# A MISSED PERSISTENCE PROPERTY FOR THE EULER EQUATIONS, AND ITS EFFECT ON INVISCID LIMITS

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**Abstract**

We consider the problem of the *strong* convergence, as the viscosity goes to zero, of the solutions to the three-dimensional evolutionary Navier-Stokes equations under a Navier slip-type boundary condition to the solution of the Euler equations under the zero-flux boundary condition. In spite of the arbitrarily strong convergence results proved in the flat boundary case, see [4], it was shown in reference [5] that the result is false in general, by constructing an explicit family of smooth initial data in the sphere, for which the result fails. Our aim here is to present a more general, simpler and incisive proof. In particular, counterexamples can be displayed in arbitrary, smooth, domains. As in [5], the proof is reduced to the lack of a suitable persistence property for the Euler equations. This negative result is proved by a completely different approach.

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## 1 Introduction and some results.

We investigate *strong convergence, up to the boundary*, as $\nu \to 0$, of the solutions $\underline{u}^\nu$ of the Navier-Stokes equations

$$(1.1)\qquad \begin{cases} \partial_t \underline{u}^\nu + (\underline{u}^\nu \cdot \nabla)\, \underline{u}^\nu + \nabla p = \nu\, \Delta\, \underline{u}^\nu\,, \\ div\, \underline{u}^\nu = 0\,, \\ \underline{u}^\nu(0) = \underline{a}\,, \end{cases}$$

under the boundary condition

$$(1.2)\qquad \begin{cases} \underline{u}^\nu \cdot \underline{n} = 0, \\ \underline{\omega}^\nu \times \underline{n} = 0\,, \end{cases}$$

to the solution $\underline{u}$ of the Euler equations

$$(1.3)\qquad \begin{cases} \partial_t \underline{u} + (\underline{u} \cdot \nabla)\, \underline{u} + \nabla p = 0, \\ div\, \underline{u} = 0\,, \\ \underline{u}(0) = \underline{a}\,, \end{cases}$$

under the zero-flux boundary condition

$$(1.4)\qquad \underline{u} \cdot \underline{n} = 0\,.$$

We are interested in the three-dimensional situation. Here, and in the sequel, we use the notation $\underline{\omega} = curl\, \underline{u}$.

**Definition 1.1.** *We say that a vector field* $\underline{a}$ *is* admissible *if it is smooth and divergence free in* $\overline{\Omega}$*, and satisfies the boundary condition*

$$\begin{cases} \underline{a} \cdot \underline{n} = 0, \\ \underline{b} \times \underline{n} = 0\,, \end{cases} \tag{1.5}$$

*where*

$$\underline{b} = curl\, \underline{a}\,. \tag{1.6}$$

The above condition is usually referred to as a Navier slip-type boundary condition.

Recently the vanishing viscosity limit problem under the above or similar Navier type conditions has been studied in [2], [8], [12], [14], in the 2D case, and in [2], [3], [5], [7], [21], in the 3D case. See also [23] for the magnetohydrodynamic system and [6] for a different approach to the inviscid limit for the slip-type boundary value problem.

*The domain* $\Omega$ : In the sequel $\Omega$ is a bounded open set in $\mathbb{R}^3$ locally situated on one side of its boundary, a smooth manifold $\Gamma$. The boundary $\Gamma$ may consist of a finite number of disjoint, connected components, $\Gamma_j$, $j = 0, 1, ..., m$, $m \geq 0$. $\Gamma_0$ denotes the "external boundary". If $\Gamma$ is not simply-connected we assume the typical existence of $N$ mutually disjoint and transversal cuts, after which $\Omega$ becomes simply-connected (see [10] and [19] for details).

We denote by $\underline{n} = (n_1, n_2, n_3)$ the unit outward normal to $\Gamma$, and denote by $\kappa_j(s)$, $j = 1, 2$, the *principal curvatures* of $\Gamma$ at a point $s$. We set

$$\Sigma = \{ s \in \Gamma : \kappa_j(s) \neq 0\,, \quad j = 1, 2 \}\,. \tag{1.7}$$

$\Sigma$ is the subset of boundary-points where the Gaussian curvature $\kappa_1 \kappa_2$ does not vanish. It is worth noting that, for $\Omega$ as above, $\Sigma$ is never empty. Mostly, $\Sigma$ coincides with $\Gamma$ itself.

We recall that application of the operator *curl* to the first equation (1.3) leads to the well known Euler vorticity equation

$$\partial_t\, \underline{\omega} - curl\, (\underline{u} \times \underline{\omega}) = 0\,. \tag{1.8}$$

**Definition 1.2.** We say that the Euler equations (1.3), under the boundary condition (1.4), satisfy the *persistence property* (with respect to the boundary condition $\underline{\omega} \times \underline{n} = 0$, and to the initial data $\underline{a}$), if $\underline{\omega}(0) \times \underline{n} = 0$ on $\Gamma$ implies the existence of some $t_0 > 0$ (which may depend on $\underline{a}$) such that $\underline{\omega}(t) \times \underline{n} = 0$ on $\Gamma$, for each $t \in (0, t_0)$. Furthermore, we say that the persistence property holds, if it holds for all (smooth) initial data $\underline{a}$.

**Definition 1.3.** By *strong convergence* we mean any (sufficiently strong) convergence in $(0, T) \times \Omega$ such that if $\underline{u}^\nu$ converges to $\underline{u}$ with respect to this convergence, and if $\underline{\omega}^\nu \times \underline{n} = 0$ on $\Gamma$, then necessarily $\underline{\omega} \times \underline{n} = 0$ on $(0, T) \times \Gamma$. *Strong inviscid limit* is defined accordingly.

Examples of strong convergence are, for instance, convergence in $L^1(0,\,T;W^{s,q})$, for some $q>\,1$, and some $s>\,1+\frac{1}{q}$, and convergence in $L^1(0,\,T;W^{2,1})$. Recall that $L^1(0,\,T)$ convergence implies a.e. convergence in $(0,\,T)$, for suitable "sub-sequences".

A strong inviscid limit result, without a spatial boundary, was proved in [11]. See also the more recent papers [1] and [15]. In [21], [2], [3], and [4], strong inviscid limit results are proved under a flat-boundary assumption. However, in the case of non-flat boundaries the problem remained open. The arbitrarily strong convergence results proved in [4], some estimates proved for non-flat boundaries in [2] and [3], and the strong convergence results available in the two-dimensional case, led to the conviction that strong convergence results held in the general three-dimensional case, at least in "moderately strong" topologies. In spite of this guess, in reference [5] we have shown that the result is false in a sphere.

In reference [5], section 2, the following result is proved.

**Theorem 1.1.** *Let the initial data* $\underline{a}$ *be admissible. Then:*

*a) If a strong inviscid limit result holds, then necessarily the Euler equations* (1.3), (1.4) *enjoy the persistence property.*

*b) If the persistence property holds, then necessarily*

$$curl\,(\underline{a}\times\,\underline{b})\,\times\,\underline{n}=\,0 \tag{1.9}$$

*everywhere on* $\Gamma$.

The proof of this particularly useful result is astonishingly simple. Actually, a strong inviscid limit result immediately implies the persistence property for the Euler equations. Assume now the persistence property. External multiplication of (1.8) by $\underline{n}$ gives

$$\partial_t\,(\underline{\omega}\times\,\underline{n}\,)-\,curl\,(\underline{u}\times\underline{\omega})\,\times\underline{n}=\,0\,. \tag{1.10}$$

Since the persistency property holds, the time derivative in the above equation must vanish on $\Gamma$, at time $t=\,0$. So, the second term must verify this same property. That such a simple short-cut remained hidden may be due to its being extremely elementary.

For convenience, we state the above result in the following equivalent form.

**Theorem 1.2.** *Let* $\underline{a}$ *be an admissible vector field. Then:*

*a) If, in some point* $x_0\in\,\Gamma$, *the inequality*

$$curl\,(\underline{a}\times\,\underline{b})\,\times\,\underline{n}\neq\,0 \tag{1.11}$$

*holds, then the persistence property, with respect to the initial data* $\underline{a}$, *fails.*

*b) If the persistence property fails then, necessarily, any strong inviscid limit result fails.*

Clearly, if the condition (1.11) holds in $x_0$, it holds in a $\Gamma-$neighborhood of this same point.

It follows from the above theorem that in order to prove the failure of the persistence property and, a fortiori, that of strong inviscid limit results, it is sufficient to show the existence of admissible vector fields $\underline{a}$ which satisfy (1.11) somewhere in $\Gamma$. We will show that, given $\Omega$ as above, there exist a large family of such $\underline{a}$. For fixing ideas, we state our main result in its simplest form. Actually, our argument leads to more precise and deeper versions of Theorem 1.3 below, as the interested reader may verify. However, to be clear and concise, we limit ourselves to the following statement.

**Theorem 1.3.** *Let $x_0$ be a boundary point where the Gaussian curvature does not vanish, that is*

$$\kappa_1(x_0)\,\kappa_2(x_0) \neq 0\,. \tag{1.12}$$

*Then, there are admissible vector fields $\underline{a}$ for which the inequality* (1.11) *holds at a sequence of boundary points $x_n$, convergent to $x_0$. So, persistence property and strong vanishing limit results fail in general.*

By the way, note that when (1.11) or (1.12) hold in some point, they hold in a neighborhood of this same point.

# 2 Remarks

Boundedness of $\Omega$ is not essential here. The existence of points $x_0 \in \Gamma$ where the Gaussian curvature does not vanish is sufficient to apply our argument.

We overlook to consider boundary points where only one of the two principal curvatures does not vanish. It could be of some interest to study this case, by taking into account equation (3.3) below. This situation applies, for instance, to the case in which $\Omega$ is a cylinder, and slip boundary conditions are prescribed only on the lateral boundary (a developable surface), and periodicity is assumed in the ruling's direction.

We prove the identity (3.3) in the case of non umbilical points $x_0$. If the principal curvatures coincide at $x_0$, our proof needs some modification since the local system of coordinates used in the proof may not exist (for instance, it does not exist if $\Gamma$ is spherical near $x_0$). We leave to the interested reader the proof of (3.3) under the assumption $\kappa_1 = \kappa_2$. Note, however, that this particular situation can be bypassed here since if in a connected, closed, smooth surface all points are umbilical, the surface is a sphere. And this case was already treated in reference [3].

We note that our negative result does not exclude inviscid limit results in weaker spatial norms, such as $H^1(\Omega)$. In this regard we refer to paper [22], where this kind of convergence is proved under one of the following additional assumptions : $(\underline{\omega}^\nu - \underline{\omega}) \times \underline{n} = 0$ on $(0,T) \times \Gamma$, where $\underline{\omega}^\nu$ and $\underline{\omega}$ are the vorticity for the Navier-Stokes and Euler equations respectively, or $\underline{b} = 0$ on $\Gamma$.

**Remark 2.1.** On flat portions of the boundary, the slip boundary condition (1.2) coincides with the classical Navier boundary condition

$$\begin{cases} \underline{u} \cdot \underline{n} = 0, \\ \underline{t} \cdot \underline{\tau} = 0\,, \end{cases} \tag{2.1}$$

where $\underline{\tau}$ stands for any arbitrary unit tangential vector. Here $\underline{t}$ is the stress vector defined by $\underline{t} = \mathcal{T} \cdot \underline{n}$, where the stress tensor $\mathcal{T}$ is defined by

$$\mathcal{T} = -\,p\,I + \frac{\nu}{2}\,(\nabla \underline{u} + \nabla \underline{u}^T)\,.$$

These conditions were introduced by Navier in [17], and derived by Maxwell in [16] from the kinetic theory of gases. For general boundaries

$$\underline{t} \cdot \underline{\tau} = \frac{\nu}{2}\,(\underline{\omega} \times \underline{n}) \cdot \underline{\tau} - \nu\,\mathcal{K}_\tau\,\underline{u} \cdot \underline{\tau}\,, \tag{2.2}$$

where $\mathcal{K}_\tau$ is the principal curvature in the $\underline{\tau}$ direction, positive if the corresponding center of curvature lies inside $\Omega$.

Note that our counter-example in [5] and the results presented here do not exclude that strong inviscid limit results hold under the Navier boundary condition (2.1) in the non-flat boundary case. To prove or disprove this kind of result remains a challenging open problem.

# 3 A main identity

The Theorem 1.2 places the non-linear term $curl\,(\underline{a} \times \underline{b}) \times \underline{n}$ in a central position. So, reducing the order of this second order term is here very helpful. This reduction is done by proving the identity (3.3). In our context, this identity has another valuable merit. It makes explicit a precise dependence on curvature, which is essential in the sequel.

Before stating Theorem 3.1 we introduce some notation.

Given $x_0 \in \Gamma_0$ we introduce, in a sufficiently small neighborhood of $x_0$, a suitable system of orthogonal curvilinear coordinates $(\xi_1, \xi_2, \xi_3)$. See, for instance, [13], in particular, chapter 8, paragraph 89. The surface $\Gamma$ is locally described by the equation $\xi_3 = 0$, moreover the surfaces $\xi_3 = constant$ are parallel to $\Gamma$ in the usual sense. They are located at distance $|\,\xi_3\,|$ from $\Gamma$. The coordinate $\xi_3$ increases outside $\Omega$. Further, on each parallel surface the lines $\xi_j = constant$, $j = 1, 2$ are lines of curvature, hence tangent to a principal direction. Recall that the lines of curvature on parallel surfaces correspond to each other. The point $x_0$ has zero coordinates. We denote by $\underline{i}_j$ the unit vector, tangent to the $\xi_j$ line, and pointing in the direction of increasing $\xi_j$. Hence, at each point of a parallel surface, the vectors $\underline{i}_j$, $j = 1, 2$, are tangent to the principal directions. The corresponding normal curvatures, the so called principal curvatures, are denoted by $\kappa_j$, $j = 1, 2$. They take the maximum and the minimum of the set of all the normal curvatures. The unit vector $\underline{i}_3$ coincides with the normal $\underline{n}$. Roughly speaking, concerning signs, it is sufficient to remark that the curvature of a normal section of $\Gamma$ at a point $x$ is positive whenever the normal section of $\Omega$ is convex at $x$.

Components of vector fields are with respect to the orthogonal basis $\underline{i}_j$, $j = 1, 2, 3$. For instance $\underline{a} = a_1\,\underline{i}_1 + a_2\,\underline{i}_2 + a_3\,\underline{i}_3$.

A point is *umbilical* if $\kappa_1 = \kappa_2$. If $\kappa_1 = \kappa_2 = 0$ the point is a *planar* (or parabolic umbilical) point. As already remarked we assume, for convenience, that $x_0$ is not umbilical. In this case some modifications are needed.

At each point, the ordered orthogonal basis $\underline{i}_1$, $\underline{i}_2$, $\underline{i}_3$ is assumed to be positively (right-handed) oriented. If $s(\xi_j)$ denotes the arc length along a $\xi_j$-line, the (positive) $h_j$ scale functions are defined by

$$h_j = \frac{d\,s(\xi_j)}{d\,\xi_j}\,.$$

Note that $h_3 = 1$ everywhere. In particular,

$$\frac{\partial\,h_3}{\partial\,\xi_j} = 0\,, \quad j = 1,\,2\,. \tag{3.1}$$

We recall that

$$\kappa_1 = \frac{1}{h_3\,h_1}\,\frac{\partial\,h_1}{\partial\,\xi_3}\,, \quad \kappa_2 = \frac{1}{h_3\,h_2}\,\frac{\partial\,h_2}{\partial\,\xi_3}\,. \tag{3.2}$$

**Theorem 3.1.** *Let* $\underline{a}$ *be an admissible vector field. Then, the identity*

$$curl\,(\underline{a} \times \underline{b}) \times \underline{n} = -\,2\,b_3\,(\kappa_2\,a_2\,\underline{i}_1 - \kappa_1\,a_1\,\underline{i}_2\,) \tag{3.3}$$

*holds on* $\Gamma$.

*Proof.* We recall the following expression for the *curl* of a vector field $\underline{v}$ in curvilinear, orthogonal, coordinates:

$$\begin{aligned} curl\,\underline{v} = \frac{1}{h_2\,h_3}\Big[\frac{\partial\,(h_3\,v_3)}{\partial\,\xi_2} - \frac{\partial\,(h_2\,v_2)}{\partial\,\xi_3}\Big]\,\underline{i}_1 + \\ \frac{1}{h_3\,h_1}\Big[\frac{\partial\,(h_1\,v_1)}{\partial\,\xi_3} - \frac{\partial\,(h_3\,v_3)}{\partial\,\xi_1}\Big]\,\underline{i}_2 + \frac{1}{h_1\,h_2}\Big[\frac{\partial\,(h_2\,v_2)}{\partial\,\xi_1} - \frac{\partial\,(h_1\,v_1)}{\partial\,\xi_2}\Big]\,\underline{i}_3\,. \end{aligned} \tag{3.4}$$

Since (recall that $\underline{b} = curl\,\underline{a}$)

$$curl\,(\underline{a} \times \underline{b}) \times \underline{n} = [curl\,(\underline{a} \times \underline{b})]_2\,\underline{i}_1 - [curl\,(\underline{a} \times \underline{b})]_1\,\underline{i}_2\,, \tag{3.5}$$

we are interested in the two tangential components of $curl\,(\underline{a} \times \underline{b})$. Hence, by setting $\underline{v} = \underline{a} \times \underline{b}$ in (3.4), we want to determine the two first terms on the left hand side of (3.4). Due to the similarity of these two terms it is sufficient to treat one of them. We consider the first one. This leads to

$$[curl\,(\underline{a}\times \underline{b})]_1 = \frac{1}{h_2\,h_3}\,\frac{\partial}{\partial\,\xi_2}\,[\,h_3\,(\underline{a}_1\,\underline{b}_2 - \underline{a}_2\,\underline{b}_1\,)\,] - \frac{1}{h_2\,h_3}\,\frac{\partial}{\partial\,\xi_3}\,[\,h_2\,(\underline{a}_3\,\underline{b}_1 - \underline{a}_1\,\underline{b}_3\,)\,]\,. \tag{3.6}$$

Note that

$$a_3 = b_1 = b_2 = 0$$

for $\xi_3 = 0$, hence

$$\frac{\partial\,a_3}{\partial\,\xi_j} = \frac{\partial\,b_i}{\partial\,\xi_j} = 0\,,$$

for $i,\,j = 1,\,2$. It follows that the first term on the right hand side of equation (3.6) and the "first half" of the second term vanish on $\Gamma$. So,

$$[curl\,(\underline{a} \times \underline{b})]_1 = \frac{1}{h_2\,h_3}\,\frac{\partial}{\partial\,\xi_3}\,(\,h_2\,a_1\,b_3\,).$$

Consequently,

$$[curl\,(\underline{a} \times \underline{b})]_1 = \kappa_2\,a_1\,b_3 + \frac{1}{h_3}\,b_3\,\frac{\partial\,a_1}{\partial\,\xi_3} + \frac{1}{h_3}\,a_1\,\frac{\partial\,b_3}{\partial\,\xi_3}\,. \tag{3.7}$$

Since $b_2 = 0$ on $\Gamma$, it follows from (3.4) applied to $\underline{a}$ that

$$\frac{\partial a_1}{\partial \xi_3} = -\frac{a_1}{h_1} \frac{\partial h_1}{\partial \xi_3} . \tag{3.8}$$

We have appealed to $a_3 = \frac{\partial a_3}{\partial \xi_1} = 0$ on $\Gamma$, and to (3.1).

Next

$$div\, \underline{b} = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial (h_2 h_3 b_1)}{\partial \xi_1} + \frac{\partial (h_3 h_1 b_2)}{\partial \xi_2} + \frac{\partial (h_1 h_2 b_3)}{\partial \xi_3} \right\} = 0 , \tag{3.9}$$

in $\Omega$. From $b_i = \frac{\partial b_i}{\partial \xi_j} = 0$ on $\Gamma$, $i, j = 1, 2$, one gets

$$\frac{1}{h_1 h_2 h_3} \frac{\partial (h_1 h_2 b_3)}{\partial \xi_3} = 0 ,$$

on $\Gamma$. This leads to

$$\frac{1}{h_3} \frac{\partial b_3}{\partial \xi_3} = -(\kappa_1 + \kappa_2)\, b_3 . \tag{3.10}$$

From (3.7), (3.8), and (3.10), it readily follows that

$$[curl\, (\underline{a} \times \underline{b})]_1 = -2\, \kappa_1\, a_1\, b_3 . \tag{3.11}$$

Analogously we may prove that

$$[curl\, (\underline{a} \times \underline{b})]_2 = -2\, \kappa_2\, a_2\, b_3 . \tag{3.12}$$

□

Note that the above result has a local character. In fact the proof immediately applies to show the following. Let $U$ be an arbitrary open set such that $\Gamma_0 = U \cap \Gamma$ is not empty. Further, assume that $\underline{a}$ is a smooth divergence free vector field in $\overline{\Omega}_0$, where $\Omega_0 = U \cap \Omega$, which satisfies the boundary conditions (1.5) on $\Gamma_0$. Then, the identity (3.3) holds on $\Gamma_0$.

The next result follows from the theorems 1.2 and 3.1.

**Corollary 3.1.** *Let* $\underline{a}$ *be admissible. If, at some point* $x_0 \in \Gamma$,

$$b_3 \neq 0 , \tag{3.13}$$

*and if (at least) one of the two following conditions*

$$\kappa_1\, a_1 \neq 0 \quad or \quad \kappa_2\, a_2 \neq 0 , \tag{3.14}$$

*hold, then the inequality* (1.11) *takes place. In particular, the persistence property fails. Consequently, any strong inviscid limit result is false.*

Corollary 3.1 leads us to look for points $x_0 \in \Gamma$ for which (3.13) and (3.14) hold simultaneously. Hence, to show that these two inequalities are not independent is here of great help. The following result holds.

**Proposition 3.1.** *Let* $\underline{a}$ *be admissible, and assume that* (3.13) *holds in some point* $x_0 \in \Gamma$. *Then, there is a sequence of boundary points* $x_n \in \Gamma$, *convergent to* $x_0$, *and such that*

$$a_j(x_n) \neq 0\,, \tag{3.15}$$

*for, at least, one of the two index* $j$. *So, if* $x_0 \in \Sigma$,

$$b_3\, \kappa_j\, a_j \neq 0\,, \tag{3.16}$$

*at* $x_n$, *at least for sufficient large values of* $n$.

*Proof.* If there are neighborhoods $U_j$, $j = 1, 2$, where (3.15) does not hold, then $a_1 = a_2 = 0$ in $U_1 \cap U_2$. On the other hand, equation (3.4) shows that

$$b_3 = \frac{1}{h_1\, h_2} \Big( \frac{\partial\,(h_2\, a_2)}{\partial\, \xi_1} - \frac{\partial\,(h_1\, a_1)}{\partial\, \xi_2} \Big)\,. \tag{3.17}$$

It follows that $b_3 = 0$ on $U_1 \cap U_2$. This contradicts (3.13). ☐

From proposition 3.1 it follows that to prove the theorem 1.3 it is sufficient to show that, given any $x_0 \in \Sigma$, there is an admissible vector field $\underline{a}$ such that $b_3 \equiv b \times \underline{n} \neq 0$ at $x_0$. This is the aim of the next section.

# 4 Proof of Theorem 1.3

In this section we show an elementary way to explicitly construct admissible vector fields $\underline{a}$ for which (3.13) holds at any fixed $x_0 \in \Gamma$. By choosing $x_0 \in \Sigma$, we prove the theorem 1.3. In the following, topological properties of subsets of the boundary $\Gamma$ concern the $\Gamma$ topology (and not the $\overline{\Omega}$ topology).

Let $\beta(s)$ be a smooth real function on $\Gamma$ such that

$$\int_{\Gamma_j} \beta(s)\, ds = 0\,, \quad j = 0, ..., m\,, \tag{4.1}$$

and define

$$\underline{b}(s) = b(s)\, \underline{n}\,. \tag{4.2}$$

Clearly,

$$\int_{\Gamma_j} \underline{b}(s) \cdot \underline{n}\, ds = 0 \quad , j = 0, ..., m\,. \tag{4.3}$$

It is well known that, under assumption (4.3), there exists in $\overline{\Omega}$ an extension $\underline{b}(x)$ of $\underline{b}(s)$ such that

$$div\, \underline{b}(x) = 0\,. \tag{4.4}$$

On the other hand, it is well known that, under the constraints (4.3) and (4.4), the linear problem

$$\begin{cases} div\, \underline{a} = 0\,, \\ curl\, \underline{a} = \underline{b}, \quad \text{in} \quad \Omega\,, \\ \underline{a} \cdot \underline{n} = 0\,, \quad \text{on} \quad \Gamma \end{cases} \tag{4.5}$$

is always solvable. This existence result is sufficient to our purposes. However we recall that the solution is unique if $\Omega$ is simply connected and that, in general, the kernel (set $\underline{b} = 0$) of the related linear map has dimension $N$. For a very clear and complete treatment of this, and related, problems we refer the reader to the section 1, in reference [10].

The following result shows a crucial advantage of our approach.

**Lemma 4.1.** *For each $\beta(s)$ as above, the vector field $\underline{a}$ is admissible.*

In fact, (4.2) together with the second equation (4.5), implies the second boundary condition (1.5).

Given $\beta(s)$ as above we denote by $\Lambda[\beta]$ the set of boundary points defined by

$$\Lambda[\beta] = \{s \in \Gamma \,:\, \beta(s) \neq 0\}. \tag{4.6}$$

The following result is obvious.

**Lemma 4.2.** *We may choose functions $\beta$ for which $\Lambda[\beta] = \Gamma$, except for $m + 1$ arbitrary, closed simple curves, $C_j \subset \Gamma_j$, $j = 0, ..., m$. These curves may be arbitrarily chosen.*

*Proof of Theorem 1.3*: Let $x_0$ be an arbitrary, but fixed, boundary-point where the Gaussian curvature does not vanish. Taking into account Lemma 4.2, we fix a real function $\beta(s)$ such that $\beta(x_0) \neq 0$, and construct $\underline{a}$ as above. From proposition 3.1, and theorems 3.1 and 1.2, the thesis follows.

**Remark 4.1.** *Define (recall* (1.11)*)*

$$K[\beta] = \{s \in \Gamma \,:\, curl\,(\underline{a} \times \underline{b}) \times \underline{n} \neq 0\}.$$

*By appealing to our argument, we may prove that the set $K[\beta]$ is dense in $\Sigma \cap \Lambda[\beta]$, and that there are functions $\beta(s)$ for which $K[\beta]$ is dense in $\Sigma$.*

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# References

[1] H. Beirão da Veiga, On the sharp vanishing viscosity limit of viscous incompressible fluid flows, to appear in a volume to the memory of A. Kazhikhov in the Birkhaueser book series "Advances in Mathematical Fluid Mechanics".

[2] H. Beirão da Veiga and F. Crispo, Sharp inviscid limit results under Navier type boundary conditions. An $L^p$ theory, *J. Math. Fluid Mech.*, **12** (2010), 397–411.

[3] H. Beirão da Veiga and F. Crispo, Concerning the $W^{k,p}-$ inviscid limit for $3-D$ flows under a slip boundary condition, *J. Math. Fluid Mech.*, DOI 10.1007/s00021-009-0012-3.

[4] H. Beirão da Veiga, F. Crispo and C. R. Grisanti, Reducing slip boundary value problems from the half to the whole space. Applications to inviscid limits and to non-Newtonian fluids, *J. Math. Anal. Appl.*, DOI 10.1016/j.jmaa.2010.10.045.

[5] H. Beirão da Veiga and F. Crispo, The 3-D inviscid limit result under slip boundary conditions. A negative answer, *J. Math. Fluid Mech.*, in press.

[6] H. Bellout, J. Neustupa and P. Penel, On a $\nu$-continuous family of strong solutions to the Euler or Navier-Stokes equations with the Navier-type boundary condition, *Discrete Cont. Dyn. Syst.*, **27** (2010), 1353-1373.

[7] G.-Q. Chen and Z. Qian, A study of the Navier-Stokes equations with the kinematic and Navier boundary conditions, arXiv:0901.0147v1.

[8] T. Clopeau, A. Mikelic and R. Robert, On the vanishing viscosity limit for the 2-D incompressible Navier-Stokes equations with the friction type boundary conditions, *Nonlinearity*, **11** (1998), 1625–1636.

[9] F. Crispo, On the zero-viscosity limit for $3D$ Navier-Stokes equations under slip boundary conditions, *Riv. Mat. Univ. Parma*, **3** (2010).

[10] C. Foias and R. Temam, Remarques sur les equations de Navier-Stokes stationaires et les phenomenes successifs de bifurcation, *Ann. Sc. Norm. Sup. Pisa*, **5** (1978), 29–63.

[11] T. Kato, Quasi-linear equations of evolution with applications to partial differential equations, in "Spectral Theory and Differential Equations", *Lecture Notes in Mathematics*, **448**, Springer-Verlag (1975).

[12] J. P. Kelliher, Navier-Stokes equations with Navier boundary conditions for a bounded domain in the plane, *SIAM J. Math. Anal.*, **38** (2006), 210-232.

[13] E. Kreyszig, Differential Geometry, *University of Toronto press*, Toronto (1959).

[14] M. C. Lopes Filho, H. J. Nussenzveig Lopes and G. Planas, On the inviscid limit for two-dimensional incompressible flow with Navier friction condition, *SIAM J. Math. Anal.*, **36** (2005), 1130-1141.

[15] N. Masmoudi, Remarks about the Inviscid Limit of the NavierStokes System, *Commun. Math. Phys.*, **270** (2007), 777–788.

[16] J. C. Maxell, On stresses in rarefied gases arising from inequalities of temperature, *Phil. Trans. Roy. Soc.*, **170** (1879), 249–256.

[17] C. L. M. Navier, Mémoire sur les lois du mouvement des fluides, *Mémoires de l'Académie Royale des Sciences de l'Institut de France* (2), 6 (1823).

[18] R. Temam, On the Euler equations of incompressible perfect fluids, *J. Func. Anal.*, **20** (1975), 32–43.

[19] R. Temam, Navier-Stokes equations. Theory and numerical analysis. Third edition. North-Holland Publishing Co., Amsterdam, 1984.

[20] J. Watanabe, On incompressible viscous fluid flows with slip boundary conditions, *J. Comput. Appl. Math.*, **159** (2003), 161–172.

[21] Y. Xiao and Z. Xin, On the vanishing viscosity limit for the 3-D Navier-Stokes equations with a slip boundary condition, *Comm. Pure Appl. Math.*, **60** (2007), 1027–1055.

[22] Y. Xiao and Z. Xin, Remarks on the vanishing viscosity limits for the 3-D Navier-Stokes equations with a slip boundary condition, *Dedicated to R. Temam on the occasion of his 70th birthday.*

[23] Y. Xiao, Z. Xin and J. Wu, Vanishing viscosity limit for the 3D magnetohydrodynamic system with a slip boundary condition, *J. Funct. Anal.*, **257** (2009), 3375-3394.