

HIERARCHICAL ZONOTOPAL POWER IDEALS

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ABSTRACT. This work unifies and generalizes results by Ardila-Postnikov on power ideals and by Holtz-Ron and Holtz-Ron-Xu on (hierarchical) zonotopal algebra.

Zonotopal algebra deals with ideals and vector spaces of polynomials that are related to several combinatorial and geometric structures defined by a finite sequence of vectors. Given such a sequence X , an integer $k \geq -1$ and an upper set in the lattice of flats of the matroid defined by X , we define and study the associated *hierarchical zonotopal power ideal*. This ideal is generated by powers of linear forms. Its Hilbert series depends only on the matroid structure of X . It is related to various other matroid invariants, *e. g.* the shelling polynomial and the characteristic polynomial.

1. INTRODUCTION

Let $X = (x_1, \dots, x_N) \subseteq \mathbb{R}^r$ be a sequence of vectors that span \mathbb{R}^r . For a vector η , let $m(\eta)$ denote the number of vectors in X that are *not* perpendicular to η . A vector $v \in \mathbb{R}^r$ defines a linear functional $p_v := \sum_i v_i t_i \in \mathbb{R}[t_1, \dots, t_r]$. For $Y \subseteq X$, let $p_Y := \prod_{x \in Y} p_x$. Then define

$$\mathcal{P}(X) := \text{span}\{p_Y : X \setminus Y \text{ spans } \mathbb{R}^r\} \quad (1.1)$$

$$\mathcal{I}(X) := \text{ideal}\{p_\eta^{m(\eta)} : \eta \neq 0\} \quad (1.2)$$

The following theorem and several generalizations are well known:

Theorem 1.1 ([2, 8, 16]).

$$\mathcal{P}(X) = \ker \mathcal{I}(X) := \text{span}\{q \in \mathbb{R}[t_1, \dots, t_r] : f(D)q = 0 \text{ for all } f \in \mathcal{I}(X)\} \quad (1.3)$$

where $f(D) := f\left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_r}\right)$. In addition, $\mathcal{I}(X)$ is equal to the ideal

$$\mathcal{I}'(X) := \{p_\eta^{m(\eta)} : \text{The vectors perpendicular to } \eta \text{ span a hyperplane}\}$$

In this paper, we show that a statement as in Theorem 1.1 holds in a far more general setting: we study the kernel of the *hierarchical zonotopal power ideal*

$$\mathcal{I}(X, k, J) := \text{ideal}\{p_\eta^{m(\eta)+k+\chi_J(\eta)} : \eta \neq 0\} \quad (1.4)$$

Date: November 5, 2010.

2010 Mathematics Subject Classification. Primary: 05A15, 05B35, 05C31, 13B25, 16S32; Secondary: 05C15, 05C21, 13P99, 41A15, 41A63, 47F05, 47L20, 52B20, 52C22.

Key words and phrases. zonotopal algebra, matroids, Tutte polynomials, power ideals, kernels of differential operators, Hilbert series.

This work was supported by a Sofia Kovalevskaya Research Prize of Alexander von Humboldt Foundation awarded to Olga Holtz.

where $k \geq -1$ is an integer and χ_J is the indicator function of an upper set J in the lattice of flats of the matroid defined by X . We study those spaces in a slightly more abstract setting, *e.g.* $\mathcal{P}(X, k, J)$ is contained in the symmetric algebra over an r -dimensional \mathbb{K} -vector space, where \mathbb{K} is a field of characteristic zero.

The choice of a sequence of vectors X defines a large number of objects in various mathematical fields which are all related to zonotopal algebra [16]. Examples include combinatorics (matroids, matroid and graph polynomials, generalized parking functions if X is graphic [12, 17, 22]), discrete geometry (hyperplane arrangements, zonotopes and tilings of zonotopes), approximation theory (box splines, least map interpolation) and algebraic geometry (Cox rings, fat point ideals [2, 23]).

Central \mathcal{P} -spaces (in our terminology the kernel of $\mathcal{I}'(X, 0, \{X\})$) were introduced in the literature on approximation theory around 1990 [1, 8, 13]. A dual space called $\mathcal{D}(X)$ appeared almost 30 years ago [6]. See [16, Section 1.2] for a historic survey and the book [7] for a treatment of polynomial spaces appearing in the theory of box splines.

Recently, Olga Holtz and Amos Ron coined the term *zonotopal algebra* [16]. They introduced internal ($k = -1$) and external ($k = +1$) \mathcal{P} -spaces and \mathcal{D} -spaces. Federico Ardila and Alexander Postnikov [2] constructed \mathcal{P} -spaces for arbitrary integers $k \geq -1$. Olga Holtz, Amos Ron, and Zhiqiang Xu [18] introduced hierarchical zonotopal spaces, *i.e.* structures that depend on the choice of an upper set J in addition to X and k . They studied semi-internal and semi-external spaces (*i.e.* $k = -1$ and $k = 0$ and some special upper sets J). The central case was treated in an algebraic setting in [10, 11]. Other related results include [3, 24].

The *least map* [9] assigns to a finite set $S \subseteq \mathbb{R}^r$ of cardinality m an m -dimensional space of homogeneous polynomials in $\mathbb{R}[t_1, \dots, t_r]$. Holtz and Ron [16] showed that in the internal, central and external case, \mathcal{P} -spaces can be obtained via the least map if X is unimodular. In those cases, the \mathcal{P} -spaces are obtained by choosing the set S as a certain subset of the set of lattice points of the zonotope $Z(X) := \{\sum_i \lambda_i x_i : 0 \leq \lambda_i \leq 1\}$. In a follow-up paper, we will investigate a similar correspondence in the hierarchical setting. The condition of unimodularity can be dropped in the discrete theory, where differential operators are replaced by difference operators [11, 20].

As an example for the connections between zonotopal algebra and combinatorics, we now explain various relationships between zonotopal spaces and matroid/graph polynomials. They can be deduced from the fact that both, the matroid/graph polynomials [15] and the Hilbert series of the zonotopal spaces [2] are evaluations of the Tutte polynomial (see also equations (5.9) and (5.10)).

$\text{Hilb}(\mathcal{P}(X, 0, \{X\}), t)$ equals the shelling polynomial [4] $h_{\Delta(X^*)}(t)$ of the matroid dual to X with the coefficients reversed. The t^{N-r-i} coefficient of $h_{\Delta(X^*)}(t+1)$ equals the number of independent sets of cardinality i in the matroid X . Let X_G denote the reduced oriented incidence matrix of a connected graph G , *i.e.* the matrix that is obtained from the oriented incidence matrix by removing one row so that it has full rank. The Hilbert series of $\mathcal{P}(X_G, -1, \{X_G\})$ is related to the flow polynomial ϕ_G . By duality it is also related to the chromatic polynomial of the graph resp. the characteristic polynomial of the matroid in the general case. The connection is as follows: if G is connected, then $\phi_G(t) = (t-1)^{N-r} \text{Hilb}(\mathcal{P}(X_G, -1, \{X_G\}), 1/(1-t))$. The four color theorem is equivalent to the following statement: if G is a planar graph, then $\text{Hilb}(\mathcal{P}(X_{G^*}, -1, \{X_{G^*}\}), -1/3) > 0$, where G^* denotes the graph dual to G .

Organization of the article. In Section 2, we introduce our notation and review the mathematical background. In Section 3, we describe the kernels of the ideals $\mathcal{I}(X, k, \chi)$ and define a subideal $\mathcal{I}'(X, k, \chi)$ with finitely many generators. We show that for $k \leq 0$, the two ideals are equal.

In Section 4, we construct bases for the vector spaces $\mathcal{P}(X, k, J)$. We deduce formulas for the Hilbert series of the spaces $\mathcal{P}(X, k, J)$ in Section 5. Those formulas depend only on the matroid structure of X but not on the representation. In Section 6, we apply our results to prove a statement about zonotopal Cox modules that were defined by Bernd Sturmfels and Zhiqiang Xu [23]. Finally, in Section 7, we give plenty of examples.

Acknowledgments. I would like to thank my advisor Olga Holtz for fruitful discussions of this work and Federico Ardila for suggesting the possibility of the results in Section 6.

2. PRELIMINARIES

2.1. Notation. The following notation is used throughout this paper: $\mathbb{N} := \{0, 1, 2, 3, \dots\}$. For $n \in \mathbb{N}$, let $[n] := \{1, \dots, n\}$. \mathbb{K} is a fixed field of characteristic zero. V denotes a finite-dimensional \mathbb{K} -vector space of dimension $r \geq 1$ and $U := V^*$ its dual. Our main object of study is a finite sequence $X = (x_1, \dots, x_N) \subseteq U$ whose elements span U . We slightly abuse notation by using the symbol \subseteq for subsequences. For $Y \subseteq X$, the deletion $X \setminus Y$ denotes the deletion of a subsequence and not the deletion of a subset, *i. e.* $(x_1, x_2) \setminus (x_1) = (x_2)$ even if $x_1 = x_2$. The order of the elements in X is irrelevant for us except in a few cases, where this is explicitly mentioned.

2.2. Matroids and posets. Let $X = (x_1, \dots, x_N)$ be a finite sequence whose elements span U . Let $\mathfrak{M}(X) := \{I \subseteq \{1, \dots, N\} : \{x_i : i \in I\} \text{ linearly independent}\}$. Then $\mathfrak{M}(X)$ is a matroid of rank r on N elements. X is called a \mathbb{K} -*representation* of the matroid $\mathfrak{M}(X)$. For more information on matroids, see Oxley's book [21].

We now introduce some additional matroid theoretic concepts. To facilitate notation, we always write X instead of $\mathfrak{M}(X)$.

The *rank* of $Y \subseteq X$ is defined as the cardinality of a maximal independent set contained in Y . It is denoted $\text{rk}(Y)$. The *closure* of Y in X is defined as $\text{cl}_X(Y) := \{x \in X : \text{rk}(Y \cup x) = \text{rk}(Y)\}$. $C \subseteq X$ is called a *flat* if $C = \text{cl}(C)$. A *hyperplane* is a flat of rank $r - 1$. The set of all hyperplanes in X is denoted by $\mathcal{H} = \mathcal{H}(X)$.

Given a flat $C \subseteq X$, we call $\eta \in V$ a *defining normal* for C if $C = \{x \in X : \eta(x) = 0\}$. Note that for hyperplanes, there is a unique defining normal (up to scaling). The set of bases of the matroid X (*i. e.* the subsequences of X of cardinality r and rank r) is denoted $\mathbb{B}(X)$. If $x = 0$, then x is called a *loop*. If $\text{rk}(X \setminus x) = r - 1$, then x is called a *coloop*.

The set of flats of a given matroid X ordered by inclusion forms a lattice (*i. e.* a poset with joins and meets) called the *lattice of flats* $\mathcal{L}(X)$. An *upper set* $J \subseteq \mathcal{L}(X)$ is an upward closed set, *i. e.* $C \subseteq C'$, $C \in J$ implies $C' \in J$. We call $C \in \mathcal{L}(X)$ a *maximal missing flat* if $C \notin J$ and C is maximal with this property.

The *Tutte polynomial* [5] $T_X(x, y) := \sum_{A \subseteq X} (x - 1)^{r - \text{rk}(A)} (y - 1)^{|A| - \text{rk}(A)}$ captures a lot of information about the matroid $\mathfrak{M}(X)$.

2.3. Algebra. $\text{Sym}(V)$ denotes the *symmetric algebra* over V . This is a base-free version of the ring of polynomials over V . The choice of a basis $B = \{b_1, \dots, b_n\} \subseteq V$ yields an isomorphism $\text{Sym}(V) \cong \mathbb{K}[b_1, \dots, b_n]$. For a definition and more background on algebra, see [11] or [14].

A *derivation* on $\text{Sym}(V)$ is a \mathbb{K} -linear map D satisfying Leibniz's law, *i. e.* $D(fg) = D(f)g + fD(g)$ for $f, g \in \text{Sym}(V)$. For $v \in V$, we define the *directional derivative* in direction v , $D_v : \text{Sym}(U) \rightarrow \text{Sym}(U)$ as the unique derivation which satisfies $D_v(u) = v(u)$ for all $u \in U$. For $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$, this definition agrees with the analytic definition of the directional derivative. It follows that $\text{Sym}(V)$ can be identified with the ring of differential operators on U .

Now we define a pairing $\langle \cdot, \cdot \rangle : \text{Sym}(U) \times \text{Sym}(V) \rightarrow \mathbb{K}$ by $\langle f, q \rangle := (q(D)f)(0)$ where $q(D)f$ means that q acts as a differential operator on f .

A *graded vector space* is a vector space V that decomposes into a direct sum $V = \bigoplus_{i \geq 0} V_i$. A graded linear map $f : V \rightarrow W$ preserves the grade, *i. e.* $f(V_i) \subseteq W_i$. For a graded vector space, we define its *Hilbert series* as the formal power series $\text{Hilb}(V, t) := \sum_{i \geq 0} \dim(V_i)t^i$. A *graded algebra* V has the additional property $V_i V_j \subseteq V_{i+j}$. We use the symmetric algebra with its natural grading. This grading is characterized by the property that the degree 1 elements are exactly the ones that are contained in V . A \mathbb{Z}^n -*multigraded ring* R is defined similarly: R decomposes into a direct sum $R = \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} R_{\mathbf{a}}$ and $R_{\mathbf{a}} R_{\mathbf{b}} \subseteq R_{\mathbf{a}+\mathbf{b}}$.

A linear map $f : V \rightarrow W$ induces an algebra homomorphism $\text{Sym}(f) : \text{Sym}(V) \rightarrow \text{Sym}(W)$.

2.4. Homogeneous ideals and their kernels. An ideal $\mathcal{I} \subseteq \text{Sym}(V)$ is called a *power ideal* [2] if $\mathcal{I} = \text{ideal}\{D_{\eta}^{e_{\eta}} : \eta \in Z\}$ for some $Z \subseteq V \setminus \{0\}$ and $e \in \mathbb{N}^Z$.¹ D_{η} denotes the image of η under canonical injection $V \hookrightarrow \text{Sym}(V)$. By definition, power ideals are homogeneous.

Definition 2.1. Let $\mathcal{I} \subseteq \text{Sym}(V)$ be a homogeneous ideal. Its *kernel* (or inverse system) is defined as

$$\ker \mathcal{I} := \{f \in \text{Sym}(U) : \langle f, q \rangle = 0 \text{ for all } q \in \mathcal{I}\} \quad (2.1)$$

Let G be a set of generators for \mathcal{I} . It can easily be seen that $\ker \mathcal{I} = \{f \in \text{Sym}(U) : g(D)f = 0 \text{ for all } g \in G\}$. It is known that for a homogeneous ideal $\mathcal{I} \subseteq \text{Sym}(V)$ of finite codimension the Hilbert series of $\ker \mathcal{I}$ and $\text{Sym}(V)/\mathcal{I}$ are equal. For instance, this follows from [11, Theorem 5.1.5].²

2.5. A remark on the notation. As zonotopal spaces were studied by people from different fields, the notation and the level of abstraction used in the literature varies. Authors with a background in spline theory usually work over \mathbb{R}^n and identify it with its dual space via the canonical inner product. \mathcal{P} -spaces and \mathcal{I} -ideals are then both subsets of the polynomial ring $\mathbb{R}[x_1, \dots, x_r]$. Other authors work in an abstract setting as we do. Since the Euclidean setting captures all the important parts of the theory, a reader with no background in abstract algebra may safely do

¹In the original definition in [2], 1 is added to every exponent.

² $\ker \mathcal{I}$ is sometimes defined slightly different in the literature: first note that $\text{Sym}(U)$ (\approx polynomials) is a subspace of $\text{Sym}(V)^*$ (\approx formal power series). The pairing $\langle \bullet, \bullet \rangle$ is defined on $\text{Sym}(V)^* \times \text{Sym}(V)$ and $\ker \mathcal{I}$ is the subset of $\text{Sym}(V)^*$ that is annihilated by \mathcal{I} . It is then proved that if \mathcal{I} has finite codimension, $\ker \mathcal{I}$ is contained in $\text{Sym}(U)$, *i. e.* in this case, both definitions yield the same space.

the following substitutions: $\mathbb{K} = \mathbb{R}$, $U \cong V \cong \mathbb{R}^n$. $f \in \mathbb{R}[x_1, \dots, x_r] = \text{Sym}(V)$ acts on $\mathbb{R}[x_1, \dots, x_r] = \text{Sym}(U)$ as the differential operator that is obtained from f by substituting $x_i \mapsto \frac{\partial}{\partial x_i}$. We make those substitutions in Section 7 (examples).

Some authors work in the dual setting and consider a central hyperplane arrangement instead of a finite sequence of vectors X . While hierarchical zonotopal power ideals can be defined in both settings, it is natural for us to work with vectors as we are also interested in the zonotope $Z(X)$.

3. HIERARCHICAL ZONOTOPAL POWER IDEALS AND THEIR KERNELS

In this section, we define hierarchical zonotopal power ideals and show that their kernels have a nice description as \mathcal{P} -spaces.

The first subsection contains the definitions and the statement of the Main Theorem. In the second subsection, we prove some simple facts and give explicit formulas for the \mathcal{P} -spaces in two simple cases. In the third subsection, we define deletion and contraction for pairs consisting of a matroid and an upper set in its lattice of flats. This is then used to give an inductive proof of the Main Theorem.

3.1. Definitions and the Main Theorem. Recall that $U = V^*$ denotes an r -dimensional vector space over a field \mathbb{K} of characteristic zero and $X = (x_1, \dots, x_N)$ denotes a finite sequence whose elements span U .

$\eta \in V$ defines a flat $C \subseteq X$. Define $m_X(C) = m_X(\eta) := |X \setminus C|$. Sometimes, we write $m(C)$ instead of $m_X(C)$. Given an upper set $J \subseteq \mathcal{L}(X)$, $\chi_J : \mathcal{L}(X) \rightarrow \{0, 1\}$ denotes its indicator function. Again, the index is omitted if it is clear which upper set is meant. χ can be extended to the powerset of X by $\chi(A) := \chi(\text{cl}(A))$ for $A \subseteq X$.

For a given $x \in U$, we denote the image of X under the canonical injection $U \hookrightarrow \text{Sym}(U)$ by p_x . For $Y \subseteq X$, we define $p_Y := \prod_{x \in Y} p_x$. For $\eta \in V$, we write D_η for the image of η under the canonical injection $V \hookrightarrow \text{Sym}(V)$ in order to stress the fact that we mostly think of $\text{Sym}(V)$ as the algebra generated by the directional derivatives on $\text{Sym}(U)$.

Definition 3.1 (Hierarchical zonotopal power ideals and \mathcal{P} -spaces). Let \mathbb{K} be a field of characteristic zero, V be a finite-dimensional \mathbb{K} -vector space of dimension $r \geq 1$ and $U = V^*$. Let $X = (x_1, \dots, x_N) \subseteq U$ be a finite sequence whose elements span U . Let $k \geq -1$ be an integer and let $J \subseteq \mathcal{L}(X)$ be a non-empty upper set. $\mathcal{L}(X)$ denotes the lattice of flats of the matroid defined by X .

Let $\chi : \mathcal{L}(X) \rightarrow \{0, 1\}$ denote the indicator function of J . Let $E : \mathcal{L}(X) \rightarrow V$ be a function that assigns a defining normal to every flat. Now define

$$\mathcal{I}'(X, k, J, E) := \text{ideal} \left\{ D_{E(C)}^{m(C)+k+\chi(C)} : C \text{ hyperplane or maximal missing flat} \right\} \quad (3.1)$$

$$\mathcal{I}(X, k, J) := \text{ideal} \left\{ D_\eta^{m(\eta)+k+\chi(\eta)} : \eta \in V \setminus \{0\} \right\} \subseteq \text{Sym}(V) \quad (3.2)$$

$$\mathcal{P}(X, k, J) := \text{span } S(X, k, J) \subseteq \text{Sym}(U) \quad (3.3)$$

where

$$S(X, k, J) := \{f p_Y : Y \subseteq X, 0 \leq \deg f \leq \chi(X \setminus Y) + k - 1\} \quad \text{for } k \geq 1 \quad (3.4)$$

$$S(X, 0, J) := \{p_Y : Y \subseteq X, \text{cl}(X \setminus Y) \in J\} \quad (3.5)$$

$$S(X, -1, J) := \{p_Y : |Y \setminus C| < m(C) - 1 + \chi(C) \text{ for all } C \in \mathcal{L}(X) \setminus \{X\}\} \quad (3.6)$$

Note that the definition of $S(X, 0, J)$ can be seen as a special case of the definition of $S(X, k, J)$ for $k \geq 1$. Therefore, we distinguish only the two cases $k \geq 0$ and $k = -1$ in the proofs.

The condition $X \in J$ is only relevant in the case $k = 0$. Then it ensures $1 \in S(X, 0, J)$. One can easily see that in the definition of $S(X, -1, J)$, it is sufficient to check only the inequalities associated to hyperplanes and to maximal missing flats. If x is a coloop and $X \setminus x \notin J$, then $S(X, -1, J) = \emptyset$.

For examples, see Section 7, Remark 3.9, and Proposition 3.10.

Theorem 3.2 (Main Theorem). *We use the same terminology as in Definition 3.1. For $k = -1$, we assume in addition that $J \supseteq \mathcal{H}$, i. e. J contains all hyperplanes in X . Then,*

$$\mathcal{P}(X, k, J) = \ker \mathcal{I}(X, k, J) \subseteq \ker \mathcal{I}'(X, k, J, E) \quad (3.7)$$

Furthermore, for $k \in \{-1, 0\}$, $\mathcal{I}'(X, k, J, E)$ is independent of the choice of E and

$$\mathcal{P}(X, k, J) = \ker \mathcal{I}(X, k, J) = \ker \mathcal{I}'(X, k, J, E) \quad (3.8)$$

Example 7.3 explains why there is an additional condition for $k = -1$ (see also Remark 3.18). Holtz, Ron, and Xu [18] define a different semi-internal structure. For a fixed $C_0 \in \mathcal{L}(X)$ and $J_{C_0} := \{C \in \mathcal{L}(X) : C \supseteq C_0\}$, they show $\ker \mathcal{I}'(X, -1, J_{C_0}) = \bigcap_{x \in C_0} \mathcal{P}(X \setminus x, 0, \{X\})$. However, they do not have a canonical generating set for this space. See Subsection 5.3 for more details.

From the Main Theorem and the results in Section 5, one can easily deduce the following two corollaries:

Corollary 3.3. *In the setting of the Main Theorem,*

$$\mathcal{P}(X, k, \mathcal{L}(X)) = \mathcal{P}(X, k + 1, \{X\}) \quad (3.9)$$

Corollary 3.4. *The Hilbert series of $\mathcal{P}(X, k, J)$ depends only on the matroid $\mathfrak{M}(X)$, but not on the representation X .*

Remark 3.5. One might wonder if similar Theorems can be proved for $k \leq -2$. One would of course need to impose extra conditions on X to ensure that the exponents appearing in the definition of the ideals are non-negative. It is easy to see that \mathcal{I} and \mathcal{I}' are equal in this case (Lemma 3.11). However, we do not know how to construct generating sets for them. A different approach would be required: in general, they are not spanned by a set of polynomials of type p_Y for some $Y \subseteq X$ [2].

3.2. Basic results. In this subsection, we prove three lemmas that are needed later on and we prove the Main Theorem in two special cases that are the base cases for the inductive proof in the next subsection.

Lemma 3.6. *Let $Y \subseteq X$ and let $\eta \in V$ be the defining normal for $C \subseteq X$. Then,*

$$D_\eta p_Y = p_{Y \cap C} D_\eta p_{Y \setminus C} \quad (3.10)$$

Proof. This is a direct consequence of Leibniz's law. \square

Lemma 3.7. *Let $u_1, \dots, u_s \in U$ and let $k \in \mathbb{N}$. Then,*

$$\text{span}\{(\alpha_1 u_1 + \dots + \alpha_s u_s)^k : \alpha_i \in \mathbb{K} \setminus \{0\}\} = \text{span}\left\{\prod_{i=1}^s u_i^{a_i} : \sum_{i=1}^s a_i = k, a_i \in \mathbb{N}\right\}$$

Proof. “ \subseteq ” is clear. For “ \supseteq ”, order the L monomials of the form $\prod_{i=1}^s u_i^{a_i}$ ($\sum_i a_i = k$, $a_i \in \mathbb{N}$) lexicographically. By induction, we can see that there are polynomials p_1, \dots, p_L contained in the set on the left s.t. the leading term of p_i is the i th monomial. This implies that all those monomials are contained in the set on the left side. \square

Lemma 3.8. $\mathcal{P}(X, k, J) \subseteq \ker \mathcal{I}(X, k, J) \subseteq \ker \mathcal{I}'(X, k, J)$ holds for all $k \geq -1$ and $J \subseteq \mathcal{L}(X)$.

Proof. The second inequality is clear. For the first, we generalize the proof of [16, Theorem 3.5]: it suffices to prove that every generator of $\mathcal{I}(X, k, J)$ annihilates every element of $S(X, k, J)$. For $k = -1$, this is obvious. Now consider the case $k \geq 0$. Let C be a flat, η a defining normal for C , $Y \subseteq X$, $\deg f \leq k + \chi(X \setminus Y) - 1$. Set $e(C) := m(C) + k + \chi(C)$. By Lemma 3.6,

$$D_\eta^{e(C)} f p_Y = p_{Y \cap C} \sum_{i=0}^{e(C)} \binom{e(C)}{i} D_\eta^i f D_\eta^{e(C)-i} p_{Y \setminus C} \quad (3.11)$$

$$\stackrel{(*)}{=} p_{Y \cap C} \sum_{i=k+\chi(C)}^{k+\chi(X \setminus Y)-1} \binom{e(C)}{i} D_\eta^i f D_\eta^{e(C)-i} p_{Y \setminus C} \quad (3.12)$$

(*) holds because f does not survive $k + \chi(X \setminus Y)$ differentiations and $p_{Y \setminus C}$ is annihilated by $m(C) + 1$ differentiations. Suppose the term in (3.12) is not zero. Then $\chi(X \setminus Y) = 1$ and $\chi(C) = 0$. Furthermore, $m(C)$ differentiations in direction η do not annihilate $p_{Y \setminus C}$. This is only possible if $Y \setminus C = X \setminus C$. This implies $X \setminus Y \subseteq C$. Then $\chi(X \setminus Y) \leq \chi(C)$. This is a contradiction. \square

Now we give explicit formulas for $\mathcal{P}(X, k, J)$ and $\mathcal{I}(X, k, J)$ in two particularly simple cases:

Remark 3.9. Suppose that $\dim U = 1$ and that X contains m non-zero entries. Let $x \in U$ and $y \in V$ be non-zero vectors. Then $\mathcal{I}'(X, k, J) = \mathcal{I}(X, k, J) = \text{ideal}\{D_y^{m+k+\chi(\emptyset)}\}$ and $\mathcal{P}(X, k, J) = \text{span}\{p_x^i : i \in \{0, 1, \dots, m-1+k+\chi(\emptyset)\}\}$.

Proposition 3.10. We use the same terminology as in Definition 3.1. Let $X = (x_1, \dots, x_r)$ be a basis for U . Let (y_1, \dots, y_r) denote the dual basis of V .

Then $\mathcal{P}(X, k, J) = \ker \mathcal{I}(X, k, J) \subseteq \ker \mathcal{I}'(X, k, J, E)$. Furthermore, for $k \in \{-1, 0\}$, $\ker \mathcal{I}(X, k, J) = \ker \mathcal{I}'(X, k, J, E)$ for arbitrary E .

More precisely, writing $p_i := p_{x_i}$ and $D_i := D_{y_i}$ as shorthand notation, we get

$$\mathcal{I}(X, k, J) = \text{ideal} \left\{ \prod_{i \in I} D_i^{a_i+1} : I \subseteq [r], a_i \in \mathbb{N}, \sum_{i \in I} a_i = k + \chi(X \setminus \{x_i : i \in I\}) \right\} \quad (3.13)$$

$$\mathcal{P}(X, k, J) = \text{span} \left\{ \prod_{i \in I} p_i^{a_i+1} : I \subseteq [r], a_i \in \mathbb{N}, \sum_{i \in I} a_i \leq k + \chi(X \setminus \{x_i : i \in I\}) - 1 \right\} \quad (3.14)$$

For $k = 0$, this specializes to $\mathcal{P}(X, 0, J) = \text{span}\{p_Y : X \setminus Y \in J\}$. For $k = -1$, $\mathcal{I}(X, -1, J) = \text{ideal}\{D_1, \dots, D_r\}$ if $J \supseteq \mathcal{H}$ and $\mathcal{I}(X, -1, J) = \text{ideal}\{1\}$ otherwise.

For a two-dimensional example of this construction, see Example 7.1 and Figure 1.

Proof. This proof generalizes the proof of Proposition 4.3 in [2]. The statements about $k = -1$ are trivial.

Every flat of X can be written as $C = X \setminus \{x_i : i \in I\}$ for some $I \subseteq [r]$. The set of defining normals for C is given by $\{\sum_{i \in I} \alpha_i y_i : \alpha_i \in \mathbb{K} \setminus \{0\}\}$.

First, we show that for $k = 0$, $\mathcal{I}(X, k, J) = \mathcal{I}'(X, k, J, E)$. y_i is the defining normal for the hyperplane $X \setminus x_i$. Hence, $D_i^{1+\chi(X \setminus x_i)} \in \mathcal{I}'(X, k, J, E)$ for $i = 1, \dots, n$ (*).

Let $D_{\eta_C}^{m(C)+\chi(C)}$ be a generator of $\mathcal{I}(X, k, J)$. We prove now that $D_{\eta_C}^{m(C)+\chi(C)}$ is contained in $\mathcal{I}'(X, k, J, E)$.

Case 1: $C \in J$. Hence, $D_{\eta_C}^{m(C)+\chi(C)} = (\sum_{i \in I} \alpha_i D_i)^{|I|+1}$ for $I \subseteq [r]$ and $\alpha_i \in \mathbb{K} \setminus \{0\}$. In the monomial expansion of this term, every monomial contains a square. By (*), this polynomial is contained in $\mathcal{I}'(X, k, J, E)$.

Case 2: $C \notin J$. Let C' be a maximal missing flat that contains C . Then $D_{E(C')}^{m(C')+\chi(C')} = (\sum_{x_i \notin C'} \lambda_i D_i)^{m(C')} \in \mathcal{I}'(X, k, J, E)$. In the monomial expansion of this polynomial, there is only one monomial that does not contain a square: $q := \prod_{x_i \notin C'} D_i$. It follows from the definition of $\mathcal{I}'(X, k, J, E)$ and (*) that $q \in \mathcal{I}'(X, k, J, E)$. In the monomial expansion of $D_{\eta_C}^{m(C)+\chi(C)}$, there are only monomials containing squares and a monomial that is a multiple of q . Hence, $D_{\eta_C}^{m(C)+\chi(C)} \in \mathcal{I}'(X, k, J, E)$.

By comparing with (3.4) and taking into account that X is a basis for U , we see that (3.14) describes the \mathcal{P} -space. Using Lemma 3.7, we can calculate $\mathcal{I}(X, k, J)$:

$$\mathcal{I}(X, k, J) = \text{ideal} \left\{ \left(\sum_{i \in I} \alpha_i D_i \right)^{|I|+k+\chi(X \setminus \{x_i : i \in I\})} : I \subseteq [r], \alpha_i \in \mathbb{K} \setminus \{0\} \right\} \quad (3.15)$$

$$= \text{ideal} \left\{ \prod_{i \in I} D_i^{a_i+1} : I \subseteq [r], a_i \in \mathbb{N}, \sum_{i \in I} a_i = k + \chi(X \setminus \{x_i : i \in I\}) \right\} \quad (3.16)$$

It is now clear that $\ker \mathcal{I}(X, k, J) = \mathcal{P}(X, k, J)$. \square

The following Lemma implies $\mathcal{I}(X, k, J) = \mathcal{I}'(X, k, J, E)$ for $k \leq 0$, using the Main Theorem for $k = 0$ as base case (cf. Remark 3.5).

Lemma 3.11. *Let $J \subseteq \mathcal{L}(X)$ be an arbitrary upper set and k be an arbitrary integer. If $\mathcal{I}(X, k, J)$ is contained in $\text{Sym}(V)$ (i. e. $m(C) \geq k$ for all flats C) and $\mathcal{I}'(X, k, J, E) = \mathcal{I}(X, k, J)$ then $\mathcal{I}'(X, l, J, E) = \mathcal{I}(X, l, J)$ for all $l \leq k$ which satisfy $\mathcal{I}(X, l, J) \subseteq \text{Sym}(V)$.*

Proof. Let $D_{\eta}^{m(\eta)+l+\chi(\eta)}$ be a generator of $\mathcal{I}(X, l, J)$. We show that this generator is contained in $\mathcal{I}'(X, l, J, E)$. By induction, we may suppose that $D_{\eta}^{m(\eta)+l+1+\chi(\eta)} \in \mathcal{I}'(X, l+1, J)$, i. e. there exist $q_i \in \text{Sym}(V)$ and $D_{\eta_i}^{m(\eta_i)+l+1+\chi(\eta_i)}$ generators of $\mathcal{I}'(X, l+1, J)$ s. t.

$$D_{\eta}^{m(\eta)+l+1+\chi(\eta)} = \sum_i q_i D_{\eta_i}^{m(\eta_i)+l+1+\chi(\eta_i)} \quad (3.17)$$

Let $u \in U$ be a vector s.t. $\eta(u) = 1$. We consider p_u as a differential operator on $\text{Sym}(V)$. By applying p_u to (3.17), we see that $D_\eta^{m(\eta)+l+\chi(\eta)}$ is contained in $\mathcal{I}'(X, l, J, E)$. \square

3.3. Deletion and Contraction. In the third paragraph of this subsection, we prove the Main theorem. The proof is inductive using deletion and contraction. In the first paragraph, we define those two operations for representable matroids. In the second paragraph, we define them for upper sets.

3.3.1. Matroids under deletion and contraction. Two important constructions for matroids are deletion and contraction of an element. Let $x \in X$. The *deletion* of x is the matroid defined by the sequence $X \setminus x$.

For the rest of this paragraph, fix an element $x \in X$ that is not a loop. Let $\pi_x : U \rightarrow U/x$ denote the projection to the quotient space. The *contraction* of x is the matroid defined by the sequence X/x which contains the images of the elements of $X \setminus x$ under π_x .

We want to be able to see $\text{Sym}(U/x)$ as a subspace of $\text{Sym}(U)$. For that, pick a basis $B = \{b_1, \dots, b_r\} \subseteq U$ with $b_r = x$. Let $W := \text{span}\{b_1, \dots, b_{r-1}\}$. Then we have an isomorphism $U/x \cong W$ which extends to an isomorphism $\text{Sym}(U/x) \cong \text{Sym}(W) \subseteq \text{Sym}(U)$. Under this identification, $\text{Sym}(\pi_x)$ becomes the map that sends p_x to zero and maps all other basis vectors to themselves. Then $\text{Sym}(U) \cong \text{Sym}(W) \oplus p_x \text{Sym}(U)$.

Let $Y \subseteq X \setminus x$. We write \bar{Y} to denote the subsequence of X/x with the same index set as Y and vice versa. Let $\bar{C} \subseteq X/x \subseteq W$ be a flat and $\eta \in W^*$ be a defining normal for \bar{C} . Since $W^* \cong x^\perp := \{v \in V : v(x) = 0\}$, η is also a defining normal for the flat $C \cup \{x\} \subseteq X$.

3.3.2. The lattice of flats under deletion and contraction. In this paragraph, we discuss how the lattice of flats of a matroid behaves under deletion and contraction and for a given upper set J we define upper sets $J \setminus x \subseteq \mathcal{L}(X \setminus x)$ and $J/x \subseteq \mathcal{L}(X/x)$.

For the whole paragraph, fix an element $x \in X$ that is not a loop. First, we exhibit the relations between the lattices of flats of X , $X \setminus x$ and X/x . There are two bijective maps:

$$L_x : \mathcal{L}(X \setminus x) \rightarrow \{C \in \mathcal{L}(X) : C = \text{cl}(C \setminus x)\} \quad (3.18)$$

$$L^x : \mathcal{L}(X/x) \rightarrow \{C \in \mathcal{L}(X) : x \in C\} \quad (3.19)$$

The maps are given by $L_x(C) := \text{cl}_X(C)$, $L_x^{-1}(C) := C \setminus x$, $L^x(\bar{C}) := C \cup x$ and $(L^x)^{-1}(C) := \overline{C \setminus x}$.

Definition 3.12. Let $J \subseteq \mathcal{L}(X)$ be an upper set. Then define

$$J \setminus x := \{C \setminus x : C \in J \text{ and } C = \text{cl}(C \setminus x)\} = L_x^{-1}(J \cap L_x(\mathcal{L}(X \setminus x))) \subseteq \mathcal{L}(X \setminus x)$$

$$J/x := \{\overline{C \setminus x} : x \in C \in J\} = (L^x)^{-1}(J \cap L^x(\mathcal{L}(X/x))) \subseteq \mathcal{L}(X/x)$$

It is easy to check that those two sets are upper sets. The following statement on the indicator functions is also easy to prove:

Lemma 3.13. *Let $x \notin Y \subseteq X$. Then $\chi_{J \setminus x}(Y) = \chi_J(Y)$ and $\chi_{J/x}(\bar{Y}) = \chi_J(Y \cup x)$.*

From this, we can deduce the following:

Lemma 3.14. *If $C \subseteq X \setminus x$ is a maximal missing flat for $J \setminus x$ then C or $C \cup x$ is a maximal missing flat for J .*

If $\bar{C} \subseteq X/x$ is a maximal missing flat for J/x then $C \cup x$ is a maximal missing flat for J .

We also need the following two facts:

Remark 3.15. Let $x \in X$ be neither a loop nor a coloop. Suppose that $J \supseteq \mathcal{H}(X)$. Then,

- (1) $J \setminus x \supseteq \mathcal{H}(X \setminus x)$. This follows from the fact that $\mathcal{L}(X \setminus x)$ contains exactly the flats C that satisfy $\text{rk}(C) = \text{rk}(C \setminus x)$.
- (2) $J/x \supseteq \mathcal{H}(X/x)$. This follows from the fact that $\mathcal{L}(X/x)$ contains exactly the flats containing x and contraction reduces the rank of a flat containing x by one.

3.3.3. Proof of the Main Theorem. In this paragraph, we prove the Main Theorem. The following proposition is a side product of the deletion-contraction proof:

Proposition 3.16. *We use the same terminology as in Definition 3.1. Suppose that $x \in X$ is neither a loop nor a coloop. For $k = -1$, we assume in addition that $J \supseteq \mathcal{H}$ or $J = \{X\}$.*

Then the following is an exact sequence of graded vector spaces:

$$0 \rightarrow \ker \mathcal{I}(X \setminus x, k, J \setminus x)(-1) \xrightarrow{\cdot p_x} \ker \mathcal{I}(X, k, J) \xrightarrow{\text{Sym}(\pi_x)} \ker \mathcal{I}(X/x, k, J/x) \rightarrow 0 \quad (3.20)$$

If $x \in X$ is a loop, then $\ker \mathcal{I}(X \setminus x, k, J \setminus x) = \ker \mathcal{I}(X, k, J)$. For $k \in \{-1, 0\}$, both statements also hold if we replace \mathcal{I} by \mathcal{I}' .

Here, $(\cdot)(-1)$ denotes the graded vector space (\cdot) with the degree shifted up by one and $\text{Sym}(\pi_x)$ denotes the algebra homomorphism that maps p_v to $p_{\pi_x(v)}$.

The proof of this proposition is inductive. It uses the following lemma:

Lemma 3.17. *Suppose that we are in the same setting as in Proposition 3.16. Let $x \in X$ be neither a loop nor a coloop. Suppose that $\mathcal{P}(X \setminus x, k, J \setminus x) = \ker \mathcal{I}(X \setminus x, k, J \setminus x)$ and $\mathcal{P}(X/x, k, J/x) = \ker \mathcal{I}(X/x, k, J/x)$. Then,*

- (1) *the following sequence is exact:*

$$0 \rightarrow \ker \mathcal{I}(X \setminus x, k, J \setminus x)(-1) \xrightarrow{\cdot p_x} \ker \mathcal{I}(X, k, J) \xrightarrow{\text{Sym}(\pi_x)} \ker \mathcal{I}(X/x, k, J/x) \rightarrow 0 \quad (3.21)$$

- (2) *If in addition $\mathcal{I}'(X \setminus x, k, J \setminus x, E') = \mathcal{I}(X \setminus x, k, J \setminus x)$ and $\mathcal{I}(X/x, k, J/x, E'') = \mathcal{I}(X/x, k, J/x)$ for arbitrary E' and E'' , then the following sequence is exact:*

$$0 \longrightarrow \ker \mathcal{I}'(X \setminus x, k, J \setminus x, E')(-1) \xrightarrow{\cdot p_x} \ker \mathcal{I}'(X, k, J, E) \xrightarrow{\text{Sym}(\pi_x)} \ker \mathcal{I}'(X/x, k, J/x, E'') \longrightarrow 0 \quad (3.22)$$

Proof of Lemma 3.17. We only prove part (ii). The reader will notice, that the same proof with some obvious modifications can be used to prove part (i), unless $k = -1$ and $J = \{X\}$. In that case, Lemma 3.11 can be used to deduce (1) from (2).

Before starting with the proof, we introduce some additional notation, which is only used here. For a flat C defined by η , we write $e_X(\eta) = e_X(C) := m_X(C) + k + \chi(C)$. As described above, we fix a subspace $W \subseteq U$ complementary to span x and identify U/x with W . Hence, $\ker \mathcal{I}'(X/x, k, J/x, E'') \subseteq \text{Sym}(W)$.

Let $f \in \text{Sym}(U)$ and $v, \eta \in V$. Let t be a formal symbol. Then $f(v + t\eta) \in \mathbb{K}[t]$ and the following Taylor expansion formula holds: $f(v + t\eta) = \sum_{k \geq 0} \frac{D_\eta^k}{k!} f(v) t^k$. Now define $\rho_f : V \rightarrow \mathbb{N}$, the *directional degree function* of f [2], as the function which assigns to η the degree of the univariate polynomial $f(v + t\eta) \in \mathbb{K}[t]$ for generic v . We obtain $\rho_{fg} = \rho_f + \rho_g$ by comparing the Taylor expansion of $f \cdot g$ with the product of the Taylor expansions of f and g . $\rho_f(\eta)$ tells us how many derivations f survives in direction η . Hence, ρ can be used to describe $\ker \mathcal{I}(X, k, J)$ and $\ker \mathcal{I}'(X, k, J, E)$. Namely,

$$\ker \mathcal{I}(X, k, J) = \{f \in \text{Sym}(U) : \rho_f(\eta) < e(\eta), \eta \in V \setminus \{0\}\} \quad (3.23)$$

Now we come to the main part of the proof. It is split into five parts:

(i) $\cdot p_x$ is well defined, i. e. really maps to $\ker \mathcal{I}'(X, k, J, E)$: due to Lemma 3.8, it suffices to prove $S(X \setminus x, k, J \setminus x) \xrightarrow{p_x} S(X, k, J)$. For $k \geq 0$, this follows directly from Lemma 3.13. For $k = -1$, consider $p_Y \in S(X \setminus x, -1, J \setminus x)$ and $C \in \mathcal{L}(X)$. Then $C \setminus x \in \mathcal{L}(X \setminus x)$ and $\chi_{J \setminus x}(C \setminus x) \leq \chi_J(C)$. One can easily deduce $|(Y \cup x) \setminus C| < m(C) - 1 + \chi_J(C)$ from the corresponding inequality for $C \setminus x$.

(ii) $\text{Sym}(\pi_x)$ is well defined: let $g \in \ker \mathcal{I}'(X, k, J, E)$ and let $h := (\text{Sym}(\pi_x))(g) \in \text{Sym}(W)$. Let $\bar{C} \in \mathcal{L}(X/x)$ be a maximal missing flat or a hyperplane, respectively. By Lemma 3.14, $C \cup x \in \mathcal{L}(X)$ is a maximal missing flat or a hyperplane, respectively. Let $\eta := E_X(C \cup x)$. As $\eta \in x^\perp$, we may suppose $E''(\bar{C}) = \eta$. We need to prove $\rho_h(\eta) < e_{X/x}(\bar{C})$.

Note that $m_{X/x}(\bar{C}) = m_X(C \cup x)$ and by Lemma 3.13, $\chi_{J/x}(\bar{C}) = \chi_J(C \cup x)$. Hence, $e_{X/x}(\bar{C}) = e_X(C \cup x)$. g can be uniquely written as $g = h + p_x g_1$ for some $g_1 \in \text{Sym}(U)$. For all $k \in \mathbb{N}$, $D_\eta^k g = D_\eta^k h + p_x D_\eta^k g_1$. As p_x does not divide h , this implies $\rho_h(\eta) \leq \rho_g(\eta)$. In summary, we get

$$e_{X/x}(\bar{C}) = e_X(C \cup x) > \rho_g(\eta) \geq \rho_h(\eta) \quad (3.24)$$

(iii) *Injectivity of $\cdot p_x$* : clear.

(iv) *Exactness in the middle*: let $g \in \ker \mathcal{I}'(X, k, J, E)$ and $\text{Sym}(\pi_x)(g) = 0$. This implies that g can be written as $g = p_x h$ for some $h \in \text{Sym}(U)$. We need to show that $h \in \ker \mathcal{I}'(X \setminus x, k, J \setminus x, E') = \ker \mathcal{I}(X \setminus x, k, J \setminus x)$.

Let C be a maximal missing flat (resp. hyperplane) in $X \setminus x$. By Lemma 3.14, $C' = C$ or $C' = C \cup x$ is a maximal missing flat (resp. hyperplane) in X . Let $\eta := E(C')$. η is also a defining normal for $C \subseteq X \setminus x$. WLOG, $\eta = E'(C)$. We show now that $\rho_h(\eta) < e_{X \setminus x}(C)$.

If $x \in C'$, then $\rho_{p_x} = 0$, $m_{X \setminus x}(C) = m_X(C')$, and $\chi_{J \setminus x}(C) = \chi_J(C')$. If $x \notin C'$, then $\rho_{p_x}(\eta) = 1$, $m_{X \setminus x}(C) + 1 = m_X(C')$, and $\chi_{J \setminus x}(C) = \chi_J(C')$. So, in both cases, $e_{X \setminus x}(\eta) + \rho_{p_x}(\eta) = e_X(\eta)$. This implies

$$e_{X \setminus x}(\eta) = e_X(\eta) - \rho_{p_x}(\eta) > \rho_{p_x h}(\eta) - \rho_{p_x}(\eta) = \rho_h(\eta) \quad (3.25)$$

(v) *Surjectivity of $\text{Sym}(\pi_x)$* : we consider the case $k \geq 0$ first. Let $fp_{\bar{Y}} \in S(X/x, k, J/x)$. It suffices to prove that $fp_Y \in S(X, k, J)$. Since $x \notin Y$, by Lemma 3.13, $\chi_{J/x}((X/x) \setminus \bar{Y}) = \chi_J(X \setminus Y)$. This implies $fp_Y \in S(X, k, J)$.

Now consider the case $k = -1$. This requires a little more work. There are two subcases:

(a) $J \supseteq \mathcal{H}$: let $p_{\bar{Y}} \in S(X/x, -1, J/x)$. We show now that $p_Y \in S(X, -1, J)$. Let $C \in \mathcal{L}(X) \setminus \{X\}$. Suppose first that $x \in C$ or $\text{codim } C \geq 2$. Then $D := \text{cl}(C \cup x) \neq X$. By assumption, $|\bar{Y} \setminus (\overline{D \setminus x})| < m_{X/x}(\overline{D \setminus x}) - 1 + \chi_{J \setminus x}(\overline{D \setminus x})$. By Lemma 3.13, this implies

$$|Y \setminus D| < m_X(D) - 1 + \chi_J(D) \quad (3.26)$$

Since $x \in D \setminus C$ and $x \notin Y$, $|Y \setminus C| - |Y \setminus D| \leq m_X(C) - m_X(D) - 1$. Adding this inequality to (3.26), we obtain the desired inequality:

$$|Y \setminus C| < m_X(C) - 1 + \chi_J(C) \quad (3.27)$$

Now suppose that C is a hyperplane and $x \notin C$. By assumption, $\chi_J(C) = 1$. Since $x \notin Y \cup C$, we can deduce that (3.27) holds for this C .

(b) $J = \{X\}$: so far, we have shown that (3.22) is exact on the left and in the middle. By elementary linear algebra and Lemma 5.7, we get:

$$\begin{aligned} \dim \text{im}(\text{Sym}(\pi_x)) &= \dim \ker \mathcal{I}'(X, -1, \{X\}, E) - \dim \ker \mathcal{I}'(X \setminus x, -1, \{X \setminus x\}, E') \\ &= \dim \ker \mathcal{I}'(X/x, -1, \{X/x\}, E'') \end{aligned}$$

This implies surjectivity of $\text{Sym}(\pi_x)$. □

Proof of Proposition 3.16 and of the Main Theorem. This proof generalizes the proof of [2, Propositions 4.4. and 4.5.].

Loops can safely be ignored: they are contained in every flat C , thus $m_X(C) = m_{X \setminus x}(C)$ and $\mathcal{L}(X) \cong \mathcal{L}(X \setminus x)$. From now on, we suppose that X does not contain loops.

We prove both statements by induction on the number of elements of X that are not coloops. The reader should check that our argumentation also works for $k = -1$, although in that case, \mathcal{P} -spaces might be zero. Remark 3.15 ensures that an upper set that contains all hyperplanes preserves this structure under deletion and contraction.

If X contains only coloops the Main Theorem follows from Proposition 3.10. Now suppose that $x \in X$ is not a coloop and that the Main Theorem holds for X/x and $X \setminus x$. In addition, we assume $\dim U \geq 2$. If $\dim U = 1$, the statement follows from Remark 3.9.

By Lemma 3.17, the following sequence is exact:

$$0 \rightarrow \ker \mathcal{I}(X \setminus x, k, J \setminus x)(-1) \xrightarrow{p_x} \ker \mathcal{I}(X, k, J) \xrightarrow{\text{Sym}(\pi_x)} \ker \mathcal{I}(X/x, k, J/x) \rightarrow 0 \quad (3.28)$$

Every short exact sequence of vector spaces splits, hence $\ker \mathcal{I}(X, k, J) = p_x \cdot \ker \mathcal{I}(X \setminus x, k, J \setminus x) \oplus \ker \mathcal{I}(X/x, k, J/x)$. For $k \in \{-1, 0\}$, the same argumentation also works for $\mathcal{I}'(X, k, J, E)$. To conclude, we recall the following two statements, that were shown in the proof of Lemma 3.17: (i) $p_x \cdot S(X \setminus x, k, J \setminus x) \subseteq S(X, k, J)$ and (ii) $\text{Sym}(\pi_x) : S(X, k, J) \rightarrow S(X/x, k, J/x)$ is surjective. □

Remark 3.18. For $k = -1$ and $J \not\supseteq \mathcal{H}$, in general, statement (ii) in the proof of the Main Theorem does not hold (cf. Example 7.4). Proposition 3.16 is false for arbitrary $J \not\supseteq \mathcal{H}$ (cf. Example 7.3). The difficulty of the case $k = -1$ was already

observed by Holtz and Ron. They conjectured that the Main Theorem holds in the internal case *i. e.* for $k = -1$ and $J = \{X\}$, but they were unable to prove it [16, Conjecture 6.1].

4. BASES FOR \mathcal{P} -SPACES

In this section, we show how to select a basis for $\mathcal{P}(X, k, J)$ from $S(X, k, J)$ for $k \geq 0$. This construction depends on the order on X . This order is used to define internal and external activity with respect to a given basis (see [5, Section 6.6.] for a reference). Our result is a generalization of [2, Proposition 4.21] to hierarchical spaces. At the end of this section, there is a remark on the case $k = -1$.

Recall that $\mathbb{B}(X)$ denotes the set of all bases $B \subseteq X$. Fix a basis $B \in \mathbb{B}(X)$. $b \in B$ is called *internally active* if $b = \max(X \setminus \text{cl}(B \setminus b))$, *i. e.* b is the maximal element of the unique cocircuit contained in $(X \setminus B) \cup b$. The set of internally active elements in B is denoted $I(B)$. $x \in X \setminus B$ is called *externally active* if $x \in \text{cl}\{b \in B : b \leq x\}$, *i. e.* x is the maximal element of the unique circuit contained in $B \cup x$. The set of externally active elements with respect to B is denoted $E(B)$.

Definition 4.1. We use the same terminology as in Definition 3.1. In addition, let $k \geq 0$. Then define

$$\Gamma(X, k, J) := \left\{ (B, I, \mathbf{a}_I) : B \in \mathbb{B}(X), I \subseteq I(B), \mathbf{a}_I \in \mathbb{N}^I, \sum_{x \in I} a_x \leq k + \chi((B \cup E(B)) \setminus I) - 1 \right\} \quad (4.1)$$

$$\mathcal{B}(X, k, J) := \left\{ p_{X \setminus (B \cup E(B))} \prod_{x \in I} p_x^{a_x + 1} : (B, I, \mathbf{a}_I) \in \Gamma(X, k, J) \right\} \subseteq \text{Sym}(U) \quad (4.2)$$

For $k = 0$, this specializes to

$$\mathcal{B}(X, 0, J) = \left\{ p_{(X \setminus (B \cup E(B))) \cup I} : B \in \mathbb{B}(X), I \subseteq I(B), \text{cl}((B \cup E(B)) \setminus I) \in J \right\}$$

Note that *a priori*, it is unclear whether $\Gamma(X, k, J)$ has the same cardinality as $\mathcal{B}(X, k, J)$ as we do not know if distinct elements of $\Gamma(X, k, J)$ correspond to distinct polynomials in $\mathcal{B}(X, k, J)$. This desired property only becomes clear in the proof of the following theorem:

Theorem 4.2 (Basis Theorem). *We use the same terminology as in Definition 3.1. In addition, let $k \geq 0$.*

Then $\mathcal{B}(X, k, J)$ is a basis for $\mathcal{P}(X, k, J)$.

Proof. As in the proof of the Main Theorem, we may suppose that X does not contain any loops: if x is a loop, it is not contained in any basis, but x is contained in every flat and always externally active. Hence, the removal of a loop changes neither $\mathcal{P}(X, k, J)$ nor $\Gamma(X, k, J)$.

The remainder of this proof is split into four parts:

(i) Let $x \in X$ be the minimal element. Let $B, B' \in \mathbb{B}(X)$ with $x \notin B$ and $x \in B'$. x is externally active with respect to B if and only if x is a loop and x is internally active in B' if and only if x is a coloop.

(ii) $|\Gamma(X, k, J)| = \dim \mathcal{P}(X, k, J)$: we prove this by induction over the number of elements that are not coloops. Suppose that X contains only coloops. In this case there is only one basis and all its elements are internally active. The spanning set given in (3.14) is a basis and it coincides with $\mathcal{B}(X, k, J)$.

Now suppose that there is at least one element in x which is not a coloop. In addition, we may assume $\dim U \geq 1$. If $\dim U = 1$, the statement follows from Remark 3.9. As $\dim \mathcal{P}(X, k, J)$ and by induction also $|\Gamma(X/x, k, J/x)|$ and $|\Gamma(X \setminus x, k, J \setminus x)|$ are invariant of the order on X , we may assume that x is the minimal element.

$\mathbb{B}(X)$ can be partitioned as $\mathbb{B}(X) = \mathbb{B}(X \setminus x) \dot{\cup} \iota(\mathbb{B}(X/x))$, where ι denotes the map that sends a basis $\bar{B} \in \mathbb{B}(X/x)$ to $B \cup x$. It follows from (i) and Lemma 3.13 that $\Gamma(X, k, J)$ can also be partitioned as $\Gamma(X, k, J) = \Gamma(X \setminus x, k, J \setminus x) \dot{\cup} \iota_1(\Gamma(X/x, k, J/x))$, where ι_1 denotes the map that sends $(\bar{B}, \bar{I}, \mathbf{a}_{\bar{I}})$ to $(B \cup x, I, \mathbf{a}_I)$. Comparing this with Proposition 3.16, we see that $|\Gamma(X, k, J)| = \dim \mathcal{P}(X, k, J)$.

(iii) $\mathcal{B}(X, k, J) \subseteq \mathcal{S}(X, k, J) \subseteq \mathcal{P}(X, k, J)$: if $Y = (X \setminus (B \cup E(B))) \cup I$, then $X \setminus Y = (B \cup E(B)) \setminus I$. Hence, by comparison of (4.1) and (3.4), the statement follows.

(iv) $\mathcal{B}(X, k, J)$ is linearly independent: by [2, Proposition 4.21], $\mathcal{B}(X, k, \mathcal{L}(X)) = \mathcal{B}(X, k+1, \{X\})$ is linearly independent. As $\mathcal{B}(X, k, J)$ is contained in this set, it is also linearly independent. \square

Remark 4.3. We do not know if there is a simple method to construct bases for $\mathcal{P}(X, -1, J)$. This difficulty was already observed for the internal case by Holtz and Ron [16]. In Section 5.3, we define a set of semi-internal bases $\mathbb{B}_-(X, J) \subseteq \mathbb{B}(X)$ whose cardinality is in some cases equal to the dimension of $\mathcal{P}(X, -1, J)$. A natural candidate for $\mathcal{B}(X, -1, J)$ would be the set $\tilde{\mathcal{B}}(X, -1, J) := \{p_{X \setminus (B \cup E(B))} : B \in \mathbb{B}_-(X, J)\}$. In some cases, this is indeed a basis, but in general it has the wrong cardinality or it fails to be contained in $\mathcal{P}(X, -1, J)$ (see Example 7.2).

5. HILBERT SERIES

In this section, we give several formulas for the Hilbert series of $\mathcal{P}(X, k, J)$. The formulas in the first subsection are recursive. In the second subsection, we give combinatorial formulas for the case $k \geq 0$. The last subsection is devoted to the case $k = -1$. All formulas only depend on the matroid $\mathfrak{M}(X)$, the integer k , and the upper set J , but not on the representation X .

5.1. Recursive formulas. In this subsection, we give recursive formulas for the calculation of $\text{Hilb}(\mathcal{P}(X, k, J), t)$. The following statement is a direct consequence of Proposition 3.16 and of the Main Theorem:

Corollary 5.1. *We use the same terminology as in Definition 3.1. Let $x \in X$ be an element that is not a coloop. For $k = -1$, we assume in addition that $J \supseteq \mathcal{H}$ or $J = \{X\}$, i. e. J contains either all or no hyperplanes. Then,*

$$\text{Hilb}(\mathcal{P}(X, k, J), t) = \begin{cases} \text{Hilb}(\mathcal{P}(X \setminus x, k, J \setminus x), t) & \text{if } x \text{ is a loop} \\ t \text{Hilb}(\mathcal{P}(X \setminus x, k, J \setminus x), t) \\ + \text{Hilb}(\mathcal{P}(X/x, k, J/x), t) & \text{otherwise} \end{cases}$$

For coloops, the situation is more complicated and requires an additional definition. Fix a coloop $x \in X$. Then, $X \setminus x$ is a hyperplane and the following is an upper set:

$$\widehat{J/x} := \{\bar{C} : x \notin C \in J\} \cup \{\overline{X \setminus x}\} \subseteq \mathcal{L}(X/x) \quad (5.1)$$

$\widehat{J/x}$ forgets about the flats containing x , whereas J/x forgets about the flats not containing x . While the latter is always an upper set in $\mathcal{L}(X/x)$, some elements of $\widehat{J/x}$ are not closed unless $X \setminus x$ is a hyperplane.

Theorem 5.2. *We use the same terminology as in Definition 3.1. Let $x \in X$ be a coloop and $k \geq 0$. Then,*

$$\text{Hilb}(\mathcal{P}(X, k, J), t) = \begin{cases} \text{Hilb}(\mathcal{P}(X/x, k, J/x), t) \\ + \sum_{j=0}^k t^{j+1} \text{Hilb}(\mathcal{P}(X/x, k-j, \widehat{J/x}), t) & \text{if } X \setminus x \in J \\ \text{Hilb}(\mathcal{P}(X/x, k, J/x), t) \\ + \sum_{j=0}^{k-1} t^{j+1} \text{Hilb}(\mathcal{P}(X/x, k-j, \widehat{J/x}), t) & \text{if } X \setminus x \notin J \end{cases}$$

For $k = -1$, we have

$$\text{Hilb}(\ker \mathcal{I}(X, -1, J), t) = \begin{cases} \text{Hilb}(\ker \mathcal{I}(X/x, k, J/x), t) & \text{if } X \setminus x \in J \\ 0 & \text{if } X \setminus x \notin J \end{cases} \quad (5.2)$$

This formula holds for arbitrary non-empty upper sets $J \subseteq \mathcal{L}(X)$.

For an example, see Example 7.1. We actually prove a more general statement, namely decomposition formulas for the \mathcal{P} -spaces of type $\mathcal{P}(X, k, J) \cong \mathcal{P}(X/x, k, J/x) \oplus \bigoplus_j p_x^{j+1} \mathcal{P}(X/x, k-j, \widehat{J/x})$.

Proof of Theorem 5.2. We first prove the equation for $k \geq 0$ using Theorem 4.2 by showing that there exists a bijection between the bases of the \mathcal{P} -spaces appearing on each side.

Fix a basis $B \in \mathbb{B}(X)$. Let $\Gamma_B(X, k, J) := \{(B, I, \mathbf{a}_I) \in \Gamma(X, k, J)\}$. Since x is a coloop, x is contained in every basis and always internally active.

Consider the following map:

$$\Phi_B : \Gamma_B(X, k, J) \rightarrow \Gamma_B(X/x, k, J/x) \dot{\cup} \bigcup_{j=0}^{k-\varepsilon} \Gamma_B(X/x, k-j, \widehat{J/x}) \quad (5.3)$$

$$(B, I, \mathbf{a}_I) \mapsto \begin{cases} (\overline{B \setminus x}, \overline{I}, \mathbf{a}_{\overline{I}}) \in \Gamma_B(X/x, k, \widehat{J/x}) & \text{if } x \notin I \\ (\overline{B \setminus x}, \overline{I \setminus x}, \mathbf{a}_{\overline{I}}) \in \Gamma_B(X/x, k - a_x, \widehat{J/x}) & \text{if } x \in I \end{cases} \quad (5.4)$$

where $\varepsilon = 1$ if $X \setminus x \notin J$ and 0 otherwise and $\mathbf{a}_{\overline{I}}$ denotes the restriction of \mathbf{a}_I to $\mathbb{N}^{I \setminus x}$.

From following three facts we can deduce that Φ_B is a bijection:

- (i) If $x \notin I$, then by Lemma 3.13, $\chi_J((B \cup E(B)) \setminus I) = \chi_{J/x}((B \cup E(B)) \setminus (I \cup x))$.
- (ii) If $x \in I$ and $X \setminus x \in J$ (*i. e.* we are in the first case), then $\chi_J((B \cup E(B)) \setminus I) = \chi_{\widehat{J/x}}(\overline{(B \cup E(B)) \setminus I})$.
- (iii) If $x \in I$ and $X \setminus x \notin J$ (*i. e.* we are in the second case), then $\chi_J((B \cup E(B)) \setminus I) = \chi_{\widehat{J/x}}(\overline{(B \cup E(B)) \setminus I}) = 0$.

We have to distinguish the cases $X \setminus x \in J$ and $X \setminus x \notin J$ for the following reason: if $I = \{x\}$ and $\chi_J((B \cup E(B)) \setminus I) = 0$, then $\Gamma_B(X, k, J)$ contains no elements with $a_x = k$. However, $\Gamma_B(X, k-k, \widehat{J/x}) \neq \emptyset$, since by definition, $\chi_{\widehat{J/x}}(\overline{(B \cup E(B))}) = 1$.

Furthermore, note that the degrees of the polynomials corresponding to (B, I, \mathbf{a}_I) and $(\overline{B \setminus x}, \overline{I \setminus x}, \mathbf{a}_{\overline{I}})$ differ by $a_x + 1$ if $x \in I$. If $x \notin I$, the polynomials corresponding to (B, I, \mathbf{a}_I) and $(\overline{B \setminus x}, \overline{I}, \mathbf{a}_{\overline{I}})$ have the same degree. This completes the proof for $k \geq 0$.

Now we consider the case $k = -1$. If $X \setminus x \notin J$, then $\mathcal{I}(X, -1, J) = \text{ideal}\{1\}$. Suppose that $X \setminus x \in J$. Let η be a defining normal for $X \setminus x$. Then $D_\eta \in \mathcal{I}(X, -1, J)$ and it is easy to check that $\ker \mathcal{I}(X, -1, J) \cong \ker \mathcal{I}(X/x, -1, J/x)$. \square

5.2. Combinatorial formulas for $k \geq 0$. In this subsection, we prove several combinatorial formulas for $\text{Hilb}(\mathcal{P}(X, k, J), t)$. As in the case of the Tutte polynomial, there is a formula that depends on the internal and external activity of the bases of X . For $k = 0$, there is also a subset expansion formula and a particularly simple formula for the dimension.

Theorem 4.2 provides a method to compute the Hilbert series of a \mathcal{P} -space combinatorially:

Corollary 5.3. *We use the same terminology as in Definition 3.1. Let $k \geq 0$. Then,*

$$\text{Hilb}(\mathcal{P}(X, k, J), t) = \sum_{B \in \mathbb{B}(X)} t^{N-r-|E(B)|} \left(1 + \sum_{\emptyset \neq I \subseteq I(B)} \sum_{j=0}^{+k-1} t^{|I|+j} \binom{j+|I|-1}{|I|-1} \right)$$

where $E(B)$ and $I(B)$ denote the sets of externally resp. internally active elements. For $k = 0$, this specializes to

$$\text{Hilb}(\mathcal{P}(X, 0, J), t) = \sum_{B \in \mathbb{B}(X)} t^{N-r-|E(B)|} \left(1 + \sum_{\substack{\emptyset \neq I \subseteq I(B) \\ \chi((B \cup E(B)) \setminus I) = 1}} t^{|I|} \right) \quad (5.5)$$

From this, we can deduce a result, which relates the dimension of $\mathcal{P}(X, 0, J)$ and the number of independent sets satisfying a certain property. This was with a different method by Holtz, Ron, and Xu [18].

Corollary 5.4.

$$\dim \mathcal{P}(X, 0, J) = |\{Y \subseteq X : Y \text{ independent, } \text{cl}(Y) \in J\}| \quad (5.6)$$

Proof. Every independent set $Y \subseteq X$ can be uniquely written as $Y = B \setminus I$ for some $B \in \mathbb{B}(X)$ and $I \subseteq I(B)$. Furthermore, $\text{cl}((B \cup E(B)) \setminus I) = \text{cl}(B \setminus I)$. This follows from Proposition 7.3.6 and (7.13) in [4] (see also [2, Proposition 4.20]).

The statement then follows from Corollary 5.3:

$$\dim \mathcal{P}(X, 0, J) = \text{Hilb}(\mathcal{P}(X, k, J), 1) = \{(B, I) : I \subseteq I(B), \underbrace{\chi(B \setminus I)}_Y = 1\} \quad (5.7)$$

$$= |\{Y \subseteq X : Y \text{ independent, } \text{cl}(Y) \in J\}| \quad (5.8)$$

\square

Corollary 5.3 gives a formula in terms of the internal and external activity of the bases of X . For $k = 0$, there is also a subset expansion formula similar to the one for the Tutte polynomial. In the internal, central and external case, the Hilbert series of the \mathcal{P} -spaces are evaluations of the Tutte polynomial [2]. In particular,

$$\text{Hilb}(\mathcal{P}(X, 0, \{X\}), t) = t^{N-r} \sum_{\substack{A \subseteq X \\ \text{rk}(A)=r}} \left(\frac{1}{t} - 1\right)^{|A|-\text{rk}(A)} = t^{N-r} T_X \left(1, \frac{1}{t}\right) \quad (5.9)$$

$$\text{Hilb}(\mathcal{P}(X, 1, \{X\}), t) = t^{N-r} \sum_{A \subseteq X} t^{r-\text{rk}(A)} \left(\frac{1}{t} - 1\right)^{|A|-\text{rk}(A)} = t^{N-r} T_X \left(1 + t, \frac{1}{t}\right) \quad (5.10)$$

When looking at these two formulas, one might wonder if it is possible to find an ‘‘interpolating’’ formula for the semi-external case. Indeed, the natural guess works: if $\chi(A) = 1$, we take the corresponding summand from (5.10) and if $\chi(A) = 0$, we take the corresponding summand from (5.9). Note that the latter term is always 0. In the semi-internal case however, the analogous statement is false.

Theorem 5.5. *We use the same terminology as in Definition 3.1.*

$$\text{Hilb}(\mathcal{P}(X, 0, J), t) = t^{N-r} \sum_{\substack{A \subseteq X \\ \chi(A)=1}} t^{r-\text{rk}(A)} \left(\frac{1}{t} - 1\right)^{|A|-\text{rk}(A)} \quad (5.11)$$

Proof. As usual, we prove this statement by deletion-contraction. In this proof, we denote the polynomial on the right side of (5.11) by $T_{(X,J)}(t)$.

Let $x \in X$ be a loop and let $A \subseteq X \setminus x$. If $\chi_J(A) = 0$, A contributes neither to $T_{(X \setminus x, J \setminus x)}(t)$ nor to $T_{(X \setminus x, J \cup x)}(t)$. If $\chi_J(A) = 1$, A contributes to $T_{(X \setminus x, J \setminus x)}(t)$ the term $t^{N-r-1} t^{r-\text{rk}(A)} (1/t - 1)^{|A|-\text{rk}(A)} =: f_A$. To $T_{(X,J)}(t)$, A contributes the term $t f_A$ and $A \cup x$ contributes $t(1/t - 1) f_A$. This implies $T_{(X \setminus x, J \setminus x)}(t) = T_{(X,J)}(t)$.

From now on, we suppose that X does not contain any loops. Suppose that X contains only coloops. By Proposition 3.10,

$$\mathcal{P}(X, 0, J) = \text{span}\{p_Y : X \setminus Y \in J\} \quad (5.12)$$

It is easy to see that $T_{(X,J)}(t)$ is the Hilbert series of (5.12):

$$T_{(X,J)}(t) = t^{N-r} \sum_{\substack{A \subseteq X \\ A \in J}} t^{r-|A|} = \sum_{\substack{A \subseteq X \\ X \setminus A \in J}} t^{|A|} \quad (5.13)$$

Now suppose that $x \in X$ is neither a loop nor a coloop. By induction, we may suppose that (5.11) holds for X/x and $X \setminus x$. Using Corollary 5.1 and Lemma 3.13, we obtain

$$\begin{aligned}
\text{Hilb}(\mathcal{P}(X, 0, J), t) &= t \text{Hilb}(\mathcal{P}(X \setminus x, k, J \setminus x), t) + \text{Hilb}(\mathcal{P}(X/x, k, J/x), t) \\
&= t^{N-r} \sum_{\substack{A \subseteq X \setminus x \\ \chi_{J \setminus x}(A)=1}} t^{r-\text{rk}(A)} \left(\frac{1}{t} - 1\right)^{|A|-\text{rk}(A)} \\
&\quad + t^{N-r} \sum_{\substack{\bar{A} \in X/x \\ \chi_{J/x}(\bar{A})=1}} t^{(r-1)-\text{rk}(\bar{A})} \left(\frac{1}{t} - 1\right)^{|\bar{A}|-\text{rk}(\bar{A})} \\
&= t^{N-r} \sum_{\substack{x \notin A \\ \chi_J(A)=1}} t^{r-\text{rk}(A)} \left(\frac{1}{t} - 1\right)^{|A|-\text{rk}(A)} \\
&\quad + t^{N-r} \sum_{\substack{x \in A \\ \chi_J(A)=1}} t^{r-\text{rk}(A)} \left(\frac{1}{t} - 1\right)^{|A|-\text{rk}(A)} \\
&= t^{N-r} \sum_{\substack{A \subseteq X \\ \chi(A)=1}} t^{r-\text{rk}(A)} \left(\frac{1}{t} - 1\right)^{|A|-\text{rk}(A)} = T_{(X, J)}(t)
\end{aligned}$$

□

5.3. **The case $k = -1$.** For $k = -1$, we don't know if there is such a nice formula as in Corollary 5.3 or Theorem 5.5. As we stated in Remark 4.3, the set $\mathbb{B}_-(X, J) := \{B \in \mathbb{B}(X) : \chi(B \setminus I(B)) = 1\}$ has in some cases several nice properties, but in general the cardinality of $\mathbb{B}_-(X, J)$ depends on the order imposed on X . Consider for example a sequence X of three vectors a, b, c in general position in a two-dimensional vector space and the ideal $J = \{X, \{a\}\}$. Depending on the order, $\mathbb{B}_-(X, J)$ may have cardinality 1 or 2.

Fix $C_0 \in \mathcal{L}(X)$ and set $J_{C_0} := \{C \in \mathcal{L}(X) : C \supseteq C_0\}$. All maximal missing flats in J_{C_0} are hyperplanes. They have unique defining normals (up to scaling). Then $\ker \mathcal{I}(X, -1, J_{C_0}) = \ker \mathcal{I}'(X, -1, J_{C_0}) = \bigcap_{x \in C_0} \mathcal{P}(X \setminus x, 0, \{X \setminus x\})$. This was shown by Holtz, Ron, and Xu [18]. They also show that for a specific order on X (see below), $|\mathbb{B}_-(X, J_{C_0})| = \dim \ker \mathcal{I}(X, -1, J_{C_0})$ and that $\mathbb{B}_-(X, J_{C_0})$ can be used to calculate the Hilbert series:

Theorem 5.6 ([18, p. 20]). *We use the same terminology as in Definition 3.1. In addition, let $C_0 \in \mathcal{L}(X)$. Then,*

$$\text{Hilb}(\ker \mathcal{I}(X, -1, J_{C_0}), t) = \sum_{B \in \mathbb{B}_-(X, J_{C_0})} t^{N-r-|E(B)|} \quad (5.14)$$

The proof in [18] relies on the following construction: an independent spanning subset $I \subseteq C_0$ is fixed and the order is chosen s. t. the elements of I are maximal. This makes it difficult or impossible to adjust this proof to a more general setting.

We end this subsection with a small lemma that is needed in the proof of the Main Theorem:

Lemma 5.7. *Let $\dim U \geq 2$ and let $x \in X$ be neither a loop nor a coloop. Then,*

$$\begin{aligned} \dim \ker \mathcal{I}'(X, -1, \{X\}) &= \dim \ker \mathcal{I}'(X \setminus x, -1, \{X \setminus x\}) \\ &\quad + \dim \ker \mathcal{I}'(X/x, -1, \{X/x\}) \end{aligned}$$

Proof. We write $\mathbb{B}_-(X)$ for $\mathbb{B}_-(X, \{X\})$. It is known that $\dim \ker \mathcal{I}'(X, -1, \{X\}) = \dim \mathbb{B}_-(X)$ ([16, Theorem 5.9]). This implies that $|\mathbb{B}_-(X)|$ is independent of the order on X . Hence, we may suppose that x is the minimal element. Now it is easy to check the following equation: $|\mathbb{B}_-(X)| = |\mathbb{B}_-(X \setminus x)| + |\mathbb{B}_-(X/x)|$. \square

6. ZONOTOPAL COX RINGS

In this section, we briefly describe the zonotopal Cox rings defined by Sturmfels and Xu [23] and we show that our Main Theorem can be used to generalize a result on zonotopal Cox modules due to Ardila and Postnikov [2].

Fix m vectors $D_1, \dots, D_m \in V$ and $\mathbf{u} = (u_1, \dots, u_m) \in \mathbb{N}^m$. Sturmfels and Xu [23] introduced the Cox-Nagata ring $R^G \subseteq \mathbb{K}[s_1, \dots, s_m, t_1, \dots, t_m]$. This is the ring of polynomials that are invariant under the action of a certain group G which depends on the vectors D_1, \dots, D_m . It is multigraded with a \mathbb{Z}^{m+1} -grading. For $r \geq 3$, R^G is equal to the Cox ring of the variety X_G which is gotten from \mathbb{P}^{r-1} by blowing up the points D_1, \dots, D_m . Cox rings have received a considerable amount of attention in the recent literature in algebraic geometry. See [19] for a survey.

Cox-Nagata rings are closely related to power ideals: let $\mathcal{I}_{\mathbf{u}} := \text{ideal}\{D_1^{u_1+1}, \dots, D_m^{u_m+1}\}$ and let $\mathcal{I}_{d,\mathbf{u}}^{-1}$ denote the homogeneous component of grade d of $\ker \mathcal{I}_{\mathbf{u}}$. Then, $R_{(d,\mathbf{u})}^G$, the homogeneous component of R^G of grade (d, \mathbf{u}) , is naturally isomorphic to $\mathcal{I}_{d,\mathbf{u}}$ ([23, Proposition 2.1]).

Cox-Nagata rings are an object of great interest but in general, it is quite difficult to understand their structure. However, for some choices of the vectors D_1, \dots, D_m , we understand a natural subring of the Cox-Nagata ring very well.

Let $\mathcal{H} = \{H_1, \dots, H_m\}$ denote the set of hyperplanes in $\mathcal{L}(X)$. Let $\mathfrak{H} \in \{0, 1\}^{m \times N}$ denote the non-containment vector-hyperplane matrix, *i. e.* the 0-1 matrix whose (i, j) entry is 1 if and only if x_j is not contained in H_i .

Sturmfels and Xu defined the following structures: the *zonotopal Cox ring*

$$\mathcal{Z}(X) := \bigoplus_{(d,\mathbf{a}) \in \mathbb{N}^{N+1}} R_{(d,\mathfrak{H}\mathbf{a})}^G \quad (6.1)$$

and for $\mathbf{w} \in \mathbb{Z}^n$ the *zonotopal Cox module* of shift \mathbf{w}

$$\mathcal{Z}(X, \mathbf{w}) := \bigoplus_{(d,\mathbf{a}) \in \mathbb{N}^{N+1}} R_{(d,\mathfrak{H}\mathbf{a}+\mathbf{w})}^G \quad (6.2)$$

Let $X(\mathbf{a})$ denote sequence of $\sum_i a_i$ vectors in U that is obtained from X by replacing each x_i by a_i copies of itself and let $\mathbf{e} := (1, \dots, 1) \in \mathbb{N}^m$. Ardila and Postnikov prove the following isomorphisms [2, Proposition 6.3]:

$$R_{(d,\mathfrak{H}\mathbf{a})}^G \cong \mathcal{P}(X(\mathbf{a}), 1, \{X\})_d \quad (6.3)$$

$$R_{(d,\mathfrak{H}\mathbf{a}-\mathbf{e})}^G \cong \mathcal{P}(X(\mathbf{a}), 0, \{X\})_d \quad (6.4)$$

$$R_{(d,\mathfrak{H}\mathbf{a}-2\mathbf{e})}^G \cong \mathcal{P}(X(\mathbf{a}), -1, \{X\})_d \quad (6.5)$$

They prove those isomorphisms by showing the following lemma:

Lemma 6.1. *We use the same terminology as in Definition 3.1. Let $\mathbf{b} \in \{0, 1\}^{\mathcal{H}}$ and let $J_{\mathbf{b}} := \{C \in \mathcal{L}(X) : b_H = 1 \text{ for all } H \supseteq C\}$, i. e. the maximal missing flats in $J_{\mathbf{b}}$ are exactly the hyperplanes that satisfy $b_H = 0$.*

Suppose that $\mathcal{I}(X, k, J) = \mathcal{I}'(X, k, J, E)$ for all E . Then,

$$R_{(d, \mathfrak{H}\mathbf{a} + (k-1)\mathbf{e} + \mathbf{b})}^G \cong (\ker \mathcal{I}(X, k, J_{\mathbf{b}}))_d \quad (6.6)$$

Using the Main Theorem, we can deduce the following results about *hierarchical zonotopal Cox modules*:

Proposition 6.2. *We use the same terminology as in Lemma 6.1. For the graded components of the semi-external zonotopal Cox module $\mathcal{Z}(X, \mathfrak{H}\mathbf{a} - \mathbf{e} + \mathbf{b})$, the following holds:*

$$R_{(d, \mathfrak{H}\mathbf{a} - \mathbf{e} + \mathbf{b})}^G \cong \mathcal{P}(X(\mathbf{a}), 0, J_{\mathbf{b}})_d \quad (6.7)$$

Proposition 6.3. *We use the same terminology as in Lemma 6.1. Let $C_0 \in \mathcal{L}(X)$ be a fixed flat and $J_{C_0} := \{C \in \mathcal{L}(X) : C \supseteq C_0\}$ (cf. Subsection 5.3). If $\mathbf{b} \in \{0, 1\}^{\mathcal{H}}$ satisfies $b_H = 1$ if and only if $H \supseteq C_0$, then for the graded components of the semi-internal zonotopal Cox module $\mathcal{Z}(X, \mathfrak{H}\mathbf{a} - 2\mathbf{e} + \mathbf{b})$, the following holds:*

$$R_{(d, \mathfrak{H}\mathbf{a} - 2\mathbf{e} + \mathbf{b})}^G \cong \ker \mathcal{I}(X(\mathbf{a}), -1, J_{C_0})_d \quad (6.8)$$

Using Theorems 5.5 and 5.6, we can calculate the multigraded Hilbert series of the semi-external and the semi-internal zonotopal Cox modules:

Corollary 6.4. *In the setting of Proposition 6.2, the dimension of $R_{(d, \mathfrak{H}\mathbf{a} - \mathbf{e} + \mathbf{b})}^G$ equals the coefficient of t^d in*

$$\text{Hilb}(\mathcal{P}(X(\mathbf{a}), 0, J_{\mathbf{b}}), t) = t^{|\mathbf{a}| - r} \sum_{\substack{A \subseteq X \\ \chi(A) = 1}} t^{r - \text{rk}(A)} \sum_{\substack{1 \leq s_i \leq a_i \\ \mathbf{s} \in \mathbb{N}^A, x_i \in A}} \left(\prod_i \binom{a_i}{s_i} \right) \left(\frac{1}{t} - 1 \right)^{|\mathbf{s}| - \text{rk}(A)}$$

where $|\mathbf{a}| := \sum_i a_i$.

Proof. Apply Theorem 5.5 to $X(\mathbf{a})$. Take into account that for every $S \subseteq X(\mathbf{a})$, there is unique pair (A, \mathbf{s}) such that S is obtained from A by replacing each $x_i \in A$ by s_i copies of itself and $\text{rk}(S) = \text{rk}(A)$. For (A, \mathbf{s}) fixed, there are $\binom{a_i}{s_i}$ options to choose the vectors for every i . \square

Corollary 6.5. *In the setting of Proposition 6.3, the dimension of $R_{(d, \mathfrak{H}\mathbf{a} - 2\mathbf{e} + \mathbf{b})}^G$ equals the coefficient of t^d in*

$$\text{Hilb}(\ker \mathcal{I}(X(\mathbf{a}), -1, J_{C_0}), t) = \sum_{B \in \mathbb{B}_-(X, J_{C_0})} \sum_{\substack{0 \leq s_i \leq a_i - 1 \\ \mathbf{s} \in \mathbb{N}^B}} t^{e(B, \mathbf{s})} \quad (6.9)$$

where $e(B, \mathbf{s}) := \sum_{i: x_i \notin E(B)} a_i - r - \sum_{x_i \in B} s_i$

Proof. Apply Theorem 5.6. Choose an order on $X(\mathbf{a})$ that is compatible with the order on X , i. e. if $x', y' \in X(\mathbf{a})$ are copies of $x, y \in X$ ($x \neq y$) then $x' < y'$ if and only if $x < y$. Fix a basis $B \subseteq X$. All copies of the elements that are externally active with respect to B in X are externally active with respect to every copy of B

in $X(\mathbf{a})$. Let $x \in B \subseteq X$. If the i th copy (the maximal one being the first) of x in $X(\mathbf{a})$ is chosen, $i - 1$ copies of x are externally active in $X(\mathbf{a})$. Hence, for the copy of B in $X(\mathbf{a})$ that corresponds to \mathbf{s} , the exponent of t equals $e(B, \mathbf{s})$. \square

7. EXAMPLES

This section contains a large number of examples. In the first subsection, we give explicit examples for the various structures appearing in this paper ($X, J, S, \mathcal{P}, \mathcal{I}, \Gamma, \mathcal{B}, \mathbb{B}, \mathbb{B}_-$). In the second subsection, we give an example for deletion and contraction as defined in Section 3.3. The third paragraph exemplifies the recursive expression for the Hilbert series in Theorem 5.2. In the last subsection, we explain several problems that occur in the semi-internal case ($k = -1$).

In this section, we do the following identifications: $\text{Sym}(V) = \text{Sym}(U) = \mathbb{K}[x, y]$ respectively $\text{Sym}(V) = \text{Sym}(U) = \mathbb{K}[x, y, z]$.

7.1. Structures. Let

$$X_1 := \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = (x_1, x_2, x_3) \quad (7.1)$$

Define two ideals $J_1 := \{X\}$ and $J_2 := \{X, (x_1), (x_3)\}$. The set of bases is $\mathbb{B}(X) = \{(x_1x_2), (x_1x_3), (x_2x_3)\}$. The sets of semi-internal bases are $\mathbb{B}_-(X, J_1) = \{(x_1x_2)\}$ and $\mathbb{B}_-(X, J_2) = \{(x_1x_2), (x_1x_3)\}$.

$$\begin{aligned} S(X_1, -1, J_1) &= \{1\} & S(X_1, 0, J_1) &= \{1, p_{x_1}, p_{x_2}, p_{x_3}\} \\ \mathcal{P}(X_1, -1, J_1) &= \text{span}\{1\} & \mathcal{P}(X_1, 0, J_1) &= \text{span}\{1, x, y\} \\ \mathcal{I}(X_1, -1, J_1) &= \text{ideal}\{x, y\} & \mathcal{I}(X_1, 0, J_1) &= \text{ideal}\{x^2, xy, y^2\} \\ \Gamma(X_1, 0, J_1) &= \{((x_1x_2), \emptyset, 0), ((x_1x_3), \emptyset, 0), \\ & \quad ((x_2x_3), \emptyset, 0)\} \\ \mathcal{B}(X, 0, J_1) &= \{p_\emptyset, p_{x_2}, p_{x_1}\} \\ S(X_1, -1, J_2) &= \{1, p_{x_2}\} & S(X_1, 0, J_2) &= \{1, p_{x_1}, p_{x_2}, p_{x_3}, p_{x_1x_2}, p_{x_2x_3}\} \\ \mathcal{P}(X_1, -1, J_2) &= \text{span}\{1, y\} & \mathcal{P}(X, 0, J_2) &= \text{span}\{1, x, y, xy, y^2\} \\ \mathcal{I}(X_1, -1, J_2) &= \text{ideal}\{x, y^2\} & \mathcal{I}(X_1, 0, J_2) &= \text{ideal}\{x^2, xy^2, y^3\} \\ \Gamma(X_1, 0, J_2) &= \{((x_1x_2), \emptyset, 0), ((x_1x_3), \emptyset, 0), \\ & \quad ((x_1x_3), (x_3), 0), ((x_2x_3), \emptyset, 0), \\ & \quad ((x_2x_3), (x_2), \emptyset, 0)\} \\ \mathcal{B}(X_1, 0, J_2) &= \{p_\emptyset, p_{x_2}, p_{x_2x_3}, p_{x_1}, p_{x_1x_2}\} \end{aligned}$$

7.2. Deletion and contraction. In this subsection, we give examples explaining deletion and contraction for pairs (X, J) .

$$X_1 \setminus x_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = (x_2, x_3) \quad X_1/x_1 = [1 \quad 1] = (\bar{x}_2, \bar{x}_3) \quad (7.2)$$

$$J_1 \setminus x_1 = \{(x_2, x_3)\} \quad J_1/x_1 = \{(\bar{x}_2, \bar{x}_3)\} \quad (7.3)$$

$$J_2 \setminus x_1 = \{(x_2, x_3), (x_3)\} \quad J_2/x_1 = \{(\bar{x}_2, \bar{x}_3), \bar{\emptyset}\} = \mathcal{L}(X_1/x_1) \quad (7.4)$$

Recall that we identify $\mathbb{K}[x, y]/x$ and $\mathbb{K}[y]$. Then,

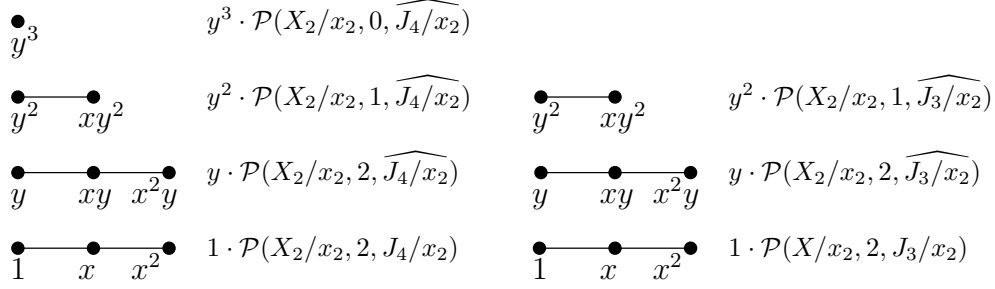


FIGURE 1. On the left, $\mathcal{P}(X_2, 2, J_4)$ and on the right $\mathcal{P}(X_2, 2, J_3)$. For both spaces, the decompositions corresponding to Theorem 5.2 are shown.

$$\begin{aligned}
 \mathcal{I}(X_1 \setminus x_1, 0, J_1 \setminus x_1) &= \text{ideal}\{x, y\} & \mathcal{I}(X_1 \setminus x_1, 0, J_2 \setminus x_1) &= \text{ideal}\{x, y^2\} \\
 \mathcal{P}(X_1 \setminus x_1, 0, J_1 \setminus x_1) &= \text{span}\{1\} & \mathcal{P}(X_1 \setminus x_1, 0, J_2 \setminus x_1) &= \text{span}\{1, y\} \\
 \mathcal{I}(X_1/x_1, 0, J_1/x_1) &= \text{ideal}\{y^2\} & \mathcal{I}(X_1/x_1, 0, J_2/x_1) &= \text{ideal}\{y^3\} \\
 \mathcal{P}(X_1/x_1, 0, J_1/x_1) &= \text{span}\{1, y\} & \mathcal{P}(X_1/x_1, 0, J_2/x_1) &= \text{span}\{1, y, y^2\}
 \end{aligned}$$

The reader should check that $\mathcal{P}(X_1, 0, J_i) = p_x \mathcal{P}(X_1 \setminus x_1, 0, J_i \setminus x_1) \oplus \mathcal{P}(X_1/x_1, 0, J_i/x_1)$ holds for $i = 1$ and $i = 2$.

7.3. Recursion for the Hilbert series.

Example 7.1. This is an example for the recursion in Theorem 5.2 and for Proposition 3.10:

$$X_2 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (x_1, x_2) \quad J_3 := \{X_2\} \quad J_4 := \{X_2, (x_1)\} \quad (7.5)$$

$$\mathcal{I}(X_2, 2, J_3) = \text{ideal}\{x^3, y^3, x^2y^2\} \quad \widehat{J_3/x_2} = \widehat{J_4/x_2} = \{(x_1)\} \quad (7.6)$$

$$\mathcal{P}(X_2, 2, J_3) = \text{span}\{1, x, y, x^2, xy, y^2, x^2y, xy^2\} \quad (7.7)$$

$$\mathcal{I}(X_2, 2, J_4) = \text{ideal}\{x^3, y^4, x^2y^2, xy^3\} \quad (7.8)$$

$$\mathcal{P}(X_2, 2, J_4) = \text{span}\{1, x, y, x^2, xy, y^2, x^2y, xy^2, y^3\} \quad (7.9)$$

For a graphic description of the \mathcal{P} -spaces involved in the decomposition, see Figure 1.

7.4. Problems in the semi-internal case.

Example 7.2 (No canonical basis for internal spaces). In Section 5.3 we defined the set of semi-internal bases $\mathbb{B}_-(X, J)$. Let $\tilde{\mathcal{B}}_-(X, J) := \{p_{X \setminus (B \cup E(B))} : B \in \mathbb{B}_-(X, J)\}$. This example shows that even in the internal case, where $\tilde{\mathcal{B}}_-(X, J)$ has

the right cardinality, it is in general not contained in $\ker \mathcal{I}(X, -1, J)$.

$$X_3 := \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} = (x_1, x_2, x_3, x_4, x_5) \quad (7.10)$$

$$\mathbb{B}_-(X_3, \{X_3\}) = \{(x_1, x_2, x_3), (x_1, x_2, x_4)\} \quad (7.11)$$

$$\mathcal{I}(X_3, -1, \{X_3\}) = \text{ideal}\{x^2, y, z\} \quad (7.12)$$

$$\mathcal{P}(X_3, -1, \{X_3\}) = \ker \mathcal{I}(X_3, -1, \{X_3\}) = \text{span}\{1, x\} \quad (7.13)$$

$$E((x_1, x_2, x_3) = \{x_4, x_5\} \quad E((x_1, x_2, x_4) = \{x_5\} \quad (7.14)$$

$$\tilde{\mathcal{B}}_-(X_3, \{X_3\}) = \{1, x + z\} \not\subseteq \text{span}\{1, x\} \quad (7.15)$$

Example 7.3. This example shows why there is an additional condition on the ideal J in the Main Theorem for $k = -1$. We use the matrix X_1 defined at the beginning of this section and the ideal $J_5 := \{X_1, \{x_1\}\}$, *i. e.* $J \not\subseteq \mathcal{H}$.

Then $J_5 \setminus x_1 = \{X_1 \setminus x_1\}$ and $J_5/x_1 = \{X_1/x_1, \emptyset\}$. This implies $S(X_1 \setminus x_1, -1, J_5 \setminus x_1) = \emptyset$, $S(X_1, -1, J_5) = \{1\}$ and $S(X_1/x_1, -1, J_5/x_1) = \{1, y\}$. Hence, the map $\text{Sym}(\pi_{x_1}) : \mathcal{P}(X_5, J) \rightarrow \mathcal{P}(X_5/x_1, J_5/x_1)$ is not surjective.

The three S -sets appearing in this example through the corresponding kernels. However, our proof of the Main Theorem fails here, since $\ker \mathcal{I}(X_1, -1, J_5) \neq p_{x_1} \ker \mathcal{I}(X_1 \setminus x_1, k, J_5 \setminus x_1) \oplus \ker \mathcal{I}(X_1/x_1, k, J_5/x_1)$, *i. e.* Proposition 3.16 does not hold. Hence, proving that a set $\tilde{S}(X_1, -1, J_5)$ spans $\ker \mathcal{I}(X_5, -1, J)$ requires a different proof technique.

Example 7.4. This example shows why our proof of the Main Theorem in general does not work in the case $J = \{X\}$ even though the theorem might still be correct (cf. Remark 3.18). It demonstrates that $\text{Sym}(\pi_x) : S(X, -1, \{X\}) \rightarrow S(X/x, -1, \{X/x\})$ is in general not surjective.

Consider the following matrix:

$$X_4 = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \quad (7.16)$$

The corresponding internal \mathcal{P} -space and ideal are:

$$\mathcal{I}(X_4, -1, \{X_4\}) = \text{ideal}\{x^3, y^3, z^2, (x-y)^3, (x-z)^2, (x-y-z)^2\} \quad (7.17)$$

$$\mathcal{P}(X_4, -1, \{X_4\}) = \text{span}\{1, x, y, z, xy + yz, xy + y^2, x^2 + xz, x^2y + xy^2 + xyz + y^2z\} \quad (7.18)$$

By deletion and contraction of x_7 we obtain:

$$\mathcal{I}(X_4 \setminus x_7, -1, X_4 \setminus x_7) = \text{ideal}\{x, y, z\} \quad (7.19)$$

$$\mathcal{P}(X_4 \setminus x_7, -1, X_4 \setminus x_7) = \text{span}\{1\} \quad (7.20)$$

$$\mathcal{I}(X_4/x_7, -1, \{X_4/x_7\}) = \text{ideal}\{x^3, y^3, (x-y)^3\} \quad (7.21)$$

$$\mathcal{P}(X_4/x_7, -1, X_4/x_7) = \text{span}\{1, x, y, x^2, xy, y^2, x^2y + xy^2\} \quad (7.22)$$

The Main Theorem and Proposition 3.16 both hold in this example.

$p_{\bar{x}_5} p_{\bar{x}_6} \in S(X_4/x_7, -1, \{X_4/x_7\})$, but $p_{x_5} p_{x_6} \notin S(X_4, -1, \{X_4\})$! No element of $S(X_4, -1, \{X_4\})$ is projected to $p_{\bar{x}_5} p_{\bar{x}_6}$. Hence, our proof of the Main Theorem does not work in this case.

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