

SURFACES WITH $p_g = 0$, $K^2 = 5$ AND BICANONICAL MAPS OF DEGREE 4

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Abstract Let S be a minimal surface of general type with $p_g(S) = 0$, $K_S^2 = 5$ and bicanonical map of degree 4. Denote by Σ the bicanonical image. If Σ is smooth, then S is a Burniat surface; and if Σ is singular, then we reduced Σ to one case and described it, furthermore S has at most one (-2) -curve.

1. INTRODUCTION

Let S be a minimal surface of general type with $p_g(S) = 0$; denote by $\phi : S \rightarrow \mathbb{P}^{K_S^2}$ its bicanonical map and by Σ the bicanonical image. It is known that the image Σ is a surface for $K_S^2 \geq 2$ (see [X1]) and ϕ is a morphism for $K_S^2 \geq 5$ (see [M], [Re]). For $K_S^2 \geq 2$, we denote by d the degree of ϕ . Mendes Lopes and Pardini proved: if $K_S^2 = 9$, then $d = 1$; if $K_S^2 = 7, 8$, then $d = 1, 2$; if $K_S^2 = 3, 4$ and $|2K_S|$ is base point free or if $K_S^2 = 5, 6$, then $d = 1, 2, 4$ (see [MP2], [MP4]). When $d > 1$, its image is relatively simple, so it is possible to describe the surface S precisely. For more details, we refer the readers to [MP1], [MP2], [MP5] and [Par2]. In particular, the surfaces with $K_S^2 = 6$ and $d = 4$ have been completely characterized: they are exactly Burniat surfaces (see [MP1]). And in [MP3], the authors proposed the following problem:

Problem 1. *Is it possible to characterize surfaces with $K^2 = 5$, $p_g = 0$ and $d = 4$?*

In this paper, we answer the problem in the case that the bicanonical image is smooth. Our main result is:

Theorem 1.1. *Let S be a smooth minimal surface of general type with $p_g(S) = q(S) = 0$, $K_S^2 = 5$ and bicanonical map of degree 4. If S has smooth bicanonical image, then it is a Burniat surface.*

Remark 1.2. *For the moduli space of the surfaces described in Theorem 1.1, by Theorem 1.2 in [BC], it is an irreducible connected component, normal and rational of dimension 3. In fact, Bauer and Catanese studied systematically the moduli spaces of Burniat surfaces with $K^2 = 4, 5, 6$. For more details, we refer the readers to [BC].*

As to the bicanonical image, we proved:

Theorem 1.3. *Let S be as in Theorem 1.1, and denote by Σ its bicanonical image. If Σ is not smooth, then it is the image of $\psi : \hat{P} \rightarrow \mathbb{P}^5$, where \hat{P} is isomorphic to the blow-up of \mathbb{P}^2 at four points P_1, P_2, P_3, P_4 such that P_1, P_2, P_3 lie on a line, P_3 is infinitely near to P_2 , P_1, P_4 are distinct from P_2, P_3 , and ψ is defined by the linear system $|-K_{\hat{P}}|$. Moreover, S contains at most one (-2) -curve, and the bicanonical map can not lift to a morphism to \hat{P} .*

Remark 1.4. *By the theorem above, S cannot be constructed by bidouble cover over \hat{P} . We failed in constructing a surface as in Theorem 1.3. But if S is such a surface, we get some information about its fibration, and write out a divisor linear equivalent to K_S (see section 7.2.2).*

The main idea of the proof is from [MP1]. To prove Theorem 1.1, the key step is to show that there exists a (-4) -curve on S which is indicated by the known examples for this case; to prove Theorem 1.3, we develop a method to make use of the ramification divisor (Lemma 2.8) and refer to computer sometimes as the calculation is very complicate.

The plan of the paper is: in Section 7.2.2, we collect some basic tools; in Section 3, 4, 5 and 6, we consider the case when the bicanonical image is smooth and prove Theorem 1.1; in Section 7, we study the the case when the bicanonical image is singular and prove Theorem 1.3.

Notations and conventions: We work over complex numbers; all varieties are assumed to be compact and algebraic. We don't distinguish between the line bundles and the divisors on a smooth variety, and we use both the additive and multiplicative notation. We say a line bundle is effective if it has a global non-zero section. Let V be a prime Weil divisor and let D be another divisor on a normal \mathbb{Q} -factorial projective variety. Then $ord_V(D)$ denotes the vanishing order of an equation for D in the local ring along the subvariety V . Let D_1 and D_2 be two \mathbb{Q} -Cartier divisor, $D_1 \leq D_2$ means $D_2 - D_1$ is an effective \mathbb{Q} -Cartier divisor; moreover if D_1, D_2 are Cartier, $D_1 \equiv D_2$ means they are linearly equivalent. Let $f : X \rightarrow Y$ be a morphism between two normal projective varieties. Let D be a \mathbb{Q} -Cartier divisor on Y and assume mD is Cartier for a positive integer m . Then $f^*D := \frac{1}{m}f^*mD$. Write $D = \sum_i a_i D_i$ where $a_i \in \mathbb{Q}$ and D_i is a reduced Weil divisor. Then $[D] := \sum_i [a_i]D_i$. Denote by S the singular locus of Y and by $i : Y - S \hookrightarrow Y$ the natural inclusion map. If moreover D is a Weil divisor, then $\mathcal{O}_Y(D) := i_*\mathcal{O}_{Y-S}(D)$, i.e., $\mathcal{O}_Y(D)_y = \{s \in K(Y) : (s^m) + mD \geq 0 \text{ in a neighbourhood of } y\}$.

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2. PRELIMINARIES

The main tools we use in this paper are from [MP1]. And we list them in the following. Let S be a smooth surface, let $D \subset S$ be a curve having at most negligible singularities (possibly empty) and let M be a line bundle on S such that $2M \equiv D$. Then there exists a double cover $\pi : Y \rightarrow S$ branched over D and such that $\pi_*\mathcal{O}_Y = \mathcal{O}_S \oplus M^{-1}$. Note that Y is smooth if D is smooth, Y has at most canonical singularities if D has at most negligible singularities, and Y is connected if and only if M is non-trivial. The invariants of Y can be calculated as follows:

$$\begin{aligned}
 K_Y^2 &= 2(K_S + M)^2 \\
 (2.1) \quad \chi(\mathcal{O}_Y) &= 2\chi(\mathcal{O}_S) + \frac{1}{2}M(K_S + M) = 2\chi(\mathcal{O}_S) + \frac{1}{2}\frac{D}{2}(K_S + \frac{D}{2}) \\
 p_g(Y) &= p_g(S) + h^0(S, K_S + M)
 \end{aligned}$$

Proposition 2.1 (Proposition 2.1 in [MP1]). *Let S be a smooth surface with $p_g(S) = q(S) = 0$, and let $\pi : Y \rightarrow S$ be a smooth double cover. Suppose that $q(Y) > 0$. Denote by $\alpha : Y \rightarrow A$ be the Albanese map. Then*

- i) *the Albanese image of Y is a curve B ;*
- ii) *there exists a fibration $g : S \rightarrow \mathbb{P}^1$ and a degree 2 map $p : B \rightarrow \mathbb{P}^1$ such that $p \circ \alpha = g \circ \pi$.*

Proposition 2.2 (Corollary 2.2 in [MP1]). *Let S be a smooth surface of general type with $p_g(S) = q(S) = 0, K_S^2 \geq 3$, and let $\pi : Y \rightarrow S$ be a smooth double cover. Then $K_Y^2 \geq 16(q(Y) - 1)$.*

Proposition 2.3 (Remark 2.3 in [MP1]). *Let S be a smooth surface with $p_g(S) = q(S) = 0$, and let $\pi : Y \rightarrow S$ be a smooth double cover. Let $g : S \rightarrow \mathbb{P}^1$ be a fibration such that the general fiber of $g \circ \pi$ is not connected. Let $\pi' \circ g' : Y \rightarrow B \rightarrow \mathbb{P}^1$ be the Stein factorization. Then one has the following commutative diagram:*

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & S \\ g' \downarrow & & g \downarrow \\ B & \xrightarrow{\pi'} & \mathbb{P}^1 \end{array}$$

where B is a smooth curve of genus b and π' is a double cover. Furthermore if k is the cardinality of the image in \mathbb{P}^1 of the branch locus of π , then g has at least $2b + 2 - k$ fibers that are divisible by 2.

Here we make a useful remark since we often have to deal with singular double cover.

Remark 2.4. *If the branch locus has negligible singularities which causes that Y has canonical singularities, then the propositions above still hold true.*

The following lemma plays an important role in the proof of Theorem 1.3, it can be proved by using the idea of the proof Lemma 5.7 in [MP1], and we omit the details.

Lemma 2.5. *Let $\pi : X \rightarrow Y$ be a finite morphism between two normal \mathbb{Q} -factorial surfaces. Assume that X and Y have equal Picard numbers. Let $u : X \rightarrow C$ of X be a fibration. Then there exists a finite morphism $\pi' : C \rightarrow B$ and a fibration $v : Y \rightarrow B$ satisfying the following commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Y \\ u \downarrow & & v \downarrow \\ C & \xrightarrow{\pi'} & B \end{array}$$

Lemma 2.6. *Let $f : X \rightarrow Y$ be a generically finite surjective morphism between two normal projective varieties. Let $D \subset Y$ and $G \subset X$ be two \mathbb{Q} -Cartier Weil divisors such that $G \geq [f^*D]$. Then $h^0(\mathcal{O}_X(G)) \geq h^0(\mathcal{O}_Y(D))$.*

Proof. Note that $H^0(\mathcal{O}_Y(D)) \cong \{s \in K(Y) \mid (s) + D \geq 0\}$. It suffices to show that $(f^*s) + [f^*D] \geq 0$ for every $s \in K(Y)$ such that $(s) + D \geq 0$. Assume mD is Cartier where m is a positive integer. By definition we have $(s^m) + mD \geq 0$, thus $(f^*s^m) + f^*mD = m((f^*s) + f^*D) \geq 0$. Since (f^*s) is Cartier, $(f^*s) + [f^*D] \geq 0$, so the lemma follows. \square

Lemma 2.6 is very useful, we can give a low bound of $h^0(\mathcal{O}_X(G))$ by calculating $h^0(\mathcal{O}_Y(D))$. The divisor that we have to deal with is often not Cartier.

The following lemma is well known to experts, but we give detailed proof here for lack of references.

Lemma 2.7. *Let $f : S \rightarrow \Delta$ be a fibration over the unit disc Δ such that $S_0 = 2M$ be the only singular fiber. Let $\pi : X \rightarrow S$ be an etale double cover given by the relation $2L \equiv \mathcal{O}_S$. Let $\Delta' \rightarrow \Delta$ be a double cover given by $t \rightarrow s^2$ branched along the point 0. If X coincides with the normalization of the fiber product $\Delta' \times_{\Delta} S$, then $L \equiv M$.*

Proof. Denote by $\pi' : X' \rightarrow S$ the double cover given by the relation $2M \equiv \mathcal{O}_S$. Considering the Stein factorizations of the composition map $f \circ \pi'$ and $f \circ \pi$, we find that both X and X' are isomorphic to the normalization of the product $S \times_{\Delta} \Delta'$, hence the double cover π coincides with π' , and thus $\pi_*\mathcal{O}_X \cong \pi_*\mathcal{O}_{X'}$. So the lemma follows from the fact that $\pi_*\mathcal{O}_X \cong \mathcal{O}_S \oplus \mathcal{O}_S(L^{-1})$ and $\pi_*\mathcal{O}_X \cong \mathcal{O}_S \oplus \mathcal{O}_S(M^{-1})$. \square

Lemma 2.8. *Let $h : X \rightarrow T$ be a generically finite morphism between two normal surfaces. Let $e \subset T$ be a reduced and irreducible \mathbb{Q} -Cartier Weil divisor such that $e^2 < 0$. Denote by R be the ramification divisor, and let R' be an effective divisor such that $R' \leq R$. Then we have $R'(h^*e) > (h^*e)^2$.*

Proof. Let $g \circ \eta : X \rightarrow Y \rightarrow T$ be the factorization of h . Write $g^*e = \sum_i a_i E_i$. Note that $(g_*E_i)e < 0$ since $e^2 < 0$. Then we have

$$\begin{aligned}
 R'(h^*e) &= (\eta_*R')(g^*e) \geq \left(\sum_i (a_i - 1)E_i\right)(g^*e) \\
 (2.2) \quad &= (g^*e - \sum_i E_i)g^*e \\
 &= (g^*e)^2 - \sum_i (g_*E_i)e > (g^*e)^2 = (h^*e)^2
 \end{aligned}$$

\square

3. EXAMPLES

In this section we study some examples and make a useful observation. Let $\rho : \Sigma \rightarrow \mathbb{P}^2$ be the blow-up of \mathbb{P}^2 at 4 points P_1, P_2, P_3, P_4 in general position. We denote by l the pull-back of a line on \mathbb{P}^2 , by e_i the exceptional curve corresponding to $P_i, i = 1, 2, 3, 4$, by e'_i the strict transform of the line joining P_j and P_k where $\{i, j, k\} = \{1, 2, 3\}$, by g_i the strict transform of the line joining P_4 and $P_i, i = 1, 2, 3$, and by m_i the strict transform of a general line through $P_i, i = 1, 2, 3, 4$. Then the Picard group of Σ is a free Abelian group generated by the classes of l, e_1, e_2, e_3, e_4 . The anticanonical class $-K_{\Sigma} \equiv 3l - e_1 - e_2 - e_3 - e_4$ is very ample. And the linear system $| -K_{\Sigma} |$ embeds Σ as a smooth surface of degree 5 in \mathbb{P}^5 , the so-called del Pezzo surface of degree 5.

As to the examples for the surfaces of general type with $p_g = 0, K^2 = 5$ and bicanonical map of degree 4, there are two known examples: example 6 in [Cat], namely, Burniat surface (see [Pet], [Bu]) and example 7 in [Cat]. We describe the two examples by giving a collection of bidouble cover data over Σ . Let $\Gamma = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, denote by $\gamma_1, \gamma_2, \gamma_3$ the nonzero elements of Γ and by $\chi_i \in \Gamma^*$ the nontrivial character orthogonal to γ_i ; by [Par1], to define a smooth Γ -cover $\phi : S \rightarrow \Sigma$ one assigns the following bidouble cover data:

i) smooth divisors $D_i, i = 1, 2, 3$ such that $D = D_1 + D_2 + D_3$ is a normal crossing divisor,

ii) line bundles $L_i, i = 1, 2, 3$ satisfying $2L_i \equiv D_k + D_j, \{i, j, k\} = \{1, 2, 3\}$.

Note that once one assigns the data *i*), the data *ii*) are determined since the Picard group of Σ is free.

Example 3.1 (Example 6 in [Cat], i.e., Burniat surfaces). *We construct a surface S by giving the following bidouble cover data:*

$$D_1 = e_3 + e'_3 + g_1 + m_1 \equiv 3l - 3e_1 - e_2 + e_3 - e_4$$

$$D_2 = e_1 + e'_1 + g_2 + m_2 \equiv 3l + e_1 - 3e_2 - e_3 - e_4$$

$$D_3 = e_2 + e'_2 + g_3 + m_3 \equiv 3l - e_1 + e_2 - 3e_3 - e_4$$

We denote by $\pi : S \rightarrow \Sigma$ the 4-to-1 covering map and by \mathcal{S} the family of surfaces constructed in the example.

Example 3.2 (Example 7 in [Cat]). *We construct a surface S' by giving the following bidouble cover data:*

$$D_1 = m_1 + e_2 + e_3 + e_4 \equiv l - e_1 + e_2 + e_3 + e_4$$

$$D_2 = e_1 + e'_1 + g_2 + m_2 \equiv 3l + e_1 - 3e_2 - e_3 - e_4$$

$$D_3 = e'_2 + g_1 + g_3 + B \equiv 5l - 3e_1 - e_2 - 3e_3 - 3e_4$$

where B is the strict transform of a conic passing through the points P_1, P_2, P_3, P_4 . We denote by $\pi : S' \rightarrow \Sigma$ the 4-to-1 covering map and by \mathcal{S}' the family of surfaces constructed in the example.

Remark 3.3. *For a surface S in the two families constructed above, by $\pi_*\mathcal{O}_S \cong \mathcal{O}_\Sigma \oplus_i L_i^{-1}$, we calculate that $p_g(S) = q(S) = 0$. Since $D \equiv -3K_\Sigma$, by the formula $2K_S \equiv \pi^*(2K_\Sigma + D)$ (cf. [Par1]), we get $2K_S \equiv \pi^*(-K_\Sigma)$, thus $K_S^2 = 5$. Checking that $h^0(\Sigma, -K_\Sigma \otimes L_i^{-1}) = 0$ for $i = 1, 2, 3$, we have $H^0(S, 2K_S) \cong H^0(\Sigma, -K_\Sigma) \oplus_i H^0(\Sigma, -K_\Sigma \otimes L_i^{-1}) \cong H^0(\Sigma, -K_\Sigma)$. This implies that the bicanonical map of S coincides with the covering map π .*

Although the two collections of bidouble cover data are of different forms, we have following observation:

Claim 3.4. *The two families \mathcal{S} and \mathcal{S}' are the same.*

To see this, first note that Σ has two types automorphisms.

Type 1: An automorphism of this type arises from a linear automorphism of \mathbb{P}^2 . Let $s \in S_4$ be a permutation of $\{1, 2, 3, 4\}$. Then there exists a unique linear automorphism $\bar{\eta}_s$ of \mathbb{P}^2 such that $\bar{\eta}_s(P_i) = P_{s(i)}$ and a unique automorphism η_s of Σ fitting in the following commutative diagram:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\eta_s} & \Sigma \\ \rho \downarrow & & \rho \downarrow \\ \mathbb{P}^2 & \xrightarrow{\bar{\eta}_s} & \mathbb{P}^2 \end{array}$$

It follows from the definition that $\eta_s^*e_{s_i} = e_i$ for $i = 1, 2, 3, 4$ and $\eta_s^*l \equiv l$.

Type 2: An automorphism of type 2 arises from a birational map from \mathbb{P}^2 to itself, the so-called Cremona transformation (cf. [Har] P.30). Here we give an example. Denote by (x_0, x_1, x_2) the homogenous coordinate of \mathbb{P}^2 . Up to a linear

transformation, we can assume the coordinates of $P_i, i = 1, 2, 3, 4$ are as follows: $P_1(1, 0, 0), P_2(0, 1, 0), P_3(0, 0, 1), P_4(1, 1, 1)$. Note that for $i = 1, 2, 3$, e'_i is the strict transform of the line defined by $x_{i-1} = 0$. A Cremona transformation $\varphi : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ is given by $\varphi : (x_0, x_1, x_2) \mapsto (x_1x_2, x_0x_2, x_0x_1)$, so it is well defined on $\mathbb{P}^2 \setminus \{P_1, P_2, P_3\}$, and it fixes P_4 . By Example 4.2.3 chap.V in [Har], to extend φ to a morphism, it suffices to blow up \mathbb{P}^2 at the points P_1, P_2, P_3 . So φ extends to a morphism $\rho' : \Sigma \rightarrow \mathbb{P}^2$ which blows down the curves $e'_i, i = 1, 2, 3$ and e_4 to the points $P_i, i = 1, 2, 3$ and P_4 respectively. Note that both ρ and ρ' are the blowing-up of \mathbb{P}^2 at the points $P_i, i = 1, 2, 3, 4$. Thus the map ρ' lifts to a morphism $\tau : \Sigma \rightarrow \Sigma$ fitting into the following diagram:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\tau} & \Sigma \\ \rho \downarrow & \searrow \rho' & \downarrow \rho \\ \mathbb{P}^2 & \xrightarrow{\varphi} & \mathbb{P}^2 \end{array}$$

We can see that τ is an automorphism of Σ . And it follows from the definition that $\tau^*e'_i = e_i, \tau^*e_i = e'_i, i = 1, 2, 3$ and $\tau^*e_4 = e_4$, hence $\tau^*l \equiv \tau^*(e'_1 + e_2 + e_3) \equiv 2l - e_1 - e_2 - e_3$. For a permutation s of $\{1, 2, 3, 4\}$, let η_s be defined as in Type 1. Then the automorphism $\eta_s^{-1} \circ \tau \circ \eta_s$ also arises from a Cremona transformation, and it fixes the curve $\eta_s^{-1}(e_4)$.

Proof of Claim 3.4. Let $S' \in \mathcal{S}'$ be a surface given by the data in Example 3.2. Consider an automorphism τ of type 2 introduced above such that

$$\tau^*e'_i = e_i, \tau^*e_i = e'_i \text{ for } i = 1, 2, 3, \tau^*e_4 = e_4, \tau^*l \equiv 2l - e_1 - e_2 - e_3.$$

It follows that

$$\tau^*g_i \equiv l - e_i - e_4, \tau^*m_i \equiv l - e_i \text{ for } i = 1, 2, 3, \tau^*B \equiv l - e_4.$$

Let $n_i = \tau^*m_i$ for $i = 1, 2, 3$ and $n_4 = \tau^*B$. Then n_i is the strict transform of a line through $P_i, i = 1, 2, 3, 4$. We obtain

$$\begin{aligned} \tau^*D_1 &= n_1 + e'_2 + e'_3 + e_4 \equiv 3l - 3e_1 - e_2 - e_3 + e_4 \\ \tau^*D_2 &= e_1 + e'_1 + g_2 + n_2 \equiv 3l + e_1 - 3e_2 - e_3 - e_4 \\ \tau^*D_3 &= e_2 + g_1 + g_3 + n_4 \equiv 3l - e_1 + e_2 - e_3 - 3e_4. \end{aligned} \tag{3.1}$$

Let $s = (34)$, and let η_s be an automorphism of type 1 defined as above. Then we have

$$\eta_s^*e_1 = e_1, \eta_s^*e_2 = e_2, \eta_s^*e_3 = e_4, \eta_s^*e_4 = e_3, \eta_s^*l \equiv l.$$

It follows that

$$\begin{aligned} \eta_s^*\tau^*D_1 &= k_1 + g_1 + e'_3 + e_3 \equiv 3l - 3e_1 - e_2 + e_3 - e_4 \\ \eta_s^*\tau^*D_2 &= e_1 + g_2 + e'_1 + k_2 \equiv 3l + e_1 - 3e_2 - e_3 - e_4 \\ \eta_s^*\tau^*D_3 &= e_2 + e'_2 + g_3 + k_3 \equiv 3l - e_1 + e_2 - 3e_3 - e_4 \end{aligned} \tag{3.2}$$

where $k_1 = \eta_s^*n_1 \equiv l - e_1$, $k_2 = \eta_s^*n_2 \equiv l - e_2$, $k_3 = \eta_s^*n_4 \equiv l - e_3$, and for $i = 1, 2, 3$, k_i is the strict transform of a line through P_i .

The data pulled back via $\tau \circ \eta_s$ is exactly one collection of bidouble cover data as in Example 3.1. Therefore the claim follows. \square

There are exactly 10 (-1) -curves on Σ in all. For every (-1) -curve, there are 3 (-1) -curves intersecting it and 6 (-1) -curves not intersecting it. An automorphism of Σ sends a (-1) -curve to a (-1) -curve, so it induces an action on the set of

(-1)-curves on Σ . Here we list some facts about the existence of the automorphism with a certain action on the set of (-1)-curves.

Fact 3.5. *Let all the notations be as above. Then we have*

- i) *For two (-1)-curves c_1, c_2 on Σ , there exists an automorphism σ of Σ such that $\sigma(c_1) = c_2$;*
- ii) *For three (-1)-curves c_1, c_2, c on Σ such that $c_1c = c_2c = 0$, there exists an automorphism σ of Σ such that $\sigma(c) = c$ and $\sigma(c_1) = c_2$;*
- iii) *For four (-1)-curves c_1, c_2, c_3, c_4 on Σ such that $c_1c_2 = c_3c_4 = 0$, there exists an automorphism σ of Σ such that $\sigma(c_1) = c_3$ and $\sigma(c_2) = c_4$.*

Proof. In the following, when we say a curve is exceptional, we mean it is ρ -exceptional.

i) If necessary, acting on c_1 by a suitable automorphism of type 2, we can assume c_1 is an exceptional curve, say $c_1 = e_1$. If c_2 is also exceptional, then there exists an automorphism of type 1 sending c_1 to c_2 . Now we assume c_2 is not exceptional. If c_2 does not intersect c_1 , say $c_2 = e'_1$, then $\tau(e_1) = e'_1 = c_2$; if c_2 intersects c_1 , say $c_2 = e'_2$, then $\eta_{(12)} \circ \tau(e_1) = \eta_{(12)}(e'_1) = e'_2 = c_2$.

ii) Thanks to *i)*, we can assume $c = e_4$. So c_1, c_2 belong to the set $\{e_i, e'_i\}_{i=1,2,3}$. Then arguing as in *i)*, we prove *ii)*.

iii) By *i)*, we can find an automorphism σ_1 such that $\sigma_1(c_1) = c_3$, then we have $\sigma_1(c_2) \cap c_3 = \sigma_1(c_2 \cap c_1) = \phi$. And since $c_4 \cap c_3 = \phi$, applying *ii)* gives an automorphism fixing c_3 and sending $\sigma_1(c_2)$ to c_4 , then *iii)* follows. \square

Remark 3.6. *Notice that in Example 3.1, e_4 is the only (-1)-curve that is not contained in the branch divisor, and the pull-back π^*e_4 is a reduced (-4)-curve while the pull-backs of the other (-1)-curves are non-reduced. This observation suggests that we should prove that S has a (-4)-curve.*

4. DIVISORS, PENCILS AND TORSION OF S

This section is the preparation for the proof of the main theorem.

Notation 4.1. *Let S be a surface with $p_g(S) = q(S) = 0, K_S^2 = 5$. Let $\phi : S \rightarrow \mathbb{P}^5$ be the bicanonical map and Σ be the bicanonical image. We assume the degree of ϕ is 4 and Σ is a smooth. By [Na], Σ is a del Pezzo surface of degree 5 in \mathbb{P}^5 . Let $\rho : \Sigma \rightarrow \mathbb{P}^2$ be the blow-up of \mathbb{P}^2 at 4 points P_1, P_2, P_3, P_4 in general position, and let $l, e_i, e'_j, g_j, i = 1, 2, 3, 4, j = 1, 2, 3$ be as at the beginning of Section 3. Set $f_i \equiv l - e_i$ and $F_i \equiv \phi^*(l - e_i)$ for $i = 1, 2, 3, 4$.*

Proposition 4.2. *Let the notations be as in 4.1. For $i = 1, 2, 3, 4$, if $f_i \in |f_i|$ is general, then ϕ^*f_i is connected, hence $|F_i|$ induces a genus 3 fibration $u_i : S \rightarrow \mathbb{P}^1$.*

Proof. By Bertini theorem, for a general element $f_i \in |f_i|$, $F_i = \phi^*f_i$ is smooth. Since $K_S F_i = 4$, it suffices to prove ϕ^*f_i is connected. Otherwise, we will get a genus 2 fibration of S . However, by [[X1] P.37], S has no genus 2 fibration. Then we get a contradiction. \square

Proposition 4.3. *Let the notations be as in 4.1. Then the bicanonical map ϕ is finite, and for $i = 1, 2, 3, 4$, the pull-back of an irreducible curve in $|f_i|$ is also irreducible (possibly non-reduced).*

Proof. By $\chi(S) = 1$ and $K_S^2 = 5$, Noether's formula gives $e(S) = 7$. Then by $p_g(S) = q(S) = 0$, we get $h^2(S) = h^2(\Sigma) = 5$. So $\phi^* : H^2(\Sigma, \mathbb{Q}) \rightarrow H^2(S, \mathbb{Q})$ is an isomorphism preserving the intersection form up to multiplication by 4. Therefore the bicanonical map ϕ is finite. For an irreducible curve $f_1 \in |f_1|$, if $\phi^* f_1$ is reducible, then it contains an irreducible component C with $C^2 < 0$. Put $D = C - \frac{C(\phi^* e_1)}{4} \phi^* f_1$. Then $D^2 = C^2 < 0$, and $D(\phi^* e_1) = 0$. And for $i = 2, 3, 4$, $(C - \frac{C(\phi^* e_1)}{4} \phi^* f_1) \phi^* e_i = 0$ since e_i is contained in one fiber of the pencil $|l - e_1|$. Then we can see that the intersection matrix of $\phi^* l, C - \frac{C(\phi^* e_1)}{4} \phi^* f_1, \phi^* e_i, i = 1, 2, 3, 4$ has rank 6. However, this contradicts $h^2(S) = 5$, thus $\phi^* f_1$ is irreducible. The proof for the other cases is similar. \square

Lemma 4.4. *Let $\phi : S \rightarrow T$ be a finite morphism between two smooth surfaces. Let h be a divisor on T such that $|\phi^* h| = \phi^* |h|$. Let M be a divisor on S such that the linear system $|M|$ has no fixed part. Suppose that $\phi^* h - M$ is effective. Then there exists a divisor $m \subset T$ such that $|M| = \phi^* |m|$. Furthermore the line bundle $h - m$ is effective.*

Proof. Choose an effective divisor N on S such that $N \equiv \phi^* h - M$. Let $M_1 \in |M|$ be a general element. Then there exists an element $h_1 \in |h|$ such that $M_1 + N = \phi^* h_1$ since $|\phi^* h| = \phi^* |h|$. We can assume $\phi(M_1)$ and $\phi(N)$ have no common components. Write $h_1 = m_1 + n$ where m_1 and n are supported on $\phi(M_1)$ and $\phi(N)$ respectively. Since ϕ is a finite morphism, $\phi^* m_1$ and $\phi^* n$ have no common components, so $\phi^* m_1 = M_1$ and $\phi^* n = N$. Since M_1 is general, we conclude that $|M| = \phi^* |m_1|$. Obviously $h - m_1$ is effective, and we are done. \square

Lemma 4.5. *There does not exist a divisor d on Σ such that $h^0(\Sigma, d) > 1$ and that the line bundle $-K_\Sigma - 2d$ is effective.*

Proof. To the contrary, suppose that there exists such a divisor d . Assume $d \equiv al - b_1 e_1 - b_2 e_2 - b_3 e_3 - b_4 e_4$. The condition that $-K_\Sigma - 2d$ is effective implies that $a \leq 1$. And $h^0(\Sigma, d) > 1$ implies that $a \geq 1$. So $a = 1$, and at most one of b_1, b_2, b_3, b_4 is positive. But then the line bundle $-K_\Sigma - 2d \equiv l - (1 - b_1)e_1 - (1 - b_2)e_2 - (1 - b_3)e_3 - (1 - b_4)e_4$ can not be effective since there is no line on \mathbb{P}^2 passing through 3 points in general position. Thus the lemma is true. \square

Lemma 4.6. *Let S and Σ be as in 4.1. Let $D \subset S$ be a divisor. If there exists a divisor d on Σ such that*

- i) $\phi^* d \equiv 2D$;
- ii) the line bundle $-K_\Sigma - d$ is effective,

then $h^0(S, D) \leq 1$.

Proof. To the contrary, suppose that $h^0(S, D) > 1$. We may write $|D| = |M| + F$ where $|M|$ is the moving part and F is the fixed part. Since $|2K_S| = |\phi^*(-K_\Sigma)| = \phi^* |-K_\Sigma|$ and $\phi^*(-K_\Sigma) - M > \phi^*(-K_\Sigma - d)$ is effective, applying Lemma 4.4 gives a divisor m on Σ such that $\phi^* |m| = |M|$. Choose an element $M_1 \in |M|$ and an effective divisor N on S such that $2M_1 + 2F + N \equiv \phi^*(-K_\Sigma)$. Note that we can find $h \in |-K_\Sigma|$ and $m_1 \in |m|$ such that $2M_1 + 2F + N = \phi^* h$ and $2M_1 = \phi^*(2m_1)$. So we conclude that $h - 2m_1$ is effective, i.e., the line bundle $-K_\Sigma - 2m$ is effective. However, this contradicts Lemma 4.5. \square

Now we analyze the pull-backs of the (-1) -curves on Σ which give much information about the surface S . We begin with one lemma from [MP1].

Lemma 4.7 (Lemma 5.1 in [MP1]). *Let $\phi : S \rightarrow \Sigma$ be as in 4.1, and let $C \subset \Sigma$ be a (-1) -curve. Then we have either*

- i) ϕ^*C is a reduced smooth rational (-4) -curve; or
- ii) $\phi^*C = 2E$ where E is an irreducible curve with $E^2 = -1, K_S E = 1$.

Lemma 4.8. *Let S be as in 4.1. Then there are at most two disjoint (-4) -curves on S .*

Proof. Let r be the cardinality of a set of smooth disjoint rational curves with self-intersection -4 . Then by [Miy], one has $r \cdot \frac{25}{12} \leq c_2(S) - \frac{1}{3}K_S^2 = \frac{16}{3}$, therefore $r \leq 2$. \square

Proposition 4.9. *Let S and Σ be as in 4.1. Then there exists at least one (-1) -curve on Σ such that its pull-back is a (-4) -curve.*

As the proof of the proposition is long, we postpone the proof to the last section.

Proposition 4.10. *Let the notations be as in 4.1, then there do not exist two (-1) -curves $C_1, C_2 \subset \Sigma$ satisfying*

- i) $C_1 \cap C_2 = \emptyset$;
- ii) both ϕ^*C_1 and ϕ^*C_2 are (-4) -curves.

Proof. Suppose on the contrary that the proposition is not true. Thanks to Fact 3.5 iii), we may assume $C_1 = e_4$ and $C_2 = e_2$. Then both $E_2 = \phi^*e_2$ and $E_4 = \phi^*e_4$ are reduced rational (-4) -curves. By Lemma 4.8, $\phi^*e'_2$ is non-reduced. And by Lemma 4.7 we may write $\phi^*e'_2 = 2E'_2$. From the formula

$$2K_S \equiv \phi^*(3l - e_1 - e_2 - e_3 - e_4) \equiv \phi^*(2g_2 + e'_2 + e_2 + e_4) \equiv 2\phi^*(g_2) + 2E'_2 + E_2 + E_4,$$

we obtain $2(K_S - \phi^*(g_2) - E'_2) \equiv E_2 + E_4$. Therefore, we get a double cover $\pi : Y \rightarrow S$ branched over E_2 and E_4 . Applying formula 2.1, we have

$$\chi(\mathcal{O}_Y) = 2 + \frac{(K_S - \phi^*(g_2) - E'_2) \cdot (2K_S - \phi^*(g_2) - E'_2)}{2} = 2,$$

$$K_Y^2 = 2(2K_S - \phi^*(g_2) - E'_2)^2 = 14,$$

$$\begin{aligned} p_g(Y) &= h^0(S, 2K_S - \phi^*(g_2) - E'_2) = h^0(S, \phi^*(g_2) + E'_2 + E_2 + E_4) \\ &= h^0(S, \phi^*(g_2 + e_2 + e_4) + E'_2) = h^0(S, \phi^*(l) + E'_2) \geq 3. \end{aligned}$$

Hence $q(Y) \geq 2$ and $K_Y^2 < 16(q(Y) - 1)$. This contradicts Proposition 2.2. \square

Since there do not exist three (-1) -curves on Σ intersecting each other, by the propositions above, we get the following corollary.

Corollary 4.11. *There are at most two (-1) -curves on Σ whose pull-back is a (-4) -curve, and if there are two such curves, then they intersect.*

Assumption-Notation 4.12. *By Fact 3.5 i) and the corollary above, up to an automorphism of Σ , we may assume $\phi^*e_4 = E_4$ where E_4 is a (-4) -curve, $\phi^*e_i = 2E_i, \phi^*e'_i = 2E'_i, i = 1, 2, 3$ and $\phi^*g_j = 2G_j, j = 1, 2$. With these assumptions, we know that $2(E_j + E'_k)$ and $2(E'_j + E_k)$ are two double fibers of $u_i : S \rightarrow \mathbb{P}^1$ where $\{i, j, k\} = \{1, 2, 3\}$. Set $\eta_i \equiv (E_j + E'_k) - (E'_j + E_k)$ where $\{i, j, k\} = \{1, 2, 3\}$, and set $\eta \equiv K_S - \sum_{j=1}^3 (E_j + E'_j)$. Then $2\eta \equiv -E_4$; and [BPV] Lemma 8.3 chap.III gives that $\eta_i \neq 0$ for $i = 1, 2, 3$, hence it is torsion of order 2.*

The following proposition corresponds to Proposition 5.9 in [MP1]. For the readers' convenience, we will give an explicit proof.

Proposition 4.13. *Let notations and assumptions be as in 4.1 and 4.12, and let $F_i \in |E_i|, i = 1, 2, 3$ be a general curve. Then $F_j |_{F_i} \equiv K_{F_i}$ if $i \neq j$.*

Proof. We show that $F_2 |_{F_1} \equiv K_{F_1}$. Note that $2K_S \equiv F_1 + 2(2E_1 + E'_3 + E'_2) - E_4$, then we have

$$2(K_S - (2E_1 + E'_3 + E'_2) + E_4) \equiv F_1 + E_4.$$

Given this relation, we get a double cover $\pi : Y \rightarrow S$ branched over F_1 and E_4 . By formula 2.1, we calculate the invariants of Y :

$$\chi(\mathcal{O}_Y) = 3, p_g(Y) = h^0(S, F_1 + 2E_1 + E'_3 + E'_2) = h^0(S, \phi^*(f_1 + e_1) + E'_3 + E'_2) \geq 3,$$

hence $q(Y) \geq 1$. By Proposition 2.1, the Albanese pencil of Y is the pull-back of a pencil $|F|$ of S such that π^*F is disconnected for a general element F in $|F|$. Since π is branched over F_1 , it follows that $F \cdot F_1 = 0$ and therefore $|F| = |F_1|$. For a general element $F_1 \in |F_1|$, the pull-back π^*F_1 is an unramified double cover of F_1 given by the relation $2(K_S - (2E_1 + E'_3 + E'_2) + E_4) |_{F_1}$. Since π^*F_1 is disconnected, we have $(K_S - (2E_1 + E'_3 + E'_2) + E_4) |_{F_1} \equiv (K_S - 2E_1) |_{F_1} \equiv (K_S - 2E_1 - 2E'_3) |_{F_1} \equiv (K_S - F_2) |_{F_1}$ is trivial, thus $F_2 |_{F_1} \equiv K_{F_1}$. \square

Lemma 4.14. *Let the notations be as in 4.1 and 4.12. Then we have:*

- i) $\chi(S, K_S + \eta + \eta_i) = 0, h^2(S, K_S + \eta + \eta_i) = 0;$
- ii) $h^0(\mathcal{O}_{F_i}(K_{F_i} + \eta)) \leq 2;$
- iii) $h^1(S, \eta - \eta_i) = 1;$
- iv) $h^0(S, K_S + \eta_i) = 1, h^1(S, K_S + \eta_i) = 0.$

Proof. i) By $2\eta \equiv -E_4$, applying Riemann-Roch formula, we get $\chi(S, K_S + \eta + \eta_i) = 0$. Using Serre duality, we have $h^2(S, K_S + \eta + \eta_i) = h^0(S, -\eta + \eta_i)$. Since $2(-\eta + \eta_i) \equiv E_4$ and E_4 is a reduced (-4) -curve, it follows that $h^0(S, -\eta + \eta_i) = 0$.

ii) We show that $h^0(\mathcal{O}_{F_1}(K_{F_1} + \eta)) \leq 2$. Since $\eta_1 |_{F_1} \equiv \mathcal{O}_{F_1}$, we get the following exact sequence:

$$0 \rightarrow \mathcal{O}_S(K_S + \eta + \eta_1) \rightarrow \mathcal{O}_S(K_S + \eta + \eta_1 + F_1) \rightarrow \mathcal{O}_{F_1}(K_{F_1} + \eta) \rightarrow 0.$$

Considering the long cohomology sequence, by the results of i), we have

$$\begin{aligned} & h^0(\mathcal{O}_{F_1}(K_{F_1} + \eta)) \\ & \leq h^0(\mathcal{O}_S(K_S + \eta + \eta_1 + F_1)) - h^0(\mathcal{O}_S(K_S + \eta + \eta_1)) + h^1(\mathcal{O}_S(K_S + \eta + \eta_1)) \\ & = h^0(\mathcal{O}_S(K_S + \eta + \eta_1 + F_1)) - \chi(\mathcal{O}_S(K_S + \eta + \eta_1)) + h^2(\mathcal{O}_S(K_S + \eta + \eta_1)) \\ & = h^0(\mathcal{O}_S(K_S + \eta + \eta_1 + F_1)). \end{aligned}$$

Note that $K_S + \eta + \eta_1 + F_1 \equiv 2K_S - \sum_{i=1}^3 (E_i + E'_i) + (E_2 + E'_3 - E_3 - E'_2) + 2(E_3 + E'_2) \equiv 2K_S - (E_1 + E'_1)$. Since the linear system $|2K_S|$ embeds $E_1 + E'_1$ as a pair of skew lines in \mathbb{P}^5 , we obtain $h^0(2K_S - (E_1 + E'_1)) = 2$, thus $h^0(\mathcal{O}_{F_1}(K_{F_1} + \eta)) \leq 2$.

iii) Note that $h^0(S, \eta - \eta_i) = 0$ since $2(\eta - \eta_i) \equiv -E_4$. Applying Riemann-Roch formula, we get $\chi(S, \eta - \eta_i) = 1$, thus $-h^1(S, \eta - \eta_i) + h^2(S, \eta - \eta_i) = 1$. Then applying Serre duality, it suffices to show $h^0(S, K_S - \eta + \eta_i) = 2$.

We show that $h^0(S, K_S - \eta + \eta_1) = 2$. Since E_4 is a rational (-4) -curve and $(2K_S + E_4)E_4 = 0$, we get the following exact sequence

$$0 \rightarrow \mathcal{O}_S(2K_S) \rightarrow \mathcal{O}_S(2K_S + E_4) \rightarrow \mathcal{O}_{E_4} \rightarrow 0.$$

From the long cohomology sequence, using $h^1(S, 2K_S) = 0$, we derive $h^0(S, 2K_S + E_4) = 7$. Since $h^0(\Sigma, 3l - e_1 - e_2 - e_3) = 7$ and $2K_S + E_4 \equiv 2(K_S - \eta + \eta_1) \equiv \phi^*(3l - e_1 - e_2 - e_3)$, we obtain $|2(K_S - \eta + \eta_1)| = \phi^*|3l - e_1 - e_2 - e_3|$. Notice that $K_S - \eta + \eta_1 \equiv \sum_{j=1}^{j=3} (E_j + E'_j) + (E_2 + E'_3 - E_3 - E'_2) \equiv F_1 + E'_1 + E_1$, we get $h^0(S, K_S - \eta + \eta_1) = h^0(S, F_1 + E'_1 + E_1) \geq 2$. We may write $|K_S - \eta + \eta_1| = |M| + F$ where $|M|$ is the moving part and F is the fixed part. Applying Lemma 4.4, we can find a divisor m on Σ such that $|M| = \phi^*|m|$. Then arguing as in the proof of Lemma 4.6, we conclude that $3l - e_1 - e_2 - e_3 - 2m$ is effective. So the only possibility is $m \equiv f_i$ for some $i \in \{1, 2, 3\}$, consequently $h^0(S, K_S - \eta + \eta_1) = h^0(S, M) = h^0(\Sigma, f_i) = 2$.

iv) Since η_i is a torsion line bundle of order 2, applying Riemann-Roch formula and Serre duality, we get $\chi(S, K_S + \eta_i) = 1$ and $h^2(S, K_S + \eta_i) = h^0(S, \eta_i) = 0$, thus $h^0(S, K_S + \eta_i) - h^1(S, K_S + \eta_i) = 1$. So it suffices to show $h^0(S, K_S + \eta_i) \leq 1$. It is a consequence of Lemma 4.6 since $2(K_S + \eta_i) \equiv \phi^*(-K_\Sigma)$. \square

Corollary 4.15. *Let S be as in 4.1. Let $F_i \in |F_i|, i = 1, 2, 3$ be a general curve. Then $-\eta + \eta_j|_{F_i} \equiv \mathcal{O}_{F_i}$, if $i \neq j$; $\eta_i|_{F_i} \equiv \mathcal{O}_{F_i}$, $-\eta + \eta_i|_{F_i} \neq \mathcal{O}_{F_i}$.*

Proof. By definition, it follows that $\eta_i|_{F_i} \equiv \mathcal{O}_{F_i}$. Using Lemma 4.13, we have $\eta|_{F_1} \equiv (K_S - (E_1 + E'_1))|_{F_1} \equiv K_{F_1} - (E_1 + E'_1)|_{F_1} \equiv (F_2 - (E_1 + E'_1))|_{F_1} \equiv (2(E_1 + E'_3) - (E_1 + E'_1))|_{F_1} \equiv (E_1 - E'_1)|_{F_1}$. Notice that $\eta_2|_{F_1} \equiv \eta_3|_{F_1} \equiv (E_1 - E'_1)|_{F_1}$ and that η_i is torsion of order 2, to prove the corollary, it suffices to show that $\eta|_{F_i} \neq \mathcal{O}_{F_i}$. It is true because otherwise we have $h^0(\mathcal{O}_{F_i}(K_{F_i} + \eta)) = h^0(\mathcal{O}_{F_i}(K_{F_i})) = 3$ which contradicts Lemma 4.14 *ii)*. \square

5. PROOF OF THEOREM 1.1

In this section, we will prove the main results. All the assumptions and notations are as in 4.1 and 4.12. Following the approach in [MP1], we give 3 involutions on S by considering its fibrations. Let's begin with a lemma.

Lemma 5.1. *Let $u : S \rightarrow \mathbb{P}^1$ be a fibration such that E_4 is contained in one fiber. Then u is induced by one of the pencils $|F_i|, i = 1, 2, 3$.*

Proof. Arguing as in the proof of Lemma 5.7 in [MP1], we get the lemma. \square

Remark 5.2. *Note that in the following proof, Step 2,4,5,6 correspond to Step 1,2,3,4 in the proof of the main result in [MP1], and the corresponding argument is nearly the same except that in Step 2. For the readers' convenience, we give all the details.*

Proof of Theorem 1.1. Let $\pi_i : Y_i \rightarrow S$ be the double cover branched over E_4 given by the relation $2(-\eta + \eta_i) = E_4$. By Lemma 4.14 *iii)*, we have $q(Y_i) = h^1(S, \eta - \eta_i) = 1$. Denote by $\alpha_i : Y_i \rightarrow B_i$ the Albanese pencil where B_i is an elliptic curve. Corollary 4.15 implies that $\eta_i \neq \eta_j$ if $i \neq j$, so π_i is different from π_j . By Proposition 2.1, there exists a fibration $h_i : S \rightarrow \mathbb{P}^1$ and a double cover $\pi'_i : B \rightarrow \mathbb{P}^1$ such that $\pi'_i \circ \alpha_i = h_i \circ \pi_i$. Since $\pi_i^{-1}(E_4)$ is a rational curve, it must be contained in one fiber of α_i , therefore E_4 is contained in one fiber of h_i . By Lemma 5.1, we can find some $s_i \in \{1, 2, 3\}$ such that $h_i = u_{s_i}$. Then we get the

following commutative diagram:

$$\begin{array}{ccc} Y_i & \xrightarrow{\pi_i} & S \\ \alpha_i \downarrow & & u_{s_i} \downarrow \\ B_i & \xrightarrow{\pi'_i} & \mathbb{P}^1 \end{array}$$

By Corollary 4.15, we have $-\eta + \eta_i|_{F_i} \neq \mathcal{O}_{F_i}$, so a general curve in $\pi_i^*|F_i|$ is connected, hence $s_i \neq i$. Since $\pi'_i : B_i \rightarrow \mathbb{P}^1$ is branched over four points and $\pi_i : Y_i \rightarrow S$ is branched over E_4 , by Proposition 2.3, we conclude that the fibration $u_{s_i} : S \rightarrow \mathbb{P}^1$ has at least three double fibers. As the proof is long, we break the proof into 6 steps.

Step 1: *The fibration $u_i : S \rightarrow \mathbb{P}^1, i = 1, 2, 3$ has at most three double fibers.*

We show that $u_3 : S \rightarrow \mathbb{P}^1$ has at most three double fibers. By the analysis above, we can assume $u_{s_3} = u_1$, so u_1 has another double fiber $2N$ aside from $2(E_2 + E'_3)$ and $2(E_3 + E'_2)$. Since the curve $\phi(N)$ is irreducible and $NE_1 = 1$, by Proposition 4.3, we conclude that N is reduced and irreducible. Moreover ϕ is ramified along N since the curve in the pencil $|f_1|$ supported on $\phi(N)$ is reduced. Now to the contrary, suppose that u_3 has two additional double fibers $2M_1, 2M_2$ aside from $2(E_1 + E'_2)$ and $2(E_2 + E'_1)$. Similarly we can see that both M_1 and M_2 are reduced and irreducible, and ϕ is ramified along M_1 and M_2 . Let R be the ramification divisor of $\phi : S \rightarrow \Sigma$. By the Hurwitz-formula $K_S \equiv \phi^*(K_\Sigma) + R$, we get $R \equiv 3K_S$. Put $R_0 = \sum_{i=1}^3 (E_i + E'_i) + G_1 + G_2 + N + M_1 + M_2$. Immediately it follows that $R_0 \leq R$ and $2R_0 \equiv \phi^*(8l - 3e_1 - 2e_2 - 3e_3 - 2e_4)$, hence $(R - R_0)G_2 = -1$ and then $G_2 \leq (R - R_0)$. However, since $\phi : S \rightarrow \Sigma$ is ramified along G_2 with branching order 2 and $G_2 \leq R_0$, G_2 cannot be a component of $R - R_0$.

By a similar argument as above or as in the proof of Lemma 5.8 in [MP1], one can show that u_1, u_2 each has at most 3 double fibers.

Step 2: *$(s_1 s_2 s_3)$ is a cyclic permutation.*

Since $s_i \neq i$, it suffices to show that $s_i \neq s_j$ if $i \neq j$.

We show that $s_1 \neq s_2$. Otherwise we have $s_1 = s_2 = 3$, and $\alpha_1 : Y_1 \rightarrow B_1$ (resp. $\alpha_2 : Y_2 \rightarrow B_2$) arises in the Stein factorization of $u_3 \circ \pi_1$ (resp. $u_3 \circ \pi_2$), i.e., the following commutative diagram holds.

$$\begin{array}{ccccc} Y_1 & \xrightarrow{\pi_1} & S & \xleftarrow{\pi_2} & Y_2 \\ \alpha_1 \downarrow & & u_3 \downarrow & & \alpha_2 \downarrow \\ B_1 & \xrightarrow{\pi'_1} & \mathbb{P}^1 & \xleftarrow{\pi'_2} & B_2 \end{array}$$

Note that for $i = 1, 2$, Y_i coincides with the normalization of the fiber product $B_i \times_{\mathbb{P}^1} S$ since π_i factors through the natural projection $B_i \times_{\mathbb{P}^1} S \rightarrow S$ which is also of degree 2, so π'_1 is different from π'_2 . Denote by $P_1, P_2, P_3, P_4 = u_3(E_4)$ the branch points of π'_1 . There exists a branch point P_5 of π'_2 which is not branched over by π'_1 . Then we find that the fibers over the points $P_i, i = 1, 2, 3, 5$ of u_3 are double fibers. This contradicts Step 1.

In the following we assume $s_1 = 2, s_2 = 3, s_3 = 1$. Furthermore we conclude that for $i = 1, 2, 3$, the fibration u_i has exactly 3 double fibers.

Step 3: *ϕ^*g_3 is not reduced.*

With the assumption above, we get the following commutative diagram:

$$\begin{array}{ccc} Y_2 & \xrightarrow{\pi_2} & S \\ \alpha_2 \downarrow & & u_3 \downarrow \\ B_2 & \xrightarrow{\pi'_2} & \mathbb{P}^1 \end{array}$$

Let $W = B_2 \times_{\mathbb{P}^1} S$, and denote by $p : W \rightarrow S$ the natural projection which is a double cover. Assume by contradiction that $G_3 = \phi^* g_3$ is reduced. Since $\pi'_2 : B_2 \rightarrow \mathbb{P}^1$ is branched over the point $u_3(G_3) = u_3(E_4)$, the map p is branched over G_3 , and W is normal along $p^{-1}G_3$. Since Y_2 is the normalization of W , the map $\pi_2 : Y_2 \rightarrow S$ is also branched over G_3 . This contradicts the fact that the branch locus of π_2 is E_4 .

Step 4: *A general element $F_i \in |F_i|$ is hyperelliptic for $i = 1, 2, 3$.*

Let $F_2 \in |F_2|$ be a general fiber of $u_2 : S \rightarrow \mathbb{P}^1$. We show that F_2 is hyperelliptic. Since $\pi_1^* F_2$ (resp. $\pi_1^* F_3$) is disconnected, we may write $\pi_1^* F_2 = \hat{F}_2 + \hat{F}_2'$ (resp. $\pi_1^* F_3 = \hat{F}_3 + \hat{F}_3'$) where the two components are disjoint. By $F_2 F_3 = 4$, we have $\hat{F}_2 \hat{F}_3 = 2$. Let $p \circ h : Y_1 \rightarrow C \rightarrow \mathbb{P}^1$ be the Stein factorization of $u_3 \circ \pi_1 : Y_1 \rightarrow \mathbb{P}^1$. Since \hat{F}_3 is one fiber of $h : Y_1 \rightarrow C$, so $\hat{F}_2 \hat{F}_3 = 2$ implies the restriction map $h|_{\hat{F}_2} : \hat{F}_2 \rightarrow C$ is a 2-to-1 map. In addition since $h : Y_1 \rightarrow C$ is not the Albanese map and $q(Y_1) = 1$, the curve C must be rational. Therefore \hat{F}_2 is hyperelliptic, and so is F_2 .

Step 5: *$\phi : S \rightarrow \Sigma$ is a Galois cover, and the Galois group $G \cong \Gamma = Z_2 \times Z_2$.*

For $i = 1, 2, 3$, we denote by γ_i the involution on S that induces the involution on the general fiber F_i . Since S is minimal, the maps γ_i are regular maps; and they belong to Γ by Proposition 4.13. So we only need to show that $\gamma_i \neq \gamma_j$ if $i \neq j$. Now we show that $\gamma_2 \neq \gamma_3$. Consider the lifted involution $\hat{\gamma}_2 : Y_1 \rightarrow Y_1$. By the construction in Step 4, the restriction of α_1 identifies $\hat{F}_3/\hat{\gamma}_2$ with B_1 . So we get $g(\hat{F}_3/\hat{\gamma}_2) = 1$. Then we can see $\gamma_2 \neq \gamma_3$ since $\hat{F}_3/\hat{\gamma}_3 \cong \mathbb{P}^1$.

Step 6: *S is a Burniat surface.*

The fibration $u_i : S \rightarrow \mathbb{P}^1, i = 1, 2, 3$ has exactly three double fibers. We denote by $2M_i$ the double fiber different from $2(E_k + E'_j)$ and $2(E'_k + E_j)$ of u_i where $\{i, j, k\} = \{1, 2, 3\}$. There exists a fiber m_i of $v_i : \Sigma \rightarrow \mathbb{P}^1$ such that $2M_i = \phi^* m_i$. Then m_i is a component of the branch locus. Denote by D be the branch divisor of ϕ . Then we have

$$-3K_\Sigma \equiv D \geq \sum_{i=1}^{i=3} (e_i + e'_i + g_i + m_i) \equiv -3K_\Sigma,$$

thus $D = \sum_{i=1}^{i=3} (e_i + e'_i + g_i + m_i)$. If we denote by D_i the image of the divisorial part of the fixed locus of γ_i , then $D = D_1 + D_2 + D_3$. By step 4, we get $D_1 = g_2 + m_2 + e_1 + e'_1; D_2 = g_3 + m_3 + e_2 + e'_2; D_3 = g_1 + m_1 + e_3 + e'_3$. Finally the theorem is proved. \square

6. PROOF OF PROPOSITION 4.9

Notations are as in Notation 4.1. We will prove the proposition by contradiction. To the contrary, in this section, by Lemma 4.7, we assume

$$\phi^* e_i = 2E_i, i = 1, 2, 3, 4; \phi^* e'_j = 2E'_j, \phi^* g_j = 2G_j, j = 1, 2, 3$$

Let R be the ramification divisor. By the formula $K_S \equiv \phi^*(K_\Sigma) + R$, we have $R \equiv 3K_S$, hence $2R \equiv \phi^*(-3K_\Sigma)$. Put $R_0 = R - (\sum_{i=1}^{i=3}(E_i + E'_i + G_i) + E_4)$. Then R_0 is effective, and none of its irreducible components are in the set $\{E_i, E'_j, G_j\}_{i=1,2,3,4;j=1,2,3}$. Notice that $2(\sum_{i=1}^{i=3}(E_i + E'_i + G_i) + E_4) \equiv \phi^*(6l - 2e_1 - 2e_2 - 2e_3 - 2e_4)$, we get

$$(6.1) \quad 2R_0 \equiv \phi^*(3l - e_1 - e_2 - e_3 - e_4) \equiv \phi^*(-K_\Sigma).$$

Since $|\phi^*(-K_\Sigma)| = \phi^*| - K_\Sigma|$, we can assume $2R_0 = \phi^*r_0$ where r_0 is an effective divisor on Σ .

Lemma 6.1. *Let $D \subset S$ be an effective divisor. If $2D \equiv \phi^*(3l - e_1 - e_2 - e_3 - e_4)$, then every irreducible component of D is contained in R .*

Proof. To the contrary, suppose that D contains a reduced and irreducible component D_1 such that ϕ is not ramified along it. Denote by d_1 be the reduced divisor supported on $\phi(D_1)$. Thus $\text{ord}_{D_1}(\phi^*d_1) = 1$. By $2D \in |2K_S| = \phi^*| - K_\Sigma|$, we find a divisor $d \in | - K_\Sigma|$ such that $2D = \phi^*d$. Note that d_1 is contained in d , and since $\text{ord}_{D_1}(\phi^*d) = \text{ord}_{D_1}(2D) \geq 2$, we have $d - 2d_1$ is effective. Then we conclude that d_1 must be a (-1) -curve. So a contradiction follows from the assumptions at the beginning of the section. \square

Lemma 6.2. *The bicanonical map $\phi : S \rightarrow \Sigma$ is ramified along R_0 with branching order 2, and either*

- i) R_0 is irreducible or
- ii) R_0 can be written as $R_0 = B + F$ where both B and F are irreducible.

Proof. Note that by Formula 6.1, we have $R_0E_i = R_0E'_j = R_0G_j = 1$ for $i = 1, 2, 3, 4; j = 1, 2, 3$. To show that $\phi : S \rightarrow \Sigma$ is ramified along R_0 with branching order 2, it suffices to show that R_0 is reduced. Let D be an irreducible component of R_0 . Write $R_0 = mD + R'$ where $m = \text{ord}_D(R_0)$. Since the elements in $\{E_i, E'_j, G_j\}_{i=1,2,3,4;j=1,2,3}$ are integral and span $H^2(S, \mathbb{Q})$, we can find a curve C in $\{E_i, E'_j, G_j\}_{i=1,2,3,4;j=1,2,3}$ such that DC is a non-zero integer. Note that since C is not a component of R_0 , we have $DC > 0$ and $R'C \geq 0$. Then $R_0C = 1$ implies that $m = 1$ and $DC = 1$, therefore R_0 is reduced. If we denote by d and c the reduced divisors supported on $\phi(D)$ and $\phi(C)$ respectively, then we have $d \leq r_0$ and $\phi^*c = 2C$. Write $\phi^*d = 2D + D'$. By $0 < d \cdot c \leq r_0 \cdot c = 1$, we deduce that $d \cdot c = 1$, and then $\phi^*d \cdot \phi^*c = 4 = (2D)(2C)$. This implies that $\phi^{-1}(c \cap d) = C \cap D$ and $D'C = 0$. Note that the divisor D' , if it is not zero, then it is mapped onto d_1 , hence it contains the point $C \cap D$. Then we can see $D' = 0$, because otherwise $D'C > 0$. Therefore we have $\phi^*d = 2D$.

If R_0 is reducible, write $R_0 = D_1 + \dots + D_k$ where D_1, \dots, D_k are distinct reduced irreducible divisors. Denote by $d_i, i = 1, 2, \dots, k$ the reduced divisor on Σ with support $\phi(D_i)$. Then we have $2D_i = \phi^*d_i$. Since d_i is not in the set $\{e_i, e'_j, g_j\}_{i=1,2,3,4;j=1,2,3}$ and $r_0 = \sum_i d_i \equiv 3l - e_1 - e_2 - e_3 - e_4$, using the method used in the proof of Lemma 4.5, we deduce that $k = 2$, and either

- i) $d_1 \equiv l, d_2 \equiv 2l - e_1 - e_2 - e_3 - e_4$ or
- ii) $d_1 \equiv l - e_i, d_2 \equiv 2l - e_j - e_k - e_l$ for some $i \in \{1, 2, 3, 4\}$ and $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

\square

By the proof of the lemma above, if R_0 is reducible, then there exist two reduced and irreducible divisors b and f on Σ such that $\phi^*b = 2B$, $\phi^*f = 2F$ and $r_0 = b + f$. Recall that in Section 3, we introduced an automorphism τ of Σ such that $\tau^*l \equiv 2l - e_1 - e_2 - e_3$ and $\tau^*(2l - e_1 - e_2 - e_3 - e_4) \equiv l - e_4$. By symmetry, from now on, if R_0 is reducible, then we assume $b \equiv 2l - e_1 - e_2 - e_3$ and $f \equiv l - e_4$.

Lemma 6.3. *With the assumptions above, we have*

$$K_S \equiv E_4 + G_1 + G_3 + E_1 + E'_2 + E_3 - E_2$$

Proof. Put $R_1 = E_4 + G_1 + G_3 + E_1 + E'_2 + E_3$. Then we get

$$2(R_1 - E_2) \equiv 2K_S.$$

Suppose that the lemma is not true, then the line bundle $K_S - (R_1 - E_2)$ is torsion of order 2. Immediately we have $h^0(S, K_S - (R_1 - E_2)) = 0$ and $\chi(S, K_S - (R_1 - E_2)) = 1$, hence $h^2(S, K_S - (R_1 - E_2)) = h^0(S, R_1 - E_2) \geq 1$. So there exists an effective divisor D such that $D \equiv R_1 - E_2$ and a divisor d on Σ such that $\phi^*d = 2D$. By Lemma 6.1, every irreducible component of D is contained in R . We will prove that $R_1 - E_2$ cannot be effective by considering all the possibilities of D . Let's begin with a claim.

Claim 6.4. *D and R_1 do not have two common irreducible components C_1, C_2 such that $C_1C_2 = 1$.*

Proof of Claim 6.4. If there exist two such curves C_1, C_2 , then we can find $c_1, c_2 \in \{e_4, g_1, g_3, e_1, e_3, e'_2\}$ such that $\phi^*c_1 = 2C_1$ and $\phi^*c_2 = 2C_2$. Immediately we have $c_1c_2 = 1$ and $c_1 + c_2 \equiv l - e_i$ for some $i \in \{1, 3, 4\}$ fixed, thus $2(R_1 - C_1 - C_2) \equiv 2l - e_j - e_k$ where $\{i, j, k\} = \{1, 3, 4\}$. Notice that $-K_\Sigma - (2l - e_j - e_k) \equiv l - e_i - e_2$ is effective, thus applying Lemma 4.6, we obtain that $h^0(S, R_1 - C_1 - C_2) = 1$. However, this contradicts the fact that $R_1 - C_1 - C_2$ and $D + E_2 - C_1 - C_2$ are two different linearly equivalent effective divisors. Then the claim follows. \square

By the formula $K_S \equiv \phi^*(K_\Sigma) + R$, we obtain

$$(6.2) \quad \begin{aligned} K_S &\equiv -2(R_1 - E_2) + R \equiv R - R_1 + E_2 - (R_1 - E_2) \\ &\equiv R_0 + 2E_2 + G_2 + E'_1 + E'_3 - D \end{aligned}$$

So $R_0 + 2E_2 + G_2 + E'_1 + E'_3 - D$ is not effective since $p_g(S) = 0$.

According to whether D has an irreducible component contained in R_0 or R_1 , the possibilities for D fall into the two cases:

Case 1: No irreducible component of D is contained in R_0 , and at least one irreducible component of D is contained in R_1 .

Case 2: At least one irreducible component of D is contained in R_0 , or no irreducible component of D is contained in R_1 .

Case 1: The following argument is not involved with any information about the divisor R_0 . Note that $\phi(R_1)$ contains exactly the six (-1) -curves not intersecting e_2 . So by Fact 3.5 *ii*), up to an automorphism of Σ fixing e_2 , we may assume E_4 is a common component of D and R_1 . Write $D = E_4 + D'$, then $D'E_4 = 2$. Notice that G_1, G_2, G_3 are the only elements in $\{E_i, E'_j, G_j\}_{i=1,2,3,4; j=1,2,3}$ that intersect E_4 , so at least one of the G_i 's is a component of D' . By Claim 6.4, neither G_1 nor G_3 is a component of D , so the only possibility is $D' = 2G_2 + E'_2 + E_2$, i.e., $D = E_4 + 2G_2 + E'_2 + E_2$. We get $E_4 + G_1 + G_3 + E_1 + E'_2 + E_3 \equiv 2G_2 + E'_2 + E_4 + 2E_2$, equivalently $G_1 + G_3 + E_1 + E_3 \equiv 2G_2 + 2E_2$. Then we have $E_1 + G_1 \mid_{E_1 + G_1} \equiv$

$G_1 + G_3 + E_1 + E_3 |_{E_1+G_1} \equiv 2(E_2 + G_2) |_{E_1+G_1}$ is trivial. By [BPV] Lemma 8.3 chap.III, this contradicts the fact that $2(G_1 + E_1)$ is a double fiber of u_4 .

Case 2: First we list all the possibilities for D . We consider the divisor d . Note that every irreducible component of d is either a (-1) -curve or is contained in r_0 . We break the possibilities into following three cases:

- $d = r_0$
- $d \neq r_0$, but d and r_0 have at least one common component.
- d and r_0 have no common components

If we are in the first case, then we get that

$$I : d = r_0, D = R_0.$$

If d is as in the second case, then r_0 is reducible. By assumption, $r_0 = b + f$ where b and f are reduced and irreducible divisors such that $b \equiv 2l - e_1 - e_2 - e_3$ and $f \equiv l - e_4$. Either b or f is contained in d . First if $b \leq d$, then the divisor $d - b \equiv l - e_4$, and its irreducible components are all (-1) -curves. The pencil $|l - e_4|$ has exactly 3 reducible elements $g_i + e_i, i = 1, 2, 3$. So correspondingly, we get the possibilities:

$$II(i) : d = b + g_i + e_i, D = B + G_i + E_i, i = 1, 2, 3.$$

If $f \leq d$, then similarly the divisor $d - f \equiv 2l - e_1 - e_2 - e_3$, and its irreducible components are all (-1) -curves. Considering the coefficient of l appearing in the equation $d - f \equiv 2l - e_1 - e_2 - e_3$, we conclude that with multiplicity considered, the divisor $d - f$ contains exactly two irreducible non-exceptional (-1) -curves, hence the divisor $d - f$ contains at least one exceptional (-1) -curve. So if e_4 is contained in $d - f$, then we obtain the possibilities

$$III(i) : d = f + g_i + e'_i + e_4, D = F + G_i + E'_i + E_4, i = 1, 2, 3.$$

And for $i = 1, 2, 3$, if e_i is contained in $d - f$, then we get the possibilities

$$IV(i) : d = f + e_i + e'_j + e'_k, D = F + E_i + E'_j + E'_k \text{ where } \{i, j, k\} = \{1, 2, 3\}.$$

If d falls into the third case, equivalently, D and R_0 have no common components, then the assumptions of Case 2 implies that D and R_1 have no common components either. Thus the components of d belong to the set $\{e_2, e'_1, e'_3, g_2\}$. Then we get following possibilities:

$$V : d = g_2 + e'_1 + e'_3 + 2e_2, D = G_2 + E'_1 + E'_3 + 2E_2.$$

Since $R_0 + 2E_2 + G_2 + E'_1 + E'_3 - D$ is not effective, the cases $I, II(2), IV(2)$ and V cannot happen. And by Claim 6.4, we exclude the cases $II(1, 3), III(1, 3), IV(1, 3)$. For the remaining case $III(2)$, i.e., $D = F + G_2 + E'_2 + E_4$, we have $E_4 + G_1 + G_2 + E_1 + E'_2 + E_3 - E_2 \equiv F + G_2 + E'_2 + E_4$, equivalently $G_1 + G_3 + E_1 + E_3 \equiv F + G_2 + E_2$. Restricting the two divisors to the double fiber $2(G_1 + E_1)$ of u_4 , by similar argument as in Case 1, we get a contradiction.

In conclusion the lemma is true. \square

Proof of Proposition 4.9. By the same argument as in the proof of Lemma 6.3, we can prove

$$K_S \equiv E_4 + G_2 + G_3 + E_2 + E'_1 + E_3 - E_1.$$

Consequently we get

$$E_4 + G_2 + G_3 + E_2 + E'_1 + E_3 - E_1 \equiv E_4 + G_1 + G_3 + E_1 + E'_2 + E_3 - E_2,$$

equivalently $G_2 + 2E_2 + E'_1 \equiv G_1 + 2E_1 + E'_2$. This implies $h^0(S, G_2 + 2E_2 + E'_1) \geq 2$. Then a contradiction follows from Lemma 4.6 since $2(G_2 + 2E_2 + E'_1) \equiv \phi^*(2l -$

$e_4 - e_3$) and $-K_\Sigma - (2l - e_4 - e_3) \equiv l - e_1 - e_2 \equiv e'_3$. Finally we finish the proof of the proposition. \square

7. THE CASE WHEN THE BICANONICAL IMAGE IS SINGULAR

In this section, we consider the case when the bicanonical image is singular. First we reduce the bicanonical image to 6 cases, then study them case by case.

7.1. Preparations. Let S be a minimal surface of general type with $K_S^2 = 5$ and $p_g = q = 0$; let $\phi : S \rightarrow \Sigma \subset \mathbb{P}^5$ be the bicanonical map which is a morphism by [Re]. We assume the degree of ϕ is 4, then Σ is a linearly normal surface of degree 5 in \mathbb{P}^5 . By [Na], Σ is the image of $\psi : \hat{P} \rightarrow \mathbb{P}^5$ where \hat{P} is the blow-up of \mathbb{P}^2 at four points P_1, P_2, P_3, P_4 such that $|-K_{\hat{P}}|$ has no fixed components and ψ is given by the linear system $|-K_{\hat{P}}|$. The P_i 's can be infinitely near, but it is impossible that two of them are distinct and both infinitely near to another one. We denote by e_i the exceptional divisor over P_i . In the previous sections, we have studied the case when Σ is smooth, from now on, we assume Σ is singular. First we list all the the possibilities below.

- \hat{P}_1 : P_1, P_2, P_3 are distinct and lie on a line, and P_4 is distinct from them;
- \hat{P}_2 : P_1, P_2, P_3 lie on a line, P_3 is infinitely near to P_2 , and P_1, P_4 are distinct from them and each other;
- \hat{P}_3 : P_1, P_2, P_3 lie on a line, P_2 is infinitely near to P_1 , P_3 is infinitely near to P_2 , P_4 is distinct from them;
- \hat{P}_4 : P_1, P_2, P_3 are distinct and lie on a line, and P_4 is infinitely near to P_3 ;
- \hat{P}_5 : P_1, P_2, P_3 lie on a line, P_3 is infinitely near to P_2 , P_4 is infinitely near to P_3 , P_1 is distinct from them;
- \hat{P}_6 : P_1, P_2, P_3 lie on a line, P_2 is infinitely near to P_1 , P_3 is infinitely near to P_2 , and P_4 is infinitely near to P_3 ;
- \hat{P}'_3 : P_1, P_2, P_3 lie on a line, P_1 and P_2 are distinct, P_3 is infinitely near to P_2 , P_4 is infinitely near to P_1 ;
- \hat{P}'_1 : no three points lie on a line, P_1, P_2, P_3 are distinct, and P_4 is infinitely near to P_3 ;
- \hat{P}'_2 : no three points lie on a line, P_1 and P_3 are distinct, P_4 is infinitely near to P_3 , and P_2 is infinitely near to P_1 ;
- \hat{P}'_4 : no three points lie on a line, P_3 is infinitely near to P_2 , P_4 is infinitely near to P_3 , and P_1 is distinct from them;
- \hat{P}'_5 : no three points lie on a line, P_2 is infinitely near to P_1 , P_3 is infinitely near to P_2 , and P_4 is infinitely near to P_3 .

Claim 7.1. *Let the notations be as above. Then for $i = 1, 2, 3, 4, 5$, the surface \hat{P}_i is isomorphic to \hat{P}'_i .*

Proof. We identify \hat{P}_i and \hat{P}'_i by considering their blowing down map to \mathbb{P}^2 . For simplicity, we focus on the cases $i = 1, 2$.

For the surface \hat{P}'_1 , we denote by e'_1, e'_2, e'_3 the strict transforms of the lines through the points P_2 and P_3 , P_1 and P_3 , P_1 and P_2 respectively, then they are all (-1) -curves. We get a plane after contracting the curves e'_1, e'_2, e'_3, e_4 . Note that e_3 is mapped to a line, and e'_1, e'_2, e_4 are mapped to three points on the line. So \hat{P}'_1 can be obtained by blowing up of \mathbb{P}^2 at 4 points with three of which lying on a line. Then we can identify \hat{P}'_1 with \hat{P}_1 .

For the surface \hat{P}'_2 , we denote by e'_3, e'_4 the strict transform of the lines through P_1 and P_3, P_3 and P_4 respectively. Contracting the curves e_2, e_3, e_4 , we get a plane. Similarly we identify \hat{P}'_2 with \hat{P}_2 .

For the other cases, the proof is similar. \square

So in the following, we only need to consider the cases $\hat{P} = \hat{P}_i, i = 1, 2, \dots, 6$.

Notation 7.2. Let S, \hat{P}, Σ be as above, let $\bar{\phi} \circ \eta : S \rightarrow \bar{S} \rightarrow \Sigma$ be the Stein factorization of $\phi : S \rightarrow \Sigma$. So \bar{S} is the canonical model of S . For a divisor $D \subset S$ and $e \subset \hat{P}$, denote by \bar{D} the divisor $\eta_* D$ on \bar{S} and by \bar{e} the divisor $\psi_* e$ on Σ .

Lemma 7.3. Let $e \subset \hat{P}$ be a (-1) -curve. Denote by E the strict transform of \bar{e} with respect to ϕ . Then $K_S E = 2$ and either:

- i) E is reduced and irreducible; or
- ii) E is non-reduced and $E = 2E'$; or
- iii) $E = A + B$ where both A and B are irreducible reduced divisors such that $K_S A = K_S B = 1$.

Moreover if E is reduced, then $E^2 \geq -6$.

Proof. Immediately $K_S E = 2$ follows from $-K_\Sigma \bar{e} = 1$. If A is a reduced and irreducible component of E , then $K_S A > 0$. If E is not reduced, then $E = 2E'$ with E' irreducible and reduced. If E is reducible, then $E = A + B$ where A and B are reduced and irreducible divisors such that $K_S A = K_S B = 1$, moreover we have $A^2 \geq -3$ and $B^2 \geq -3$. And it is easy to see that $E^2 \geq -6$ if E is reduced. \square

Proposition 7.4. Let \hat{P} be as in the previous section, and let $\hat{g} : \hat{P} \rightarrow \mathbb{P}^1$ be a fibration induced by the pencil $|f|$ such that every (-2) -curve is contained in one fiber. Then a general fiber $f \in |f|$ is a rational curve. Precisely we have:

- i) if $\hat{P} = \hat{P}_1$, then $|f| = |l - e_i|$ for some $i = 1, 2, 3$;
- ii) if $\hat{P} = \hat{P}_2$, then $|f| = |l - e_1|$;
- iii) if $\hat{P} = \hat{P}_4$, then $|f| = |l - e_i|$ for some $i = 1, 2$;
- iv) if $\hat{P} = \hat{P}_5$, then $|f| = |l - e_1|$;
- v) if $\hat{P} = \hat{P}_3$ or $\hat{P} = \hat{P}_6$, there does not exist such a fibration.

Proof. For a general element $f \in |f|$, by $-K_{\hat{P}} f > 0$, we get $(K_{\hat{P}} + f)f = 2g(f) - 2 < 0$, thus it follows that:

- 1) f is a rational curve such that $f^2 = 0$ and $-K_{\hat{P}} f = 2$;
- 2) for a (-2) -curve C , $fC = 0$;
- 3) $|f|$ has no fixed part, thus for every effective divisor D , $fD \geq 0$.

Note that the Picard group of \hat{P}_i is a free group generated by l, e_1, e_2, e_3, e_4 , so we can set $f \equiv al - b_1 e_1 - b_2 e_2 - b_3 e_3 - b_4 e_4$. By 1), we obtain $(a, b_1, b_2, b_3, b_4) = 1$; and by 3), we get $a > 0, b_i \geq 0$ for $i = 1, 2, 3, 4$.

We show the proposition for case ii). In this case, the equations $fc = 0, fe_2 = 0$ and $-K_{\hat{P}} f = 2$ yield $a - b_1 - b_3 = 0, 2b_2 - b_3 = 0$ and $3a - b_1 - b_3 - b_4 = 2$. In turn we have $a = b_1 + 2b_2, b_3 = 2b_2$ and $b_4 = 2b_1 + 4b_2 - 2$, thus $f \equiv (b_1 + 2b_2)l - b_1 e_1 - b_2 e_2 - 2b_2 e_3 - b_4 e_4$.

Denote by e'_4 the strict transform of the line through the point P_1, P_4 , then $e'_4 \equiv l - e_1 - e_4$. By $fe'_4 \geq 0$, we obtain

$$a = b_1 + 2b_2 \geq b_1 + b_4 \Rightarrow b_4 \leq 2b_2 \Rightarrow b_4 = 2b_1 + 4b_2 - 2 \leq 2b_2 \Rightarrow b_2 \leq 1 - b_1$$

We have either $b_1 = 1, b_2 = b_3 = b_4 = 0$ or $b_1 = 0, b_2 = 1$. The latter is impossible since then $f \equiv 2l - e_2 - 2e_3 - 2e_4$, which contradicts $f^2 = 0$.

The other cases can be proven by similar calculations. \square

Remark 7.5. *If the cardinalities of the (-2) -curves on S and \hat{P} are equal, then \bar{S} and Σ have equal Picard numbers. If $g : S \rightarrow \mathbb{P}^1$ is a fibration such that every (-2) -curve on S is contained in one fiber, then it factorizes a fibration of $\bar{g} : \bar{S} \rightarrow \mathbb{P}^1$. So by Lemma 2.5, there exists a fibration $\bar{u} : \Sigma \rightarrow \mathbb{P}^1$ of Σ and a fibration $u : \hat{P} \rightarrow \mathbb{P}^1$ of \hat{P} such that $u = \bar{u} \circ \psi$ and $g = \bar{u} \circ \phi$.*

7.2. Analyze all the cases. All the notations are as in Section 7.1. And if P_1, P_2, P_3 lie on one line, then we denote by c the strict transform of the line.

7.2.1. *The case $\hat{P} = \hat{P}_1$.*

Notation 7.6. *Let $Q = \psi(c)$ be the A_1 -singularity on Σ . Denote by l_i the strict transform of a general line through P_i and by l_{i4} the strict transform of the line through P_i and P_4 for $i = 1, 2, 3$, then $l_i \equiv l - e_i$, $c \equiv l - e_1 - e_2 - e_3$ and $l_{i4} \equiv l - e_i - e_4$. Denote by L_i (resp. $E_1, E_2, E_3, E_4, L_{i4}$) the strict transform of \bar{l}_i (resp. $\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4, l_{i4}$) with respect to ϕ .*

With these notations, it follows that

$$(7.1) \quad \begin{aligned} -K_{\hat{P}} &\equiv 3l - e_1 - e_2 - e_3 - e_4 \\ &\equiv 2l_4 + c + e_4 \\ &\equiv l_{i4} + 2e_i + l_i + c \end{aligned}$$

consequently

$$(7.2) \quad \begin{aligned} -K_{\Sigma} &\equiv 2\bar{l}_4 + \bar{e}_4 \\ &\equiv \bar{l}_{i4} + 2\bar{e}_i + \bar{l}_i \end{aligned}$$

Note that there exists at most one (-2) -curve on S since $\rho(S) - \rho(\Sigma) = 1$.

Lemma 7.7. $E_4 = 2E'_4$ for some divisor E'_4 on S .

Proof. By $-K_{\Sigma} \equiv 2\bar{l}_4 + \bar{e}_4$, we may write

$$2K_S \equiv 2L_4 + E_4 + Z$$

where Z is zero or supported on a (-2) -curve. We write $Z = 2Z' + Z''$ where Z'' is reduced.

Suppose otherwise that E_4 is reduced. We have $2(K_S - L_4 - Z') \equiv E_4 + Z''$, and it gives a double cover $\pi : Y \rightarrow S$ branched along E_4 and Z'' . Observing that $E_4 + Z'' = \phi^*e_4 + Z_1$ where Z_1 is zero or supported on some (-2) -curves, we conclude that $(E_4 + Z'')^2 = (\phi^*e_4)^2 + Z_1^2 \leq -4$ and equality holds if and only if $E_4 + Z'' = \phi^*e_4$. Note that if ϕ^*e_4 contains some (-2) -curves, i.e. $E_4 \neq \phi^*e_4$, then $E_4^2 < -4$, hence $E_4^2 = -6$ by Lemma 7.3. Using Formula 2.1 and Lemma 2.6, we

calculate the invariants of Y :

$$\begin{aligned}
\chi(Y) &= 2 + \frac{1 + \frac{(E_4 + Z'')^2}{4}}{2} \leq 2 \\
p_g(Y) &= h^0(S, 2K_S - L_4 - Z') \\
(7.3) \quad &= h^0(S, \phi^*(\bar{l}_4 + \bar{e}_4)) \\
&\geq h^0(\mathcal{O}_\Sigma(\bar{l}_4 + \bar{e}_4)) \geq h^0(\mathcal{O}_{\hat{P}}(\psi^*(\bar{l}_4 + \bar{e}_4))) \geq h^0(\hat{P}, l_4 + e_4) = 3 \\
q(Y) &\geq 2
\end{aligned}$$

Note that $(E_4 + Z'')^2 \geq E_4^2 + Z''^2 \geq -8$ since $Z''^2 \geq -2$. To guarantee that $\chi(Y)$ is an integer, we have $(E_4 + Z'')^2 = -4$ and thus $E_4 + Z'' = \phi^*e_4$. In turn we get that either

- i) $\phi^*e_4 = E_4$ is a smooth rational (-4) -curve; or
- ii) $E_4^2 = -6, Z''^2 = -2, E_4Z'' = 2$, thus $E_4 = A + B$ where A, B are smooth rational (-3) -curves and Z'' is composed with a (-2) -curve.

In any case, $E_4 + Z''$ has at most simple $[3]$ -points or double points which are negligible singularities. So Y has at most canonical singularities. Then we can apply Proposition 2.2 and get a contradiction since $K_Y^2 = 2(K_S + \frac{E_4 + Z''}{2})^2 \leq 12 < 16(q(Y) - 1)$. Hence E_4 is reduced, and we can write $E_4 = 2E'_4$. \square

Case 1: There's a (-2) -curve θ on S .

Step 1: Obtain a double cover and a fibration of S .

As in the proof of the lemma above, we have $2K_S \equiv 2L_4 + E_4 + Z$ and write $Z = 2Z' + Z''$ where Z'' is reduced, then get a relation

$$2(K_S - L_4 - E'_4 - Z') = Z''$$

which gives a double cover $\pi : Y \rightarrow S$. Using Formula 2.1, calculate the invariants of Y :

$$\begin{aligned}
\chi(Y) &= 2 + \frac{Z''^2}{4} \leq 2 \\
(7.4) \quad p_g(Y) &= h^0(S, 2K_S - L_4 - E'_4 - Z') = h^0(S, L_4 + E'_4 + Z' + Z'') \geq 2 \\
q(Y) &\geq 1.
\end{aligned}$$

Remark that $Z'' = 0$ to guarantee that $\chi(Y)$ is an integer. Then applying Proposition 2.1, we get a commutative diagram as follows:

$$\begin{array}{ccc}
Y & \xrightarrow{\pi} & S \\
\alpha \downarrow & & g \downarrow \\
B & \xrightarrow{\pi'} & \mathbb{P}^1
\end{array}$$

where $\alpha : Y \rightarrow B$ is the Albanese pencil and $\pi' : B \rightarrow \mathbb{P}^1$ is a double cover. Since $\pi^*\theta$ is composed with 2 (-2) -curves, θ is contained in one fiber of $g : S \rightarrow \mathbb{P}^1$. Then Remark 7.5 gives a fibration $v : \Sigma \rightarrow \mathbb{P}^1$ such that $g = v \circ \phi$. By Proposition 7.4, we may assume that g is induced by the pencil $|L_1| = \phi^*|\bar{l}_1|$.

Step 2: Analyze the ramification divisor of $\bar{\phi}$.

Now consider the ramification divisor R of the map $\bar{\phi} : \bar{S} \rightarrow \Sigma$. We have $R \equiv 3K_{\bar{S}}$. Since $\pi' : B \rightarrow \mathbb{P}^1$ is branched over 4 points, $g : S \rightarrow \mathbb{P}^1$ has at least 4 double fibers, we select 4 and denote them by $2M_1, 2M_2, 2M_3, 2M_4$. Note

that since every curve in $|\bar{l}_1|$ is reduced, $\bar{\phi}$ is ramified along $\bar{M}_i, i = 1, 2, 3, 4$, thus $\bar{M}_1 + \bar{M}_2 + \bar{M}_3 + \bar{M}_4 \leq R$. Then we have $(R - (\bar{M}_1 + \bar{M}_2 + \bar{M}_3 + \bar{M}_4))(\bar{\phi}^* e_1) = -2 = (\bar{\phi}^* e_1)^2$ which contradicts Lemma 2.8. Therefore this case does not occur.

Remark 7.8. *The process in Step 2 above will be frequently used in the following proof, and we call it ARDP, namely analyzing the ramification divisor process.*

Case 2: There's no (-2) -curve on S .

In this case, $E_i = \phi^* \bar{e}_i$ and $L_1 = \phi^* \bar{l}_1$ are all Cartier divisors. By $E_i^2 = -2$ for $i = 1, 2, 3$, applying Lemma 7.3, we conclude that the E_i 's are reduced.

Claim 7.9. $\phi^* l_{i4} = L_{i4} = 2L'_{i4}$ is non-reduced for $i = 1, 2, 3$.

Proof. To the contrary, suppose that L_{i4} is reduced. By $K_\Sigma \equiv \bar{l}_{i4} + 2\bar{e}_i + \bar{l}_i$, we have $2K_S \equiv L_{i4} + 2E_i + L_i$, thus

$$2(K_S - E_1) \equiv L_i + L_{i4}.$$

Then by the relation above, we get a double $\pi : Y \rightarrow S$. Calculating the invariants of Y , we obtain

$$\chi(Y) = 3, p_g(Y) = 4, q(Y) = 2.$$

By Proposition 2.1, we get the following commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & S \\ \alpha \downarrow & & g \downarrow \\ B & \xrightarrow{\pi'} & \mathbb{P}^1 \end{array}$$

where $\alpha : Y \rightarrow B$ is the Albanese pencil and $\pi' : B \rightarrow \mathbb{P}^1$ is a double cover. Since $L_i^2 = 0$, $g : S \rightarrow \mathbb{P}^1$ must be induced by the pencil $|L_i|$. The map $\pi' : B \rightarrow \mathbb{P}^1$ is branched along 6 points, so there exist at least 4 double fibers, and we choose four double fibers: $2M_1, 2M_2, 2M_3, 2M_4$. Then considering the ramification divisor of the map $\bar{\phi} : \bar{S} \rightarrow \Sigma$ and going process ARDP, we get a contradiction. \square

By the Claim above, we get a relation $2(K_S - L'_{i4} - E_i) \equiv L_i$, and then a double $\pi : Y \rightarrow S$. Calculating the invariants of Y , we obtain

$$\chi(Y) = 3, p_g(Y) = 3, q(Y) = 1.$$

By Proposition 2.1, we get the following commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & S \\ \alpha \downarrow & & g \downarrow \\ B & \xrightarrow{\pi'} & \mathbb{P}^1 \end{array}$$

where $\alpha : Y \rightarrow B$ is the Albanese pencil and $\pi' : B \rightarrow \mathbb{P}^1$ is a double cover. Similarly we can see $g : S \rightarrow \mathbb{P}^1$ is induced by the pencil $|L_i|$. The map $\pi' : B \rightarrow \mathbb{P}^1$ is branched at 4 points, so there exist at least 3 double fibers. Select two double fibers $2M_{i_1}, 2M_{i_2}$ different from $2(L'_{i_4} + E'_4)$. For $i_1 \neq i_2$, if M_{i_1} and M_{i_2} have common components, then the image of the common components under the bicanonical map must be contained in some fibers of the pencils $|\bar{l}_{i_1}|$ and $|\bar{l}_{i_2}|$. Note that the fiber containing $E_i, i = 1, 2, 3$ must be reduced since $\phi^* e_i = E_i$ is reduced in this case. So the image of M_{i_j} does not contain $\bar{e}_i, i = 1, 2, 3, 4$, as an easy consequence, any two of the M_{i_j} 's have no common components, thus

$R_1 = \sum_{i=1}^{i=3}(M_{i1} + M_{i2} + L'_{i4}) + E'_4 \leq R$. Put $R_1 = \sum_{i=1}^{i=3}(M_{i1} + M_{i2} + L'_{i4}) + E'_4$. It follows that $2R_1 \equiv \phi^*(9\bar{l} - 3\bar{e}_1 - 3\bar{e}_2 - 3\bar{e}_3 - 2\bar{e}_4)$ and $R_1 \leq R$. Notice that $2R \equiv 6K_S \equiv \phi^*(9\bar{l} - 3\bar{e}_1 - 3\bar{e}_2 - 3\bar{e}_3 - 3\bar{e}_4)$ which contradicts $R_1 \leq R$, so we are done.

In conclusion, we have $\hat{P} \neq \hat{P}_1$.

7.2.2. *The case $\hat{P} = \hat{P}_2$.* We fail to exclude this case, but we can describe the fibration of S and give an effective divisor linearly equivalent to K_S .

Notation 7.10. *Let $Q = \psi(c)$ be the A_1 -singularity on Σ . Denote by l_1 the strict transform of a general line through P_1 , by \hat{e}_1 the strict transform of the line through P_4 and P_2 , by \hat{e}_4 the strict transform of the line through P_4 and P_1 . Then $l_1 \equiv l - e_1, c \equiv l - e_1 - e_2 - 2e_3, \hat{e}_1 \equiv l - e_2 - e_3 - e_4, \hat{e}_4 \equiv l - e_1 - e_4$. Denote by L_1 (resp. $\hat{E}_1, \hat{E}_4, E_3, E_4$) the strict transform of \bar{l}_1 (resp. $\bar{e}_1, \bar{e}_4, \bar{e}_3, \bar{e}_4$) w.r.t. ϕ .*

Immediately we have

$$\begin{aligned} -K_{\hat{P}} &\equiv 3l - e_1 - e_2 - 2e_3 - e_4 \\ (7.5) \quad &\equiv 2l_4 + c + e_4 \\ &\equiv 3l_1 + 2e_1 - e_2 - 2e_3 - e_4 \end{aligned}$$

and then

$$\begin{aligned} -K_{\Sigma} &\equiv 2\bar{l}_4 + \bar{e}_4 \\ (7.6) \quad &\equiv 3\bar{l}_1 + 2\bar{e}_1 - 2\bar{e}_3 - \bar{e}_4 \end{aligned}$$

Note that there exist at most two (-2) -curves on S since $\rho(S) - \rho(\Sigma) = 2$.

Lemma 7.11. *Let the notations be as above. Then $\phi^*\bar{e}_4 = 2E'_4$ for some divisor E'_4 on S .*

Proof. By $-K_{\Sigma} \equiv 2\bar{l}_4 + \bar{e}_4$, we may write

$$2K_S \equiv 2L_4 + E_4 + Z$$

where Z is zero or supported on a (-2) -curve. Note that Z arises from $\phi^*\bar{e}_4$ or $\phi^*2\bar{l}_4$. We write $Z = 2Z' + Z''$ where Z'' is reduced.

Since Z'' contains at most two (-2) -curves, so $Z''^2 \geq -4$. Arguing as in the proof of Section 7.2.1 Lemma 7.7, we get that E_4 is reduced.

We write $E_4 = 2D_4$ and then write $\phi^*e_4 = 2D_4 + 2Z'_1 + Z''_1$. We still need to prove $Z''_1 = 0$. Considering the relation $2(K_S - L_4 - D_4 - Z') = Z''$, since $K_S(K_S - L_4 - D_4 - Z') = 0$ is even, $(K_S - L_4 - D_4 - Z')^2$ is even, hence $8|Z''^2$. It is only possible that $Z'' = 0$. Note that since L_4 moves, we can assume Z''_1 is not contained in $\phi^*\bar{l}_4$. So $Z'' = 0$ implies $Z''_1 = 0$, then we are done. \square

Lemma 7.12. *Let $g : S \rightarrow \mathbb{P}^1$ be the fibration induced by the pencil $|L_1|$. Then g has at most 4 double fibers.*

Proof. Otherwise $\bar{\phi}$ is ramified along at least 4 double fibers since the fibration $u : \Sigma \rightarrow \mathbb{P}^1$ induced by $|\bar{f}_1|$ has unique non-reduced fiber $2\bar{e}_3$. Considering the ramification divisor of the map $\bar{\phi} : \bar{S} \rightarrow \Sigma$ and going process ARDP, we get a contradiction. \square

By $-K_{\Sigma} \equiv 3\bar{l}_1 + 2\bar{e}_1 - 2\bar{e}_3 - \bar{e}_4$, we have $2K_S \equiv 3L_1 + 2E_1 - 2E_3 - 2E'_4 + Z$ where $Z = \phi^*(2\bar{e}_1 - 2\phi^*\bar{e}_3) - 2E_1 + 2E_3$. Write $Z = 2Z' + Z''$, we get the relation $2(K_S - L_1 - E_1 + E_3 + E'_4 - Z') \equiv Z'' + L_1$, and then a double cover $\pi : Y \rightarrow S$

branched along L_1 and Z'' . Note that $L_1 Z'' = 0$ and $L_1^2 = 0$. By Formula 2.1 and Lemma 2.6, we calculate the invariants of Y :

(7.7)

$$\begin{aligned}\chi(Y) &= 2 + \frac{2 + \frac{(L_1 + Z'')^2}{4}}{2} = 2 + \frac{2 + \frac{Z''^2}{4}}{2} \\ p_g(Y) &= h^0(S, 2K_S - L_1 - E_1 + E_3 + E'_4 - Z') \\ &\geq h^0(S, 2L_1 + E_1 - E_3 - E'_4 - Z') \geq h^0(\mathcal{O}_\Sigma(2\bar{l}_1 + \bar{e}_1 - \bar{e}_3 - \bar{e}_4)) \\ &\geq h^0(\mathcal{O}_{\hat{P}}(\psi^*(2\bar{l}_1 + \bar{e}_1 - \bar{e}_3 - \bar{e}_4))) \geq h^0(\mathcal{O}_{\hat{P}}(2l - e_1 - e_2 - e_3 - e_4)) = 3\end{aligned}$$

and deduce that $Z'' = 0$ to guarantee that $\chi(Y)$ be an integer, thus $\chi(Y) = 3$ and $q(Y) \geq 1$, moreover we have

$$(7.8) \quad \phi^*(\bar{e}_1 - \bar{e}_3) = E_1 - E_3 + Z'$$

By Proposition 2.1, we get a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & S \\ \alpha \downarrow & & g \downarrow \\ B & \xrightarrow{\pi'} & \mathbb{P}^1 \end{array}$$

where $\alpha : Y \rightarrow B$ is the Albanese pencil and $\pi' : B \rightarrow \mathbb{P}^1$ is a double cover. By Lemma 7.12, we get that $g(B) = 1$ because otherwise g will have at least 5 double fibers. Denote by $P_i, i = 1, 2, 3, 4$ the branch points of π' , and assume $g^*P_i = 2M_i, i = 1, 2, 3$ and $g^*P_4 = L_1$. Since Y can be obtained by the normalization of fiber product $S \times_{\mathbb{P}^1} B$, by Lemma 2.7, we have $K_S - L_1 - E_1 + E_3 + E'_4 - Z'|_F = 0$ if F is a fiber of g different from $2M_i$, and $K_S - L_1 - E_1 + E_3 + E'_4 - Z'|_{M_i} = M_i|_{M_i}$. Considering the Chern classes, we conclude that

$$(7.9) \quad K_S - L_1 - E_1 + E_3 + E'_4 - Z' \equiv M_1 + M_2 - M_3$$

where the $2M_i$'s are double fibers of g .

Similarly, we get another relation

$$(7.10) \quad K_S - L_1 - \hat{E}_1 + E_3 + \hat{E}'_4 - \hat{Z}' \equiv \hat{M}_1 + \hat{M}_2 - \hat{M}_3$$

where the $2\hat{M}_i$'s are double fibers of g and

$$(7.11) \quad \phi^*(\hat{e}_1 - \hat{e}_3) = \hat{E}_1 - E_3 + \hat{Z}'$$

Since g has at most 4 double fibers, we may assume that $\hat{M}_i = M_i$ for $i = 1, 2$. It follows that

$$M_1 + M_2 \equiv K_S - L_1 - E_1 + E_3 + E'_4 - Z' + M_3 \equiv K_S - L_1 - \hat{E}_1 + E_3 + \hat{E}'_4 - \hat{Z}' + \hat{M}_3,$$

thus

$$-E_1 + E'_4 - Z' + M_3 \equiv -\hat{E}_1 + \hat{E}'_4 - \hat{Z}' + \hat{M}_3.$$

The equation 7.8 minus 7.11 yields $\phi^*\bar{e}_1 - \phi^*\hat{e}_1 = \hat{E}_1 + \hat{Z}' - (E_1 - Z')$, thus $\hat{M}_3 + \phi^*\bar{e}_1 + \hat{E}'_4 \equiv M_3 + \phi^*\bar{e}_1 + E'_4$. Since $2(\hat{M}_3 + \phi^*\bar{e}_1 + \hat{E}'_4) \equiv 2K_S$ and $|2K_S| = \phi^*| -K_\Sigma|$, $\hat{M}_3 + \hat{E}'_4 + E'_4 + M_3 + \phi^*(\bar{e}_1 + \bar{e}_1)$ must be the pull-back of some element in $| -K_\Sigma|$. Since $2\hat{M}_3, 2(\hat{E}'_4 + E'_4), 2M_3$ are 3 double fibers (maybe the same), ϕ is

ramified along them unless one of them is $\phi^*\bar{e}_3$. We assume $M_3 = \phi^*\bar{e}_3$, then the only possibility is $\hat{M}_3 = \hat{E}'_4 + E'_4$. By 7.9, we have

$$K_S \equiv \phi^*\bar{e}_1 - E'_4 + M_1 + M_2.$$

Claim 7.13. *S has at most one (-2) -curve, and the bicanonical map $\phi : S \rightarrow \Sigma$ can not lift to a morphism to \hat{P} .*

Proof. Consider the double cover $\pi : X \rightarrow S$ given by the relation $2(M_1 - M_2) \equiv \mathcal{O}_S$. By $K_S \equiv \phi^*\bar{e}_1 - E'_4 + M_1 + M_2$, using Formula 2.1 and Lemma 2.6, we get the invariants of X are as follows:

$$(7.12) \quad \begin{aligned} \chi(X) &= 2 \\ p_g(X) &= h^0(S, \phi^*\bar{e}_1 - E'_4 + M_1 + M_2 + (M_1 - M_2)) = h^0(S, \phi^*\bar{e}_1 - E'_4 + L_1) \\ &\geq h^0(\mathcal{O}_\Sigma(\bar{l}_1 + \bar{e}_1 - \bar{e}_4)) \geq h^0(\mathcal{O}_{\hat{P}}(l - e_4)) = 2 \\ q(X) &\geq 1 \end{aligned}$$

Therefore we get a commutative diagram as follows:

$$\begin{array}{ccc} X & \xrightarrow{\pi} & S \\ \alpha \downarrow & & f \downarrow \\ B & \xrightarrow{\pi'} & \mathbb{P}^1 \end{array}$$

where $\alpha : Y \rightarrow B$ is the Albanese pencil and $\pi' : B \rightarrow \mathbb{P}^1$ is a double cover. We claim that the fibration $f : S \rightarrow \mathbb{P}^1$ is different from the fibration $g : S \rightarrow \mathbb{P}^1$ defined by the pencil $|L_1|$. Indeed, considering the Stein factorization $\pi'' \circ h : X \rightarrow C \rightarrow \mathbb{P}^1$ of the map $g \circ \pi$, π'' is a double cover branched along the two points $g(M_1), g(M_2)$, so f is different from g . If S has two (-2) curves, then applying Remark 7.5, Proposition 7.4 tells that f coincides with g , thus a contradiction follows.

If ϕ lifts to a morphism $\hat{\phi} : S \rightarrow \hat{P}$, then $\hat{\phi}^*cK_S = \hat{\phi}^*e_2K_S = 0$, thus ϕ contracts at least two (-2) -curves. For the same reason, this is impossible. \square

In conclusion, with the notations and assumptions above, we have

- i) The pencil $|L_1|$ induces a genus 3 fibration with exactly 4 double fibers:
 $M_1, M_2, M_3 = \phi^*\bar{e}_3, \hat{M}_3 = 2(\hat{E}'_4 + E'_4)$;
- ii) $K_S \equiv \phi^*\bar{e}_1 - E'_4 + M_1 + M_2$;
- iii) S contains at most one (-2) -curve, and the bicanonical map $\phi : S \rightarrow \Sigma$ can not lift to a morphism to \hat{P} .

7.2.3. *The case $\hat{P} = \hat{P}_4$.*

Notation 7.14. *Denote by l_3 the strict transform of a general line through P_3 and by l_1 the strict transform of a general line through P_1 . Let $Q = \psi(c + e_3)$ be the A_2 -singularity on Σ . Denote by L_3 (resp. L_1, E_1, E_4, E_2) the strict transform of \bar{l}_3 (resp. $\bar{l}_1, \bar{e}_1, \bar{e}_4, \bar{e}_2$) with respect to ϕ .*

With the notations above, it follows that

$$-K_{\hat{P}} \equiv 3l - e_1 - e_2 - e_3 - 2e_4 \equiv 2l_3 + e_4 + c + 2e_3$$

thus

$$-K_\Sigma \equiv 2\bar{l}_3 + \bar{e}_4$$

Immediately we have

$$\begin{aligned}\psi^* \bar{l}_3 &= l_3 + \frac{2}{3}e_3 + \frac{1}{3}c, \quad \psi^* \bar{e}_4 = e_4 + \frac{2}{3}e_3 + \frac{1}{3}c \\ \psi^* \bar{e}_2 &= e_2 + \frac{2}{3}c + \frac{1}{3}e_3, \quad \psi^* \bar{e}_1 = e_1 + \frac{2}{3}c + \frac{1}{3}e_3\end{aligned}$$

and then

$$\bar{l}_3 \bar{e}_1 = \bar{e}_4 \bar{e}_1 = \frac{1}{3}, \quad \bar{e}_2 \bar{e}_1 = \frac{2}{3}, \quad \bar{e}_1^2 = -\frac{1}{3}.$$

Note that there exist at most two (-2) -curves on S since $\rho(S) - \rho(\Sigma) = 2$. First we introduce the following claim.

Claim 7.15. *There are exactly two (-2) -curves which we denote by θ_1, θ_2 such that $\theta_1 \theta_2 = 1$ and $\phi^{-1}Q = \theta_1 \cup \theta_2$.*

Proof. By $\bar{e}_1^2 = -\frac{1}{3}$, we have $(\phi^* \bar{e}_1)^2 = -\frac{4}{3}$. Write $\phi^* \bar{e}_1 = E_1 + x\theta_1 + y\theta_2$ where θ_i is a (-2) -curve or zero and x, y are rational numbers. Note that if θ_i is nonzero then so is its coefficient. We set the coefficient of θ_i to be zero if θ_i is zero, so it makes sense to set $\theta_i^2 = -2$. Then we have

$$(\phi^* \bar{e}_1)^2 = E_1^2 + 2xE_1\theta_1 + 2yE_1\theta_2 - 2x^2 + 2xy\theta_1\theta_2 - 2y^2 = -\frac{4}{3},$$

therefore one of the denominators of x, y is divided by 3. By $\phi^* \bar{e}_1 \theta_1 = \phi^* \bar{e}_1 \theta_2 = 0$, we get

$$(7.13) \quad \begin{aligned}2x - \theta_1 \theta_2 y &= E_1 \theta_1 \\ -\theta_1 \theta_2 x - 2y &= E_2 \theta_2\end{aligned}$$

Applying Crammar's rule, we get that $3|4 - \theta_1 \theta_2$, hence $\theta_1 \theta_2 = 1$. So the claim is true. \square

Denote by θ_1, θ_2 the two (-2) -curves on S . By $-K_\Sigma \equiv 2\bar{l}_3 + \bar{e}_4$, we may write

$$2K_S = 2L_3 + E_4 + Z,$$

then $L_3 Z > 0, E_4 Z > 0$ since \bar{l}_3 and \bar{e}_4 pass through the singular point on Σ . Set $Z = a\theta_1 + b\theta_2$ where a, b are positive integers. We assume $b \leq a$. By

$$K_S L_3 = 4, K_S E_4 = 2, K_S \theta_i = 0 \quad i = 1, 2,$$

we get that

$$(7.14) \quad 8 = 2K_S L_3 = 2L_3^2 + L_3 E_4 + L_3 Z \Rightarrow L_3 Z = 8 - 2L_3^2 - L_3 E_4 > 0$$

$$(7.15) \quad 4 = 2K_S E_4 = 2L_3 E_4 + E_4^2 + E_4 Z \Rightarrow E_4 Z = 4 - E_4^2 - 2L_3 E_4 > 0$$

$$(7.16) \quad \begin{aligned}0 &= 2K_S Z = 2L_3 Z + E_4 Z + Z^2 \\ &\Rightarrow -Z^2 = 2L_3 Z + E_4 Z = 20 - 4L_3^2 - 4L_3 E_4 - E_4^2 \\ &\Rightarrow a^2 + b^2 - ab = 10 - 2L_3^2 - 2L_3 E_4 - \frac{E_4^2}{2}\end{aligned}$$

By $E_4 \theta_i \geq 0, L_3 \theta_i \geq 0$, we have $Z \theta_i \leq 0$ which implies

$$(7.17) \quad b \leq a \leq 2b$$

Applying the argument in Section 7.2.1 Case 1 to the relation $2K_S = 2L_3 + E_4 + Z$, we exclude the case when E_4 is non-reduced. From now on, we assume that E_4 is reduced.

By $K_S L_3 = 4$, applying Hodge index theorem, we have $L_3^2 = 0$ or 2 . By Lemma 7.3, we have $E_4^2 = -2$ or -4 or -6 . By $L_3 Z = 8 - 2L_3^2 - L_3 E_4 \leq 8$, we have

$$(7.18) \quad \min\{a, b\} \leq 8$$

With the equation 7.16 and the inequalities 7.14, 7.15, 7.17, 7.18, going a computer program, we list all the possibilities:

a	b	L_3^2	$L_3 E_4$	E_4^2	$E_4 Z$
2	1	2	2	-2	2
2	2	2	2	-4	4
3	2	0	2	-2	2
3	2	2	0	-2	6
3	3	0	1	-2	4
4	2	0	0	-4	8
2	1	2	3	-6	4
3	2	0	3	-6	4
3	2	2	1	-6	8
3	3	0	2	-6	6
3	3	2	0	-6	10
4	3	0	0	-6	10

Write $Z = 2Z' + Z''$ where Z'' is reduced. Then by $2(K_S - L_3 - Z') \equiv E_4 + Z''$, we get a double cover $\pi : Y \rightarrow S$ branched over E_4 and Z'' . Check case by case listed in the table that $E_4 + Z''$ has at most negligible singularities as follows: say for the case in the first row, we have $Z'' = \theta_2$, and by $\theta_2 Z = 0$, we deduce $E_4 Z'' = 0$, then since E_4 has arithmetic genus 1, we can see that $E_4 + Z''$ has at most negligible singularities; say for the case in the 9th row, we have $Z'' = \theta_1$ and $E_4 \theta_1 \leq 2$, then since E_4 is composed with 2 disjoint smooth (-3) -curves, $E_4 \cap Z''$ are the singularities of $E_4 + Z''$, thus $E_4 + Z''$ has at most double points, so we are done. So Y has at most canonical singularities. Then we can use Formula 2.1 to calculate the invariants of Y :

$$\chi(Y) = 2 + \frac{1}{2}(K_S + \frac{E_4 + Z''}{2}) \frac{E_4 + Z''}{2} = 2 + \frac{1}{2}(1 + \frac{E_4^2 + Z''^2 + 2E_4 Z''}{4})$$

$$p_g(Y) = h^0(S, 2K_S - L_3 - Z') = h^0(S, L_3 + Z' + E_4 + Z'') \geq 2.$$

Claim 7.16. $\chi(Y) \leq 2$ and $q(Y) \geq 1$.

Proof. We only need to show that $\frac{1}{2}(1 + \frac{E_4^2 + Z''^2 + 2E_4 Z''}{4}) < 1$, i.e., $E_4^2 + Z''^2 + 2E_4 Z'' < 4$. If $Z'' = 0$, then we are done. So we may assume $Z'' \neq 0$, then $Z''^2 = -2$, thus it suffices to show $E_4 Z'' < 3 - \frac{E_4^2}{2}$. From the table above, it follows from the observation that $E_4 Z'' \leq \frac{E_4 Z}{b}$. \square

We denote by $\alpha : Y \rightarrow B$ the Albanese pencil. By proposition 2.2, we get a fibration $g : S \rightarrow \mathbb{P}^1$ and a double cover $\pi' : B \rightarrow \mathbb{P}^1$ such that $g \circ \pi = \pi' \circ \alpha$. Checking that $(E_4 + Z'')\theta_i < 4$, then using the Riemann-Hurwitz formula, $\pi^{-1}\theta_i$ is composed with one or two rational curves (maybe singular) thus contained in

some fibers of α , hence θ_i is contained in one fiber of g . Since $\rho(\bar{S}) = \rho(\Sigma)$, by Remark 7.5, we may assume g is induced by the pencil $|L_1| = \phi^*|\bar{l}_1|$. Since $\pi' : B \rightarrow \mathbb{P}^1$ is branched over at least 4 points, so $\bar{g} : \bar{S} \rightarrow \mathbb{P}^1$ has at least 3 double fibers $2\bar{M}_1, 2\bar{M}_2, 2\bar{M}_3$ which does not contain \bar{E}_4 as a component. Since every fiber of \bar{g}' is reduced, $\bar{\phi}$ is ramified along $\bar{M}_1, \bar{M}_2, \bar{M}_3$.

Claim 7.17. *With the assumptions above, E_2 is non-reduced.*

Proof. First remark that $\pi' : B \rightarrow \mathbb{P}^1$ is branched over the point $g(E_4) = g(E_2)$. Let $Z = B \times_{\mathbb{P}^1} S$, and denote by $p : Z \rightarrow S$ the projection. If E_2 is reduced, then the map p is branched along E_2 , and Z is normal along the locus over E_2 . Note that there is a natural birational morphism $h : Y \rightarrow Z$ such that $\pi = p \circ h$, so the map $\pi : Y \rightarrow S$ is also branched over E_2 . But the branch locus of π does not contain E_2 , and we are done. \square

By the claim, we may assume $E_2 = 2E'_2$, then $\bar{\phi}$ is ramified along \bar{E}'_2 . Denote the ramification divisor of $\bar{\phi}$ by R , and put $R_1 = \bar{M}_1 + \bar{M}_2 + \bar{M}_3 + \bar{E}'_2$. Note that $M_i(\bar{\phi}^*\bar{e}_1) = 2, i = 1, 2, 3$ and $\bar{E}'_2(\bar{\phi}^*\bar{e}_1) = \frac{4}{3}$ since $\bar{e}_2\bar{e}_1 = \frac{2}{3}$. We get a contradiction from Lemma 2.8 since $(R - R_1)\bar{E}_1 = -\frac{4}{3} = (\bar{\phi}^*\bar{e}_1)^2$.

In conclusion, we prove $\hat{P} \neq \hat{P}_4$.

7.2.4. *The case $\hat{P} = \hat{P}_3$.*

Notation 7.18. *We denote by l_4 the strict transform of a general line through P_4 , by l_1 the strict transform of a general line through P_1 . Let $Q_1 = \psi(c)$ and $Q_2 = \psi(e_1 + e_2)$. Denote by L_4 (resp. L_1, E_3, E_4) the strict transforms of \bar{l}_4 (resp. $\bar{l}_1, \bar{e}_3, \bar{e}_4$) with respect to ϕ .*

Immediately, it follows that $l_4 \equiv l - e_4, l_1 \equiv l - e_1 - e_2 - e_3$ and $c \equiv l - e_1 - 2e_2 - 3e_3$. So we have

$$\begin{aligned} -K_{\hat{P}} &\equiv 3l - e_1 - 2e_2 - 3e_3 - e_4 \\ (7.19) \quad &\equiv 3l_4 + 2e_4 - e_1 - 2e_2 - 3e_3 \\ &\equiv 2l_4 + e_4 + c \end{aligned}$$

hence

$$\begin{aligned} -K_{\Sigma} &\equiv 3\bar{l}_4 + 2\bar{e}_4 - 3\bar{e}_3 \\ (7.20) \quad &\equiv 2\bar{l}_4 + \bar{e}_4 \end{aligned}$$

Considering the pull-backs

$$\psi^*\bar{l}_4 = l_4 + \frac{1}{2}c, \quad \psi^*\bar{l}_1 = l_1 + \frac{2}{3}e_1 + \frac{1}{3}e_2, \quad \psi^*\bar{e}_3 = e_3 + \frac{2}{3}e_2 + \frac{1}{3}e_1 + \frac{1}{2}c$$

we get

$$\bar{l}_4^{-2} = \frac{1}{2}, \quad \bar{l}_1^{-2} = \frac{2}{3}, \quad \bar{e}_3^{-2} = \frac{1}{6}.$$

Using the fact $\bar{l}_1^{-2} = \frac{2}{3}$, similar argument as in Section 7.2.3 Claim 7.15 shows

Claim 7.19. $\phi^{-1}Q_2 = \theta_2 \cup \theta_3$ where θ_2 and θ_3 are (-2) -curves such that $\theta_2\theta_3 = 1$.

Then there exists at most one (-2) -curve on S except for θ_2, θ_3 since $\rho(S) - \rho(\Sigma) = 3$.

Case 1: There's no (-2) -curve aside from θ_2, θ_3 , consequently $\phi^{-1}Q_1$ is composed with two points.

We write $\phi^*3\bar{e}_3 = 3E_3 + Z$ where $Z = a\theta_2 + b\theta_3$ with a, b are positive integers, and assume $a \geq b$.

Note that in this case, $\phi^*\bar{l}_4, \phi^*\bar{e}_4$ are Cartier divisors that do not contain (-2) -curves. Then by $-K_\Sigma \equiv 3\bar{l}_4 + 2\bar{e}_4 - 3\bar{e}_3$, we have

$$2K_S \equiv 3L_4 + 2\phi^*\bar{e}_4 - 3E_3 - Z$$

Subcase 1: E_3 is reduced.

In this case, by Lemma 7.3, since $(\phi^*\bar{e}_3)^2 = \frac{2}{3}$ and $E_3^2 \leq (\phi^*\bar{e}_3)^2$, it is possible that $E_3^2 = 0$ or -2 or -4 or -6 . Combining the two formulas $(\phi^*3\bar{e}_3)^2 = 6$ and $(\phi^*\bar{e}_3)\theta_2 = (\phi^*\bar{e}_3)\theta_3 = 0$, we get that:

$$\begin{aligned} 9E_3^2 + 6aE_3\theta_2 + 6bE_3\theta_3 - 2a^2 + 2ab - 2b^2 &= 6 \\ 3E_3\theta_2 - 2a + b &= 3E_3\theta_3 + a - 2b = 0 \end{aligned}$$

Resolving these equations, we get that either

$$E_3^2 = 0, Z = 2\theta_2 + \theta_3, E_3\theta_2 = 1, E_3\theta_3 = 0$$

or

$$E_3^2 = -2, Z = 4\theta_2 + 2\theta_3, E_3\theta_2 = 2, E_3\theta_3 = 0$$

or

$$E_3^2 = -4, Z = 5\theta_2 + 4\theta_3, E_3\theta_2 = 2, E_3\theta_3 = 1.$$

If we write $Z = 2Z' + Z''$ where Z'' is reduced, then we have

$$2(K_S - L_4 - \phi^*\bar{e}_4 + 2E_3 + Z' + Z'') \equiv L_4 + E_3 + Z''$$

By the relation, we get a double cover $\pi : Y \rightarrow S$. Note that $L_4^2 = 4\bar{l}_4^2 = 2$, $L_4Z = 0$ and $L_4E_3 = 2$. Checking that $[-\phi^*\bar{e}_3] = [-E_3 - \frac{Z}{3}] \leq -E_3 - Z'$, then by use of Formula 2.1 and Lemma 2.6, we calculate the invariants of Y as follows:

(7.21)

$$\begin{aligned} \chi(Y) &= 2 + \frac{(K + \frac{L_4 + E_3 + Z''}{2})(\frac{L_4 + E_3 + Z''}{2})}{2} \\ &= 2 + \frac{3 + \frac{L_4^2 + 2L_4E_3 + Z''^2 + E_3^2 + 2E_3Z''}{4}}{2} = 4 \end{aligned}$$

$$\begin{aligned} p_g(Y) &= h^0(S, 2K_S - L_4 - \phi^*\bar{e}_4 + 2E_3 + Z' + Z'') = h^0(S, 2L_4 + \phi^*\bar{e}_4 - E_3 - Z') \\ &\geq h^0(\mathcal{O}_\Sigma(2\bar{l}_4 + \bar{e}_4 - \bar{e}_3)) \geq h^0(\hat{P}, 2l - e_1 - e_2 - e_3 - e_4) \geq 4, \end{aligned}$$

thus $q(Y) \geq 1$. By Proposition 2.1, we obtain

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & S \\ \alpha \downarrow & & g \downarrow \\ B & \xrightarrow{\pi'} & \mathbb{P}^1 \end{array}$$

where $\alpha : Y \rightarrow B$ is the Albanese pencil and $\pi' : B \rightarrow \mathbb{P}^1$ is a double cover. Note that L_4 is contained in one fiber since it is branched over by $\pi : Y \rightarrow S$. This contradicts $L_4^2 = 2$, so this case does not occur.

Subcase 2: E_3 is non-reduced.

By Lemma 7.3, we can assume $E_3 = 2E'_3$. Then we get:

$$2(K_S - L_4 - \phi^*\bar{e}_4 + 3E'_3 + Z' + Z'') \equiv L_4 + Z''.$$

Considering the double cover $\pi : Y \rightarrow S$ given by the relation, and calculating the invariants of Y , we obtain:

$$\begin{aligned}
\chi(Y) &= 2 + \frac{(K + \frac{L_4 + Z''}{2})(\frac{L_4 + Z''}{2})}{2} \\
&= 2 + \frac{2 + \frac{L_4^2 + Z''^2}{4}}{2} \leq 3 \\
(7.22) \quad p_g(Y) &= h^0(S, 2K_S - L_4 - \phi^* \bar{e}_4 + 3E'_3 + Z' + Z'') \\
&= h^0(S, 2L_4 + \phi^* \bar{e}_4 - 3E'_3 - Z') \\
&\geq h^0(\mathcal{O}_\Sigma(2\bar{l}_4 + \bar{e}_4 - 2\bar{e}_3)) \geq h^0(\hat{P}, 2l - e_1 - 2e_2 - 2e_3 - e_4) = 3,
\end{aligned}$$

and thus $q(Y) \geq 1$. By similar argument as above, we get a contradiction.

Case 2: There exists another (-2) -curve on S except for θ_2, θ_3 .

Since there are 3 (-2) -curves on S , $\rho(\bar{S}) = \rho(\Sigma) = 2$. By $-K_\Sigma \equiv 2\bar{l}_4 + \bar{e}_4$, we have

$$2K_S \equiv 2L_4 + E_4 + Z$$

where Z is zero or supported on exactly one (-2) -curve since \bar{l}_4 and \bar{e}_4 do not contain the A_2 -singularity Q_2 . We may write $Z = 2Z' + Z''$ where Z'' is reduced. No matter whether E_4 is reduced or not, by the double covering trick, arguing as in the proof of Lemma 7.7 and Case 1 in Section 7.2.1, it is easy to show that S has a fibration such that every (-2) -curve is contained in one fiber. We omit the details. Then by Remark 7.5, there exists a fibration of \hat{P} such that every (-2) -curve is contained in one fiber. However, this contradicts Proposition 7.4.

In conclusion we prove $\hat{P} \neq \hat{P}_3$

7.2.5. *The case $\hat{P} = \hat{P}_5$.*

Notation 7.20. Denote by l_2 the strict transform of a general line through P_2 . Let $Q = \psi(c + e_2 + e_3)$ be the A_3 -singularity on Σ . Denote by L_2 (resp. E_1, E_4) the strict transforms of \bar{l}_2 (resp. \bar{e}_1, \bar{e}_4) with respect to ϕ .

It follows that $l_2 \equiv l - e_2 - e_3 - e_4$, $c \equiv l - e_1 - e_2 - 2e_3 - 2e_4$, and then

$$-K_{\hat{P}} \equiv 3l - e_1 - e_2 - 2e_3 - 3e_4 \equiv 2l_2 + e_4 + c + 2e_2 + 2e_3$$

hence

$$-K_\Sigma \equiv 2\bar{l}_2 + \bar{e}_4$$

Considering the pull-backs

$$\begin{aligned}
\psi^* \bar{l}_2 &= l_2 + \frac{3}{4}e_2 + \frac{1}{2}e_3 + \frac{1}{4}c \equiv \frac{5}{4}l - \frac{1}{4}e_1 - \frac{1}{2}e_2 - e_3 - \frac{3}{2}e_4 \\
\psi^* \bar{e}_4 &= e_4 + e_3 + \frac{1}{2}(e_2 + c) \equiv \frac{1}{2}(l - e_1) \\
\psi^* \bar{e}_1 &= e_1 + \frac{3}{4}c + \frac{1}{2}e_3 + \frac{1}{4}e_2 \equiv \frac{3}{4}l + \frac{1}{4}e_1 - \frac{1}{2}e_2 - e_3 - \frac{3}{2}e_4 \equiv \bar{l}_2 - \bar{e}_4,
\end{aligned}$$

we get

$$\bar{l}_2^2 = \frac{3}{4}, \quad \bar{e}_4^2 = 0, \quad \bar{l}_2 \bar{e}_1 = \frac{1}{4}, \quad \bar{e}_4 \bar{e}_1 = \frac{1}{2}, \quad \bar{l}_2 \bar{e}_4 = \frac{1}{2}, \quad \bar{e}_1^2 = -\frac{1}{4}.$$

Since $\rho(S) - \rho(\Sigma) = 3$, S has at most three (-2) -curves, precisely we have the following claim.

Claim 7.21. $\phi^{-1}(Q)$ is composed with either three (-2) -curves which we denote by $\theta_1, \theta_2, \theta_3$ such that $\theta_1\theta_2 = \theta_2\theta_3 = 1$ or 2 disjoint (-2) -curves.

Proof. Set $\phi^*\bar{e}_1 = E_1 + x\theta_1 + y\theta_2 + z\theta_4$ and $\phi^*\bar{e}_4 = D_4 + u\theta_1 + v\theta_2 + w\theta_4$ where θ_i is either zero or a (-2) -curve mapped to Q . Note that if θ_i is non-zero, then so is its coefficient appearing in $\phi^*\bar{e}_j, j = 1, 4$. We set the coefficient of θ_i to be zero if $\theta_i = 0$, so we can assume $\theta_i^2 = -2$ in the following calculations. Set: $E_1\theta_i = k_i, E_4\theta_i = n_i$ for $i = 1, 2, 3; \theta_1\theta_3 = 0, \theta_1\theta_2 = \alpha, \theta_2\theta_3 = \beta$ where $\alpha, \beta = 0$ or 1.

The equations $(\phi^*\bar{e}_1)\theta_i = 0$ and $(\phi^*\bar{e}_4)\theta_i = 0$ yield

$$\begin{aligned} 2x - \alpha y &= k_1 & 2u - \alpha v &= n_1 \\ -\alpha x + 2y - \beta z &= k_2 & -\alpha u + 2v - \beta w &= n_2 \\ -\beta y + 2z &= k_3 & \beta v + 2w &= n_3 \end{aligned} \quad \text{and}$$

If $\alpha = \beta = 0$, then the equations $(\phi^*\bar{e}_1)^2 = -1, (\phi^*\bar{e}_4)^2 = 0, (\phi^*\bar{e}_1)(\phi^*\bar{e}_4) = 2$ yield

$$(7.23) \quad \begin{aligned} E_1^2 + \frac{1}{2}(k_1^2 + k_2^2 + k_3^2) &= -1 \\ E_4^2 + \frac{1}{2}(n_1^2 + n_2^2 + n_3^2) &= 0 \\ E_1E_4 + \frac{1}{2}(n_1k_1 + n_2k_2 + n_3k_3) &= 2 \end{aligned}$$

Since E_1^2 is even, the first equation implies two of k_1, k_2, k_3 are odds, we may assume k_1, k_2 are odds, so θ_1, θ_2 are non-zero, and $n_1, n_2 > 0$ since then $u, v > 0$. Since E_4^2 is even, the second equation implies that n_1, n_2, n_3 are even simultaneously. Since $E_1E_4 \geq 0$, the third equation implies $k_1 = k_2 = 1, n_1 = n_2 = 2, k_3 = n_3 = 0$. Then we conclude that $\phi^{-1}Q$ is two disjoint (-2) -curves.

If $\alpha = 1, \beta = 0$, then $(\phi^*\bar{e}_1)^2 = -1$ yields:

$$E_1^2 + \frac{2}{3}(k_1^2 + k_2^2 + k_1k_2) + \frac{1}{2}k_3^2 = -1$$

To guarantee the left hand side is an integer, k_3 should be even, but then the left side of the equation is even.

If $\alpha = \beta = 1$, then $\phi^{-1}(Q)$ is composed with 3 (-2) -curves and $\theta_1\theta_2 = \theta_2\theta_3 = 1$, and we are done. \square

Case 1: $\phi^{-1}(Q)$ is composed with 3 (-2) -curves.

By $-K_\Sigma \equiv 2\bar{L}_2 + \bar{e}_4$, we may write $2K_S = 2L_2 + E_4 + Z$, and write $Z = 2Z' + Z''$ where Z'' is reduced.

First if E_4 is non-reduced, we may write $E_4 = 2E_4'$. Consider the double cover of S given by the relation $2(K_S - L_2 - E - 4' - Z') \equiv Z''$. Note that $\bar{E}_4 = 2\bar{E}_4'$, so $\bar{E}_4' \leq \bar{R}$. Then arguing as Section 7.2.1 case 1, we get a contradiction.

Now we assume that E_4 is reduced. Since $\bar{e}_4^2 = 0$ and $\phi^*\bar{e}_4$ contains (-2) -curves, so $E_4^2 < 0$. Then Lemma 7.3 tells that

$$(7.24) \quad E_4^2 = -2 \text{ or } -4 \text{ or } -6.$$

Assume $Z = a\theta_1 + b\theta_2 + c\theta_3$ and $a \geq c$. Then equation $2K_S = 2L_2 + E_4 + Z$ intersecting E_4 yields $0 < E_4Z = 4 - E_4^2 - 2L_2E_4 \leq 10$ which implies

$$(7.25) \quad L_2E_4 < 5, \min\{a, b, c\} \leq 10, a \geq c$$

By $Z \equiv 2K_S - 2L_2 - E_4$, calculating Z^2 , we get that

$$(7.26) \quad -\frac{Z^2}{2} = a^2 + b^2 + c^2 - ab - bc = 10 - 2L_2^2 - 2L_2E_4 - \frac{E_4^2}{2}$$

And $Z\theta_i \leq 0, i = 1, 2, 3$ yields that

$$(7.27) \quad 2a \geq b, 2b \geq a + c, 2c \geq b$$

Note that since $L_2^2 < 4\bar{l}_2^2 = 3$, we get

$$(7.28) \quad L^2 = 0 \text{ or } 2$$

Then by 7.24, 7.25, 7.26, 7.27 and 7.28, going a computer program, we get the following possibilities:

a	b	c	L_2^2	L_2E_4	E_4^2	E_4Z
1	1	1	2	4	-6	2
1	2	1	2	3	-4	2
2	2	1	2	2	-2	2
2	3	2	2	1	-2	4
3	2	1	0	3	-4	2
3	2	1	2	1	-4	6
3	3	2	2	0	-2	6
3	3	2	0	2	-2	2
3	3	3	0	1	-2	4
3	4	2	0	1	-2	4
3	4	3	0	1	-4	2
4	3	2	0	0	-2	6
2	4	2	0	2	-4	4
4	4	2	0	0	-4	8
2	4	2	2	0	-4	8
3	2	1	3	0	-4	8
2	2	1	2	3	-6	4
3	4	2	0	2	-6	6
3	4	2	2	0	-6	10
4	4	3	0	0	-6	10

Consider the double cover $\pi : Y \rightarrow S$ branched over E_4 and Z'' given by $2(K_S - L_2 - Z') \equiv E_4 + Z''$ and check that $E_4 + Z''$ has at most negligible singularities. So Y has at most canonical singularities, applying Formula 2.1, we get

$$\chi(Y) = 2 + \frac{1}{2}(K_S + \frac{E_4 + Z''}{2}) \frac{E_4 + Z''}{2} = 2 + \frac{1}{2}(1 + \frac{E_4^2 + Z''^2 + 2E_4Z''}{2}),$$

$$p_g(Y) = h^0(S, 2K_S - L_2 - Z') = h^0(S, L_2 + Z' + E_4 + Z'') \geq 2.$$

Claim 7.22. $\chi(Y) \leq 2$ and $q(Y) \geq 1$.

Proof. We only need to show that $\frac{1}{2}(1 + \frac{E_4^2 + Z''^2 + 2E_4Z''}{4}) < 1$, i.e., $E_4^2 + Z''^2 + 2E_4Z'' < 4$. If $Z'' = 0$, then we are done. Now we assume $Z'' \neq 0$, then $Z''^2 = -2$ or -4 , so it suffices to show that $E_4Z'' < 3 - \frac{E_4^2}{2}$. Observing that $E_4Z'' \leq \frac{E_4Z}{\min\{a,b,c\}}$, we exclude all the possibilities listed above except the case $Z = 3\theta_1 + 2\theta_2 + \theta_3$, $E_4Z = 6$ and $Z'' = \theta_1 + \theta_3$. Immediately we get $Z\theta_3 = 0$ which implies that $E_4\theta_3 = 0$. By $E_4Z = 6$, we have $E_4\theta_1 \leq 2$, hence $E_4Z'' \leq 2 < 4$. Then the claim holds true. \square

Denote by $\alpha : Y \rightarrow B$ the Albanese map to the image. By Proposition 2.1, we get the following commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & S \\ \alpha \downarrow & & g \downarrow \\ B & \xrightarrow{\pi'} & \mathbb{P}^1 \end{array}$$

where $\pi' : B \rightarrow \mathbb{P}^1$ is a double cover. Checking that $(E_4 + Z'')\theta_i \leq 3$, we can see the inverse image $\phi^{-1}\theta_i$ is composed with 1 or 2 rational curves for $i = 1, 2, 3$, hence θ_i is contained in one fiber of g . By Remark 7.5, applying Proposition 7.4, the fibration $g : S \rightarrow \mathbb{P}^1$ is given by the pencil $|F| = \phi^*|\bar{l}_1|$. Let $P = g(E_4)$, then $g^*P_4 = 2E_4 + D$ since $2\bar{e}_4 \equiv \bar{l}_1 - \bar{e}_1$. Let $Z = B \times_{\mathbb{P}^1} S$, and let $\tilde{Z} \rightarrow Z$ be the normalization. Since $g^*P = 2E_4 + D$ and π' is branched over P , the map $p : \tilde{Z} \rightarrow S$ is not branched over E_4 . There is a morphism $h : Y \rightarrow \tilde{Z}$ such that $\pi = p \circ h$. Note that $h : Y \rightarrow \tilde{Z}$ is birational morphism since both π and p have the same degree 2. So the map $\pi : Y \rightarrow S$ is not branched over E_4 which contradicts the construction. Therefore this case does not occur.

Case 2: $\phi^{-1}Q$ is composed with two disjoint (-2) -curves.

Assume that $\phi^{-1}Q = \theta_1 \cup \theta_2$ where θ_1 and θ_2 are two disjoint (-2) -curves. Remark that there might be another (-2) -curve θ_3 disjoint from θ_1, θ_2 mapped to \bar{e}_1 via ϕ .

Subcase 1: There is no such a (-2) -curve θ_3 that is disjoint from θ_1, θ_2 and mapped to \bar{e}_1 via ϕ .

By the proof of the claim 7.21, we have $\phi^*\bar{e}_1 = E_1 + \frac{1}{2}(\theta_1 + \theta_2)$ where a is an integer and is set to be zero if there is no such a (-2) -curve, $\phi^*\bar{l}_2 = L_2 + \frac{1}{2}(\theta_1 + \theta_2)$, $E_1^2 = -2$, $L_2^2 = 2$ and $E_1L_2 = 0$, in particular, E_1 is reduced.

By $-K_{\hat{P}} \equiv 3l - e_1 - e_2 - 2e_3 - 3e_4 \equiv 3\bar{l}_2 - e_1 + 2e_2 + e_3$, we have $-K_{\Sigma} \equiv 3\bar{l}_2 - \bar{e}_1$. So we have $2K_S \equiv 3L_2 - E_1 + \theta_1 + \theta_2$, thus $2(K_S - L_2 + E_1) \equiv L_2 + E_1 + \theta_1 + \theta_2$. In turn we get a double cover $\pi : Y \rightarrow S$. Similarly, check that Y has at most canonical singularities. Calculating the invariants of Y , we obtain

$$\chi(Y) = 4, \quad p_g(Y) \geq 4, \quad q(Y) \geq 1.$$

Note that $p_g(Y) \geq 4$ is due to $p_g(Y) \geq h^0(\mathcal{O}_{\Sigma}(2\bar{l}_2)) = h^0(\hat{P}, 2l - e_2 - e_3 - 2e_4) = 4$. By Proposition 2.1, we get the following commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & S \\ \alpha \downarrow & & g \downarrow \\ B & \xrightarrow{\pi'} & \mathbb{P}^1 \end{array}$$

where $\alpha : Y \rightarrow B$ is the Albanese map and $\pi' : B \rightarrow \mathbb{P}^1$ is a double cover. L_2 must be contained in one fiber, but this contradicts $L_2^2 = 2 > 0$. So this case does not occur.

Subcase 2: There is a (-2) -curve θ_3 disjoint from θ_1, θ_2 mapped to \bar{e}_1 via ϕ .

Note that θ_3 is not mapped to \bar{e}_4 and \bar{l}_2 , so $\phi^*\bar{e}_4 = E_4 + (\theta_1 + \theta_2)$ and $\phi^*\bar{l}_2 = L_2 + \frac{1}{2}(\theta_1 + \theta_2)$. Therefore we get the relation $2K_S \equiv 2L_2 + E_4 + 2(\theta_1 + \theta_2)$. Then similar argument as in Case 1 shows that this case does not occur.

In conclusion, we prove that $\hat{P} \neq \hat{P}_5$

7.2.6. *The case $\hat{P} = \hat{P}_6$.*

Notation 7.23. *Let Q be the A_4 -singularity on Σ . We denote by l_1 the strict transform of a general line through P_1 and by c the strict transform of the line through P_1, P_2, P_3 .*

Immediately it follows that $l_1 \equiv l - e_1 - e_2 - e_3 - e_4$ and $c \equiv l - e_1 - 2e_2 - 3e_3 - 3e_4$. Then we have:

$$-K_{\hat{P}} \equiv 3l - e_1 - 2e_2 - 3e_3 - 4e_4 \equiv 2l_1 + e_4 + c + 2e_1 + 2e_2 + 2e_3$$

and

$$-K_{\Sigma} \equiv 2\bar{l}_1 + \bar{e}_4$$

About the (-2) -curves on S , we have the following claim:

Claim 7.24. $\phi^{-1}Q$ is composed with 4 (-2) -curves: $\theta_1, \theta_2, \theta_3, \theta_4$ such that $\theta_i\theta_{i+1} = 1$ for $i = 1, 2, 3$.

Proof. Considering the self-intersection $\bar{l}_1^2 = \frac{4}{5}$, then the claim follows from similar argument as in the proof of Claim 7.15 in Section 7.2.3. \square

By the claim, \hat{P} and S has the same Picard number. Denote by L_1, E_4 the strict transforms of \bar{l}_1, \bar{e}_4 respectively. We can write $2K_S = 2L_1 + E_4 + Z$, and assume $Z = a\theta_1 + b\theta_2 + c\theta_3 + d\theta_4$ where a, b, c, d are positive integers and $a \geq d$.

Claim 7.25. *There exists a fibration $g : S \rightarrow \mathbb{P}^1$ such that every (-2) -curve is contained in a fiber.*

Proof. Case 1: E_4 is reduced.

Arguing as in Section 7.2.5 Case 1, for the readers' convenience, we list all the possibilities for Z and the the intersection numbers of the divisors involved in the

proof:

a	b	c	d	L_1^2	L_1E_4	E_4^2	E_4Z
1	2	2	1	2	3	-4	2
2	2	2	1	2	2	-2	2
2	2	2	2	2	2	-4	4
2	3	2	1	2	2	-4	4
2	3	3	2	2	1	-2	4
2	3	4	2	0	2	-2	2
2	3	4	2	2	0	-2	6
2	4	3	2	2	0	-2	6
2	4	4	2	0	2	-4	4
2	4	4	2	2	0	-4	8
3	3	2	1	0	3	-4	2
3	3	2	1	2	1	-4	6
3	3	3	2	0	2	-2	2
3	3	3	3	0	1	-2	4
3	4	3	2	2	0	-4	8
3	4	3	2	0	2	-4	4
3	4	4	2	0	1	-2	4
3	4	4	3	0	1	-4	6
3	4	5	3	0	0	-4	8
3	5	4	2	0	0	-2	6
3	5	4	3	0	0	-4	8
4	3	2	1	0	1	-4	6
4	4	3	2	0	0	-2	6
4	4	4	2	0	0	-4	8
2	3	3	2	0	4	-6	2
1	1	1	1	2	4	-6	2
2	2	2	1	2	3	-6	4
2	3	3	2	2	2	-6	6
2	3	4	2	0	3	-6	4
2	3	4	2	2	1	-6	8
2	4	3	2	0	3	-6	4
2	4	3	2	2	1	-6	8
3	3	3	2	0	3	-6	4
3	3	3	2	2	1	-6	8
3	3	3	3	0	2	-6	6
3	3	3	3	2	0	-6	10
3	4	4	2	0	2	-6	6
3	4	4	2	2	0	-6	10
3	5	4	2	0	1	-6	8
3	5	5	3	0	0	-6	10
4	4	3	2	0	1	-6	8
4	4	4	3	0	0	-6	10
4	5	4	2	0	0	-6	10

Then going similar progress as in Section 7.2.5 Case 1, we get a fibration $g : S \rightarrow \mathbb{P}^1$ such that every (-2) -curve is contained in a fiber.

Case 2: E_4 is non-reduced.

Considering the relation $2K_S = 2L_1 + E_4 + Z$, using double cover trick, it is easy to prove that there exists such a fibration. \square

By the claim, Remark 7.5 tells that there is a fibration $\hat{g} : \hat{P} \rightarrow \mathbb{P}^1$ of \hat{P} such that every (-2) -curve is contained in one fiber. Then a contradiction follows from Proposition 7.4, and we are done.

7.2.7. *Conclusion.* Finally, we complete the proof of Theorem 1.3.

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