

THE HADAMARD TYPE INEQUALITIES FOR m -CONVEX FUNCTIONS

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ABSTRACT. In this paper we obtained some new Hadamard-Type inequalities for functions whose derivatives absolute values m -convex. Some applications to special means of real numbers are given.

1. INTRODUCTION

Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$, with $a < b$. The following inequality is well known as the Hermite-Hadamard inequality for convex functions,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}.$$

In recent years many authors have established several inequalities connected to Hermite-Hadamard's inequality. For recent results, refinements, counterparts, generalizations and new Hadamard-Type inequalities see [3], [4] and [5].

A function $f : I \rightarrow \mathbb{R}$ is said to be convex function on I if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

In [2], G. Toader defined m -convexity as the following:

Definition 1. *The function $f : [0, b] \rightarrow \mathbb{R}$, $b > 0$, is said to be m -convex where $m \in [0, 1]$, if we have*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$. We say that f is m -concave if $-f$ is m -convex.

For recent results related to above definitions we refer interest of readers to [6],[7],[8].

The following theorems which was obtained by Kavurmacı et al. contains the Hadamard-Type integral inequalities in [1].

Theorem 1. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is convex function on $[a, b]$, then the*

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following inequality holds:

$$(1.1) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \frac{b-a}{12} \left[\left| f' \left(\frac{a+b}{2} \right) \right| + |f'(a)| + |f'(b)| \right].$$

Theorem 2. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is convex function on $[a, b]$, for some fixed $q \geq 1$, then the following inequality holds:

$$(1.2) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \frac{b-a}{8} \left[\left(\frac{|f'(\frac{a+b}{2})|^q + 2|f'(a)|^q}{3} \right)^{\frac{1}{q}} + \left(\frac{|f'(\frac{a+b}{2})|^q + 2|f'(b)|^q}{3} \right)^{\frac{1}{q}} \right].$$

The main purpose of this paper is to establish refinements inequalities of right-hand side of Hadamard's type for m -convex functions.

2. MAIN RESULTS

In [1], in order to prove some inequalities related to Hermite-Hadamard's inequality Kavurmacı et al. used the following lemma.

Lemma 1. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:

$$\begin{aligned} & \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \\ &= \frac{(x-a)^2}{b-a} \int_a^b (1-t)f'(tx + (1-t)a) dt + \frac{(b-x)^2}{b-a} \int_a^b (t-1)f'(tx + (1-t)b) dt. \end{aligned}$$

Theorem 3. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$. If $|f'|$ is m -convex function on $[a, b]$ for some fixed $m \in (0, 1]$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| \\ & \leq \frac{(x-a)^2}{b-a} \left[\frac{|f'(x)| + 2m|f'(\frac{a}{m})|}{6} \right] + \frac{(b-x)^2}{b-a} \left[\frac{|f'(x)| + 2m|f'(\frac{b}{m})|}{6} \right] \end{aligned}$$

where $m \in (0, 1]$.

Proof. From Lemma 1 and using property of absolute value, we get;

$$\begin{aligned} & \left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u)du \right| \\ & \leq \frac{(x-a)^2}{b-a} \int_0^1 (1-t) |f'(tx + (1-t)a)| dt + \frac{(b-x)^2}{b-a} \int_0^1 (1-t) |f'(tx + (1-t)b)| dt. \end{aligned}$$

Since $|f'|$ is m -convex we can write;

$$\begin{aligned} & \left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u)du \right| \\ & \leq \frac{(x-a)^2}{b-a} \int_0^1 (1-t) \left[t |f'(x)| + m(1-t) \left| f'\left(\frac{a}{m}\right) \right| \right] dt \\ & \quad + \frac{(b-x)^2}{b-a} \int_0^1 (1-t) \left[t |f'(x)| + m(1-t) \left| f'\left(\frac{b}{m}\right) \right| \right] dt \\ & = \frac{(x-a)^2}{b-a} \left[|f'(x)| \int_0^1 (t-t^2) dt + m \left| f'\left(\frac{a}{m}\right) \right| \int_0^1 (1-t)^2 dt \right] \\ & \quad + \frac{(b-x)^2}{b-a} \left[|f'(x)| \int_0^1 (t-t^2) dt + m \left| f'\left(\frac{b}{m}\right) \right| \int_0^1 (1-t)^2 dt \right] \\ & = \frac{(x-a)^2}{b-a} \left[\frac{|f'(x)| + 2m |f'(\frac{a}{m})|}{6} \right] + \frac{(b-x)^2}{b-a} \left[\frac{|f'(x)| + 2m |f'(\frac{b}{m})|}{6} \right]. \end{aligned}$$

This completes the proof. \square

Corollary 1. In Theorem 3, if we choose $x = \frac{a+b}{2}$, we have

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(u)du \right| \leq \frac{b-a}{12} \left[\left| f'\left(\frac{a+b}{2}\right) \right| + m \left| f'\left(\frac{a}{m}\right) \right| + m \left| f'\left(\frac{b}{m}\right) \right| \right].$$

Remark 1. In Corollary 1, if we choose $m = 1$, (1.1) inequality is obtained.

Theorem 4. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$. If $|f'|^q$ is m -convex function on $[a, b]$, then the following inequality

holds:

$$\begin{aligned} & \left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| \\ & \leq \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[\frac{(x-a)^2}{b-a} \left(\frac{|f'(x)|^q + m |f'(\frac{a}{m})|^q}{2} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \frac{(b-x)^2}{b-a} \left(\frac{|f'(x)|^q + m |f'(\frac{b}{m})|^q}{2} \right)^{\frac{1}{q}} \right] \end{aligned}$$

where $m \in (0, 1]$.

Proof. From Lemma 1, we can write;

$$\begin{aligned} & \left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| \\ & \leq \frac{(x-a)^2}{b-a} \int_0^1 (1-t) |f'(tx + (1-t)a)| dt \\ & \quad + \frac{(b-x)^2}{b-a} \int_0^1 (1-t) |f'(tx + (1-t)b)| dt. \end{aligned}$$

Using the Hölder's inequality we have;

$$\begin{aligned} & \left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| \\ & \leq \frac{(x-a)^2}{b-a} \left(\int_0^1 (1-t)^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'(tx + (1-t)a)|^q dt \right)^{\frac{1}{q}} \\ & \quad + \frac{(b-x)^2}{b-a} \left(\int_0^1 (1-t)^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'(tx + (1-t)b)|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

Since $|f'|^q$ is m -convex function;

$$\begin{aligned}
& \left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u)du \right| \\
& \leq \frac{(x-a)^2}{b-a} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\int_0^1 [t|f'(x)|^q + m(1-t)|f'(\frac{a}{m})|^q] dt \right)^{\frac{1}{q}} \\
& \quad + \frac{(b-x)^2}{b-a} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\int_0^1 [t|f'(x)|^q + m(1-t)|f'(\frac{b}{m})|^q] dt \right)^{\frac{1}{q}} \\
& = \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[\frac{(x-a)^2}{b-a} \left(\frac{|f'(x)|^q + m|f'(\frac{a}{m})|^q}{2} \right)^{\frac{1}{q}} \right. \\
& \quad \left. + \frac{(b-x)^2}{b-a} \left(\frac{|f'(x)|^q + m|f'(\frac{b}{m})|^q}{2} \right)^{\frac{1}{q}} \right].
\end{aligned}$$

This completes the proof. \square

Corollary 2. In Theorem 4, if we choose $x = \frac{a+b}{2}$, we have

$$\begin{aligned}
& \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(u)du \right| \\
& \leq \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \cdot \frac{b-a}{4} \left[\left(\frac{|f'(\frac{a+b}{2})|^q + m|f'(\frac{a}{m})|^q}{2} \right)^{\frac{1}{q}} + \left(\frac{|f'(\frac{a+b}{2})|^q + m|f'(\frac{b}{m})|^q}{2} \right)^{\frac{1}{q}} \right].
\end{aligned}$$

In Corollary 2, if we choose $m = 1$ and $\left(\frac{1}{p+1} \right)^{\frac{1}{p}} \leq 1$, we obtain

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(u)du \right| \leq \frac{b-a}{4} \left[\left(\frac{|f'(\frac{a+b}{2})|^q + |f'(a)|^q}{2} \right)^{\frac{1}{q}} + \left(\frac{|f'(\frac{a+b}{2})|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}} \right].$$

Theorem 5. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$. If $|f'|^q$ is m -convex function on $[a, b]$ for some fixed $m \in (0, 1]$, then the following inequality holds:

$$\begin{aligned}
& \left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u)du \right| \\
& \leq \frac{(x-a)^2}{2(b-a)} \left(\frac{|f'(x)|^q + 2m|f'(\frac{a}{m})|^q}{3} \right)^{\frac{1}{q}} \\
& \quad + \frac{(b-x)^2}{2(b-a)} \left(\frac{|f'(x)|^q + 2m|f'(\frac{b}{m})|^q}{3} \right)^{\frac{1}{q}}.
\end{aligned}$$

Proof. Using Lemma 1, we get;

$$\begin{aligned} & \left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u)du \right| \\ & \leq \frac{(x-a)^2}{b-a} \int_0^1 (1-t) |f'(tx + (1-t)a)| dt + \frac{(b-x)^2}{b-a} \int_0^1 (1-t) |f'(tx + (1-t)b)| dt. \end{aligned}$$

By the Power mean inequality, we can write;

$$\begin{aligned} & \left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u)du \right| \\ & \leq \frac{(x-a)^2}{b-a} \left(\int_0^1 (1-t) dt \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t) |f'(tx + (1-t)a)|^q dt \right)^{\frac{1}{q}} \\ & \quad + \frac{(b-x)^2}{b-a} \left(\int_0^1 (1-t) dt \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t) |f'(tx + (1-t)b)|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

Since m -convexity;

$$\begin{aligned} & \left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u)du \right| \\ & \leq \frac{(x-a)^2}{b-a} \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t) \left[t |f'(x)|^q + m(1-t) \left| f' \left(\frac{a}{m} \right) \right|^q \right] dt \right)^{\frac{1}{q}} \\ & \quad + \frac{(b-x)^2}{b-a} \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t) \left[t |f'(x)|^q + m(1-t) \left| f' \left(\frac{b}{m} \right) \right|^q \right] dt \right)^{\frac{1}{q}} \\ & = \frac{(x-a)^2}{b-a} \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \left(|f'(x)|^q \int_0^1 (t-t^2) dt + m \left| f' \left(\frac{a}{m} \right) \right|^q \int_0^1 (1-t)^2 dt \right)^{\frac{1}{q}} \\ & \quad + \frac{(b-x)^2}{b-a} \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \left(|f'(x)|^q \int_0^1 (t-t^2) dt + m \left| f' \left(\frac{b}{m} \right) \right|^q \int_0^1 (1-t)^2 dt \right)^{\frac{1}{q}} \\ & = \frac{(x-a)^2}{2(b-a)} \left(\frac{|f'(x)|^q + 2m \left| f' \left(\frac{a}{m} \right) \right|^q}{3} \right)^{\frac{1}{q}} \\ & \quad + \frac{(b-x)^2}{2(b-a)} \left(\frac{|f'(x)|^q + 2m \left| f' \left(\frac{b}{m} \right) \right|^q}{3} \right)^{\frac{1}{q}}. \end{aligned}$$

This completes the proof. \square

Corollary 3. In Theorem 5, if we choose $x = \frac{a+b}{2}$, we get

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \frac{b-a}{8} \left[\left(\frac{|f'(\frac{a+b}{2})|^q + 2m |f'(\frac{a}{m})|^q}{3} \right)^{\frac{1}{q}} + \left(\frac{|f'(\frac{a+b}{2})|^q + 2m |f'(\frac{b}{m})|^q}{3} \right)^{\frac{1}{q}} \right].$$

Remark 2. In Corollary 3, if we choose $m = 1$, (1.2) inequality is obtained.

3. APPLICATIONS TO SPECIAL MEANS

We now consider the means for arbitrary real numbers α, β ($\alpha \neq \beta$). We take

(1) *Arithmetic mean* :

$$A(\alpha, \beta) = \frac{\alpha + \beta}{2}, \quad \alpha, \beta \in \mathbb{R}^+.$$

(2) *Logarithmic mean*:

$$L(\alpha, \beta) = \frac{\alpha - \beta}{\ln |\alpha| - \ln |\beta|}, \quad |\alpha| \neq |\beta|, \quad \alpha, \beta \neq 0, \quad \alpha, \beta \in \mathbb{R}^+.$$

(3) *Generalized log - mean*:

$$L_n(\alpha, \beta) = \left[\frac{\beta^{n+1} - \alpha^{n+1}}{(n+1)(\beta - \alpha)} \right]^{\frac{1}{n}}, \quad n \in \mathbb{Z} \setminus \{-1, 0\}, \quad \alpha, \beta \in \mathbb{R}^+.$$

Now using the results of Section 2, we give some applications for special means of real numbers.

Proposition 1. Let $a, b \in \mathbb{R}^+$, $a < b$, $m \in [0, 1]$ and $n \in \mathbb{Z}$, $n > 1$. Then, we have

$$|A(a^n, b^n) - L_n^n(a, b)| \leq n \cdot \frac{b-a}{12} \left[\left| \frac{a+b}{2} \right|^{n-1} + m \left| \frac{a}{m} \right|^{n-1} + m \left| \frac{b}{m} \right|^{n-1} \right].$$

If we choose $m = 1$, we obtain

$$|A(a^n, b^n) - L_n^n(a, b)| \leq n \cdot \frac{b-a}{12} \left[\left| \frac{a+b}{2} \right|^{n-1} + |a|^{n-1} + |b|^{n-1} \right].$$

Proof. The assertion follows from Corollary 1 applied to the m -convex mapping $f(x) = x^n$, $x \in \mathbb{R}$, $n \in \mathbb{Z}$. \square

Proposition 2. Let $a, b \in \mathbb{R}^+$, $a < b$, $m \in [0, 1]$ and $n \in \mathbb{Z} \setminus \{-1, 0\}$. Then, for all $q > 1$, we have

$$|A(a^n, b^n) - L_n^n(a, b)| \leq n \cdot \frac{b-a}{8} \left(\left[\frac{|\frac{a+b}{2}|^{q(n-1)} + 2m |\frac{a}{m}|^{q(n-1)}}{3} \right]^{\frac{1}{q}} + \left[\frac{|\frac{a+b}{2}|^{q(n-1)} + 2m |\frac{b}{m}|^{q(n-1)}}{3} \right]^{\frac{1}{q}} \right).$$

If we choose $m = 1$, we obtain

$$|A(a^n, b^n) - L_n^n(a, b)| \leq n \cdot \frac{b-a}{8} \left(\left[\frac{|\frac{a+b}{2}|^{q(n-1)} + 2|a|^{q(n-1)}}{3} \right]^{\frac{1}{q}} + \left[\frac{|\frac{a+b}{2}|^{q(n-1)} + 2|b|^{q(n-1)}}{3} \right]^{\frac{1}{q}} \right).$$

Proof. The assertion follows from Corollary 3 applied to the m -convex mapping $f(x) = x^n$, $x \in \mathbb{R}$, $n \in \mathbb{Z}$. \square

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