

Integrate and Fire Neural Networks, Piecewise Contractive Maps and Limit Cycles

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Abstract

We study the global dynamics of integrate and fire neural networks composed of an arbitrary number of identical neurons interacting by inhibition and excitation. We prove that if the interactions are strong enough, then the support of the stable asymptotic dynamics consists of limit cycles. We also find sufficient conditions for the synchronization of networks containing excitatory neurons. The proofs are based on the analysis of the equivalent dynamics of a piecewise continuous Poincaré map associated to the system. We show that for strong interactions the Poincaré map is piecewise contractive. Using this contraction property, we prove that there exist limit cycles attracting all the orbits dropping into the stable subset of the phase space. This result applies not only to the Poincaré map under study, but also to a wide class of general n -dimensional piecewise contractive maps.

Keywords: Integrate and fire neural networks, piecewise contractive maps, limit cycles, synchronization.

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1 Introduction

Numerous physical or biological systems can be seen as composed of a large number of units in interaction. In many occasions, their time evolution is modelled by a system of coupled differential equations, or by a high dimensional discrete time dynamical system, that suppose a proper individual dynamics for each unit and a coupling between units which may depend on the state of the whole system. Typical example of such models are coupled oscillators (continuous time) and coupled map lattices (discrete time) [9]. They usually assume well mathematically characterized individual dynamics, the main question under study being how the coupling of units can generate the collective behaviors observed in physical and biological systems. Nevertheless, from a mathematical point of view, these systems are a source of open problems and most mathematical results have been proved under the assumption of weak coupling and/or focus on particular solutions [25].

Pulse-coupled oscillators appear frequently in biological sciences, in particular in neuroscience to model neural networks [20]. In this context, the state of each oscillator describes the difference of electrical potential between the inside and the outside of a neuron's membrane. An archetype of pulse-coupled neural network appears in literature [17] in the following form:

$$\dot{V}_i = f_i(V_i) + \sum_{j=1}^n h_{ji}(V_i)\delta(t - t_j) \quad \forall i \in \{1, \dots, n\}. \quad (1)$$

The solutions of the non-coupled equation $\dot{V}_i = f_i(V_i)$ define the individual dynamics of the membrane potential of the neuron i . The following additional rule is assumed: if the potential V_i reaches the so called

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threshold potential $\theta > 0$ at an instant denoted t_i , then the neuron i is said to fire (or to emit a spike) and its potential is reset to zero. The term $h_{ji}(V_i)\delta(t-t_j)$ is a short hand notation meaning that at time t_j the potential of the neuron i suffers a discontinuity jump of amplitude $h_{ji}(V_i(t_j^-))$ produced by the firing of the neuron j . If the jump is negative ($h_{ji} < 0$) the interaction is said inhibitory and if it is positive ($h_{ji} > 0$) it is said excitatory.

When weak interactions are assumed, and extra assumptions on f_i are posed, it is possible to reduce (1), and also more realistic neural models, to a canonical system of phase coupled oscillators [15, 16]. This opens the possibility to get some insight in the dynamics of a huge class of weakly coupled neural networks by studying for example the existence and the stability of synchronized states [11, 15, 17]. Further insight in the dynamics of neural networks can be obtained by considering specific models. In this respect (leaky) Integrate and Fire (IF) neural networks [13] are certainly the most popular. For these networks f_i is a real affine function. Many mathematical works on IF neural networks deal with the dependence of particular solutions on the parameters describing the interactions (that are not necessarily impulsive). The effect of the velocity of the interactions on the stability of synchronized and anti-synchronized states is detailed for weak excitatory and inhibitory interactions in the case of two [23, 10] or more neurons [24] and in presence of delay [10, 2]. Synchronized solutions and more generally phase-locked solutions are also studied in the case of strong coupling for different architectures of network [2].

Although important results about the phenomenology of IF neural networks have been obtained, they principally focus on particular solutions, and there is still few mathematical results about their *global dynamics*, possibly letting unknown important features of neural networks. The purpose of this paper is precisely to give a mathematical description, developing analytical proofs, of the global dynamics of IF neural networks. In spite of IF neural networks being continuous time dynamical systems, as far as we know, previous studies of their global dynamics develop methods of discrete time dynamical systems. In the seminal work [21], Mirollo and Strogatz study a Poincaré return map to prove that for the system (1) with homogeneous constant excitatory interactions ($h_{ji} = cte > 0$) and homogeneous individual dynamics ($f_i = f$), almost all orbits become synchronized. In [8] a discrete time IF neural network, which may be seen as a discretization of (1) by a formal Euler scheme, is studied. The corresponding dynamical system is defined by the iterations of a general piecewise affine map. It is proved that for generic values of the parameters, the global asymptotic dynamics is supported on a finite number of stable periodic orbits. It is also proved that for non generic values of the parameters, asymptotic dynamics is sensitive to initial conditions.

Motivated by the rigorous results of [8], which are proved for discrete time system, the question we address in this paper is if they are also true for continuous time IF neural networks. Although elaborated integration schemes, especially designed for the simulation of neural networks have been developed [3], there is no way eliminating completely numerical errors and the results of simulations can drastically depend on the used integration strategy [22]. This motivates the interest of developing rigorous mathematical proofs also in the continuous time case. In this respect, the previous works [5, 7, 18] proved that periodic orbits attract almost all initial conditions, but under the assumption that the interactions are all inhibitory, while arbitrary interactions are considered in [8].

The main property allowing the proof of these results is that the (return) map is *piecewise contractive* in the whole phase space. In Section 2, we derive the return map of (1), assuming $f_i(V_i) = -\gamma V_i + K$, the independence of h_{ji} on V_i , and other hypothesis ((H1) and (H2), stated in the same section) giving a precise meaning to (1). In Section 3, we investigate the contraction properties of the return map. We prove that, unexpectedly, it is not piecewise contractive in the whole phase space for an open region of the interactions values, if some of them are excitatory (Theorem 2). Nevertheless, we find also an open subset in the space of parameters (strong interactions), such that this return map is piecewise contractive in the whole phase space, with respect to an adapted metric (Theorem 3). This parameters subset is defined by hypothesis (H3) and (H4) stated in Section 3.

We state now our result about the global dynamics of IF neural networks:

Theorem 1. *Under the hypothesis (H1), (H2), (H3) and (H4):*

1) *If the neural network is completely excitatory and the number of neurons is sufficiently large, then all the orbits are eventually periodic and synchronized.*

2) If the IF neural network contains inhibitory neurons, then the stable orbits are attracted to (at most countable many) limit cycles.

The part 1) is Theorem 5 which is proved in Section 4. The part 2) follows from Theorem 6 in Section 5. This last theorem states that the stable asymptotic dynamics of the return map is supported by periodic orbits. Actually, its proof does not use the particular formulation of the return map, and applies to a wide class of piecewise contractive maps (see Definition 5.3). Previous results [1, 4, 6, 8, 12] stating the existence of periodic attractors for piecewise contractive maps are proved in a different context. In [1] the maps are one dimensional and injective. In the works studying higher dimensional dynamics, only affine maps [8, 4] or injective maps [6] are considered. In our case, none of these hypothesis is assumed since the proof applies to n -dimensional maps, that are neither necessarily piecewise affine nor globally injective.

2 Integrate and fire neural network

We propose to study the global dynamics of leaky integrate and fire neural networks. Our working model is a standard IF neural network considering an arbitrary number of neurons connected by inhibitory and excitatory synapses. This system, which is defined precisely in Subsection 2.1, is the model (1) with $f_i(V_i) = -\gamma V_i + K$ and where $h_{ji}(V_i) = H_{ji}$ is independent of V_i . Its global dynamics is studied in the next sections via a Poincaré map which is derived in Subsection 2.2.

2.1 Definition of the model

At each time $t \in \mathbb{R}$, the state of a neuron $i \in I := \{1, \dots, n\}$ is described by its membrane potential $V_i(t)$ and the state of the network is represented by the vector $\mathbf{V}(t) = (V_1(t), \dots, V_n(t))$. The time evolution of the network has two regimes: a subthreshold regime and a firing regime.

The subthreshold regime occurs when $V_i(t) < \theta$ for all $i \in I$, where $\theta > 0$ is called the threshold potential. In such a regime, the state of the network satisfies the system of differential equations defined by:

$$\dot{V}_i(t) = -\gamma V_i(t) + K \quad \forall i \in I. \quad (2)$$

The constant γ stands for $1/RC$ where R and C are respectively the resistivity and the capacity of the neural membrane and K is proportional to a constant external current. According to Equation (2), the potential of the neurons tends to the equilibrium value $\beta = K/\gamma$.

If we suppose $\beta > \theta$, then there is an instant (supposed to be the smallest one) when the potential of (at least) one neuron reaches the threshold and the network enters in the firing regime. At this instant, the neuron emits a spike that induces a change in the potential of all the neurons to which it is connected, and its potential is reset to a smaller value than the threshold, chosen equal to 0 (without loss of generality).

Formally, when the network enters in the firing regime, its state suffers a discontinuity due to the reset of the firing neurons and to the change of potential of the neurons receiving the spikes of the firing neurons. If $J \subset I$ is the set of all the neurons that reach θ at time t_0 , the state of the network satisfies

$$\lim_{t \rightarrow t_0^+} V_i(t) = 0 \quad \text{if } i \in J \quad \text{and} \quad \lim_{t \rightarrow t_0^+} V_i(t) = V_i(t_0) + \sum_{j \in J} H_{ji} \quad \text{if } i \notin J. \quad (3)$$

The constant H_{ji} represents the synaptic interaction due to a spike coming from neuron j to neuron i . It is positive for an excitatory synapse, negative for an inhibitory synapse, and equal to 0 if the neurons are not connected. We will say that a neuron j is excitatory (inhibitory) if all its synapses are excitatory (inhibitory), i.e $H_{ji} \geq 0$ ($H_{ji} \leq 0$) for all $i \in I$, and we will say that a neuron is “mixed” if it is neither excitatory nor inhibitory.

Remark 2.1. Note that in model (1), if an excitatory neuron fires at time t_0 , then the potential of others neurons may become larger than θ at time t_0^+ and these neurons should also fire. To avoid possible ambiguities in the definition of the model, we have to assume that all the neurons that reach the threshold because of the firing of other(s) neuron(s) fire simultaneously at time t_0 and belong to the set J . We also

have to assume that a neuron cannot fire twice in the same instant. All these characteristics of the firing regime are typical of neural networks modeled by Equation (1) and containing excitatory neurons [17, 21]. They are reflected in the definition of the set J , which for convenience is detailed in Subsection 2.2.

To prove Theorem 1 and other results obtained along this paper, we will make the following hypothesis.

(H1) The membrane potential of the neurons has a lower bound, i.e there exists $\alpha < 0$ such that for all $i \in I$ and $t \in \mathbb{R}$, we have $V_i(t) \geq \alpha$. We will suppose that $\alpha > -\beta$.

(H2) If a neuron suffers inhibitory and excitatory interactions at the same time and if the sum of the excitatory interactions is enough to make it reach the threshold potential, then the inhibitory interactions are not taken into account and this neuron emits a spike.

The hypothesis (H1) fits with physical bounds of the electric potentials of real biological or electronic IF neural networks. The potential of a neuron can not be arbitrarily small and always remains in the order of magnitude of the characteristic constants of the model such as β and θ . When supposing instantaneous interactions, we collapse the time interval when the neurons are interacting. However, during this time interval the spikes are coming in a given order. The hypothesis (H2) assumes that positive interactions are faster than negative ones.

2.2 The Poincaré return map

In order to analyze the global dynamics of the IF neural network, we reduce it to an equivalent discrete time dynamical system, namely a first return Poincaré map. We first define a Poincaré section in the phase space and then we derive the corresponding Poincaré return map.

Since the potential of a neuron is always larger than α and always smaller than θ , the states of the network always belong to the n -dimensional space $Q = [\alpha, \theta]^n$. By definition of the model, the network never stops to emit spikes (since $\beta > \theta$). It exists then arbitrarily large times such that the potential of a neuron is reset to zero. In other words, any solution of the model returns infinitely many times to the set:

$$\Sigma = \bigcup_{j=1}^n \hat{\Sigma}_j \quad \text{where} \quad \hat{\Sigma}_j = \{\mathbf{V} \in Q : V_j = 0\}. \quad (4)$$

The set Σ is the Poincaré section that we will consider. The topology we use is the one induced by the embedding $\Sigma \subset \mathbb{R}^n$. Specifically we consider in Σ the metric derived from the supremum norm of \mathbb{R}^n , denoted $\|\cdot\|$ in this paper and defined by $\|\mathbf{V}\| = \max_{i \in I} |V_i|$.

The first step to compute the return map is to introduce a cover of Σ by n pieces. A piece of this cover will consist in the points of Σ such that the potential of a specified neuron reaches the threshold in a smaller time than those of all the other neurons. Solving the system (2) leads to the time t map $\phi^t = (\phi_1^t, \dots, \phi_n^t)$ where for each $i \in I$ and $t \in \mathbb{R}$

$$\phi_i^t(\mathbf{V}) = (V_i - \beta)e^{-\gamma t} + \beta \quad \forall \mathbf{V} \in \mathbb{R}^n. \quad (5)$$

If at time $t = 0$ the network is in the state $\mathbf{V} \in Q$, then the network enters in the firing regime at time:

$$\bar{t}(\mathbf{V}) := \min_{i \in \{1, \dots, n\}} t_i(\mathbf{V}) \quad \text{where} \quad t_i(\mathbf{V}) := \inf\{t \geq 0 : \phi_i^t(\mathbf{V}) \geq \theta\}.$$

The cover of Σ that we consider is:

$$\Sigma = \bigcup_{i=1}^n \Sigma_i \quad \text{where} \quad \Sigma_i = \{\mathbf{V} \in \Sigma : \bar{t}(\mathbf{V}) = t_i(\mathbf{V})\}. \quad (6)$$

As required, the set Σ_i is the set of points of Σ such that the first neuron to fire is the neuron i . These sets are not disjoint (because several neurons may reach θ at the same time) but have disjoint interiors.

Now, we introduce a partition of Σ . Each atom of this partition will consist of the points $\mathbf{V} \in \Sigma$ such that the neurons which fire a time $\bar{t}(\mathbf{V})$ are those of a specified subset J of I . We first need to identify

for all $\mathbf{V} \in \Sigma$ the set $J(\mathbf{V})$ of the neurons that fire at time $\bar{t}(\mathbf{V})$ taking into account Remark 2.1 and the hypothesis (H2). Let $\mathbf{V} \in \Sigma$, at time $\bar{t}(\mathbf{V})$ we may distinguish between two types of neurons. The neurons of the first type are those which potential reaches the threshold by action of the time t map (5). They belong to the set of indexes:

$$J_0(\mathbf{V}) = \{i \in I : \mathbf{V} \in \Sigma_i\}.$$

The neurons of the second type are those that fire as a consequence of the interactions with other neurons, in particular with that of $J_0(\mathbf{V})$. To obtain them, we define recursively a finite sequence of sets of indexes $\{J_m(\mathbf{V})\}_{m \geq 0}$. We have already defined the set $J_0(\mathbf{V})$. We define $J_m(\mathbf{V})$ by induction, for $m \geq 1$, as follows:

$$J_m(\mathbf{V}) = J_{m-1}(\mathbf{V}) \cup \{k \in I \setminus J_{m-1}(\mathbf{V}) : \phi_k^{\bar{t}(\mathbf{V})}(\mathbf{V}) + \sum_{i \in J_{m-1}(\mathbf{V}) : H_{ik} > 0} H_{ik} \geq \theta\} \quad \forall m \geq 1.$$

The second set of the union that defines $J_m(\mathbf{V})$ consists of the neurons that emit a spike as a result of the spiking of the neurons of $J_{m-1}(\mathbf{V})$, taking into account the hypothesis (H2). For instance, the neurons of $J_1(\mathbf{V}) \setminus J_0(\mathbf{V})$ (if not empty) fire by interaction with the neurons of the first type $J_0(\mathbf{V})$.

The set of the neurons that emit a spike at time $\bar{t}(\mathbf{V})$ can be written as:

$$J(\mathbf{V}) = \bigcup_{m \in \mathbb{N}} J_m(\mathbf{V}).$$

If there is no excitatory neurons in $J_0(\mathbf{V})$, then $J_m(\mathbf{V}) = J_0(\mathbf{V})$ for all $m \in \mathbb{N}$ and $J(\mathbf{V}) = J_0(\mathbf{V})$. If $J_0(\mathbf{V})$ contains excitatory neurons, then $J(\mathbf{V})$ may contain more neurons than $J_0(\mathbf{V})$. Note that $J(\mathbf{V})$ is finite since I is finite.

The set J used in the definition (3) of the model depends then on the state of the network (and on all the parameters). To every point $\mathbf{V} \in \Sigma$ it corresponds a set $J(\mathbf{V})$ of neurons that fire at the time $\bar{t}(\mathbf{V})$ when the network enters in the firing regime. The pre-images of the parts of I by $J(\cdot)$ can be used to get a partition \mathcal{P} of Σ :

$$\mathcal{P} = \{\Sigma_J\}_{J \in P(I)} \quad \text{where} \quad \Sigma_J = \{\mathbf{V} \in \Sigma : J(\mathbf{V}) = J\} \quad (7)$$

and $P(I)$ denotes the set of all the non empty parts of I . According to the interactions H_{ji} , some Σ_J may be empty. Nevertheless, there is always a nonempty Σ_J .

According to (3) and hypothesis (H1), the components ρ_1, \dots, ρ_n of the first return map $\rho : \Sigma \rightarrow \Sigma$ are defined in each Σ_J by:

$$\rho_i(\mathbf{V}) = 0 \quad \text{if} \quad i \in J \quad \text{and} \quad \rho_i(\mathbf{V}) = \max\{\alpha, \phi_i^{\bar{t}(\mathbf{V})}(\mathbf{V}) + \sum_{j \in J} H_{ij}\} \quad \text{if} \quad i \notin J. \quad (8)$$

Note that the return map does not satisfy standard hypothesis of differentiability or continuity in its entire domain. Actually, it is a piecewise continuous map whose continuity pieces are the sets Σ_J . On the set $\partial P = \bigcup_{J \in P(I)} \partial \Sigma_J$, where $\partial \Sigma_J$ denotes the boundary of Σ_J , the return map is not continuous and some perturbation may change drastically the temporal evolution of the network. A study of the irregularities of ρ is given in Section 5.

3 Contraction properties of the return map

Many mathematical studies of the global dynamics of neural networks consider models presenting a contraction property in the whole phase space [5, 7, 8, 18]. This property is the basis of the proof of the existence of limit cycles supporting the asymptotic dynamics. Here, we will prove that for IF neural network as defined in Section 2, the presence of excitatory interactions prevent the return map from being contractive in the whole phase space, at least for an open region of the parameters. However, we also show that when the interactions are sufficiently strong (inhibitory as well as excitatory) there exists a metric and a partition of Σ such that the return map is contractive in each of the piece of this partition.

Definition 3.1. Given a subset $\Sigma_c^* \subset \Sigma$ and a finite partition \mathcal{P}^* of Σ_c^* , we say that ρ is piecewise contractive in Σ_c^* with respect to \mathcal{P}^* if there exist a constant $0 < \lambda^* < 1$ and a norm $\|\cdot\|^*$ such that

$$\|\rho(\mathbf{V}) - \rho(\mathbf{W})\|^* \leq \lambda^* \|\mathbf{V} - \mathbf{W}\|^*$$

for all \mathbf{V} and \mathbf{W} in a same piece of \mathcal{P}^* . In such a case, we say that Σ_c^* is a contractive zone with respect to \mathcal{P}^* .

Note that if ρ is piecewise contractive in Σ_c^* with respect to \mathcal{P}^* , then ρ is Lipschitz continuous in each piece of \mathcal{P}^* . Thus, \mathcal{P}^* has to be a refined partition of \mathcal{P} .

3.1 Arbitrary interactions

The question we address now, is the existence of a norm $\|\cdot\|^*$ such that the whole Poincaré section Σ is a contractive zone with respect to the natural partition \mathcal{P} . The following theorem shows that it is not a general property of the model.

Theorem 2. Under the hypothesis (H1) and (H2), there exists an open region of the values of the interactions such that for all norm $\|\cdot\|^*$ the section Σ is not a contractive zone with respect to \mathcal{P} .

In [7] it is proved that if all interactions are inhibitory, then the return map is piecewise contractive. Thus, the region of parameters values satisfying the thesis of Theorem 2, must contain excitatory interactions. In other words, the introduction of excitatory neurons can avoid piecewise contractivity of the return map in the whole phase space, for certain values of parameters.

To prove the theorem we introduce the quantity $c^* = \beta - \sqrt{\beta(\beta - \theta)} \in (\theta/2, \theta)$ and we define for all $i \neq j \in I$ the sets

$$\Gamma_{ij} = \{\mathbf{V} \in \Sigma_i : V_i > c^*, V_k = 0 \forall k \neq i\} \cap \Sigma_{I \setminus \{j\}}.$$

Lemma 3.2. Let $i \neq j \in I$ and suppose there exist $\mathbf{V} \neq \mathbf{W} \in \Gamma_{ij} \cap \rho^{-1}(\Gamma_{ji})$, then

$$|\rho_j(\mathbf{W}) - \rho_j(\mathbf{V})| > |V_i - W_i|.$$

Proof: Since $\mathbf{V} \in \Gamma_{ij} \subset \Sigma_{I \setminus \{j\}} \cap \Sigma_i$, we have $\rho_j(\mathbf{V}) = \max\{\alpha, \phi_j^{t_i(\mathbf{V})}(\mathbf{V}) + \sum_{k \in I \setminus \{j\}} H_{kj}\}$, and since $\rho(\mathbf{V}) \in \Gamma_{ji}$, it follows that $\rho_j(\mathbf{V}) > c^* > \alpha$. As a consequence,

$$\rho_j(\mathbf{V}) = \phi_j^{t_i(\mathbf{V})}(\mathbf{V}) + \sum_{k \in I \setminus \{j\}} H_{kj} = -\frac{\beta(\beta - \theta)}{\beta - V_i} + \beta + \sum_{k \in I \setminus \{j\}} H_{kj},$$

where the second equality is obtained computing $t_i(\mathbf{V})$ and using $V_j = 0$ (by definition of Γ_{ij}). The same computation being true for $\rho_j(\mathbf{W})$ we deduce

$$|\rho_j(\mathbf{V}) - \rho_j(\mathbf{W})| = \left| \frac{\beta(\beta - \theta)}{(\beta - W_i)(\beta - V_i)} (W_i - V_i) \right| > \frac{\beta(\beta - \theta)}{(\beta - c^*)^2} |V_i - W_i| = |V_i - W_i|.$$

□

Lemma 3.3. Let $i \neq j \in I$. For an open region of values of the interactions, the sets Γ_{ij} and Γ_{ji} are not empty and there exists $(a, b) \subset (c^*, \theta)$ such that for all $\mathbf{V} \in \Gamma_{ij}$ satisfying $V_i \in (a, b)$, we have $\rho(\mathbf{V}) \in \Gamma_{ji}$ and $\rho^2(\mathbf{V}) \in \Gamma_{ji}$.

Proof: See appendix 7.1. □

Suppose that the values of parameters are such that Lemma 3.3 is true. Then, there exist two points $\mathbf{V}' \neq \mathbf{W}'$ belonging to $\Gamma_{ij} \cap \rho^{-1}(\Gamma_{ji}) \cap \rho^{-2}(\Gamma_{ij})$. Applying Lemma 3.2 twice we obtain:

$$|\rho_i^2(\mathbf{V}') - \rho_i^2(\mathbf{W}')| > |V'_i - W'_i|. \quad (9)$$

Proof of Theorem 2: We show that the existence of \mathbf{V}' and \mathbf{W}' is incompatible with the existence of a norm $\|\cdot\|^*$ such that Σ is a contractive zone with respect to \mathcal{P} . If we suppose such a norm exists, then

$$\|\rho^2(\mathbf{V}') - \rho^2(\mathbf{W}')\|^* \leq \lambda^{*2} \|\mathbf{V}' - \mathbf{W}'\|^* < \|\mathbf{V}' - \mathbf{W}'\|^*, \quad (10)$$

since \mathbf{V}' and $\mathbf{W}' \in B_{I \setminus \{i\}} \cap \rho^{-1}(B_{I \setminus \{j\}})$ and $\lambda^* < 1$.

Let's consider the norm $\|\cdot\|_i^*$ defined by $\|\mathbf{V}\|_i^* = \|\mathbf{V}\|^*$ for all $\mathbf{V} \in \{0\}^{i-1} \times \mathbb{R} \times \{0\}^{n-i} \supset \Gamma_{ij}$. On one hand, $\|\cdot\|_i^*$ being a norm in a vector space isomorphic to \mathbb{R} , there exists $\mu_i > 0$ such that $\|\mathbf{V}\|_i^* = \mu_i |V_i|$. On the other hand, recalling that $\rho^2(\mathbf{V}')$ and $\rho^2(\mathbf{W}')$ $\in \Gamma_{ij}$, the inequalities (10) are also true with the norm $\|\cdot\|_i^*$. We deduce that

$$|\rho_i^2(\mathbf{V}') - \rho_i^2(\mathbf{W}')| < |V_i' - W_i'|$$

which contradicts (9). \square

The Poincaré section is generally not a contractive zone with respect to \mathcal{P} . However, when $\beta < \frac{4}{3}\theta$, it is possible to find a subset of Σ which is a contractive zone.

Proposition 3.4. *Suppose $\beta < \frac{4}{3}\theta$ and let $c = \beta - 2\sqrt{\beta(\beta - \theta)}$. Then, the subset $\Sigma_c \subset \Sigma$ defined by*

$$\Sigma_c = \{\mathbf{V} \in \Sigma : \alpha \leq V_i \leq c, i \in I\}$$

is a contractive zone with respect to \mathcal{P} .

Proof of Proposition 3.4: We show that there exists a constant $0 < \lambda_\alpha < 1$ such that for any $J \in \mathcal{P}(I)$:

$$\|\rho(\mathbf{V}) - \rho(\mathbf{W})\| \leq \lambda_\alpha \|\mathbf{V} - \mathbf{W}\| \quad \forall \mathbf{V}, \mathbf{W} \in \Sigma_J \cap \Sigma_c. \quad (11)$$

Suppose $\mathbf{V}, \mathbf{W} \in \Sigma_J \cap \Sigma_c$ and let $i, l \in J$ be such that $\mathbf{V} \in \Sigma_i$ and $\mathbf{W} \in \Sigma_l$.

Let $k \in I$. If $k \in J$ then by definition of ρ

$$|\rho_k(\mathbf{V}) - \rho_k(\mathbf{W})| = 0. \quad (12)$$

If $k \notin J$ we have to consider 4 cases:

Case 1: If $\rho_k(\mathbf{V}) = \rho_k(\mathbf{W}) = \alpha$ then (12) is true.

Case 2: If $\rho_k(\mathbf{V}) > \alpha$ and $\rho_k(\mathbf{W}) > \alpha$ then, using the explicit expression (5) of ϕ^t at times $\bar{t}(\mathbf{V}) = t_i(\mathbf{V})$ and $\bar{t}(\mathbf{W}) = t_l(\mathbf{W})$, we obtain:

$$\begin{aligned} |\rho_k(\mathbf{V}) - \rho_k(\mathbf{W})| &= \left| \frac{(\beta - W_k)(\beta - \theta)}{\beta - W_l} - \frac{(\beta - V_k)(\beta - \theta)}{\beta - V_i} \right| \\ &= \left| \frac{(\beta - W_k)(\beta - \theta)}{\beta - W_l} - \frac{(\beta - V_k)(\beta - \theta)}{\beta - W_l} + \frac{(\beta - V_k)(\beta - \theta)}{\beta - W_l} - \frac{(\beta - V_k)(\beta - \theta)}{\beta - V_i} \right| \\ &= \left| \frac{\beta - \theta}{\beta - W_l} (V_k - W_k) + \frac{(\beta - V_k)(\beta - \theta)}{(\beta - W_l)(\beta - V_i)} (W_l - V_i) \right| \\ &\leq \frac{\beta - \theta}{\beta - c} |V_k - W_k| + \frac{(\beta - \alpha)(\beta - \theta)}{(\beta - c)^2} |V_i - W_l|. \end{aligned}$$

Suppose $V_i \leq W_l$. Since $t_i(\mathbf{V}) = \bar{t}(\mathbf{V})$, we have $V_i \leq V_i$, which implies $|V_i - W_l| \leq |V_i - W_l|$, and

$$\begin{aligned} |\rho_k(\mathbf{V}) - \rho_k(\mathbf{W})| &\leq \frac{\beta - \theta}{\beta - c} |V_k - W_k| + \frac{(\beta - \alpha)(\beta - \theta)}{(\beta - c)^2} |V_i - W_l| \\ &\leq \frac{\beta - \theta}{\beta - c} \left(1 + \frac{\beta - \alpha}{\beta - c} \right) \|\mathbf{V} - \mathbf{W}\|. \end{aligned} \quad (13)$$

If $W_l < V_i$ then, using this time that $t_l(\mathbf{W}) = \bar{t}(\mathbf{W})$ implies $W_l \geq W_i$, we obtain $|V_i - W_l| \leq |V_i - W_i|$ and the inequality (13) follows.

Case 3: If $\rho_k(\mathbf{V}) > \alpha$ and $\rho_k(\mathbf{W}) = \alpha$ then

$$|\rho_k(\mathbf{V}) - \rho_k(\mathbf{W})| = \phi_k^{\bar{i}(\mathbf{V})}(\mathbf{V}) + \sum_{j \in J} H_{jk} - \alpha \leq \phi_k^{\bar{i}(\mathbf{V})}(\mathbf{V}) + \sum_{j \in J} H_{jk} - \phi_k^{\bar{i}(\mathbf{W})}(\mathbf{W}) - \sum_{j \in J} H_{jk} = \phi_k^{\bar{i}(\mathbf{V})}(\mathbf{V}) - \phi_k^{\bar{i}(\mathbf{W})}(\mathbf{W})$$

and we obtain (13) by the same calculation as in the case 2.

Case 4: If $\rho_k(\mathbf{V}) = \alpha$ and $\rho_k(\mathbf{W}) > \alpha$ then, substituting \mathbf{V} for \mathbf{W} and \mathbf{W} for \mathbf{V} in the case 3, we obtain the inequality (13).

We conclude that for all $k \in I$ either (12) or (13) is true. It follows:

$$\|\rho(\mathbf{V}) - \rho(\mathbf{W})\| \leq \lambda_\alpha \|\mathbf{V} - \mathbf{W}\| \quad \text{where} \quad \lambda_\alpha = \frac{\beta - \theta}{\beta - c} \left(1 + \frac{\beta - \alpha}{\beta - c} \right)$$

and the proposition is proved since by definition of c and hypothesis (H1)

$$\lambda_\alpha < \lambda_{-\beta} = \frac{1}{2} \left(\sqrt{\frac{\beta - \theta}{\beta}} + 1 \right) < 1.$$

□

3.2 Strong interactions

In the previous subsection we have shown that Σ is generally not a contractive zone with respect to \mathcal{P} . Nevertheless, we have shown in Proposition 3.4 that under the hypothesis $\beta < \frac{4}{3}\theta$ there is a subset Σ_c of Σ which is a contractive zone. Here, we give values of the parameters such that this contractive zone is forward invariant. Moreover, if the network contains inhibitory neurons, then any orbit drops into Σ_c after a uniform number of iterations. It allows us to prove that there is a metric making Σ a contractive zone, with respect to a refined partition of \mathcal{P} .

From now on, we will make the following assumptions on the parameters:

(H3) Denoting $\delta = \beta - \theta > 0$, we assume $\delta < \frac{\theta}{3}$ and the interactions H_{ji} sufficiently strong such that

$$\min_{j \neq i} |H_{ji}| > 2\sqrt{(\theta + \delta)\delta} = 2\sqrt{\beta(\beta - \theta)} = \epsilon.$$

(H4) We assume the Dale's principle (see [15] page 7): a neuron is either excitatory or inhibitory. In other words, the network does not contain mixed neurons.

In Section 2.2 we noted that for every $\mathbf{V} \in \Sigma$ such that $J_0(\mathbf{V})$ contains only inhibitory neurons $J(\mathbf{V}) = J_0(\mathbf{V})$. In the contractive zone we have in addition the following property:

Lemma 3.5. *If $\mathbf{V} \in \Sigma_c$ and $J_0(\mathbf{V})$ contains an excitatory neuron then $J(\mathbf{V}) = I$.*

This lemma, gives us a first result about the dynamics of the network. If $\mathbf{V} \in \Sigma_c \cap \Sigma_i$, where i is an excitatory neuron, then $\rho(\mathbf{V}) = 0$. The origin of the phase space being a fixed point of ρ , it is a periodic orbit of the model. Moreover, this orbit corresponds to a synchronized state of the network, i.e the potential of all the neurons is the same at each time.

Proof of the lemma: Suppose $\mathbf{V} \in \Sigma_c$. Let $i \in J_0(\mathbf{V})$ and take $k \in I \setminus J_0(\mathbf{V})$. Then,

$$\phi_k^{\bar{i}(\mathbf{V})}(\mathbf{V}) = \beta - \frac{(\beta - V_k)(\beta - \theta)}{\beta - V_i} \geq \beta - \frac{(\beta - \alpha)(\beta - \theta)}{\beta - c} = \beta - \frac{\beta - \alpha}{2\beta} \sqrt{\beta(\beta - \theta)} > c,$$

since $\alpha > -\beta$. Thus, since $J_0(\mathbf{V})$ contains an excitatory neuron, it follows that

$$\phi_k^{\bar{i}(\mathbf{V})}(\mathbf{V}) + \sum_{j \in J_0(\mathbf{V}) : H_{jk} > 0} H_{jk} > c + \epsilon = \beta > \theta.$$

We conclude that $k \in J_1(\mathbf{V}) \subset J(\mathbf{V})$. □

Now we use the lemma to prove that the contractive zone is a forward invariant set of the return map:

Proposition 3.6. *If $\mathbf{V} \in \Sigma_c$ then $\rho(\mathbf{V}) \in \Sigma_c$.*

Proof: Let $\mathbf{V} \in \Sigma_c$ and $k \in I$. By hypothesis (H1), we know that $\alpha \leq \rho_k(\mathbf{V})$, so we only have to prove that $\rho_k(\mathbf{V}) \leq c$.

If $k \in J(\mathbf{V})$ then $\rho_k(\mathbf{V}) = 0 \leq c$. If $k \notin J(\mathbf{V})$, by lemma 3.5, $J_0(\mathbf{V})$ contains only inhibitory neurons. It implies that $J(\mathbf{V}) = J_0(\mathbf{V})$ contains only inhibitory neurons. Together with hypothesis (H3) and (H4), it leads to:

$$\phi_k^{\bar{i}(\mathbf{V})}(\mathbf{V}) + \sum_{j \in J(\mathbf{V})} H_{jk} < \theta - \epsilon < c.$$

Looking at the definition (8) we conclude $\rho_k(\mathbf{V}) < c$. \square

This last proposition ensures that if an orbit of the return map falls in the contractive zone, then it stays forever in this region. The following proposition states that if the network contains an inhibitory neuron, then any orbit falls in the contractive zone.

Proposition 3.7. *If the network contains an inhibitory neuron, then there exists $p \in \mathbb{N}$ such that $\rho^p(\Sigma) \subset \Sigma_c$.*

To prove the proposition, let's define for each $p \in \mathbb{N}$ the set Z_p of the points of the Poincaré section that stay outside of the contractive zone during p iterations:

$$Z_p = \{\mathbf{V} \in \Sigma : \rho^j(\mathbf{V}) \in \Sigma \setminus \Sigma_c, 1 \leq j \leq p\}.$$

Lemma 3.8. *If the network contains an inhibitory neuron and if there exists $p \geq 1$ such that Z_p is non empty, then $\alpha + p\epsilon < \theta$.*

Proof: Let us suppose there exists $\mathbf{V} \in Z_p$ for a $p \geq 1$. We first prove that for all $0 \leq j \leq p-1$ the set $J(\rho^j(\mathbf{V}))$ contains only excitatory neurons. We begin supposing $p = 1$. By definition of Σ_c , if $\mathbf{V} \in Z_1$ then $\rho_k(\mathbf{V}) > c$ for a $k \in I$. This k cannot belong to $J(\mathbf{V})$ (if it does then $\rho_k(\mathbf{V}) = 0$), therefore

$$\phi_k^{\bar{i}(\mathbf{V})}(\mathbf{V}) + \sum_{j \in J(\mathbf{V}) : H_{jk} > 0} H_{jk} < \theta.$$

It follows that

$$\rho_k(\mathbf{V}) = \phi_k^{\bar{i}(\mathbf{V})}(\mathbf{V}) + \sum_{j \in J(\mathbf{V}) : H_{jk} > 0} H_{jk} + \sum_{j \in J(\mathbf{V}) : H_{jk} < 0} H_{jk} < \theta + \sum_{j \in J(\mathbf{V}) : H_{jk} < 0} H_{jk},$$

and if $J(\mathbf{V})$ contains an inhibitory neuron $\rho_k(\mathbf{V}) < \theta - \epsilon < c$, which is a contradiction. Suppose $p > 1$ and let $0 \leq j \leq p-1$ then $\rho^j(\mathbf{V}) \in Z_1$, therefore $J(\rho^j(\mathbf{V}))$ contains only excitatory neurons.

Let us suppose now that the network contains an inhibitory neuron denoted i . Since $J(\mathbf{V})$ contains only excitatory neurons, $i \notin J(\mathbf{V})$ and the component i of $\rho(\mathbf{V})$ satisfies:

$$\rho_i(\mathbf{V}) = \phi_i^{\bar{i}(\mathbf{V})}(\mathbf{V}) + \sum_{j \in J(\mathbf{V})} H_{ji} = (V_i - \beta)e^{-\gamma \bar{i}(\mathbf{V})} + \beta + \sum_{j \in J(\mathbf{V})} H_{ji} > V_i + \epsilon.$$

Knowing that $J(\rho^j(\mathbf{V}))$ contains only excitatory neurons for all $0 \leq j \leq p-1$ and as $\phi_i^{\bar{i}(\mathbf{V})}(\mathbf{V}) > V_i$ for all $V \in \Sigma$, we deduce by induction that $\theta > \rho_i^p(\mathbf{V}) > V_i + p\epsilon > \alpha + p\epsilon$. \square

Proof of Proposition 3.7: Let $p \in \mathbb{N}$ such that $\alpha + p\epsilon > \theta$. According to Lemma 3.8 either Z_p is empty or the network does not contain inhibitory neurons. By hypothesis of the proposition the network contains inhibitory neurons, therefore Z_p is empty and $\rho^p(\Sigma) \subset \Sigma_c$. \square

Corollary 3.9. *If there exists $\mathbf{V} \in \Sigma$ such that $\rho^p(\mathbf{V}) \notin \Sigma_c$ for all $p \in \mathbb{N}$ then the network contains only excitatory neurons.*

Proposition 3.6 and Proposition 3.7 allow to prove Lemma 7.1 in appendix 7.2, which states that ρ is “eventually piecewise contractive”. This is a generalization for piecewise continuous maps of the definition of eventually contractive maps (see for instance Definition 2.6.11 in [14]). From Lemma 7.1 the classical arguments to prove the existence of an adapted metric can be reproduced for piecewise continuous maps.

Theorem 3. *Under the hypothesis (H1), (H2), (H3) and (H4), there exists an adapted metric and a partition \mathcal{P}^* of Σ such that ρ is piecewise contractive in Σ with respect to \mathcal{P}^* .*

Proof: See appendix 7.2.

Now we sum up the results of this section and we point out their important consequences.

Under the hypotheses (H3) and (H4), the Poincaré section of the model can be decomposed in two regions: the contractive zone (Σ_c), which is a forward invariant set of the return map, and the set of the orbits that never fall in the contractive zone. This last set can be non-empty only for networks composed exclusively of excitatory neurons (cf Corollary 3.9). We study these networks in the Section 4.

In the case of the networks containing inhibitory neurons, all orbits drop into the contractive zone after a finite time (Proposition 3.7). The study of the asymptotic dynamics reduces then to the analysis of the dynamics of the return map in the contractive zone. This is the purpose of Section 5.

4 Synchronization

In this section we prove the part (1) of Theorem 1 stating sufficient conditions for the synchronization of the system.

Definition 4.1. We say that an orbit $\{\mathbf{V}(t)\}_{t \in \mathbb{R}^+}$ is a *synchronized orbit* of the system if $V_i(t) = V_j(t)$ for all $i, j \in I$ and $t \in \mathbb{R}^+$. We say that the system is *globally synchronized* if for any initial state $\mathbf{V}(0)$ there exists $t_0 \in \mathbb{R}^+$ such that $\{\mathbf{W}(t) = \mathbf{V}(t + t_0)\}_{t \in \mathbb{R}^+}$ is a synchronized orbit. We say that the system is *globally periodic and synchronized* if besides $\{\mathbf{W}(t)\}_{t \in \mathbb{R}^+}$ is periodic.

Since for all $\mathbf{V} \in \Sigma$ at least one component of \mathbf{V} is equal to 0 (see (4)), a synchronized orbit intersects the Poincaré section at the origin. Therefore, we say that the origin is the synchronization state of the Poincaré section. Note that the synchronization state is a fixed point of ρ . We call $\mathbf{V} \in \Sigma$ a state of eventual synchronization if there exists $l \in \mathbb{N}$ such that $\rho^l(\mathbf{V}) = \mathbf{0}$. Note that a network globally synchronizes if and only if all the points of the Poincaré section are states of eventual synchronization.

Theorem 4. *Under the hypothesis (H1), (H2) and (H3), if the network is exclusively composed of excitatory neurons and the number of neurons $n > \left(\frac{\theta - \alpha}{\epsilon} + 1\right)^2$, then the system is globally periodic and synchronized.*

The hypothesis of Theorem 4 ensures that the synchronization state is stable. Indeed, any sufficiently small perturbation of the synchronization state, will lead some neurons to reach the threshold before the other ones. However, since all the neurons are excitatory and the interactions are strong (hypothesis (H3)), the first spiking neurons will induce the others to spike in turn. Therefore, the system is back to the synchronization state $\mathbf{0} \in \Sigma$ in the first iteration of the return map from the perturbed state.

Proof of Theorem 4: Let $\mathbf{V} \in \Sigma$. We prove that there exists $l \in \mathbb{N}$ such that $\rho^l(\mathbf{V}) = \mathbf{0}$. To this aim, we introduce $p = \lfloor \sqrt{n} \rfloor + 1$ where $\lfloor \cdot \rfloor$ is the integer part function.

Claim: We have $\bigcup_{j=0}^{p-1} J(\rho^j(\mathbf{V})) = I$.

Indeed, let $i \in I$. If $i \in \bigcup_{j=0}^{p-2} J(\rho^j(\mathbf{V}))$ then $i \in \bigcup_{j=0}^{p-1} J(\rho^j(\mathbf{V}))$. If $i \notin \bigcup_{j=0}^{p-2} J(\rho^j(\mathbf{V}))$ then by definition of ρ

$$\rho_i^j(\mathbf{V}) = \phi_i^{\bar{i}(\rho^{j-1}(\mathbf{V}))}(\rho^{j-1}(\mathbf{V})) + \sum_{k \in J(\rho^{j-1}(\mathbf{V}))} H_{ki} > \rho_i^{j-1}(\mathbf{V}) + \epsilon \quad \forall j \in \{1, \dots, p-1\},$$

where the inequality is obtained using that $\phi_i^{\bar{i}(\mathbf{V})}(\mathbf{V}) > V_i$ for all $\mathbf{V} \in \Sigma$ and $i \in I$ and the fact that all the interactions are excitatory. By induction we obtain $\rho_i^{p-1}(\mathbf{V}) > V_i + (p-1)\epsilon$. We deduce that

$$\phi_i^{\bar{i}(\rho^{p-1}(\mathbf{V}))}(\rho^{p-1}(\mathbf{V})) + \sum_{k \in J(\rho^{p-1}(\mathbf{V}))} H_{ki} > \rho_i^{p-1}(\mathbf{V}) + \epsilon > V_i + p\epsilon > \alpha + \sqrt{n}\epsilon > \theta,$$

and therefore $i \in J(\rho^{p-1}(\mathbf{V}))$, which proves the claim.

Suppose now that for all $j \in \{0, \dots, p-1\}$ the cardinal $\#J(\rho^j(\mathbf{V})) < \frac{\theta - \alpha}{\epsilon}$, then

$$\# \bigcup_{j=0}^{p-1} J(\rho^j(\mathbf{V})) \leq \sum_{j=0}^{p-1} \#J(\rho^j(\mathbf{V})) < \frac{p(\theta - \alpha)}{\epsilon}.$$

Applying the claim, the last inequality implies $n < \frac{p(\theta - \alpha)}{\epsilon} < (\sqrt{n} + 1)(\sqrt{n} - 1) = n - 1$, which is a contradiction. Thus, we proved that there exists $j \in \{0, \dots, p-1\}$ such that

$$\#J(\rho^j(\mathbf{V})) \geq \frac{\theta - \alpha}{\epsilon}.$$

Then for all $i \in I$,

$$\phi_i^{\bar{t}(\rho^j(\mathbf{V}))}(\rho^j(\mathbf{V})) + \sum_{k \in J(\rho^j(\mathbf{V}))} H_{ki} > \alpha + \epsilon \frac{\theta - \alpha}{\epsilon} = \theta,$$

and therefore $I = J(\rho^j(\mathbf{V}))$. In other words, if we take $l = j + 1$ then $\rho^l(\mathbf{V}) = \mathbf{0}$. \square

It is a non trivial exercise to construct a simple example for which $n \leq \left(\frac{\theta - \alpha}{\epsilon} + 1\right)^2$ and there exists an orbit which is not synchronized.

Definition 4.2. We say that $\mathbf{V} \in \Sigma$ is a state of eventual death of the neuron i , if there exists $p \in \mathbb{N}$ such that $i \notin J(\rho^j(\mathbf{V}))$ for all $j \geq p$.

In other words, a state of eventual death of a neuron is a state such that the neuron stops to emit spikes after a certain time. Therefore, we define a state of the (continuous time) model of eventual death as a state whose orbit intersects the Poincaré section in a state of eventual death.

Theorem 5. Under the hypothesis (H1), (H2), (H3) and (H4), if the network is composed of excitatory and inhibitory neurons, then the states of the network are either states of eventual synchronization or states of eventual death of all the excitatory neurons.

Proof: Let i be an excitatory neuron and $\mathbf{V} \in \Sigma$. By Proposition 3.7 we can assume $\mathbf{V} \in \Sigma_c$ without loss of generality. We have, either $i \notin J(\rho^j(\mathbf{V}))$ for all $j \in \mathbb{N}$ or there exists $j \in \mathbb{N}$ such that $i \in J(\rho^j(\mathbf{V}))$. In the first case \mathbf{V} is a state of eventual death of the neuron i . In the second case, by Proposition 3.6 it follows $\rho^j(\mathbf{V}) \in \Sigma_c$ for all $j \in \mathbb{N}$, and then we can apply Lemma 3.5 to deduce that $J(\rho^j(\mathbf{V})) = I$. This implies $\rho^{j+1}(\mathbf{V}) = \mathbf{0}$. \square

5 Asymptotic dynamics of networks with inhibitory neurons

In the previous section we proved that networks composed exclusively of excitatory neurons globally synchronize, whenever their size is sufficiently large (Theorem 4). Now we focus on networks containing both excitatory and inhibitory neurons. We already know that they synchronize if an excitatory neuron fires (Lemma 3.5 and Theorem 5). We are now interesting in their general asymptotic dynamics, including outside of the synchronization regime.

5.1 Continuity pieces of the return map

In Proposition 3.7 we have shown that any orbit of a network containing inhibitory neurons finally drops into the forward invariant set Σ_c , called the contractive zone. Then, it is not restrictive to consider this space as our new phase space¹. The partition \mathcal{P} (defined in (7)) formed by the continuity pieces of the return map becomes

$$\mathcal{P}_c := \{\Sigma_{c,J}\}_{J \in \mathcal{P}(I)} \quad \text{where} \quad \Sigma_{c,J} = \Sigma_J \cap \Sigma_c.$$

¹From now on, ρ will denote the return map restricted to Σ_c .

In the sequel we give a detailed description of this partition. We show that it is made of open pieces, where the return map is continuous, and of closed sets which are the boundary of the open pieces, where the return map generically presents a discontinuity jump.

By Lemma 3.5, if $J \neq I$ and contains excitatory neurons then $\Sigma_{c,J} = \emptyset$. As a consequence, \mathcal{P}_c is actually composed of the set $\Sigma_{c,I}$ and of the sets $\Sigma_{c,J}$ such that J contains only inhibitory neurons. It follows that

$$\Sigma_c = \left(\bigcup_{J \in P(I^-)} \Sigma_{c,J} \right) \bigcup \Sigma_{c,I},$$

where $I^- \subset I$ is the set of the inhibitory neurons.

The sets $\Sigma_{c,J}$ forming the partition \mathcal{P}_c admit explicit formulations useful to study their topological properties. On one hand, if $J \in P(I^-)$ then for all $\mathbf{V} \in \Sigma_{c,J}$ we have $J(\mathbf{V}) = J_0(\mathbf{V})$, since the firing of an inhibitory neuron cannot lead another neuron to fire. We deduce that $\mathbf{V} \in \Sigma_{c,J}$ if and only if $t_i(\mathbf{V}) = \bar{t}(\mathbf{V})$ for all $i \in J$ and $t_k(\mathbf{V}) > \bar{t}(\mathbf{V})$ for all $k \notin J$. This leads to the following expression:

$$\Sigma_{c,J} = \{ \mathbf{V} \in \Sigma_c : V_i = V_j > V_k \quad \forall i, j \in J \text{ and } \forall k \notin J \} \quad \forall J \in P(I^-). \quad (14)$$

On the other hand, $\mathbf{V} \in \Sigma_{c,I}$ if and only if $J_0(\mathbf{V})$ contains an excitatory neuron. Then, $\mathbf{V} \in \Sigma_{c,I}$ if and only if $t_i(\mathbf{V}) = \bar{t}(\mathbf{V})$ for some i in the set I^+ of the excitatory neurons. So, we have:

$$\Sigma_{c,I} = \{ \mathbf{V} \in \Sigma_c : \max_{i \in I^+} V_i \geq V_k \quad \forall k \in I^- \}. \quad (15)$$

Let us consider the relative topology induced by \mathbb{R}^n in Σ . In the case where $J \in P(I^-)$ is a singleton ($J = \{i\}$), the set $\Sigma_{c,J}$ is open, since (14) writes:

$$\Sigma_{c,\{i\}} = \{ \mathbf{V} \in \Sigma_c : V_i > V_k \quad \forall k \neq i \} \quad \forall i \in I^-.$$

The topological boundary of $\Sigma_{c,\{i\}}$ is the set

$$\partial \Sigma_{c,\{i\}} = \{ \mathbf{V} \in \Sigma_c : V_i \geq V_k \quad \forall k \in I \text{ and } \exists j \neq i : V_j = V_i \}. \quad (16)$$

According to (15), the interior of $\Sigma_{c,I}$ is

$$\Sigma_{c,\{0\}} := \{ \mathbf{V} \in \Sigma_c : \max_{i \in I^+} V_i > V_k \quad \forall k \in I^- \},$$

and its boundary is

$$\partial \Sigma_{c,\{0\}} = \{ \mathbf{V} \in \Sigma_c : \max_{i \in I^+} V_i \geq V_k \quad \forall k \in I^- \text{ and } \exists j \in I^- : V_j = \max_{i \in I^+} V_i \}. \quad (17)$$

From (16) and (17) it results that $\partial \Sigma_{c,\{0\}} \subset \bigcup_{i \in I^-} \partial \Sigma_{c,\{i\}}$. On one hand, $\Sigma_{c,\{0\}}$ is the interior of $\Sigma_{c,I}$, and on the other hand if $J \in P(I^-)$ is not a singleton and contains i , then $\Sigma_{c,J}$ is included in the boundary of $\Sigma_{c,\{i\}}$. Therefore, the partition \mathcal{P}_c of Σ_c consists of the open sets $\{ \Sigma_{c,\{i\}} \}_{i \in I^- \cup \{0\}}$ and of the boundaries $\{ \partial \Sigma_{c,\{i\}} \}_{i \in I^-}$. To sum up,

$$\Sigma_c = \left(\bigcup_{i \in I_0^-} \Sigma_{c,\{i\}} \right) \bigcup \partial \mathcal{P}_c, \quad \text{where } I_0^- := I^- \cup \{0\} \quad \text{and} \quad \partial \mathcal{P}_c := \bigcup_{i \in I^-} \partial \Sigma_{c,\{i\}} = \bigcup_{\substack{J \in P(I^-) \\ \#J \geq 2}} \Sigma_{c,J}.$$

On one hand, for all $i \in I_0^-$, the return map ρ is continuous in $\Sigma_{c,\{i\}}$, since this set is an open continuity piece, see (8). On the other hand, the following lemma shows that ρ is generically discontinuous in $\partial \mathcal{P}_c$.

Lemma 5.1. *If $\mathbf{V} \in \partial \mathcal{P}_c$, then there exists a sequence $\{\mathbf{U}^m\}_{m \in \mathbb{N}}$ of points of Σ_c such that $\lim_{m \rightarrow \infty} \mathbf{U}^m = \mathbf{V}$ and $\lim_{m \rightarrow \infty} \|\rho(\mathbf{U}^m) - \rho(\mathbf{V})\| \geq \min\{|\alpha|, \min_{i \neq j} |H_{ij} + \theta|\}$.*

Proof: If $\mathbf{V} \in \partial\mathcal{P}_c$ then there exists a subset J of I^- such that $\#J \geq 2$ and $\mathbf{V} \in \Sigma_{c,J}$. Let $i \neq j \in J$. Since $\Sigma_{c,J} \subset \partial\Sigma_{c,\{i\}}$, by definition of boundary, there exists a sequence $\{\mathbf{U}^m\}_{m \in \mathbb{N}}$ of points of $\Sigma_{c,\{i\}}$ such that $\lim_{m \rightarrow \infty} \mathbf{U}^m = \mathbf{V}$. Moreover,

$$\lim_{m \rightarrow \infty} \rho_j(\mathbf{U}^m) = \lim_{m \rightarrow \infty} \max\{\alpha, \beta - (\beta - U_j^m) \frac{\beta - \theta}{\beta - U_i^m} + H_{ij}\} = \max\{\alpha, \beta - (\beta - V_j) \frac{\beta - \theta}{\beta - V_i} + H_{ij}\}.$$

By (14) we have $V_i = V_j$ and therefore $\lim_{m \rightarrow \infty} \rho_j(\mathbf{U}^m) = \max\{\alpha, H_{ij} + \theta\}$. Since $j \in J$ and $\mathbf{V} \in \Sigma_{c,J}$, we have $\rho_j(\mathbf{V}) = 0$, and we obtain that

$$\lim_{m \rightarrow \infty} \|\rho(\mathbf{V}) - \rho(\mathbf{U}^m)\| \geq \lim_{m \rightarrow \infty} |\rho_j(\mathbf{V}) - \rho_j(\mathbf{U}^m)| = |\max\{\alpha, H_{ij} + \theta\}| \geq \min\{|\alpha|, \min_{i \neq j} |H_{ij} + \theta|\}.$$

□

Lemma 5.1 implies that ρ is discontinuous in any point of $\partial\mathcal{P}_c$, if $\min_{i \neq j} |H_{ij} + \theta| \neq 0$ (recall, that $\alpha < 0$). In this case ρ admits a discontinuity jump in $\partial\mathcal{P}_c$ which is not smaller than $\min\{|\alpha|, \min_{i \neq j} |H_{ij} + \theta|\} > 0$.

In the sequel we will assume the generic condition $H_{ij} \neq -\theta$ for all $i \neq j \in I$ and denote

$$\nu' := \min\{|\alpha|, \min_{i \neq j} |H_{ij} + \theta|\}/2. \quad (18)$$

Remark 5.2. From the arguments above the return map $\rho : \Sigma_c \mapsto \Sigma_c$ presents the following properties:

- 1) There exists a finite family of pairwise disjoint open sets $\{\Sigma_{c,\{i\}}\}_{i \in I_0^-}$ such that $\Sigma_c = \bigcup_{i \in I_0^-} \overline{\Sigma_{c,\{i\}}}$.
- 2) There exists $\nu' > 0$ such that ρ admits a discontinuity jump larger than ν' for all $\mathbf{V} \in \bigcup_{i \in I_0^-} \partial\Sigma_{c,\{i\}}$.
- 3) For all $i \in I_0^-$ the map ρ restricted to $\Sigma_{c,\{i\}}$ is a contraction.

Now we induce an abstract concept from the properties of the return map ρ :

Definition 5.3. Let Σ_c be a compact subset of n -dimensional real space. A map $\rho : \Sigma_c \mapsto \Sigma_c$ is said *piecewise contractive* if it satisfies the conditions 1), 2) and 3) of Remark 5.2.

The results of the sequel of this section apply to any piecewise contractive map.

5.2 The stable and the sensitive sets.

In order to study the asymptotic dynamics we divide the contractive zone Σ_c in two complementary sets, the stable set S and the sensitive set $C = \Sigma_c \setminus S$. The set S is formed by the states whose future orbit changes continuously under small perturbations of the initial state. The set C is formed by the states whose future orbit changes drastically under arbitrarily small perturbations of the initial state.

Definition 5.4. Stable set. A point $\mathbf{V} \in \Sigma_c$ is stable, if for all $\nu > 0$ there exists $\delta > 0$ such that:

$$\text{if } p \geq 0 \text{ and } \|\rho^p(\mathbf{V}) - \mathbf{W}\| < \delta \text{ then } \|\rho^k(\rho^p(\mathbf{V})) - \rho^k(\mathbf{W})\| < \nu \quad \forall k \geq 1. \quad (19)$$

We call *stable set* S the set of all the stable points. We denote $S_{\nu,\delta}$ the set of the points $\mathbf{V} \in \Sigma_c$ such that (19) holds for given $\nu > 0$ and $\delta > 0$.

Note that the stable set can be written as follows:

$$S = \bigcap_{\nu > 0} \bigcup_{\delta > 0} S_{\nu,\delta} = \bigcap_{\nu > 0} \bigcup_{h=1}^{\infty} S_{\nu, \frac{1}{h}} \quad (20)$$

and that S is not empty if and only if for all $\nu > 0$ there exists $\delta > 0$ such that $S_{\nu,\delta} \neq \emptyset$.

Definition 5.5. Sensitive set. A point $\mathbf{V} \in \Sigma_c$ is sensitive, if it is not stable. That is, there exists $\nu > 0$ such that for all $\delta > 0$ there are $p \in \mathbb{N}$ and $\mathbf{W} \in \Sigma_c$ satisfying:

$$\|\rho^p(\mathbf{V}) - \mathbf{W}\| < \delta \quad \text{and} \quad \|\rho^k(\rho^p(\mathbf{V})) - \rho^k(\mathbf{W})\| \geq \nu \quad \text{for some } k \geq 1.$$

We call *sensitive set* $C = \Sigma_c \setminus S$ the set of all the sensitive points.

Proposition 5.6. *The stable set S is forward invariant, i.e $\rho(S) \subset S$, and the sensitive set C is backward invariant, i.e $\rho^{-1}(C) \subset C$.*

Proof: Let us first prove that $\rho(S) \subset S$. Take $\mathbf{V} \in S$. Then for all $\nu > 0$ there exists $\delta > 0$ such that for all $p \geq 1$

$$\text{if } \|\rho^{p-1}(\rho(\mathbf{V})) - \mathbf{W}\| < \delta \quad \text{then} \quad \|\rho^k(\rho^{p-1}(\rho(\mathbf{V}))) - \rho^k(\mathbf{W})\| \leq \nu \quad \forall k \geq 1.$$

The last assertion is the definition of stable point applied to $\rho(\mathbf{V})$. Therefore $\rho(S) \subset S$. As C is the complement in Σ_c of S , and $\rho(S) \subset S$ we have $\rho^{-1}(C) \subset C$. \square

The same argument proves that $\rho(S_{\nu,\delta}) \subset S_{\nu,\delta}$ for all $\nu > 0$ and $\delta > 0$.

Lemma 5.7. *If $\mathbf{V} \in \partial\mathcal{P}_c$ then it is a sensitive point. Even more, for all $\delta > 0$ there exists $\mathbf{W} \in \Sigma_c$ such that:*

$$\|\mathbf{V} - \mathbf{W}\| < \delta \quad \text{and} \quad \|\rho(\mathbf{V}) - \rho(\mathbf{W})\| \geq \nu',$$

where ν' is defined in (18).

Proof: Let $\mathbf{V} \in \partial\mathcal{P}_c$. From Lemma 5.1 there exists a sequence of points $\{\mathbf{U}^m\}_{m \in \mathbb{N}}$ which converges to \mathbf{V} and such that $\lim_{m \rightarrow \infty} \|\rho(\mathbf{U}^m) - \rho(\mathbf{V})\| \geq 2\nu' > \nu'$. Given $\delta > 0$, there exists $m \geq 1$ large enough, such that \mathbf{U}^m satisfies $\|\mathbf{U}^m - \mathbf{V}\| < \delta$ and $\|\rho(\mathbf{U}^m) - \rho(\mathbf{V})\| \geq \nu'$. Therefore, taking $\mathbf{W} = \mathbf{U}^m$ the point \mathbf{V} satisfies the definition of sensitive point with $p = 0$ and $k = 1$. \square

Remark 5.8. Lemma 5.7 and Proposition 5.6 state that the sensitive set C contains the points of $\partial\mathcal{P}_c$ and their pre-images. Thus, the stable set satisfies:

$$S \subset \Sigma_c \setminus \bigcup_{k=0}^{\infty} \rho^{-k}(\partial\mathcal{P}_c)$$

and as a consequence, the orbit of any stable point is contained in $\bigcup_{i \in I_0^-} \Sigma_{c,\{i\}}$. Moreover, if $0 < \nu < \nu'$ then this last inclusion is also true for the set $S_{\nu,\delta}$. In other words,

$$\rho^k(S_{\nu,\delta}) \subset \bigcup_{i \in I_0^-} \Sigma_{c,\{i\}} \quad \forall k \in \mathbb{N}. \quad (21)$$

5.3 Asymptotically periodic dynamics

The purpose of this subsection is to study the dynamics on the stable set. Theorem 6 states that the limit sets attracting the stable points is only composed of limit cycles. The proof essentially relies on the contraction property of the return map. Since the map is not continuous, it is not possible to apply directly a classic fixed point theorem for contractive maps. Nevertheless, we obtain a generalization of the Banach fixed point theorem for piecewise contractive maps. It states the existence of periodic orbits - maybe of period different from one and not necessarily unique - that attract all the stable points.

Definition 5.9. ω -limit set. The ω -limit set $\omega(\mathbf{V}) \in \Sigma_c$ of a point $\mathbf{V} \in \Sigma_c$ is the set of the limit points of the future orbit of \mathbf{V} , that is:

$$\omega(\mathbf{V}) = \{\mathbf{W} \in \Sigma_c : \exists \{p_k\}_{k \in \mathbb{N}} : \lim_{k \rightarrow \infty} p_k = +\infty \text{ and } \lim_{k \rightarrow \infty} \rho^{p_k}(\mathbf{V}) = \mathbf{W}\}$$

As the the phase space Σ_c is compact, the ω -limit set of any point is not empty. The Poincaré map ρ is not continuous, so $\omega(\mathbf{V})$ is not necessarily forward invariant. Nevertheless, the ω -limit set is the same for all the points in the same orbit.

Definition 5.10. Limit cycle. A set $L \in \Sigma_c$ is a *limit cycle* (also called a sink in the case of the discrete dynamics of the Poincaré map) if it is a periodic orbit whose basin of attraction

$$\mathcal{B}(L) = \{\mathbf{W} \in \Sigma_c : \omega(\mathbf{W}) = L\}$$

contains an open neighborhood of L .

Theorem 6. *If the stable set S is not empty, then there exists limit cycles such that the union of their basins of attraction contains S .*

Suppose that $S \neq \emptyset$. Then for all $\mu > 0$ there exists $\delta > 0$ such that $S_{\mu,\delta} \neq \emptyset$. We fix $\mu = \nu'/2$, being ν' defined in (18), and an arbitrary $\delta > 0$ such that $S_{\mu,\delta} \neq \emptyset$. To prove the theorem, we introduce the so called *atoms* of $S_{\mu,\delta}$.

For all $i \in I_0^-$, we define $F_i : P(\Sigma_c) \rightarrow P(\Sigma_c)$, where $P(\Sigma_c)$ denotes the set of the parts of Σ_c , by:

$$F_i(E) = \rho(E \cap \Sigma_{c,\{i\}} \cap S_{\mu,\delta}) \quad \forall E \subset \Sigma_c.$$

Given $k \in \mathbb{N}$ and $(i_1, i_2, \dots, i_k) \in (I_0^-)^k$, we call *atom of generation k* the set

$$A_{i_1 i_2 \dots i_k} = F_{i_k} \circ F_{i_{k-1}} \circ \dots \circ F_{i_1}(\Sigma_{c,\{i_1\}})$$

and we call *family of the atoms of generation k* the set $\mathcal{A}_k = \{A_{i_1 i_2 \dots i_k}, (i_1, i_2, \dots, i_k) \in (I_0^-)^k\}$. Note that the forward invariance of $S_{\mu,\delta}$ by ρ ensures that any atom is contained in $S_{\mu,\delta}$.

Lemma 5.11. *Let $k \in \mathbb{N}$. i) If $\mathbf{V} \in S_{\mu,\delta}$, then $\rho^k(\mathbf{V})$ belongs to an atom of generation k .*

ii) Any atom of generation $k+1$ is contained in an atom of generation k .

iii) Let $d_k = \max_{A \in \mathcal{A}_k} \text{diam}(A)$, where $\text{diam}(A)$ denotes the diameter of A . We have $\lim_{k \rightarrow \infty} d_k = 0$.

Proof: i) It follows from the forward invariance of $S_{\mu,\delta}$ and from the inclusion $S_{\mu,\delta} \subset \bigcup_{i \in I_0^-} \Sigma_{c,\{i\}}$ resulting from (21). Suppose $\mathbf{V} \in S_{\mu,\delta}$, it exists then $i_1 \in I_0^-$ such that $\mathbf{V} \in \Sigma_{c,\{i_1\}} \cap S_{\mu,\delta}$. It follows that

$$\rho(\mathbf{V}) \in \rho(\Sigma_{c,\{i_1\}} \cap S_{\mu,\delta}) = F_{i_1}(\Sigma_{c,\{i_1\}}) \in \mathcal{A}_1.$$

By induction, suppose that $\rho^k(\mathbf{V})$ belongs to $A_{i_1 \dots i_k} \in \mathcal{A}_k$. The forward invariance of $S_{\mu,\delta}$ ensures that $\rho^k(\mathbf{V}) \in S_{\mu,\delta}$. It exists then $i_{k+1} \in I_0^-$ such that $\rho^k(\mathbf{V}) \in \Sigma_{c,\{i_{k+1}\}} \cap S_{\mu,\delta}$. Using the induction hypothesis we obtain

$$\rho^{k+1}(\mathbf{V}) \in \rho(\Sigma_{c,\{i_{k+1}\}} \cap S_{\mu,\delta} \cap A_{i_1 \dots i_k}) = F_{i_{k+1}} \circ \dots \circ F_{i_1}(\Sigma_{c,\{i_1\}}) \in \mathcal{A}_{k+1}.$$

ii) Let $A_{i_1 i_2 \dots i_{k+1}}$ be an atom of generation $k+1$. Then,

$$A_{i_1 i_2 \dots i_{k+1}} = F_{i_{k+1}} \circ \dots \circ F_{i_2} \circ F_{i_1}(\Sigma_{c,\{i_1\}}) \subset F_{i_{k+1}} \circ \dots \circ F_{i_2}(\Sigma_c) = F_{i_{k+1}} \circ F_{i_k} \circ \dots \circ F_{i_2}(\Sigma_{c,\{i_2\}}) \in \mathcal{A}_k.$$

iii) It is enough to show by induction that

$$\text{diam}(A) \leq \lambda_\alpha^{k-1} \text{diam}(\Sigma_c) \quad \forall A \in \mathcal{A}_k. \quad (22)$$

Since any atom is a subset of Σ_c , it is true for $k = 1$. Assume (22) is true for $k = k_0$. Take $A = A_{i_1 i_2 \dots i_{k_0+1}} \in \mathcal{A}_{k_0+1}$ and $\mathbf{V}, \mathbf{W} \in A$. Then $A = \rho(A' \cap \Sigma_{c,\{i_{k_0+1}\}} \cap S_{\mu,\delta})$ where $A' = A_{i_1 i_2 \dots i_{k_0}} \in \mathcal{A}_{k_0}$. Therefore, there exists $\mathbf{V}', \mathbf{W}' \in \Sigma_{c,\{i_{k_0+1}\}} \cap A'$ such that $\mathbf{V} = \rho(\mathbf{V}')$ and $\mathbf{W} = \rho(\mathbf{W}')$. Applying the contraction property and the induction hypothesis we obtain:

$$\|\mathbf{V} - \mathbf{W}\| = \|\rho(\mathbf{V}') - \rho(\mathbf{W}')\| \leq \lambda_\alpha \|\mathbf{V}' - \mathbf{W}'\| \leq \lambda_\alpha \text{diam}(A') \leq \lambda_\alpha^{k_0} \text{diam}(\Sigma_c),$$

which implies the desired result, since \mathbf{V} and \mathbf{W} are arbitrary in the atom A . \square

Lemma 5.12. *There exists $k \geq 1$ such that if $A \in \mathcal{A}_k$ then $\bar{A} \subset \Sigma_{c,\{i\}}$ for some $i \in I_0^-$.*

Proof: We first prove that if $\mathbf{W} \in S_{\mu,\delta}$ and $\mathbf{V} \in \partial\mathcal{P}_c$ then $\|\mathbf{V} - \mathbf{W}\| \geq \delta/2$. Since $\mathbf{V} \in \partial\mathcal{P}_c$, by Lemma 5.7 there exists $\mathbf{U} \in \Sigma_c$ such that

$$\|\mathbf{U} - \mathbf{V}\| < \frac{\delta}{2} \quad \text{and} \quad \|\rho(\mathbf{U}) - \rho(\mathbf{V})\| \geq \nu' = 2\mu. \quad (23)$$

By contradiction suppose $\|\mathbf{V} - \mathbf{W}\| < \delta/2$. Then, on one hand, since $\mathbf{W} \in S_{\mu,\delta}$, by definition, we have $\|\rho(\mathbf{V}) - \rho(\mathbf{W})\| < \mu$ and from (23) we obtain:

$$\|\rho(\mathbf{U}) - \rho(\mathbf{W})\| \geq \|\rho(\mathbf{U}) - \rho(\mathbf{V})\| - \|\rho(\mathbf{V}) - \rho(\mathbf{W})\| > 2\mu - \mu = \mu. \quad (24)$$

On the other hand, from (23) we have $\|\mathbf{U} - \mathbf{W}\| \leq \|\mathbf{U} - \mathbf{V}\| + \|\mathbf{V} - \mathbf{W}\| < \delta$ which together with (24) is in contradiction with $\mathbf{W} \in S_{\mu,\delta}$.

Let $k \geq 1$ such that $d_k < \delta/4$ and $A \in \mathcal{A}_k$ not empty. Take $\mathbf{W} \in A$, then there exists $i \in I_0^-$ such that $\mathbf{W} \in \Sigma_{c,\{i\}}$ (recall $A \subset S_{\mu,\delta}$ and (21)). Let $E = \Sigma_c \setminus \Sigma_{c,\{i\}}$ and let ∂E be its boundary. We denote $\text{dist}(\mathbf{W}, E)$ the distance between the point \mathbf{W} and the set E , defined from the norm $\|\cdot\|$. We have:

$$\text{dist}(\mathbf{W}, E) = \text{dist}(\mathbf{W}, \partial E) = \text{dist}(\mathbf{W}, \partial\Sigma_{c,\{i\}}) \geq \text{dist}(\mathbf{W}, \partial\mathcal{P}_c) \geq \frac{\delta}{2}.$$

Let $\mathbf{W}' \in \bar{A}$ then $\text{dist}(\mathbf{W}', \mathbf{W}) \leq \text{diam } \bar{A} \leq d_k < \delta/4$ and from

$$\text{dist}(\mathbf{W}', E) \geq \text{dist}(\mathbf{W}, E) - \text{dist}(\mathbf{W}', \mathbf{W}) \geq \frac{\delta}{2} - \frac{\delta}{4} > 0$$

we deduce that $\mathbf{W}' \notin E$. Then $\mathbf{W}' \in \Sigma_{c,\{i\}}$. □

Lemma 5.13. *If there exist $(i_0, i_1, \dots, i_{p-1}) \in (I_0^-)^p$ and a family of sets B_0, B_1, \dots, B_{p-1} satisfying*

i) $\bar{B}_k \subset \Sigma_{c,\{i_k\}}$ for all $k \in \{0, \dots, p-1\}$ and

ii) $\rho(B_{p-1}) \subset B_0$ and $\rho(B_{k-1}) \subset B_k$ for all $k \in \{1, \dots, p-1\}$,

then there exists a unique periodic point of period p in \bar{B}_0 whose orbit is the ω -limit set of any point contained in the union of the B_k 's.

Proof: Since each \bar{B}_k is contained in a contractive piece of ρ , it follows that ρ^p is contractive \bar{B}_0 and $\rho^p(\bar{B}_0) \subset \bar{B}_0$. Then, by the fixed point theorem of Banach, we deduce the existence of a unique periodic point $\tilde{\mathbf{V}}$ of period p in \bar{B}_0 .

We prove now that if $\mathbf{V} \in B_k$, then $\omega(\mathbf{V})$ is the orbit L of $\tilde{\mathbf{V}}$. Without loss of generality we can assume $\mathbf{V} \in B_0$. It is enough to show that $\omega(\mathbf{V}) = \omega(\tilde{\mathbf{V}})$, since the ω -limit set of a periodic point coincides with its orbit.

Let $\{p_j\}_{j \in \mathbb{N}}$ be a sequence of natural numbers such that $\lim_{j \rightarrow \infty} p_j = +\infty$ and either $\lim_{j \rightarrow \infty} \rho^{p_j}(\mathbf{V})$ or $\lim_{j \rightarrow \infty} \rho^{p_j}(\tilde{\mathbf{V}})$ exists. Since \mathbf{V} and $\tilde{\mathbf{V}}$ belong both to \bar{B}_0 , by hypothesis $\rho^{p_j}(\mathbf{V})$ and $\rho^{p_j}(\tilde{\mathbf{V}})$ belong to the same continuity piece for all $j \in \mathbb{N}$. Using the contraction property we obtain:

$$\lim_{j \rightarrow \infty} \|\rho^{p_j}(\mathbf{V}) - \rho^{p_j}(\tilde{\mathbf{V}})\| \leq \lim_{j \rightarrow \infty} \lambda_\alpha^{p_j} \|\mathbf{V} - \tilde{\mathbf{V}}\| = 0.$$

It follows that both $\lim_{j \rightarrow \infty} \rho^{p_j}(\mathbf{V})$ and $\lim_{j \rightarrow \infty} \rho^{p_j}(\tilde{\mathbf{V}})$ exist and are equal. This proves that $\omega(\mathbf{V}) = \omega(\tilde{\mathbf{V}})$ as wanted. □

Proof of Theorem 6: Let $\mathbf{V} \in S$. Then, due to the definition of the stable set S , there exists $\delta > 0$ such that $\mathbf{V} \in S_{\mu,\delta}$. By equality (20), it is not restrictive to assume that $\delta = 1/h$ for some natural number $h \geq 1$ depending on the point \mathbf{V} . Therefore, to prove the existence of at most countable many limit cycles attracting the orbit of all the points in S , it is enough to prove that, for any fixed $\delta = 1/h > 0$ (in a countable set), there exists a finite number $N = N(\delta)$ of limit cycles L_1, L_2, \dots, L_N (depending on δ) such that

$$\forall \mathbf{V} \in S_{\mu,\delta}, \quad \omega(\mathbf{V}) = L_i \text{ for some } i = 1, \dots, N.$$

Step 1 Let $\tilde{k} \in \mathbb{N}$ be such that the thesis of Lemma 5.12 is true. We show that the image by ρ of any atom of $\mathcal{A}_{\tilde{k}}$ is contained in an atom of $\mathcal{A}_{\tilde{k}}$. Suppose $A \in \mathcal{A}_{\tilde{k}}$. By Lemma 5.12, there exists $i \in I_0^-$ such that $\bar{A} \subset \Sigma_{c,\{i\}}$, and since any atom is contained in $S_{\mu,\delta}$, we have $\rho(A) = \rho(A \cap \Sigma_{c,\{i\}} \cap S_{\mu,\delta}) = F_i(A)$. Then, $\rho(A) \in \mathcal{A}_{\tilde{k}+1}$, and according to *ii*) of Lemma 5.11 it is included in an atom of $\mathcal{A}_{\tilde{k}}$.

Step 2 Let $\mathbf{V} \in S_{\mu,\delta}$ and $\mathbf{W} = \rho^{\tilde{k}}(\mathbf{V})$. Then, from *i*) and *ii*) of Lemma 5.11 we deduce that each point of the orbit of \mathbf{W} belongs to an atom of $\mathcal{A}_{\tilde{k}}$. Being the number of atoms in $\mathcal{A}_{\tilde{k}}$ finite, it must exist $B_0 \in \mathcal{A}_{\tilde{k}}$ such that $\rho^{j_0}(\mathbf{W}) \in B_0$ and $\rho^{j_0+p}(\mathbf{W}) \in B_0$ for some j_0 and $p \in \mathbb{N}$. Let's denote B_1, B_2, \dots, B_{p-1} the atoms of $\mathcal{A}_{\tilde{k}}$ that contain respectively $\rho^{j_0+1}(\mathbf{W}), \rho^{j_0+2}(\mathbf{W}), \dots, \rho^{j_0+p-1}(\mathbf{W})$. By Step 1, the image by ρ of each of these atoms is contained in an atom of $\mathcal{A}_{\tilde{k}}$, so they must satisfy:

$$\rho(B_{p-1}) \subset B_0 \quad \text{and} \quad \rho(B_{k-1}) \subset B_k \quad \forall k \in \{1, \dots, p-1\}.$$

Moreover, by Lemma 5.12 there exists $(i_0, \dots, i_{p-1}) \in (I_0^-)^p$ such that $\bar{B}_k \subset \Sigma_{c,\{i_k\}}$ for all $0 \leq k \leq p-1$. Therefore, this family of atoms satisfy the hypothesis of Lemma 5.13. Since $\rho^{k+j_0}(\mathbf{V}) \in B_0$ and $\omega(\mathbf{V}) = \omega(\rho^{k+j_0}(\mathbf{V}))$, we conclude that $\omega(\mathbf{V})$ is a periodic orbit of period p contained in the union of the atoms B_k 's. Note that it may exit at most $\#\mathcal{A}_{\tilde{k}}$ different families satisfying Lemma 5.13 and thus at most $\#\mathcal{A}_{\tilde{k}}$ different periodic orbits.

Step 3 We have shown that the ω -limit set of any point of $S_{\mu,\delta}$ is a periodic orbit contained in $\bigcup_{i \in I_0^-} \Sigma_{c,\{i\}}$. We finish the proof of the theorem by showing that any periodic orbit is actually a limit cycle.

Suppose $\tilde{\mathbf{V}}$ is a periodic point of period p and let $L = \{\tilde{\mathbf{V}}, \rho(\tilde{\mathbf{V}}), \dots, \rho^{p-1}(\tilde{\mathbf{V}})\} \subset \bigcup_{i \in I_0^-} \Sigma_{c,\{i\}}$ be its orbit. We have to prove that there exists an open neighborhood of L which points have L as ω -limit set (i.e, the basin of attraction of L contains an open neighborhood of L). For all $k \in \{0, \dots, p-1\}$ we have $\rho^k(\tilde{\mathbf{V}}) \in \Sigma_{c,\{i_k\}}$ where $(i_1, \dots, i_p) \in (I_0^-)^p$. Since the continuity piece $\Sigma_{c,\{i_k\}}$ is open, there exists an open ball $B(\rho^k(\tilde{\mathbf{V}}), a_k)$ of center $\rho^k(\tilde{\mathbf{V}})$ and radius $a_k > 0$ whose closure is contained in $\Sigma_{c,\{i_k\}}$. Let $0 < a < \min\{a_k, 0 \leq k \leq p-1\}$ and B be the open neighborhood of L defined by:

$$B = \bigcup_{k=0}^{p-1} B_k \quad \text{where} \quad B_k = B(\rho^k(\tilde{\mathbf{V}}), a) \quad \forall k \in \{0, \dots, p-1\}.$$

We have $\bar{B}_k \in \Sigma_{c,\{i_k\}}$, and if we show that $\rho(B_{p-1}) \subset B_0$ and $\rho(B_{k-1}) \subset B_k$ for all $k \in \{0, \dots, p-1\}$, then we can apply Lemma 5.13 to prove that the open set B contains a unique periodic orbit (L) which is the ω -limit set of all the points of B . We will only prove that $\rho(B_{p-1}) \subset B_0$, the other inclusions admit an analogous proof. If $\mathbf{V} \in B_{p-1}$ then \mathbf{V} and $\rho^{p-1}(\tilde{\mathbf{V}})$ belong to the same piece of continuity $\Sigma_{c,\{i_{p-1}\}}$. Applying the contraction property of ρ we obtain:

$$\|\rho(\mathbf{V}) - \rho^p(\tilde{\mathbf{V}})\| \leq \lambda_\alpha \|\mathbf{V} - \rho^{p-1}(\tilde{\mathbf{V}})\| < \lambda_\alpha a < a.$$

In other words $\|\rho(\mathbf{V}) - \tilde{\mathbf{V}}\| < a$ which implies $\rho(\mathbf{V}) \in B_0$. □

6 Conclusion

In this paper we proved the existence of an open set of parameter values (strong interactions) such that the dynamics of IF neural networks is equivalent to that of a piecewise contractive return map. Then, the continuous time original system satisfies the thesis of Theorem 1 describing the asymptotic dynamics of the stable points: their orbits all converge to limit cycles, regardless of whether the interactions are all excitatory, all inhibitory or combined. Besides, we proved that the limit cycle is unique and that the system is not only periodic but also synchronized, if all the neurons are excitatory.

These results principally concern the dynamics on the stable set. This set is always invariant but its size a priori depends on the piecewise contractive map under study. Nevertheless, we can state a rough almost obvious observation about its complementary set, namely the sensitive set. Since the map is not continuous, all the sensitive points may become stable after some future iterations. In this case, the sensitive set does not contain a forward invariant set and the whole asymptotic dynamics is stable. In the other case, the

limit set of some sensitive points is chaotic in a topological sense. The study of the existence of this chaotic dynamics and its characterization, in addition to the results of this paper about the stable dynamics, would give a rather complete picture of the asymptotic dynamics of general piecewise contractive maps.

We also proved the existence of a set of parameter values such that the return map is not contractive, and where consequently the asymptotic dynamics is still unknown. We believe that the problem is non trivial and relevant at least for the mathematical science. As far as we know, there are still few results in the theory of dynamical systems, describing the global dynamics of piecewise continuous maps of arbitrary large dimensions. We can mention for example results developed for piecewise continuous weakly coupled map lattices (see [19] and reference therein), but the system is supposed expanding and does not have contractive directions. From the geometric and ergodic points of view, the abundant results on dynamical systems that hold for arbitrary large dimensions, and also their proofs, were mainly developed under assumptions of C^1 plus Hölder regularity. These assumptions are sometimes too strong for the models that appear in natural or technological sciences nowadays.

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7 Appendix

7.1 Proof of Lemma 3.3

Let $i \neq j \in I$. To prove the Lemma 3.3, we consider the open region of values of the interactions satisfying:

- (H'1) H_{ij} and H_{ji} belong to $(0, \theta - c^*)$,
- (H'2) $H_{sk} > \theta$ for all for all $s \in \{i, j\}$ and $k \notin \{i, j\}$,
- (H'3) $\sum_{l \neq s: H_{ls} > 0} H_{ls} < \theta - c^*$ for all $s \in \{i, j\}$,
- (H'4) $\sum_{l \neq s} H_{ls} > 0$ for all $s \in \{i, j\}$.

We recall that $c^* = \beta - \sqrt{\beta(\beta - \theta)} \in (\theta/2, \theta)$. The proof of the lemma can be done with other open regions of the parameters. However, it seems that this region allows the simplest proof, unless one consider mixed neurons.

Step 1: We show that in this region of parameters, Γ_{ij} and Γ_{ji} are respectively equal to the set Γ'_{ij} and Γ'_{ji} defined by

$$\Gamma'_{ij} := \{\mathbf{V} \in \Sigma : V_i \in (c^*, \theta) \text{ and } V_k = 0 \forall k \neq i\} \quad \text{and} \quad \Gamma'_{ji} := \{\mathbf{V} \in \Sigma : V_j \in (c^*, \theta) \text{ and } V_k = 0 \forall k \neq j\}.$$

A direct consequence is that Γ_{ij} and Γ_{ji} are not empty.

Let us prove that $\Gamma_{ij} = \Gamma'_{ij}$ (an analogous proof allows to show that $\Gamma_{ji} = \Gamma'_{ji}$). It obvious that $\Gamma_{ij} \subset \Gamma'_{ij}$. To show $\Gamma'_{ij} \subset \Gamma_{ij}$, it is enough to prove that $\Gamma'_{ij} \subset \Sigma_i \cap \Sigma_{I \setminus \{j\}}$. Suppose $\mathbf{V} \in \Gamma'_{ij}$. Then, we have $t_i(\mathbf{V}) = \bar{t}(\mathbf{V})$, since $V_k < V_i$ for all $k \neq i$. It implies that $\mathbf{V} \in \Sigma_i$ and $J_0(\mathbf{V}) = \{i\}$. It remains to prove that $\mathbf{V} \in \Sigma_{I \setminus \{j\}}$, that is to say $J(\mathbf{V}) = I \setminus \{j\}$.

Let $k \in I \setminus J_0(\mathbf{V}) = I \setminus \{i\}$, then

$$\phi_k^{\bar{t}(\mathbf{V})}(\mathbf{V}) + \sum_{l \in J_0(\mathbf{V}): H_{lk} > 0} H_{lk} = \phi_k^{t_i(\mathbf{V})}(\mathbf{V}) + H_{ik} = \beta - \frac{\beta(\beta - \theta)}{\beta - V_i} + H_{ik} \in (H_{ik}, c^* + H_{ik}).$$

If $k \neq j$, then $k \notin \{i, j\}$ and by hypothesis (H'2) we have $H_{ik} > \theta$. Therefore, $k \in J_1(\mathbf{V})$. If $k = j$, by hypothesis (H'1) we have $H_{ik} < c^* - \theta$ and then $k \notin J_1(\mathbf{V})$. We deduce that $J_1(\mathbf{V}) = I \setminus \{j\}$.

Let $k \in I \setminus J_1(\mathbf{V}) = \{j\}$, then

$$\phi_k^{\bar{i}(\mathbf{V})}(\mathbf{V}) + \sum_{l \in J_1(\mathbf{V}) : H_{lk} > 0} H_{lk} = \phi_j^{t_i(\mathbf{V})}(\mathbf{V}) + \sum_{l \neq j : H_{lj} > 0} H_{lj} < c^* + \theta - c^* = \theta$$

and it follows that $j \notin J_2(\mathbf{V})$. We deduce that $J_2(\mathbf{V}) = J_1(\mathbf{V}) = I \setminus \{j\}$ and then $J(\mathbf{V}) = I \setminus \{j\}$.

Step 2: We show that there exists $(a, b) \subset (c^*, \theta)$ such that for all $\mathbf{V} \in \Gamma_{ij}$ such that $V_i \in (a, b)$ we have $\rho(\mathbf{V}) \in \Gamma_{ji}$ and $\rho^2(\mathbf{V}) \in \Gamma_{ij}$.

Suppose $\mathbf{V} \in \Gamma_{ji}$, then $\rho_k(\mathbf{V}) = 0$ for all $k \neq i$ and

$$\rho_i(\mathbf{V}) = \phi_i^{t_j(\mathbf{V})}(\mathbf{V}) + \sum_{l \neq i} H_{li} = g_i(V_j) \quad \text{where} \quad g_i(x) := \beta - \frac{\beta(\beta - \theta)}{\beta - x} + \sum_{l \neq i} H_{li} \quad \forall x \neq \beta.$$

Since $g_i(c^*) = c^* + \sum_{l \neq i} H_{li} > c^*$ (by H'4) and g_i is continuous, there exists $\delta' > 0$ such that $g_i(c^*) > c^*$ for all $x \in (c^* - \delta, c^* + \delta)$. On the other hand, the function g_i being decreasing, $g_i(x) < g_i(c^*) < \theta$ (by H'3) for all $x > c^*$. It results that $g_i(x) \in (c^*, \theta)$ for all $x \in (c^*, c^* + \delta')$. Since $\rho_i(\mathbf{V}) = g_i(V_j)$, using $\Gamma_{ij} = \Gamma'_{ij}$, we obtain that

$$\text{if } \mathbf{V} \in \Gamma_{ji} \text{ and } V_j \in (c^*, c^* + \delta') \text{ then } \rho(\mathbf{V}) \in \Gamma_{ij}. \quad (25)$$

Suppose now $\mathbf{V} \in \Gamma_{ij}$, then $\rho_k(\mathbf{V}) = 0$ for all $k \neq j$ and

$$\rho_j(\mathbf{V}) = \phi_j^{t_i(\mathbf{V})}(\mathbf{V}) + \sum_{l \neq j} H_{lj} = g_j(V_i) \quad \text{where} \quad g_j(x) := \beta - \frac{\beta(\beta - \theta)}{\beta - x} + \sum_{l \neq j} H_{lj} \quad \forall x \neq \beta.$$

We have $g_j(\theta) = \sum_{l \neq j} H_{lj} < \theta - c^* < c^*$ (by H'3 and because $c^* > \theta/2$) and $g_j(c^*) = c^* + \sum_{l \neq j} H_{lj} > c^*$ (by H'4). By continuity of g_j , it exists then $b \in (c^*, \theta)$ such that $g_j(b) = c^*$ and it exists $\delta > 0$ such that $g_j(x) \in (c^* - \delta', c^* + \delta')$ for all $x \in (b - \delta, b + \delta)$. If we denote $a = \max\{c^*, b - \delta\}$, then for all $x \in (a, b)$, we have $g_j(x) \in (c^* - \delta', c^* + \delta')$ and $g_j(x) > g_j(b) = c^*$, since g_j is decreasing. It follows that $g_j(x) \in (c^*, c^* + \delta')$ for all $x \in (a, b)$. Since $\rho_j(\mathbf{V}) = g_j(V_i)$, using $\Gamma_{ij} = \Gamma'_{ij}$ and supposing $\delta' < \theta - c^*$, we obtain

$$\text{if } \mathbf{V} \in \Gamma_{ij} \text{ and } V_i \in (a, b) \text{ then } \rho_j(\mathbf{V}) \in (c^*, c^* + \delta') \text{ and } \rho(\mathbf{V}) \in \Gamma_{ji}. \quad (26)$$

This ends the proof of the Lemma 3.3, since if \mathbf{V} satisfies the hypothesis of (26), then $\mathbf{V} \in \Gamma_{ij}$, $\rho(\mathbf{V}) \in \Gamma_{ji}$ and by (25) we have also $\rho^2(\mathbf{V}) \in \Gamma_{ij}$.

7.2 Proof of Theorem 3

Lemma 7.1. *Under the hypothesis (H1),(H2),(H3) and (H4) there exists $c > 0$ such that for all $n \in \mathbb{N}$*

$$\|\rho^n(\mathbf{V}) - \rho^n(\mathbf{W})\| \leq c \lambda_\alpha^n \|\mathbf{V} - \mathbf{W}\| \quad \text{if} \quad \mathbf{V}, \mathbf{W} \in \mathcal{P}_{J_0 \dots J_n} := \bigcap_{i=0}^n \rho^{-i}(\Sigma_{J_i})$$

where $J_0, \dots, J_n \in P(I)$ and λ_α is the constant of Proposition 3.4.

Proof: Consider the integer p of Proposition 3.7 and define for all $k \in \{0, \dots, p\}$

$$c_k = \max_{J_0, \dots, J_k \in P(I)} \sup \left\{ \frac{\|\rho^k(\mathbf{V}) - \rho^k(\mathbf{W})\|}{\lambda_\alpha^k \|\mathbf{V} - \mathbf{W}\|}, \mathbf{V} \neq \mathbf{W} \in \mathcal{P}_{J_0 \dots J_k} \right\} \quad \text{where} \quad \mathcal{P}_{J_0 \dots J_k} = \bigcap_{i=0}^k \rho^{-i}(\Sigma_{J_i}).$$

Let us show that c_k is bounded for all $k \in \{0, \dots, p\}$. Fix a $k \in \{0, \dots, p\}$ and let $J_0, \dots, J_k \in P(I)$. The Poincaré map ρ being Lipchitz continuous on each Σ_{J_i} , by construction of $\mathcal{P}_{J_0 \dots J_k}$, the composition ρ^k is also Lipchitz continuous on $\mathcal{P}_{J_0 \dots J_k}$. There exists then a constant $L > 0$ such that

$$\|\rho^k(\mathbf{V}) - \rho^k(\mathbf{W})\| \leq L\|\mathbf{V} - \mathbf{W}\| \quad \forall \mathbf{V}, \mathbf{W} \in \mathcal{P}_{J_0 \dots J_k},$$

and as a consequence

$$\sup_{\mathbf{V} \neq \mathbf{W} \in \mathcal{P}_{J_0 \dots J_k}} \frac{\|\rho^k(\mathbf{V}) - \rho^k(\mathbf{W})\|}{\lambda_\alpha^k \|\mathbf{V} - \mathbf{W}\|} \leq \frac{L}{\lambda_\alpha^k}.$$

It follows that c_k is bounded for all $k \in \{0, \dots, p\}$ and $c := \max_{k \in \{0, \dots, p\}} c_k$ exists.

Let $n \in \mathbb{N}$ and $J_0, \dots, J_n \in P(I)$. If $n \leq p$, then by definition of c

$$\|\rho^n(\mathbf{V}) - \rho^n(\mathbf{W})\| \leq c\lambda_\alpha^n \|\mathbf{V} - \mathbf{W}\| \quad \text{if} \quad \mathbf{V}, \mathbf{W} \in \mathcal{P}_{J_0 \dots J_n}. \quad (27)$$

If $n > p$, take (27) as an induction hypothesis. Let $J_{n+1} \in P(I)$ and suppose $\mathbf{V}, \mathbf{W} \in \mathcal{P}_{J_0 \dots J_{n+1}}$. As $n \geq p$ by Proposition 3.7 and Proposition 3.6 we have $\rho^n(\Sigma) \subset \Sigma_c$. Therefore, $\rho^n(\mathbf{V})$ and $\rho^n(\mathbf{W}) \in \Sigma_c \cap \Sigma_{J_n}$. From Proposition 3.4 it follows

$$\|\rho^{n+1}(\mathbf{V}) - \rho^{n+1}(\mathbf{W})\| \leq \lambda_\alpha \|\rho^n(\mathbf{V}) - \rho^n(\mathbf{W})\|$$

and from the induction hypothesis we obtain

$$\|\rho^{n+1}(\mathbf{V}) - \rho^{n+1}(\mathbf{W})\| \leq c\lambda_\alpha^{n+1} \|\mathbf{V} - \mathbf{W}\|.$$

□

Let $\lambda_\alpha < \mu < 1$ and let $n_0 \in \mathbb{N}$ be such that $c \left(\frac{\lambda_\alpha}{\mu}\right)^{n_0} < 1$. Consider the metric d defined by

$$d(\mathbf{V}, \mathbf{W}) := \sum_{i=0}^{n_0-1} \frac{\|\rho^i(\mathbf{V}) - \rho^i(\mathbf{W})\|}{\mu^i} \quad \forall \mathbf{V}, \mathbf{W} \in \Sigma_c$$

and the partition \mathcal{P}^* of Σ defined by $\mathcal{P}^* := \{\mathcal{P}_{J_0 \dots J_{n_0}}, J_0 \dots J_{n_0} \in P(I)\}$. To prove the theorem it is enough to show that ρ is piecewise contractive in Σ with respect to \mathcal{P}^* for the metric d . This is the purpose of the following calculation. First, note that for all \mathbf{V}, \mathbf{W} in Σ , we have:

$$d(\rho(\mathbf{V}), \rho(\mathbf{W})) = \sum_{i=1}^{n_0} \frac{\|\rho^i(\mathbf{V}) - \rho^i(\mathbf{W})\|}{\mu^{i-1}} = \mu \left(d(\mathbf{V}, \mathbf{W}) + \frac{\|\rho^{n_0}(\mathbf{V}) - \rho^{n_0}(\mathbf{W})\|}{\mu^{n_0}} - \|\mathbf{V} - \mathbf{W}\| \right).$$

Now, suppose \mathbf{V} and \mathbf{W} in $\mathcal{P}_{J_0 \dots J_{n_0}}$ for some $J_0 \dots J_{n_0} \in P(I)$. Then, applying Lemma 7.1 we obtain:

$$d(\rho(\mathbf{V}), \rho(\mathbf{W})) \leq \mu \left(d(\mathbf{V}, \mathbf{W}) + c \left(\frac{\lambda_\alpha}{\mu}\right)^{n_0} \|\mathbf{V} - \mathbf{W}\| - \|\mathbf{V} - \mathbf{W}\| \right).$$

By definition of μ and n_0 , we have then

$$d(\rho(\mathbf{V}), \rho(\mathbf{W})) \leq \mu d(\mathbf{V}, \mathbf{W})$$

which ends the proof of Theorem 3.

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