Reduction of Multidimensional Wave Equations to Two-Dimensional Equations: Investigation of Possible Reduced Equations

Irina YEHORCHENKO

Institute of Mathematics of NAS Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine
E-mail: iyegorch@imath.kiev.ua

Abstract

We study possible Lie and non-classical reductions of multidimensional wave equations and the special classes of possible reduced equations - their symmetries and equivalence classes. Such investigation allows to find many new conditional and hidden symmetries of the original equations.

1 Why nonlinear wave equation

We study Lie and non-classical reductions of multidimensional wave equations and special classes of possible reduced equations - their symmetries and equivalence classes, as well as types of the reduced equations which represent interesting classes of two-dimensional equations - parabolic, hyperbolic and elliptic. This paper carries on the discussion in [1].

Ansatzes and methods used for reduction of the d'Alembert (n-dimensional wave) equation can be also used for arbitrary Poincaré-invariant equations. Later we will show that this seemingly simple and partial problem involves many important aspects in the studies of the PDE.

The topic we consider demonstrates relations of the symmetry methods (see e.g. [2], [3]) to other aspects of investigation of PDE - compatibility of systems of equations, methods of finding general solutions (e.g. by means of hodograph transformations).

The methods we used were not fully algorithmic - it was necessary to decide when to switch methods, and lot of hypotheses had to be tested.

We consider the multidimensional wave equation

$$\Box u = F(u), \qquad (1)$$
$$\Box \equiv \partial^2_{x_0} - \partial^2_{x_1} - \cdots - \partial^2_{x_n}, \quad u = u(x_0, x_1, \ldots, x_n)$$

It seems thoroughly studied and almost trivial. Let us list only some papers where solutions of this equation are studied specifically - [4]-[12].

However, this equation appears to have many new facets and ideas to discover. Let us point out that investigation of hyperbolic equations, both with respect to their conditional symmetry and classification, is considerably more difficult than the same problem for equations in which at least for one variable partial derivatives have only lower order than the order of equation.

2 Reduction of nonlinear wave equations - ansatz

We found conditions of reduction of the multidimensional wave equation

$$\Box u = F(u),$$

by means of the ansatz with two new independent variables.

$$u = \varphi(y, z), \qquad (2)$$

where y, z are new variables. Henceforth n is the number of independent space variables in the initial d'Alembert equation.

Reduction conditions for such ansatz are a system of the d'Alembert and three Hamilton-type equations:

$$y_\mu y_\mu = r(y, z), \quad y_\mu z_\mu = q(y, z), \quad z_\mu z_\mu = s(y, z), \qquad (3)$$
$$\Box y = R(y, z), \quad \Box z = S(y, z).$$

We proved necessary conditions for compatibility of such system of the reduction conditions . However, the resulting conditions and reduced equations needed further research.

3 General background

Symmetry reduction to equations with smaller number of independent variables or to ordinary differential equations (for the algorithms see e.g. the books by Ovsyannikov [2] or Olver [3]).

"Direct method" (giving wider classes of solutions than the symmetry reduction) was proposed by P. Clarkson and M. Kruskal [13]). See more detailed investigation of the direct reduction and conditional symmetry in [4], [13]-[19]. This method for majority of equations results in considerable difficulties as it requires investigation of compatibility and solution of cumbersome reduction conditions of the initial equation.

These reduction conditions are much more difficult for investigation and solution in the case of equations containing second and/or higher derivatives for all independent variables, and for multidimensional equations - e.g. in the situation of the nonlinear wave equations.

We would like to point out once more that the problem we consider has two specific difficulties. First, it is always more technically difficult to work with hyperbolic equations such as the nonlinear wave equation than with parabolic ones (such as evolution equations). Second, normally the methods and algorithms for working with reductions and exact solutions are designed and applied for a limited number of variables - usually two or three. Here we work with arbitrary number of variables, though we limit the number of variables for specific examples.

4 Compatibility of the reduction conditions: summary

A similar problem was considered previously for an ansatz with one independent variable

$$u = \varphi(y), \tag{4}$$

where y is a new independent variable.

Compatibility analysis of the d'Alembert–Hamilton system

$$\Box u = F(u), \quad u_\mu u_\mu = f(u) \tag{5}$$

in the three-dimensional space was done by Collins [20].

The sufficient conditions of reduction of the wave equation to an ODE and the general solution of the system (5) in the case of three spatial dimensions were found by Fushchych, Zhdanov, Revenko [21]. For discussion of previous results in this area see [22]. It is evident that the d'Alembert–Hamilton system (5) may be reduced by local transformations to the form

$$\Box u = F(u), \quad u_\mu u_\mu = \lambda, \quad \lambda = 0, \pm 1. \tag{6}$$

Statement[23]. *For the system* (6) $(u = u(x_0, x_1, x_2, x_3))$ *to be compatible it is necessary and sufficient that the function* F *have the following form:*

$$F = \frac{\lambda}{N(u + C)}, \quad N = 0, 1, 2, 3.$$

Ansatzes of the type (2) for some particular cases was studied in [24]-[27].

5 Transformations of compatibility conditions

Substitution of ansatz $u = \varphi(y, z)$ into the equation $\Box u = F(u)$ leads to the following equation (see [1]):

$$\varphi_{yy} y_\mu y_\mu + 2\varphi_{yz} z_\mu y_\mu + \varphi_{zz} z_\mu z_\mu + \varphi_y \Box y + \varphi_z \Box z = F(\varphi) \tag{7}$$
$$\left(y_\mu = \frac{\partial y}{\partial x_\mu}, \quad \varphi_y = \frac{\partial \varphi}{\partial y} \right),$$

whence we get a system of equations:

$$\begin{aligned} &y_\mu y_\mu = r(y, z), \quad y_\mu z_\mu = q(y, z), \quad z_\mu z_\mu = s(y, z), \\ &\Box y = R(y, z), \quad \Box z = S(y, z). \end{aligned} \tag{8}$$

System (8) is a reduction condition for the multidimensional wave equation (1) to the two-dimensional equation (7) by means of ansatz $u = \varphi(y, z)$.

The system of equations (8), depending on the sign of the expression $rs - q^2$, may be reduced by local transformations to one of the following types:

1) elliptic case: $rs - q^2 > 0$, $v = v(y,z)$ is a complex–valued function,

$$\Box v = V(v, v^*), \quad \Box v^* = V^*(v, v^*),$$
$$v^*_\mu v_\mu = h(v, v^*), \quad v_\mu v_\mu = 0, \quad v^*_\mu v^*_\mu = 0 \tag{9}$$

(the reduced equation is of the elliptic type);

2) hyperbolic case: $rs - q^2 < 0$, $v = v(y,z)$, $w = w(y,z)$ are real functions,

$$\Box v = V(v, w), \quad \Box w = W(v, w),$$
$$v_\mu w_\mu = h(v, w), \quad v_\mu v_\mu = 0, \quad w_\mu w_\mu = 0 \tag{10}$$

(the reduced equation is of the hyperbolic type);

3) parabolic case: $rs - q^2 = 0$, $r^2 + s^2 + q^2 \neq 0$, $v(y,z)$, $w(y,z)$ are real functions,

$$\Box v = V(v, w), \quad \Box w = W(v, w),$$
$$v_\mu w_\mu = 0, \quad v_\mu v_\mu = \lambda \ (\lambda = \pm 1), \quad w_\mu w_\mu = 0 \tag{11}$$

(if $W \neq 0$, then the reduced equation is of the parabolic type);

4) first-order equations: $(r = s = q = 0)$, $y \to v$, $z \to w$

$$v_\mu v_\mu = w_\mu w_\mu = v_\mu w_\mu = 0,$$
$$\Box v = V(v, w), \quad \Box w = W(v, w). \tag{12}$$

Elliptic case

Theorem 1. *System* (9) *is compatible if and only if*

$$V = \frac{h(v, v^*)\partial_{v^*}\Phi}{\Phi}, \quad \partial_{v^*} \equiv \frac{\partial}{\partial v^*},$$

where Φ is an arbitrary function for which the following condition is satisfied

$$(h\partial_{v^*})^{n+1}\Phi = 0.$$

The function h may be represented in the form $h = \frac{1}{R_{vv^*}}$, where R is an arbitrary sufficiently smooth function, R_v, R_{v^*} are partial derivatives by the respective variables.

Then the function Φ may be represented in the form $\Phi = \sum\limits_{k=0}^{n+1} f_k(v) R_v^k$, where $f_k(v)$ are arbitrary functions, and

$$V = \frac{\sum\limits_{k=1}^{n+1} k f_k(v) R_v^k}{\sum\limits_{k=0}^{n+1} f_k(v) R_v^k}.$$

The respective reduced equation will have the form

$$h(v, v^*) \left(2\phi_{vv^*} + \phi_v \frac{\partial_{v^*}\Phi}{\Phi} + \phi_{v^*} \frac{\partial_v \Phi^*}{\Phi^*} \right) = F(\phi). \tag{13}$$

The equation (13) may also be rewritten as an equation with two real independent variables ($v = \omega + \theta$, $v^* = \omega - \theta$):

$$\widetilde{h}(\omega, \theta)(\phi_{\omega\omega} + \phi_{\theta\theta}) + \Omega(\omega, \theta)\phi_\omega + \Theta(\omega, \theta)\phi_\theta = F(\phi).$$

Hyperbolic case

Theorem 2. *System* (10) *is compatible if and only if*

$$V = \frac{h(v, w)\partial_w \Phi}{\Phi}, \quad W = \frac{h(v, w)\partial_v \Psi}{\Psi},$$

where the functions Φ*,* Ψ *are arbitrary functions for which the following conditions are satisfied*

$$(h\partial_v)^{n+1}\Psi = 0, \quad (h\partial_w)^{n+1}\Phi = 0.$$

The function h may be presented in the form $h = \frac{1}{R_{vw}}$, where R is an arbitrary sufficiently smooth function, R_v, R_w are partial derivatives by the respective variables. Then the functions Φ, Ψ may be represented in the form

$$\Phi = \sum_{k=0}^{n+1} f_k(v) R_v^k, \quad \Psi = \sum_{k=0}^{n+1} g_k(w) R_w^k,$$

where $f_k(v)$, $g_k(w)$ are arbitrary functions,

$$V = \frac{\sum\limits_{k=1}^{n+1} k f_k(v) R_v^k}{\sum\limits_{k=0}^{n+1} f_k(v) R_v^k}, \quad W = \frac{\sum\limits_{k=1}^{n+1} k g_k(w) R_w^k}{\sum\limits_{k=0}^{n+1} g_k(w) R_w^k}.$$

The respective reduced equation will have the form

$$h(v,w)\left(2\phi_{vw} + \phi_v \frac{\partial_w \Phi}{\Phi} + \phi_w \frac{\partial_v \Psi}{\Psi}\right) = F(\phi). \tag{14}$$

The equation (14) may also be rewritten as a standard wave equation ($v = \omega + \theta$, $w = \omega - \theta$):

$$\widetilde{h}(\omega,\theta)(\phi_{\omega\omega} - \phi_{\theta\theta}) + \Omega(\omega,\theta)\phi_\omega + \Theta(\omega,\theta)\phi_\theta = F(\phi).$$

Parabolic case

Theorem 3. *System* (11) *is compatible if and only if*

$$V = \frac{\lambda \partial_v \Phi}{\Phi}, \quad \partial_v^{n+1}\Phi = 0, \quad W \equiv 0.$$

Equation $\Box u = F(u)$ by means of ansatz $u = \varphi(y,z)$ cannot be reduced to a parabolic equation – in this case one of the variables will enter the reduced ordinary differential equation of the first order as a parameter.

System (12) is compatible only in the case if $V = W \equiv 0$, that is the reduced equation may be only an algebraic equation $F(u)$=0. Thus we cannot reduce equation $\Box u = F(u)$ by means of ansatz $u = \varphi(y,z)$ to a first-order equation.

Proof of the above theorems is done by means of the well-known Hamilton–Cayley theorem, in accordance to which a matrix is a root of its characteristic polynomial.

6 Reduction and conditional symmetry

Solutions obtained by the direct reduction are related to symmetry properties of the equation – Q-conditional symmetry of this equation (symmetries of such type are also called non-classical or non-Lie symmetries. It is also possible to see from the previous papers that symmetry of the two-dimensional reduced equations is often wider than symmetry of the initial equation, that is the reduction to two-dimensional equations allows to find new non-Lie solutions and hidden symmetries of the initial equation (see e.g. papers by Abraham-Schrauner and Leach [28], [29]) The Hamilton equation may also be considered, irrespective of the reduction problem, as an additional condition for the d'Alembert equation that allows extending the symmetry of this equation.

Let us look at the wave equation in two spatial dimensions. Reduction of $\Box u = F(u)$ by our ansatz $u = \varphi(v, w)$ means Q-conditional invariance this equation under the operator

$$Q = \partial_{x_o} + \tau_1(x_o, x_1, x_2)\partial_{x_1} + \tau_2(x_o, x_1, x_2)\partial_{x_2}.$$

This equivalence of reduction and Q-conditional symmetry was proved by Zhdanov, Tsyfra and Popovych [18]. New variables v, w are invariants of the operator Q:

$$Qv = Qw = 0.$$

7 Study of the reduced equations

Equivalence of quasilinear wave equations is studied well, but we consider a particular class of such equations.

We consider the reduced equation of the form

$$h(v,w)\left(2\phi_{vw} + \phi_v\frac{\partial_w\Phi}{\Phi} + \phi_w\frac{\partial_v\Psi}{\Psi}\right) = F(\phi).$$

where the functions Φ, Ψ are arbitrary functions for which the following conditions are satisfied

$$(h\partial_v)^{n+1}\Psi = 0, \quad (h\partial_w)^{n+1}\Phi = 0.$$

Equivalence transformation of the reduced equations are only of the type

$$h(v,w) \to k(v)l(w)h(v,w), v \leftrightarrow w; \phi \to a\phi + b.$$

There will be special additional equivalence groups only for special forms of the function F. Special class of the reduced equations - $h(v,w) = k(v)l(w)$; in this case the equations can be reduced to the case $h(v,w) = const$. All symmetry reductions have $h(v,w) = const$ and linear Φ, Ψ.

We have quite narrow equivalence group of the reduced equation as we actually took a single representative of an equivalence class of hyperbolic reduced equations.

Description of all possible reductions involves classification of the reductions found and nomination of certain inequivalent representatives. Any classification problem is a description of equivalence classes under certain equivalence relations.

Selection of an equivalence group for classification may be in principle arbitrary, but as a rule one of the following is selected: either the symmetry group of the conditions describing the initial limited class or the group of automorphisms of some general class.

There is a generally accepted method for classification of symmetry reductions - by subalgebras inequivalent up to conjugacy. This method does not work for general reductions, and we have to choose another method of classification.

Another important note - if we do classification in several steps, we have to consider commutativity and associativity of classification conditions (e.g. under some equivalence group) adopted at each step.

8 Example: Solutions for the two-dimensional case

We will look for parametric solutions for the system

$$\Box v = V(v,w), \quad \Box w = W(v,w),$$
$$v_\mu w_\mu = h(v,w), \quad v_\mu v_\mu = 0, \quad w_\mu w_\mu = 0, \quad \mu = 0,1,2$$

First we construct parametric or explicit solutions for the equations $w_\mu w_\mu = 0), v_\mu v_\mu = 0$, and then use them to find solutions of other equations.

Rank 0

General solution of the equations $v_\mu v_\mu = 0$, $w_\mu w_\mu = 0$

$$
\begin{aligned}
&v = A_\mu x_\mu + B, w = C_\mu x_\mu + D;\\
&A_\mu A_\mu = 0, C_\mu C_\mu = 0.
\end{aligned}
$$

p, q are parametric functions on x,

A_μ $(\mu = 1, 2)$, B, C_μ $(\mu = 1, 2)$, D are arbitrary constants up to conditions. In this case h will be constant, $\Box v = \Box w = 0$ and we will have solutions that can be obtained by symmetry reduction.

Rank 1

General solution of the equations $v_\mu v_\mu = 0$, $w_\mu w_\mu = 0$

$$
\begin{aligned}
&v = A_\mu(p)x_\mu + B(p), w = C_\mu(q)x_\mu + D(q);\\
&A^\mu_p x_\mu + B_p = 0; C^\mu_q x_\mu + D_q = 0;\\
&A_\mu A_\mu = 0, C_\mu C_\mu = 0.
\end{aligned}
$$

p, q are parametric functions on x,

A_μ $(\mu = 1, 2)$, B, C_μ $(\mu = 1, 2)$, D are arbitrary functions up to conditions.

Rank 2

General solution of the equations $v_\mu v_\mu = 0$, $w_\mu w_\mu = 0$

$$
\begin{aligned}
&v = A_\mu(p_1, p_2)x_\mu + B(p_1, p_2), w = C_\mu(q_1, q_2)x_\mu + D(q_1, q_2);\\
&A^\mu_{p_k} x_\mu + B_{p_k} = 0; C^\mu_{q_k} x_\mu + D_{q_k} = 0;\\
&A_\mu A_\mu = 0, C_\mu C_\mu = 0.
\end{aligned}
$$

p, q are parametric functions on x,

A_μ $(\mu = 1, 2)$, B, C_μ $(\mu = 1, 2)$, D are arbitrary functions up to conditions.

It is easy to prove that that for $v_\mu w_\mu = h(v, w)$ solutions of $v_\mu v_\mu = 0$, $w_\mu w_\mu = 0$, should have the same rank. Further we can find partial parametric solutions taking the same parameter functions p for v and w. This way we will have new non-Lie solutions with hidden infinite symmetry. (For definition of hidden symmetry see [28]).

It is well-known [30] that the general solution of the system 5 with $F = f = 0$, $n = 1, 2$ can be written as

$$
\begin{aligned}
&u = A_\mu(p_1, p_2)x_\mu + B(p_1, p_2),\\
&A^\mu_{p_k} x_\mu + B_{p_k} = 0,\\
&A_\mu A_\mu = 0, A^\mu_{p_k} A^\mu_{p_m} = 0.
\end{aligned}
$$

Similarly we can construct a parametric solution for (10) with $V = W = 0, h = const.$

$$
\begin{aligned}
&v = A_\mu(p_1, p_2)x_\mu + B(p_1, p_2),\\
&A^\mu_{p_k} x_\mu + B_{p_k} = 0,\\
&A_\mu A_\mu = 0, A^\mu_{p_k} A^\mu_{p_m} = 0;\\
&w = C_\mu(p_1, p_2)x_\mu + D(p_1, p_2)\\
&C^\mu_{p_k} x_\mu + D_{p_k} = 0,\\
&C_\mu C_\mu = 0, C^\mu_{p_k} C^\mu_{p_m} = 0;\\
&A_\mu C_\mu = const.
\end{aligned}
\tag{15}
$$

The operator of Q-conditional symmetry that gives such ansatz will have the form

$$
\begin{aligned}
&Q = \partial_0 + \tau_1\partial_1 + \tau_2\partial_2,\\
&\tau_1 = \frac{C_0A_2 - A_0C_2}{A_1C_2 - A_2C_1}, \tau_2 = \frac{C_0A_1 - A_0C_1}{A_1C_2 - A_2C_1}.
\end{aligned}
$$

9 Conclusion

The topic we discuss is closely related to majority of main ideas in the symmetry analysis of PDE - direct reduction of PDE; conditional symmetry; Q-conditional symmetry; finding solutions directly using nonlocal transformations; group classification of equations and systems of equations;

Our general problem - study of reductions of the nonlinear wave equation (and of other equations in general) requires several classifications up to equivalence on the way.

At each step we have to define correctly the criteria of equivalence, and check commutativity and associativity of these equivalence conditions - or otherwise take into account lack of such properties.

10 Further research

1. Study of Lie and conditional symmetry of the system of the reduction conditions.
2. Investigation of Lie and conditional symmetry of the reduced equations. Finding exact solutions of the reduced equations.
3. Finding of places of previously found exact solutions on the general equivalence map.
4. Relation of the equivalence group of the class of the reduced equations with symmetry of the initial equation.
5. Finding and investigation of compatibility conditions and classes of the reduced equations for other types of equations, in particular, for Poincaré–invariant scalar equations.

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