{a^1_t}.
\end{aligned}$$

This implies that

$$L(\bar{a}^s_t) = s^{n/2} L(\bar{a}^1_t),$$

from which $\lim_{s\to 0} L(\bar{a}_t^s) = 0$ is immediate. □

We now wish to determine the completion of $(\mathcal{M}_x, d_x)$, which we will do by comparison with another metric. Consider the Riemannian metric $\langle \cdot, \cdot \rangle^0$ on $\mathcal{M}_x$ given by

$$\langle b, c \rangle_a^0 = \operatorname{tr}_a(bc).$$

By the work of Ebin [Ebi70, Thm. 8.9] (see also [Claa, Prop. 4.9]), we know that $(\mathcal{M}_x, \langle \cdot, \cdot \rangle^0)$ is a complete Riemannian manifold. Since the scalar product $\langle \cdot, \cdot \rangle_a$ differs from $\langle \cdot, \cdot \rangle_a^0$ only by the factor $\sqrt{A}$, one reasonably suspects that the only points that could be missing from the completion of $(\mathcal{M}_x, \langle \cdot, \cdot \rangle)$ are those with determinant zero. The next proposition confirms this hunch and makes it rigorous.

Proposition 18. *The completion of $(\mathcal{M}_x, \langle \cdot, \cdot \rangle)$ can be identified with*

$$\overline{\mathcal{M}_x} \cong \operatorname{cl}(\mathcal{M}_x) \big/ \partial \mathcal{M}_x,$$

where $\operatorname{cl}(\mathcal{M}_x)$ denotes the topological closure of $\mathcal{M}_x$ as a subspace of $\mathcal{S}_x$, and $\partial \mathcal{M}_x$ denotes the boundary in $\mathcal{S}_x$.

The topology is given by the following. Given a sequence $\{a_k\} \subset \overline{\mathcal{M}_x}$, it converges to $a_0 \in \mathcal{M}_x$ if and only if it does so in the manifold topology, and it converges to $[0] \in \mathcal{M}_x$ (the equivalence class of the zero tensor) if and only if $\det A_k \to 0$.

Proof. By the standard construction of the completion of a Riemannian manifold, we must consider all piecewise differentiable paths of the form a_t, $t \in [0,1)$, in $\mathcal{M}_x$ that have finite length with respect to $\langle \cdot, \cdot \rangle$ and show two facts. First, either $\lim_{t\to 1} a_t \in \mathcal{M}_x$ (in the topology of $\mathcal{S}_x$) or $\lim_{t\to 1} \det A_t = 0$. Second, if $\lim_{t\to 1} \det A_t = 0$ and $\tilde{a}_t$, $t \in [0,1)$, is another path in $\mathcal{M}_x$ satisfying $\lim_{t\to 1} \det \tilde{A}_t = 0$, then a_t and $\tilde{a}_t$ are equivalent in the sense that $\lim_{t\to 1} d_x(a_t, \tilde{a}_t) = 0$. (From these facts, the statements about the topology on $\overline{\mathcal{M}_x}$ follow immediately.)

The second fact, however, is immediate from Lemma 17. So to prove the first fact, suppose that $\lim_{t\to 1} \det A_t \neq 0$. By Lemma 16, one can easily see that $\det A_t$ must nevertheless converge to some limit $\eta > 0$. Furthermore, Lemma 16 implies that there exists $\epsilon > 0$ such that $\eta/2 < \det A_t < 3\eta/2$ for all $t \in [1-\epsilon, 1)$. But since $\langle \cdot, \cdot \rangle$ is equivalent to $\langle \cdot, \cdot \rangle^0$ on the subset $\{a \in \mathcal{M}_x \mid \eta/2 < \det A < 3\eta/2\}$, the completeness of $\langle \cdot, \cdot \rangle^0$ implies that $\lim_{t\to 1} a_t \in \mathcal{M}_x$, as desired. □

As in the case of $\overline{\mathcal{M}}$, the completion $\overline{\mathcal{M}_x}$ together with the metric induced from d_x (which we will again denote by d_x) is a path metric space. Furthermore, given the statements about the topology of $\overline{\mathcal{M}_x}$ from the previous proposition, it is a straightforward matter to extend the results of Lemmas 16 and 17 to elements $a_0, a_1 \in \overline{\mathcal{M}_x}$, so from now on we will assume the lemmas are stated as such.

4.2. **Geodesics on $\mathcal{M}_x$.** By the Hopf-Rinow-Cohn-Vossen Theorem for path metric spaces [BBI01, Thm. 2.5.28, Rmk. 2.5.29], the completeness and local compactness of $\overline{\mathcal{M}_x}$ implies that between any two points, there exists a minimal geodesic. Using a small result of our own, but primarily by the work of Freed–Groisser [FG89] and Gil-Medrano–Michor [GMM91] in solving the geodesic equation for $\mathcal{M}$, we can write down minimal geodesics in $\overline{\mathcal{M}_x}$ explicitly.

As explained in [FG89, Appendix], the formulas for geodesics on $\mathcal{M}_x$ follow directly from those on $\mathcal{M}$ determined by Freed–Groisser [FG89, Thm. 2.3] and Gil-Medrano–Michor

[GMM91, Thm. 3.2]. Therefore, we simply quote them here. For the remainder of the paper, we denote by $b_T := b - \frac{1}{n}(\operatorname{tr}_{a_0} b)a_0$ the traceless part of any $b \in T_{a_0}\mathcal{M}_x \cong \mathcal{S}_x$.

Theorem 19. *Let $a_0 \in \mathcal{M}_x$ and $b \in T_{a_0}\mathcal{M}_x \cong \mathcal{S}_x$. Define*

$$q(t) := 1 + \frac{t}{4}\operatorname{tr}_{a_0}(b), \quad r(t) := \frac{t}{4}\sqrt{n \operatorname{tr}_{a_0}(b_T^2)}.$$

Then the geodesic starting at a_0 with initial tangent $a_0' = b$ is given by

$$a_t = \begin{cases} (q(t)^2 + r(t)^2)^{\frac{2}{n}} a_0 \exp\left(\frac{4}{\sqrt{n \operatorname{tr}(b_T^2)}} \arctan\left(\frac{r(t)}{q(t)}\right) b_T\right) & \text{if } b_T \neq 0, \\ q(t)^{4/n} a_0 & \text{if } b_T = 0. \end{cases}$$

In particular, the change in the volume element $\sqrt{A_t}$ is given by

$$\sqrt{A_t} = (q(t)^2 + r(t)^2)\sqrt{A_0}. \tag{24}$$

For precision, we specify the range of arctan *in the above. At a point where* $\operatorname{tr}_{a_0} b \geq 0$, *it assumes values in* $(-\frac{\pi}{2}, \frac{\pi}{2})$. *At a point where* $\operatorname{tr}_{a_0} b < 0$, $\arctan(r(t)/q(t))$ *assumes values as follows, with* $t_0 := -\frac{4}{\operatorname{tr}_{a_0} b}$:

1. *in* $[0, \frac{\pi}{2})$ *if* $0 \leq t < t_0$,
2. *in* $(\frac{\pi}{2}, \pi)$ *if* $t_0 < t < \infty$,

and we set $\arctan(r(t)/q(t)) = \frac{\pi}{2}$ *if* $t = t_0$.

Finally, the geodesic is defined on the following domain. If $b_T = 0$ *and* $\operatorname{tr}_{a_0} b < 0$, *then the geodesic is defined for* $t \in [0, t_0)$. *Otherwise, the geodesic is defined on* $[0, \infty)$.

Gil-Medrano–Michor also performed a detailed analysis of the exponential mapping of $\mathcal{M}$. We quote here a portion of their results, translated into the pointwise result for $\mathcal{M}_x$.

Theorem 20 ([GMM91, §3.3, Thm. 3.4])**.** *Let $U := \mathcal{S}_x \setminus (-\infty, -4/n]\, a_0$. Then U is the maximal domain of definition of $\exp_{a_0}$, and $\exp_{a_0}$ is a diffeomorphism between U and*

$$V := \exp_{a_0}(U) = \left\{ a_0 \exp(a_0^{-1} b) \;\middle|\; \operatorname{tr}_{a_0}(b_T^2) < \frac{(4\pi)^2}{n} \right\}.$$

Here and in the following, exp *without a subscript denotes the usual exponential of a matrix or linear transformation.*

The inverse of $\exp_{a_0}$ is given by the following. For $b \in \mathcal{S}_x$, define

$$\psi(b) := \begin{cases} \frac{4}{n}\left(\exp\left(\frac{\operatorname{tr}_{a_0} b}{4}\right)\cos\left(\frac{\sqrt{n \operatorname{tr}_{a_0}(b_T^2)}}{4}\right) - 1\right) a_0 & \\ \qquad + \frac{4}{\sqrt{n \operatorname{tr}_{a_0}(b_T^2)}} \exp\left(\frac{\operatorname{tr}_{a_0} b}{4}\right) \sin\left(\frac{\sqrt{n \operatorname{tr}_{a_0}(b_T^2)}}{4}\right) b_T & \text{if } b_T \neq 0 \\ \frac{4}{n}\left(\exp\left(\frac{\operatorname{tr}_{a_0} b}{4}\right) - 1\right) a_0 & \text{if } b_T = 0. \end{cases}$$

Now, if $a_1 \in V$, write (uniquely) $a_1 = a_0 \exp(a_0^{-1} b)$ for some $b \in \mathcal{S}_x$. Then $\exp_a^{-1}(a_1) = \psi(b)$.

Given these two results, and our own lemmas from the previous subsection, we are now ready to describe minimal paths in $\mathcal{M}_x$. So let any two points $a_0, a_1 \in \mathcal{M}_x$ be given.

If a minimal path a_t, $t \in [0,1]$, between a_0 and a_1 does not intersect $[0]$, then a_t is necessarily a geodesic with respect to $\langle \cdot, \cdot \rangle$ since it is a minimal path in a Riemannian manifold. Furthermore, since by Theorem 19 $\exp_{a_0}$ is a diffeomorphism onto its image, there can be at most one minimal path between a_0 and a_1 that does not intersect $[0]$. (Of course, when speaking about uniqueness of paths here and in the following, we mean up to reparametrization.)

Now consider the case where $a_{t_0} = [0]$ for some $t_0 \in [0,1]$. By breaking the path into pieces and reversing it if necessary, we may assume for this discussion that $t_0 = 1$. By Lemma 16, we have that

$$d_x(a_0, a_1) \geq \frac{4}{\sqrt{n}} \sqrt[4]{A_0}.$$

On the other hand, it is easy to compute that the length of the path $a_t := (1-t)a_0$, $t \in [0,1]$, is exactly equal to $\frac{4}{\sqrt{n}}\sqrt[4]{A_0}$. (For $t \in [0,1)$ it is the geodesic with initial tangent vector $a_0' = -\frac{4}{n}a_0$; cf. Theorem 19). Therefore, this path is minimal, and furthermore, we claim that it is the unique minimal path between a_0 and $[0]$. This is because any minimal path between a_0 and $[0]$ must be a geodesic on the time interval $[0,1)$, and from Theorem 19 it is clear that the only geodesics with $\det A_t \to 0$ are those with initial tangents $a_0' = \lambda a_0$ for some $\lambda < 0$ (since otherwise either $q(t)$ or $r(t)$ is both positive and monotonically increasing in (24)).

Knowing now what minimal paths in $\mathcal{M}_x$ look like, we can show:

Proposition 21. *There exists a unique minimal path between any two given points $a_0, a_1 \in \overline{\mathcal{M}_x}$.*

Proof. If either a_0 or a_1 is $[0]$, then the discussion preceding the proposition shows that the minimal path described there is minimal. So we assume $a_0, a_1 \in \mathcal{M}_x$. If there exists a minimal path from a_0 to a_1 that passes through $[0]$, then by the above discussion it is the concatenation of the straight-line path from a_0 to $[0]$, followed by the straight-line path from $[0]$ to a_1, and this is the only possible minimal path passing through $[0]$. On the other hand, if there exists a geodesic a_t in $\mathcal{M}_x$ from a_0 to a_1, then by Theorem 20 there is only one such geodesic. Therefore, to prove the statement of the proposition, it suffices to prove that

$$L(a_t) < \frac{4}{\sqrt{n}} \left(\sqrt[4]{A_0} + \sqrt[4]{A_1} \right). \tag{25}$$

In this case, such a geodesic is shorter than the shortest path between a_0 and a_1 passing through $[0]$, implying that geodesics in $\mathcal{M}_x$ and paths through $[0]$ cannot simultaneously be minimal.

To show (25), write $a_1 = \exp(a_0^{-1}b)$, and let ψ be as in Theorem 20, so that $a_1 = \exp_{a_0}(\psi(b))$. One sees easily that (25) is obvious if $b_T = 0$. So we must estimate $L(a_t) = |\psi(b)|_{a_0}$ for $b_T \neq 0$, and (recalling that $\operatorname{tr}_{a_0} b_T = 0$), we first have

$$\begin{aligned}
\operatorname{tr}_{a_0}(\psi(b)^2) = \frac{16}{n^2} \Bigg[&\exp\left(\frac{\operatorname{tr}_{a_0} b}{2}\right) \cos^2\left(\frac{\sqrt{n \operatorname{tr}_{a_0}(b_T^2)}}{4}\right) \\
&- 2\exp\left(\frac{\operatorname{tr}_{a_0} b}{4}\right) \cos\left(\frac{\sqrt{n \operatorname{tr}_{a_0}(b_T^2)}}{4}\right) + 1 \Bigg] \operatorname{tr}_{a_0}(a_0^2) \\
&+ \frac{16}{n \operatorname{tr}_{a_0}(b_T^2)} \exp\left(\frac{\operatorname{tr}_{a_0} b}{2}\right) \sin^2\left(\frac{\sqrt{n \operatorname{tr}_{a_0}(b_T^2)}}{4}\right) \operatorname{tr}_{a_0}(b_T^2)
\end{aligned}$$

$$= \frac{16}{n}\left(\exp\left(\frac{\operatorname{tr}_{a_0} b}{2}\right) - 2\exp\left(\frac{\operatorname{tr}_{a_0} b}{4}\right)\cos\left(\frac{\sqrt{n \operatorname{tr}_{a_0}(b_T^2)}}{4}\right) + 1\right).$$

Since $a_1 \in V$ implies $0 < \operatorname{tr}_{a_0}(b_T^2) < (4\pi)^2/n$, the argument of cosine in the above equation is between 0 and π, and therefore we can estimate

$$\operatorname{tr}_{a_0}(\psi(b)^2) < \frac{16}{n}\left(\exp\left(\frac{\operatorname{tr}_{a_0}(b)}{4}\right) + 1\right)^2.$$

On the other hand, using the well-known formula (valid for any matrix X) $\det(\exp(X)) = \exp(\operatorname{tr}(X))$, we can use the above inequality to see

$$\begin{aligned} |\psi(b)|_{a_0}^2 &< \frac{16}{n}\left(\det(\exp(a_0^{-1}b))^{1/4} + 1\right)^2 \sqrt{\det(g(x)^{-1}a_0)} \\ &= \frac{16}{n}\left(\sqrt[4]{A_1} + \sqrt[4]{A_0}\right)^2, \end{aligned}$$

which proves (25). □

Knowing what minimal paths in $(\mathcal{M}_x, d_x)$ are, we can explicitly determine d_x and thus, by Theorem 15, d. Furthermore, since there exists a unique minimal path between any two elements in $\mathcal{M}_x$, it is clear that there is a unique minimal path between any two elements $g_0, g_1 \in \overline{\mathcal{M}}$: the path g_t that gives the minimal path $g_t(x)$ between $g_0(x)$ and $g_1(x)$ for each $x \in M$. We thus end the section with a theorem stating this result for $\overline{\mathcal{M}}$, as well as summarizing the explicit realizations of geodesics and distance that we have determined up to the this point, reformulating them for $\overline{\mathcal{M}}$.

Theorem 22. *There exists a unique minimal path g_t, $t \in [0,1]$, between any two points $g_0, g_1 \in \overline{\mathcal{M}}$, given by the following. Let h be a measurable, symmetric $(0,2)$-tensor field on M defined on the subset N where neither g_0 nor g_1 is zero, and write $g_1 = g_0 \exp(g_0^{-1}h)$ on this subset. Denote by P the subset of M where $\operatorname{tr}_{g_0}(h_T^2) < (4\pi)^2/n$ (here h_T denotes the traceless part of h). Write $k := \psi(h)$, where $\psi(h)(x)$ is given as in Theorem 20. Finally, let q_t and r_t be one-parameter families of functions on N given by*

$$q_t := 1 + \frac{t}{4}\operatorname{tr}_{g_0}(h), \quad r_t := \frac{t}{4}\sqrt{n \operatorname{tr}_{g_0}(h_T^2)}.$$

Then at points $x \in N \cap P$, we have

$$g_t(x) = \left(q_t^2(x) + r_t^2(x)\right)^{\frac{2}{n}} g_0(x) \exp\left(\frac{4}{\sqrt{n \operatorname{tr}_{g_0(x)}(h_T^2(x))}} \arctan\left(\frac{r_t(x)}{q_t(x)}\right) g_0^{-1}(x) h_T(x)\right),$$

and $g_t(x)$ does not intersect $[0]$.

At all other points of M, $g_t(x)$ passes through $[0]$, and we have

$$g_t(x) = \begin{cases} \left(1 - \frac{\sqrt[4]{G_0(x)} + \sqrt[4]{G_1(x)}}{\sqrt[4]{G_0(x)}} t\right) g_0(x), & t \in \left[0, \frac{\sqrt[4]{G_0(x)}}{\sqrt[4]{G_0(x)} + \sqrt[4]{G_1(x)}}\right], \\ \left(\frac{\sqrt[4]{G_0(x)} + \sqrt[4]{G_1(x)}}{\sqrt[4]{G_1(x)}} t - \frac{\sqrt[4]{G_0(x)}}{\sqrt[4]{G_1(x)}}\right) g_1(x), & t \in \left[\frac{\sqrt[4]{G_0(x)}}{\sqrt[4]{G_0(x)} + \sqrt[4]{G_1(x)}}, 1\right]. \end{cases}$$

The distance induced by the L^2 Riemannian metric between g_0 and g_1 is given by

$$d(g_0, g_1) = \Omega_2(g_0, g_1) = \int_M d_x(g_0(x), g_1(x))^2 \, d\mu,$$

with

$$d_x(g_0(x), g_1(x)) = \begin{cases} |\psi(h)(x)|_{g_0(x)}, & x \in N \cap P, \\ \frac{4}{\sqrt{n}} \left(\sqrt[4]{G_0(x)} + \sqrt[4]{G_1(x)} \right), & x \notin N \cap P. \end{cases}$$

Here, $|\psi(h)(x)|_{g_0(x)}$ *is given explicitly by*

$$|\psi(h)(x)|_{g_0(x)} = \frac{4}{\sqrt{n}} \left(\exp\left(\frac{\operatorname{tr}_{g_0(x)}(h(x))}{2} \right) \right. \\ \left. - 2\exp\left(\frac{\operatorname{tr}_{g_0(x)}(h(x))}{4} \right) \cos\left(\frac{\sqrt{n \operatorname{tr}_{g_0(x)}(h_T^2(x))}}{4} \right) + 1 \right)^{1/2} \sqrt[4]{G_0(x)}.$$

As noted in the introduction, the existence of geodesics in $\overline{\mathcal{M}}$ is in stark contrast to the situation for the incomplete space $\mathcal{M}$, where the image of the geodesic mapping at a point contains no open d-ball.

5. Outlook

There are a number of open questions concerning the L^2 metric. One is, as mentioned in the introduction, to find submanifolds of $\mathcal{M}$ on which convergence in the L^2 metric implies a more synthetic-geometric notion of convergence—e.g., Gromov–Hausdorff convergence, or convergence as a metric-measure space [Gro07, Sect. $3\frac{1}{2}$].

Other open questions that seem difficult to solve concern the moduli space of Riemannian metrics, also called the space of Riemannian structures or, sometimes, superspace. If $\mathcal{D}$ is the group of orientation-preserving diffeomorphisms of M, then this space is the quotient $\mathcal{M}/\mathcal{D}$, where $\mathcal{D}$ of course acts by pulling back metrics. It is not hard to see that the L^2 metric is invariant under this action, and so it projects to the quotient, which is a stratified space [Bou75].

From the metric-geometric standpoint, the first question one must ask is whether the L^2 metric induces a metric space structure on the quotient. Because the L^2 metric is weak and the quotient is singular, we cannot *a priori* exclude the situation that two orbits of the diffeomorphism group become arbitrarily close to one another. Given the very weak nature of convergence with respect to the L^2 metric, it is difficult to exclude this situation directly or through some geometric invariants (cf. [Clab, Sect. 4.3] for more on this).

Assuming the previous question is answered in the affirmative, another question one could ask about the moduli space is what its completion with respect to the L^2 metric is. Potentially, some of the very pathological degenerations that lead to losing all regularity of a limit metric in the completion of $(\mathcal{M}, d)$ come from degenerations along a diffeomorphism orbit.

Finally, as mentioned in the introduction, $(\mathcal{M}, (\cdot,\cdot))$ is nonpositively curved. It is therefore natural to wonder whether $(\overline{\mathcal{M}}, d)$ is a Hadamard space—a complete, simply-connected space that has nonpositive curvature in the synthetic sense [BBI01, Sect. 9.2]. By the generalized Cartan–Hadamard Theorem of Alexander–Bishop [AB90], the existence of unique geodesics is a necessary condition for $\overline{\mathcal{M}}$ to be a Hadamard space, so it is natural to conjecture that this might be the case.

In principle, one could explicitly determine whether $\overline{\mathcal{M}}$ has nonpositive curvature using the formulas for geodesics that we have given here. In practice, this involves very lengthy computations, and we do not have an answer at the time of this writing.

We hope, however, that some of the questions posed in this section might be more amenable to study given the explicit understanding of the L^2 metric established in this paper.

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