Dirac Equation For Cold Atoms In Artificial Curved Spacetimes

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Abstract.   We argue that the Fermi-Hubbard Hamiltonian describing the physics of ultracold atoms on optical lattices in the presence of artificial non-Abelian gauge fields, is exactly equivalent to the gauge theory Hamiltonian describing Dirac fermions in the lattice. We show that it is possible to couple the Dirac fermions to an "artificial" gravitational field, i.e. to consider the Dirac physics in a curved spacetime. We identify the special class of spacetime metrics that admit a simple realization in terms of a Fermi-Hubbard model subjected to an artificial $SU(2)$ field, corresponding to position dependent hopping matrices. As an example, we discuss in more detail the physics of the 2+1D Rindler metric, its possible experimental realization and detection.

1. Introduction

The studies of ultracold quantum matter in artificially designed external gauge fields is one of the most rapidly developing areas of physics of ultracold atoms [1, 2, 3, 4, 5]. Originally, these studies arose from investigations related to the response of superfluids, such as Bose-Einstein condensates (BEC), to rotation (cf. [6, 7]). On one hand, rotation induces quantized vortices and/or vortex Abrikosov lattices [8]. On the other, its effects are equivalent to those of an artificial constant (Abelian) magnetic field. The latter analogy has led to the idea of realizing strongly correlated quantum liquids, such as the celebrated Laughlin liquid of the fractional quantum Hall effect (FQHE) (cf. [9]) by means of rapid rotation [10, 11]. Unfortunately, reaching the FQHE regime with rotation is experimentally very difficult; it has been achieved recently, but in a system of only $1 < N \leq 10$ atoms in rotating microtraps (lattice site potential wells) [12, 13]. Several researchers proposed, thus, alternative approaches involving for instance laser-induced gauge fields that employ dark states in 3-level systems [14], or laser induced gauge fields acting on atoms confined in an optical lattice [15, 16, 17]. Other approaches concerned rotating optical lattices [18, 19], or interactions of lattice atoms embedded in a rotating BEC [20]. Interestingly, the proposals involving lasers can be relatively straightforwardly generalized to particles possessing internal "color" states subjected to artificial non-Abelian gauge fields [21, 22].

In the last two years there has been a large number of works reexamining these ideas and proposing experimental realization within the reach of the current state of the art. The NIST group employed an approach similar to [14] and used Raman (Bragg) transitions in Sodium to realize experimentally the first non-zero constant vector (corresponding to zero "artificial" magnetic field) [23]. Note that when the gauge symmetry is provided by *Nature* only gauge invariant observables are physical. In the case of artificial gauge potentials the situation is in principle different. Gauge potential are controlled by the experimenters and the measurements of quantities that depend on the choice of gauge are possible; in another words, the very process of the creation of the gauge potential is not gauge invariant, even though the resulting Hamiltonians are (for discussion see [24, 25] and [26]). NIST group was also able to generate vortices using the same scheme with a potential configuration corresponding to a non zero artificial magnetic field [27]. Several practical extensions of the scheme of [14] were discussed in [28, 29]. New schemes were proposed for alkali and earth-alkali atoms in optical lattices employing superlattice techniques [30, 31], and on atomic chips [32]. Very recently, the creation of spin-orbit couplings, a special instance of synthetic non-abelian fields, was reported [33].

Artificial non Abelian gauge fields are particularly interesting, since they provide a natural framework to simulate relativistic physics of the Dirac equation. Artificial Dirac physics has been in recent years at the center of interest in condensed matter in the context of studies of the amazing properties of graphene [34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45]. In the case of graphene the Dirac points in the dispersion relation appear due to the geometry of the underlying hexagonal lattice. This idea can be carried over to cold atoms [46]; a hexagonal optical lattice (OL) with spinor bosons has been recently realized experimentally [47]. Also, other systems not relying on a lattice have been proposed to emulate Dirac particles [48]. Other systems where Dirac physics plays a role include superfluid 3He-A [49, 50] (where the analogy to Particle physics can be extended to include gauge interaction and Standard model phenomena, for instance see [51]), trapped ions [52, 53, 54, 55, 56], or Fermi-Bose mixtures [57].

In the case of artificial non-Abelian laser-induced fields (ANALF) the connection to Dirac physics was pointed out in [58]. The Dirac equation is responsible for the anomalous

integer quantum Hall effect in artificial $SU(2)$ fields on a 2D square lattice, and topological quantum phase transitions on 2D hexagonal lattices, as well as FQHE [59]. These situations correspond to relativistic physics in 2+1 dimensional spacetime [60, 61, 62, 63]. Very recently, several proposals were formulated for the creation of ANALF in 3+1 dimensional (3D spatial) allowing for simulations of Wilson fermions [64] and axion [65] quantum electrodynamics with ultracold atoms [66, 31].

So far, most of the proposals have dealt with constant non Abelian fields strengths (i.e constant Wilson loops), staggered Abelian gauge fields [67, 68], or, in the rare cases, fields that are linear in the spacial coordinates [22, 69]. The Dirac equation, resulting in some of these situations, corresponds to a Dirac equation in a flat Minkowski space. The crucial ingredient of such studies was based on the analysis of the dispersion relation between energy and quasi-momentum, as in graphene [45]. Energy bands touch in isolated singular points, called Dirac points. In the vicinity of these points, the dispersion relation linearizes, and a cone is formed. Dirac physics can then be realized for fermions adjusting the Fermi surface to include the Dirac point, and considering *low lying* energy excitations.

In this paper we take a different perspective on this issue, employing standard concepts from Hamiltonian lattice gauge theory (HLGT) [70]. We argue that using artificial non-Abelian fields in lattices it is possible to simulate with cold atoms a Dirac spinor in the same way as it is done in HLGT. In another words, the Hubbard Hamiltonian describing the physics of atoms in artificial non-Abelian fields in lattices is exactly the HLGT version of the Dirac's Hamiltonian. That is to say, in the scheme we consider, all excitations of the fermion field in the lattice are Dirac-like, not only the low lying ones. The advantage of this point of view is that it allows one to couple the simulated Dirac-fields to external fields, or quantum fields in a straightforward way, which is not so clear in artificial graphene-like systems.

In particular, we will show that it is possible to consider coupling Dirac fermions to an artificial gravitational field, that is to consider the Dirac equation in curved space. We will identify and focus on a special class of space-time metrics that admit a simple formulation of the Dirac lattice Hamiltonian in terms of a Fermi-Hubbard model subjected to an artificial $SU(2)$ field [61], corresponding to tunneling matrices with position-dependent overall hopping rate.

We will not consider here the fermion doubling problem as it is inessential for the discussion of gravitational effects ‡. Feasible solutions of such a problem in OL simulations of 3+1 fermions have been very recently proposed in [66] and [31].

The motivation for simulating the Dirac equation in curved space-time is at least two-fold. The propagation of fermions in curved space is not *experimentally* accessible in high-energy physics/cosmology as the (piece of the) Universe that we are nowadays able to probe is practically flat [74]. The lattice realization proposed in the paper is an appropriate description of the continuum physics for processes occurring over scales of several lattice spaces. The realization with cold atoms in OL provides us with "lap-top experiments", paving the way to observe exotic effects like the Thermalization Theorem, also known as Fulling-Davies-Unruh effect [75, 76, 77] (for a review on the subject see [78]). Roughly speaking, it states that an accelerated observer perceives the Minkowski vacuum as a thermal bath. This is a manifestation as Hawking radiation [79] of the same phenomenon: namely the existence of a non-trivial Bogoliubov transformation between the Minkowski vacuum and a space-time with a horizon. The effect of the latter, in Quantum Information language, is to "trace-out" part

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‡ The mixture of the two Dirac points due to the gravitational background potentially induces a gauge field coupling to the composite fermions, (for graphene like lattice see for example [71, 42, 72, 73]). However, the field contribution is relevant in the presence of conical singularities, disclination or dislocation in the graphene language, while is negligible when the metric is smooth.

of the system, giving raise to a thermal reduced density matrix. Another exotic gravitational effect to observe might be the curved space-time version of Zitterbewegung [80], to compare with its recent observation achieved in flat space [54].

On other hand, simulating the Dirac equation in curved spacetime gives rize to the possibility of modelling the analogues of graphene ripples on a *flat, square* lattice. Such a possibility could help in disentangling the action of ripples (which admit a natural "geometric" interpretations [81] or a gauge field interpretations, e.g. [82]) on the carrier density from other contributions.

Different approaches to quantum simulation include the exact simulation of the dynamics of a strongly correlated systems by unitary gates [83], or the the creation of interesting strongly correlated states which are ground states of interesting Hamiltonians [84].

The paper is organized as follows. In section 2 we demonstrate how the lattice Dirac Hamiltonian in flat space-time [85] can be obtained as the discretization of the continuum Dirac Hamiltonian H. This Hamiltonian is identified then as the Hamiltonian of the SU(2) Fermi-Hubbard model considered for instance in [61], where the hopping terms are given by the Pauli matrices σ_i, $i = x, y$ times a constant hopping rate J. Although the discretization can be done at this stage in many different ways, we obtain it by coarse graining the "symmetric" formulation of H in the fermion field ψ and its conjugate $\psi^\dagger$. This strategy turns out to be very convenient in order to find the lattice version of the Dirac Hamiltonian in curved space-time. As the first exercise, after reviewing the form of the continuum Hamiltonian on a generic manifold (admitting a "time" isometry) in section 3, we successfully apply the above strategy to the interesting case of the 2+1 Rindler Universe [86] in section 4. The resulting lattice Hamiltonian differs from the Hubbard Hamiltonian of the flat case in the following: the hopping rates exhibit a linear dependence with position. Such a simple form is due to the cancellation of the spin-connection, once ψ and $\psi^\dagger$ are treated in the same manner: as a consequence the hopping matrices need not be locally rotated. We find that this simplification happens in fact for any static spacetime. For the details of the derivation we refer the reader to the Appendix B. We discuss under which conditions, and how the fermion propagation in such space-times can be engineered and detected on optical lattices in sections 5 and 6 that are the heart of the work. There, several different experimental ways of implementing the Hubbard model of interest are presented. We propose the density of states as the relevant observable to capture the Dirac physics. It is a measurable quantity both in graphene [87, 88, 89, 90] and in OL [91, 92, 93, 94, 95, 96, 97, 98] experiments. We compute the theoretical value of the density of states analytically in the continuum limit using perturbation theory.

We collect our concluding remarks and discuss further developments in section 7.

2. Dirac's Hamiltonian on a Lattice

In this section we show, by discretization the spacial coordinates, that Dirac's Hamiltonian in $2+1$ dimensions is a hopping Hamiltonian with non-Abelian tunnelling matrices. To this end, let us recall that Dirac's equation in 2+1 dimensions reads $\gamma^a \partial_a \psi = 0$, $a = 0, 1, 2$, where γ_a are Dirac's matrices satisfying $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$, η_{ab} is the metric, and ψ is a two component spinor. The time evolution of the field ψ is,

$$i\partial_0 \psi = \underbrace{-i\left(\gamma^0\right)^{-1} \left(\gamma^1 \partial_1 + \gamma^2 \partial_2\right)}_{\mathcal{H}} \psi \,, \tag{1}$$

from which we can easily read-off the Hamiltonian. As was first pointed out in [85], in 2+1 dimensions and on a lattice of spacing Δ the discretized version of equation (1) is

$$i\partial^0\psi = -i\frac{\left(\gamma^0\right)^{-1}}{2\Delta}\left(\gamma^1\left(\psi_{m+1,n} - \psi_{m-1,n}\right) + \gamma^2\left(\psi_{m,n+1} - \psi_{m,n-1}\right)\right),\tag{2}$$

with $\psi(\mathbf{x}) = \psi(m\Delta, n\Delta) = \psi_{m,n}$. Rewriting the Hamiltonians $H = \int dxdy\,\psi^\dagger\mathcal{H}\psi$ we obtain on the lattice,

$$H = -i\frac{\left(\gamma^0\right)^{-1}}{2\Delta}\sum_{m,n}\left(\psi^\dagger_{m+1,n}\gamma^1\psi_{m,n} + \psi^\dagger_{m,n+1}\gamma^2\psi_{m,n}\right) + h.c.\tag{3}$$

It is easy to see that (3) is nothing more than a Fermi-Hubbard Hamiltonian with non-Abelian hopping matrices where the interactions and the effect of the trap have been neglected. In the notation of [61] we have $U_x = i\gamma^{0^{-1}}\gamma^1 = i\sigma_1$ and $U_y = i\gamma^{0^{-1}}\gamma^2 = i\sigma_2$ (which implies $\gamma_0 = i\sigma_3$) and the Abelian flux $\Phi = 0$.

For further reference, we note that the lattice Hamiltonian of (3) can be alternatively obtained by computing

$$H = \int dxdy\,\psi^\dagger\mathcal{H}\psi = \frac{1}{2}\int dxdy\,(\mathcal{H}\psi)^\dagger\psi + \frac{1}{2}\int dxdy\,\psi^\dagger\mathcal{H}\psi,\tag{4}$$

with the substitution of spacial derivatives with finite differences over one lattice space Δ, $\partial_x\psi \to \frac{\psi_{m+1,n}-\psi_{m,n}}{\Delta}$ and $\partial_y\psi \to \frac{\psi_{m,n+1}-\psi_{m,n}}{\Delta}$.

3. Dirac's Equation in Curved Spacetime

In this section we review the formulation of Dirac's equation on a curved spacetime. Let M be an arbitrary curved manifold and let us define a *vielbein* e^a_μ, or set of vectors that form a basis of the tangent space T_M at each point of M. Here the index μ labels the spacetime component and a simply labels the basis vector.

Although e^a_μ is not constant in general, we require it to be covariantly constant (see for example [99]). We introduce a connection ω such that

$$D_{[\mu}e^a_{\nu]} = \partial_{[\mu}e^a_{\nu]} + \omega_{[\mu}{}^a_b e^b_{\nu]} = 0.\tag{5}$$

This defines ω and the covariant derivative D_μ. Dirac's equation for a Fermi field ψ will be

$$\gamma^\mu D_\mu\psi = 0,\tag{6}$$

where γ^μ are the curved spacetime gamma matrices. The flat-space gamma matrices γ_a and the γ^μ are related by $\gamma_\mu(x) = e^a_\mu(x)\gamma_a$. Thus, we have $\{\gamma_\mu(x), \gamma_\nu(x)\} = 2g_{\mu\nu}(x)$ if $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$ and the *vielbein* forms an orthonormal basis $e^a_\mu e^{b\mu} = \eta^{ab}$, with $g_{\mu\nu}$ and η_{ab} the curved and flat-space metrics, respectively.

By separating the time component we have, $\gamma^\mu D_\mu = \gamma^t\partial_t + \frac{1}{4}\gamma^t\omega^{ab}_t\gamma_{ab} + \gamma^i\partial_i + \frac{1}{4}\omega^{ab}_i\gamma_{ab}$. Now, we are ready to write out the time variation of ψ,

$$i\partial_t\psi = \underbrace{-i\left(\gamma^t\right)^{-1}\left(\gamma^i\partial_i + \frac{1}{4}\gamma^i\omega^{ab}_i\gamma_{ab} + \frac{1}{4}\gamma^t\omega^{ab}_t\gamma_{ab}\right)}_{\mathcal{H}}\psi,\tag{7}$$

where $\gamma_{ab} = \gamma_a\gamma_b$.

Hence, we have identified the Hamiltonian for a Dirac fermion on a curved manifold. In section 5 we will proceed to its diagonalization analogously to the flat case and discuss its implementation on an optical lattice.

4. An example: Dirac Hamiltonian in Rindler's Universe

As an example, we will derive Dirac's Hamiltonian in a particularly simple spacetime. Let us start by recalling that the Hamiltonian in curved spacetime, when a timelike killing vector is present, is the integral of the Hamiltonian density on a timelike hypersurface. In 2+1 dimensions we have,

$$H = \int d\Sigma^2 \, \bar{\psi}\gamma^t \mathcal{H}\psi \,. \tag{8}$$

The differential is the volume element on a timelike slice and includes the determinant of the metric, $d\Sigma^2 = \sqrt{-g}dxdy$.

In Rindler's space, the metric takes the form

$$ds^2 = -(ax)^2 dt^2 + dx^2 + dy^2 \,. \tag{9}$$

We are interested in this metric for two reasons. First, it is the metric seen by an observer in constant acceleration in flat spacetime. Therefore, it could have implications for earth-dwelling detectors observing cosmic background radiation. Second, it is the near-horizon metric of a Schwarzschild black hole.

The Rindler metric suggests the choice of *dreivein* $e^0 = |ax|dt$, $e^1 = dx$, $e^2 = dy$. Using (5) we may compute the spin connection, whose only non-vanishing component is $w_t^{01} = a\frac{x}{|x|}$. Dirac equation (7), greatly simplifies in this spacetime,

$$i\partial_t\psi = \underbrace{-ia|x| \left(-\gamma_2 \left(\partial_x + \frac{1}{2|x|}\right) + \gamma_1\partial_y\right)}_{\mathcal{H}} \psi \,. \tag{10}$$

where we have used that $\gamma_0\gamma_1\gamma_2 = -1$, which holds only in 2+1 dimensions. In what follows, we adopt the gamma matrices representation choice

$$\sigma_x = -\gamma_2, \quad \sigma_y = \gamma_1, \quad \sigma_z = -i\gamma_0.$$

In order to carry out the discretization analogously to how it is done for the case of a gauge theory, we note that the Hamiltonian can be written in terms of the Hamiltonian density (10) as

$$H = \int dxdy \, \psi^\dagger \mathcal{H}\psi = \frac{1}{2}\int dxdy \, (\mathcal{H}\psi)^\dagger\psi + \frac{1}{2}\int dxdy \, \psi^\dagger \mathcal{H}\psi$$
$$= \frac{ia}{2}\int dxdy \, x \left((\partial_x\psi^\dagger)\sigma_x\psi + (\partial_y\psi^\dagger)\sigma_y\psi - \psi^\dagger\sigma_x\partial_x\psi - \psi^\dagger\sigma_y\partial_y\psi\right). \tag{11}$$

In this symmetric form the spin-connection term disappears and it turns out again once integrating by parts.

The discretized version of the Hamiltonian (11) is simply obtained by the substitution $\partial_x\psi \to \frac{\psi_{m+1,n}-\psi_{m,n}}{\Delta}$ and $\partial_y\psi \to \frac{\psi_{m,n+1}-\psi_{m,n}}{\Delta}$, with $x = \Delta m$ and $y = \Delta n$. One readily gets

$$H = \frac{ia}{2}\sum_{m,n} m \left(\psi^\dagger_{m+1,n}\sigma_x\psi_{m,n} + \psi_{m,n+1}\sigma_y\psi_{m,n}\right) + H.c. \,. \tag{12}$$

Therefore, a lattice with hopping matrices given by $J_iU_i = im\sigma_i$, $i = x, y$, growing linearly in the x direction gives an appropriate description of free massless fermion in a Rindler spacetime. Such an Hamiltonian can be in principle implemented in a OL. This problem will be tackled in the next section.

5. Dirac equation in curved spaces and optical lattices

In view of the explicit realization in a OL we focus to the classes of spacetime where the massless fermion propagation can be described by the Hubbard model of the form

$$H = \frac{i}{2} \sum J_{mn} \left(\psi^\dagger_{m+1,n} \sigma_x \psi_{m,n} + \psi^\dagger_{m,n+1} \sigma_y \psi_{m,n} \right) + h.c. . \tag{13}$$

As discussed in full details in the Appendix B, we find that the lattice Hamiltonian reduces to this simple form only if the metric is static. In 2+1 dimensions it is equivalent to say that it can be written (in a certain coordinate system) in a diagonal form as

$$ds^2 = -f^2(x,y)dt^2 + f^{-2}(x,y)e^{2\Phi(x,y)}(dx^2 + dy^2). \tag{14}$$

With this choice of coordinates the hopping rate is simply given by $J_{mn} = e^{\Phi(x_m,y_n)}$.

This is not the only requirement to be satisfied to reproduce such propagation on a OL, however. Indeed, it is important to note that, even if the Dirac Hamiltonian written in the symmetric form (13) is the same for any choice of the function f in (14), the corresponding Hamiltonian system is distinct for each metric as the canonical momentum is

$$\Pi = i\sqrt{-g}\bar{\psi}\gamma^t = if^{-2}e^{2\Phi}\psi^\dagger,$$

and depends explicitly on f. Now, in optical lattice experiments the canonical momentum is fixed by the anticommutation relation to be simply $i\psi^\dagger$ that implies $f = e^\Phi$. Thus the cold fermions in the optical lattice simulate the propagation of massless fermions in a metric of the form §

$$ds^2 = -e^{2\Phi}dt^2 + dx^2 + dy^2. \tag{15}$$

The Rindler metric (9) corresponds to $\Phi = \log(ax)$.

Before moving to the explicit implementation of the Hamiltonian (13) in OL, let us briefly discuss to what extent it is a good description of the dynamics of massless fermions in a space time given by (15). There are two kind of limitations. On one hand, due to finite size of the OL we are able to cover only a finite portion of spacetime. On the other end, the lattice approximation is valid when the metric, or the function Φ, is sufficiently smooth and slowly varying over one lattice space Δ. It is worth to note that the second problem can be circumvented by using techniques from lattice gauge theory (the continuum limit is obtained by extrapolation).

These two limitations should not obscure the physical content, however, as OL with up to 300×300 sites can be achieved. This implies that the overall variation of the metric over lattice can be of order one. Further considerations in the case of Rindler space are given in section 7.

5.1. Experimental realization

In this section we discuss briefly how one can achieve the appropriate site-dependent hopping rate J_{mn} in an OL. We propose two different techniques.

Recently it was proposed in [101, 31], that the intensity of the hopping can be tailored almost at will by considering bichromatic spin-independent superlattices that trap the hyperfine states of alkali bosonic or fermionic atoms. The split of the hyperfine levels is controlled by a magnetic field. The hopping between neighboring Zeeman sublevels of the F (lower) hyperfine manifold, i.e. our "electrons", is induced via adiabatic elimination of an

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§ It is worth noting that for a generic function Φ the metric is curved and not Weyl-invariant, cfr [100].

intermediate $F = F \pm 1$ (upper) hyperfine manifold coupled to F via an off-resonant Raman transition. For the details of the scheme for fermionic ^{40}K atoms, we refer to the original proposal [31]. ^{40}K have F =9/2, and allow in principle to simulate any pseudo-spin $F' \leq$9/2, employing the splitting of the Zeeman sub-levels in a magnetic filed, and optical pumping to the relevant sub-levels. For the purposes of the present paper it sufficient to have $F' = 1/2$, or alternatively to use from the very beginning atoms with $F = 1/2$ in the ground state manifold, such as ^{6}Li.

To be more concrete, consider a ^{6}Li Fermi gas loaded on a 2D square optical lattice of size $L \times L$, where the relevant information of our quantum simulator is encoded in the Zeeman sub-levels of the hyperfine manifold F =1/2. Laser-assisted tunneling methods allow us to design arbitrary operators $U_{r\nu}$ dressing the hopping between lattice sites $r \to r + \nu$, where $r = m\hat{x} + n\hat{y}$, $m, n, \in \{1...L\}$, and $\nu \in \{\hat{x}, \hat{y}\}$. Usually, such schemes rely on Raman couplings to auxiliary states trapped in the links of the original lattice and belonging to a different hyperfine manifold. Here, following [101, 31] we use bichromatic spin-independent superlattices , and use the secondary minima of F =3/2 as bus states to mediate the hopping. Note that the individual addressing of each hopping rate is granted by the Zeeman splitting within the hyperfine manifolds, and the different detunings of the Raman lasers. These detunings can be quite large, so that the lifetimes of atoms on the lattice (limited by photon absorption and spontaneous emission) can be quite large, of order $\tau_1 \sim$ 1s. By making the Raman laser intensity/detunings and/or Zeeman splitting spatially dependent one obtains the desired spatially dependent hopping rates which is necessary for the realization of curved space-times, equation (13). On top of that, is is possible to use Feschbach resonances to turn off the atom-atom scattering, and make the system essentially non-interacting.

An alternative, and perhaps even a simpler method to realize the Fermi-Hubbard model of the form of (13) can be achieved by taking into account the finite waist of the lasers used for the generation of the hopping terms. In general, this is an undesirable feature, and it can usually be neglected. Indeed, typically Gaussian laser beams of waist w are used, characterized by a Rayleigh length —the the distance along the direction of propagation from the waist to the place where the area of the cross section is doubled— $z_R = \pi w^2 / \lambda$ with λ denoting the wavelenght. That is to say, within a volume of $w^2 \times z_r$, the ideal planar wave is a good approximation, at least around the center of the beam. For a lattice with $L = 30$, its linear size is $30 \times \lambda/2$, so that focusing a laser on the whole lattice $w \simeq 15\lambda$ leads to $z_R \simeq 700\lambda$, so that the plane wave description can be even used for an array of a few hundreds of 2D 30×30 lattices.

There are, however, no technical obstacles to focus the lasers on much smaller spots, smaller than $L\lambda/2$, keeping z_r still quite large. Actually, such spacial modulation of the intensity due to the waist of "real" lasers can be used to induce hopping terms that depend non-trivially on the position [29]. In general, the hopping rate, i.e. the modulus of the hopping term, is proportional to the intensity of the laser producing it. For instance, taking the paradigmatic example of the hopping induced by the Raman transition in Jaksch and Zoller's setup [15], an optical lattice implementing the Hamiltonian of (13) can be achieved for Raman lasers propagating *all* in the same direction, once we consider only the radial waist and neglect the waist along the beam ‖. In this case, the shape of the laser intensity will correspond to the e^Φ factor of the metric.

Comparing the two methods explained above, the former has the advantage that, in

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‖ At the moment, it is not clear to us whether it is possible to engineer the hopping matrices σ_x and σ_y with parallel and anti-parallel Raman laser. Actually, we do not know of any explicit realization of such hopping in a setup á la Jaksch and Zoller. Maybe, the dark state or slow-light method [14] is more promising. The main point that the hopping rate is controlled by the intensity, remains valid.

principle, any shape of J_{mn}, i.e. any metric of the form (15), can be engineered, but at the price of dealing with a quite involved experimental apparatus, while the latter, although it allows for a restricted choice of J_{mn} (for instance a Gaussian shape) is almost for free. The desired hopping rate profile is obtained by reverse engineering of laser waist.

6. Density of states at the Fermi level as simple observable

Let us turn to the discussion of possible detection schemes of Dirac physics in curved spacetime. A simple observable that characterizes such physics, and that contain information about the effects of the beam waist is the density of states [81, 102, 103]. This quantity is routinely measured in graphene using scanning tunnel [89, 90] and electron transmission spectroscopy [87, 88]. For ideal graphene at zero temperature the density of states is zero at the Fermi level and is proportional to $|E|$ (once we have fixed $E_F = 0$) as the charge carries are described by massless fermions propagating in flat space. In the presence of deformations of the graphene sheet its deviation from the free behavior can be analytically computed at first order from the propagator of the Dirac equation. Following [81], by modeling such deformations as perturbations of the Minkowski metric it is possible to compute the Green function treating the correction to the free equation as an interacting term V. In fact, in this section we will first reproduce the computation of [81] for a metric of the form (15), instead of the spatial deformation (as in (A.1)) considered there.

Our final goal is to determine the local density of states, defined by

$$\int \mathrm{d}w \sqrt{-g} \mathrm{d}^2 r \rho(w, \mathbf{r}) \equiv \# \text{ of states },\tag{16}$$

in terms of the Feynman propagator (see appendix) using the relation

$$\rho(w, \mathbf{r}) = \text{sign}(w) \frac{1}{\pi} \text{Im} \left[\text{Tr}\, \hat{S}_F(w, \mathbf{r}, \mathbf{r}) \gamma^t \right],\tag{17}$$

where $\mathbf{r} = (x, y)$.

In order to find $\hat{S}_F(w, \mathbf{r}, \mathbf{r})$ we start by using the defining equation for the fermion propagator

$$-i\sqrt{-g}\gamma^\mu \frac{D}{Dx^\mu} S_F(\mathbf{x}, \mathbf{x}') = \delta^3(\mathbf{x} - \mathbf{x}'),\tag{18}$$

where the $\mathbf{x}$ indicates a point of the spacetime, to a metric of the form (15). By retaining terms linear in Φ, the above equation can be written as

$$-\left(i(\gamma^\mu \frac{\partial}{\partial x^\mu})_{Flat} + V(\mathbf{x})\right) S_F(\mathbf{x}, \mathbf{x}') = \delta^3(\mathbf{x} - \mathbf{x}'),\tag{19}$$

where

$$V = i\gamma_1 \left(\Phi \partial_x + \frac{1}{2}\partial_x\Phi\right) + i\gamma_2 \left(\Phi \partial_y + \frac{1}{2}\partial_y\Phi\right),\tag{20}$$

is the effective "external potential". As we are interested to $\hat{S}_F(w, \mathbf{r}, \mathbf{r}')$ and due to the time translation invariance

$$S_F(\mathbf{x}, \mathbf{x}') = S_F(t - t', \mathbf{r}, \mathbf{r}') = \int \frac{\mathrm{d}w}{2\pi} e^{-iw(t-t')} \hat{S}_F(w, \mathbf{r}, \mathbf{r}'),$$

it is convenient to perform the Fourier transformation in time of (19)

$$\left(w\gamma_0 - i\nabla_{\mathbf{r}} \cdot \gamma - V(\mathbf{r})\right) \hat{S}_F(w, \mathbf{r}, \mathbf{r}') = \delta^2(\mathbf{r} - \mathbf{r}').\tag{21}$$

The above equation can be solved consistently within the first order approximation by

$$\hat{S}_F^1(w, \mathbf{r}, \mathbf{r}') = \int \mathrm{d}^2 r'' \hat{S}_F^0(w, \mathbf{r}, \mathbf{r}'') V(\mathbf{r}'') \hat{S}_F^0(w, \mathbf{r}'', \mathbf{r}'), \tag{22}$$

where

$$\hat{S}_F^0(w, \mathbf{r}, \mathbf{r}') = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{w\gamma_0 - \mathbf{k} \cdot \gamma}{w^2 - \mathbf{k}^2 + i\epsilon} e^{i\mathbf{k}(\mathbf{r} - \mathbf{r}')}, \tag{23}$$

is the free fermion propagator and space translation's invariance holds. By using the Fourier transformation of $\Phi(\mathbf{r}'')$

$$\Phi(\mathbf{r}'') = \int \frac{\mathrm{d}^2 p}{(2\pi)^2} e^{i\mathbf{p} \cdot \mathbf{r}''} \Phi(\mathbf{p}), \tag{24}$$

equation (22) can be explicitly computed performing the integration in $\mathbf{r}''$.

The relevant contribution to the trace turns out to be linear in w. Explicitly

$$\mathrm{Tr}[\hat{S}_F^1(w, \mathbf{r}, \mathbf{r})\gamma_0] = \int \frac{\mathrm{d}^2 p}{(2\pi)^2} e^{i\mathbf{p} \cdot \mathbf{r}} \Phi(\mathbf{p}) \Gamma(w, \mathbf{p}), \tag{25}$$

with

$$\Gamma(w, \mathbf{p}) = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{4w|\mathbf{k} - \frac{1}{2}\mathbf{p}|^2}{(w^2 - \mathbf{k}^2 + i\epsilon)(w^2 - (\mathbf{k} - \mathbf{p})^2 + i\epsilon)}]. \tag{26}$$

The above integral is logarithmically divergent, but its imaginary part is not. It is easy to show that this is the only part contributing to the density. Indeed, as $\Phi(\mathbf{p})$ is the Fourier transformation of a real function, $\Phi(\mathbf{p})^* = \Phi(-\mathbf{p})$, and $\Gamma(w, \mathbf{p})$ is even in $\mathbf{p}$, $\Gamma(w, \mathbf{p}) = \Gamma(w, -\mathbf{p})$, one finds

$$\begin{aligned} \left(\mathrm{Tr}[\hat{S}_F^1(w, \mathbf{r}, \mathbf{r})\gamma_0]\right)^* &= \int \frac{\mathrm{d}^2 p}{(2\pi)^2} e^{-i\mathbf{p} \cdot \mathbf{r}} \Phi(\mathbf{p})^* \Gamma(w, \mathbf{p})^* \\ &= \int \frac{\mathrm{d}^2 p}{(2\pi)^2} e^{i\mathbf{p} \cdot \mathbf{r}} \Phi(\mathbf{p}) \Gamma(w, \mathbf{p})^*, \end{aligned} \tag{27}$$

which immediately implies

$$\delta\rho(w) = \mathrm{sign}(w) \frac{1}{\pi} \int \mathrm{d}^2 r \int \frac{\mathrm{d}^2 p}{(2\pi)^2} e^{i\mathbf{p} \cdot \mathbf{r}} \Phi(\mathbf{p}) \, \mathrm{Im}\, \Gamma(w, \mathbf{p}). \tag{28}$$

The explicit expression for $\mathrm{Im}\, \Gamma(w, \mathbf{p})$ is

$$\mathrm{Im}\, \Gamma(w, \mathbf{p}) = 2w - \frac{w}{\pi} \, \mathrm{Im} \left[\frac{\arctan \chi(w, p)}{\chi(w, p)} \right], \tag{29}$$

where $\chi(w, p) = \frac{p}{\sqrt{|4w^2 - p^2|(\Theta(4w^2 - p^2) - i\Theta(-4w^2 + p^2))}}$, and $p \equiv |\mathbf{p}|$. More details are given in the Appendix D.

Hence, the density of states always receives a correction proportional to $\Phi(\mathbf{r})$ itself

$$\delta\rho(w) = \int \mathrm{d}^2 r \frac{2|w|}{\pi} \Phi(\mathbf{r}) - \frac{|w|}{\pi^2} \int \mathrm{d}^2 r \int \frac{\mathrm{d}^2 p}{(2\pi)^2} e^{i\mathbf{p} \cdot \mathbf{r}} \Phi(\mathbf{p}) \, \mathrm{Im} \left[\frac{\arctan \chi(w, p)}{\chi(w, p)} \right]. \tag{30}$$

6.1. Example: Density of states for a Gaussian beam

We are now able to compute the correction to the density of state when the hopping J has a Gaussian shape due to the finite laser beam waist. Under the assumption that the Raman lasers propagate along y direction, this implies that $\Phi(x,y) = -(\frac{x}{a})^2$, where a is of order 10^2 lattice spacings [2]. As $\Phi(\mathbf{p}) = +\frac{(2\pi)^2}{a^2}\delta''(p_x)\delta(p_y)$, the second term of (30) turns out to be zero and the correction to the density of state simply reduces to

$$\delta\rho(w) = \int d^2r \frac{2|w|}{\pi} \Phi(\mathbf{r}) \tag{31}$$

providing a clear experimental signature. The local density of states gets a quadratic correction in x.

Incidentally, the same cancellation happens in case of an exponential behavior of J, i.e. for a linear $\Phi(\mathbf{r})$. Consequently, the simple relation (31) applies also to this case.

6.2. Experimental detection

A recent review of the detection methods that can be applied to investigate Dirac physics with ultracold fermions in non-Abelian gauge fields is contained an article authored by one of us [91]. Here we just summarize this discussion with particular focus on density of states. Let us start by observing that for a non-interacting Fermi gas at $T = 0$, the total number of fermions $N_F = \int^\mu dE \rho(E)$, where μ is the chemical potential, equal at $T = 0$ to the Fermi energy E_F. We see that $\rho(E_F) = dN_F/dE_F$ so that measuring of the variance of N_F with E_F allows to determine $\rho(E_F)$. If the systems is confined additionally in a slowly varying harmonic potential $V(\mathbf{r})$, a local chemical potential can be introduced $\mu(\mathbf{r}) = E_F - V(\mathbf{r})$, and the corresponding local density of states, related to the local density by

$$n(\mathbf{r}) = \int dE n(E)\Theta(\mu(\mathbf{r}) - E),$$

where $\Theta(.)$ is the Heaviside (step) function. In this case we get

$$dn(\mathbf{r})/d\mu(\mathbf{r}) = \rho(\mu(\mathbf{r})),$$

i.e. a similar formula to the Streda formula used in [91] for detection of Hall conductivity. The determination of density of states can be achieved by:

- Measurements of the total number of fermions as a function of the chemical potential. Here, the best currently available methods are: direct *in situ* individual atom detection [92, 93, 94], or quantum spin polarization spectroscopy [95, 96]
- Measurements of the (coarse-grained) local density of fermions as a function of the local chemical potential. Again, the best currently available methods are: direct *in situ* individual atom detection [92, 93, 94], or quantum spin polarization spectroscopy with spatial resolution [97].
- Measurements of frequency-momentum resolved single particle excitation spectrum, such as those being done in Bragg (Raman) scattering spectroscopy (for a state-of-the-art report see [98]). The spectrum in such processes is proportional to the density of initial states of the scattering process.

Of course, many other methods, such as atomic ARPES, noise interferometry, or even absorbtion and/or phase contrast imaging can give at least indirect information about $\rho(E)$. All of these methods are well developed in experiment with ultracold atoms (see [91] and references therein).

7. Conclusions and Outlook

In this paper, we discussed the simulation of the Dirac equation in artificial curved space-time with cold atoms. We showed that using state-of-the-art techniques it is possible to simulate relativistic fermion dynamics in curved spacetimes with a *flat 2D square* lattice for an interesting class of 2+1 metrics. Moreover, we pointed out the relation between a certain class of Hubbard models and Dirac's Hamiltonian in curved backgrounds, which can be employed to make analytic computations in the continuum limit of the former. We proposed to characterize the *Nature* of Dirac fermions on the lattice by measuring the density of states at the Fermi level. This observable can be, on one hand, analytically computed in perturbation theory in terms of Dirac propagator, and, on the other hand, is accessible to measurements.

The present study opens the way to the direct observation of elusive effects such as Rindler noise. Because we deal with odd dimensional (2+1) Rindler system, the Dirac thermal noise, measured by an ideal point-like De-Witt detector as a consequence of the local acceleration, is expected to be "anomalous" (see Ch.8 of [78]), i.e. it should follow Bose-Einstein distribution. This issue is currently under investigation.

Acknowledgements

Appendix A. Dirac Hamiltonian in spatially (graphene like) deformed metric

Now we consider a different situation where the 2+1 metric is spatially non-trivial. Such a case is relevant in describing the properties of a graphene sheet with ripples. The most generic spacial deformation (at least in some patch) can be always written as

$$ds^2 = -dt^2 + e^{2\Phi(x,y)}(dx^2 + dy^2). \tag{A.1}$$

The *driebein* are

$$e^0 = dt \quad e^1 = e^{\Phi(x,y)}dx \quad e^2 = e^{\Phi(x,y)}dy \quad i = x, y \,,$$

and the spin-connection can be chosen to be non-trivial in the spacial part only

$$w^{12} = \partial_y \Phi dx - \partial_x \Phi dy. \tag{A.2}$$

It follows that the curvature is

$$\Omega = 2e^{-2\Phi}\Omega_{xy}^{12} = -2e^{-2\Phi}(\partial_x^2 + \partial_y^2)\Phi \,. \tag{A.3}$$

For instance, the slices of the metric (A.1) at constant time will be spheres or hyperboloids for Φ a positive or a negative quadratic form of x and y, respectively.

Applying (7) to this case we find

$$i\partial_i \psi = \mathcal{H}\psi = -i\gamma_0 e^{-\Phi}\left(\gamma_1(\partial_x + \frac{1}{2}\partial_y\Phi\gamma_{12}) + \gamma_2(\partial_y - \frac{1}{2}\partial_x\Phi\gamma_{12})\right)\psi \,. \tag{A.4}$$

We are tempted to interpret the above Hamiltonian as that of fermions coupled to a "geometric" non-Abelian vector potential $\mathbf{A} \equiv (\partial_y\Phi, -\partial_x\Phi)\sigma_z$. Adopting the the gamma matrices' representation of the previous section, we can write it as

$$\mathcal{H}\psi = -ie^{-\Phi}\left(\sigma_x(\partial_x + iA_x) + \sigma_y(\partial_y + iA_y)\right)\psi, \tag{A.5}$$

where the presence of $\mathbf{A}$ indicates that the rotation in the xy-plane, i.e. the $SO(2)$ subgroup of the Lorentz group, is promoted to a local symmetry in the background described by the metric (A.1). Such identification is related to the treatment of the conical defects in the graphene sheets (dislocations and disclinations) as sources of magnetic fluxes. However, that this interpretation here is misleading as the gauge group does not commute with the spacetime symmetry, as σ_z anticommutes with σ_x and σ_y.

Taking in account this fact, the Hamiltonian density can more appropriately be written as

$$\mathcal{H}_{(s.d.)} = -i\left(\sigma_x(\partial_x + \frac{1}{2}\partial_x\Phi) + \sigma_y(\partial_y + \frac{1}{2}\partial_y\Phi)\right)\psi, \tag{A.6}$$

where the symmetric role of x and y is evident.

Now we are ready to compute the total Hamiltonian. By rewritting (8) we get

$$H = \int \mathrm{d}x\mathrm{d}y\, e^{2\Phi(x,y)}\, \psi^\dagger \mathcal{H}_{(s.d.)}\psi. \tag{A.7}$$

Using the the same manipulations as in the previous section we can recast it into a form where the spin-connection is not present,

$$H = \frac{i}{2}\int \mathrm{d}x\mathrm{d}y\, e^{\Phi(x,y)} \sum_{i=x,y}\left((\partial_i\psi^\dagger)\sigma_i\psi - \psi^\dagger\sigma_i\partial_i\psi\right). \tag{A.8}$$

It follows that the discretized version of H, as in Rindler spacetime, takes the form of a SU(2) Fermi-Hubbard model with the modulus of hopping depending on the position

$$H_{FH} = \frac{i}{2}\sum_{m,n}\frac{e^{\Phi(m\Delta,n\Delta)}}{\Delta}\left(\psi^\dagger_{m+1,n}\sigma_x\psi_{m,n} + \psi_{m,n+1}\sigma_y\psi_{m,n}\right) + h.c. \tag{A.9}$$

At first sight it seems very surprising that the discretized Hamiltonian of massless fermions in Rindler geometry coincides with the one of fermions propagating in metric of the form of (A.1). Indeed, by taking $\Phi = \ln(ax)$ the expression (A.9) reduces to (12). Such an apparent contradiction disappears upon closer inspection. At the end, from the point of view of the Dirac Hamiltonian for both metrics, what has changed with respect to the flat case is the effective speed of light, or equivalently the hopping rate, which becomes position (and direction) dependent.

Although the origin of the position-dependent hopping rate is different in the two cases, it comes from the Hamiltonian density in the Rindler case while is due to the invariant measure in the other, the effect is the same. Roughly speaking, there are two possible way of modifying the effective speed of light in one spacetime direction, let us say x: one is to change g_{tt} while the other is to change g_{xx} by an inverse factor.

Nevertheless, the eigenfunctions and the spectra of the two Schrödinger problems remain different as the Hamiltonian densities in the two cases are.

Appendix B. The generic case

In order to treat the generic case let us retrace a few steps and analyze the formal expression for the Hamiltonian (8). First of all, we show that it corresponds to the Legendre transformation of the relativistic Lagrangian:

$$L = \int \mathcal{L} = i \int \sqrt{-g}\, \bar{\psi} \gamma^\mu D_\mu \psi\,, \tag{B.1}$$

once we have chosen the coordinates to have that the timelike Killing vector is $K = \partial_t$, which implies $\partial_t g_{\mu\nu} = 0$. Indeed, by defining $\mathcal{H} = \frac{\delta \mathcal{L}}{\delta \partial_t \psi}\partial_t \psi - \mathcal{L}$, the expression (8) is recovered

$$H \equiv \int h = -i \int \sqrt{-g}\, \bar{\psi} \left(\gamma^i \partial_i + \frac{1}{4}\gamma^\mu w_\mu{}^{ab}\gamma_{ab} \right)\psi\,, \qquad i = x, y. \tag{B.2}$$

Now, it is instructive to check explicitly that the Hamiltonian above is a Hermitian operator due to the existence of a timelike isometry. By using that $(\gamma^\mu)^\dagger = \gamma_0 \gamma^\mu \gamma_0$ and noticing that

$$\left(\bar{\psi}\, \gamma^\mu w_\mu{}^{ab}\gamma_{ab}\, \psi \right)^\dagger = -\bar{\psi}\, w_\mu{}^{ab}\gamma_{ab}\, \gamma^\mu \psi\,,$$

we get that

$$H^\dagger = i \int \sqrt{-g} \left((\partial_i \bar{\psi})\gamma^i - \frac{1}{4}\bar{\psi}\, w_\mu{}^{ab}\gamma_{ab}\gamma^\mu \right)\psi\,. \tag{B.3}$$

In order to compare the above expression with H it is convenient to integrate by parts and rewrite it as

$$H^\dagger = -i \int \sqrt{-g}\, \bar{\psi} \left(\gamma^i \partial_i + \frac{1}{4}w_\mu{}^{ab}\gamma_{ab}\gamma^\mu + \gamma^i \partial_i \ln\sqrt{-g} + \partial_i \gamma^i \right)\psi$$

$$= H - i \int \sqrt{-g}\, \bar{\psi} \left(\frac{1}{4}w_\mu{}^{ab}[\gamma_{ab}, \gamma^\mu] + \gamma^i \partial_i \ln\sqrt{-g} + \partial_i \gamma^i \right)\psi\,. \tag{B.4}$$

$H^\dagger$ and H are the same if and only if the metric, and consequently the driebein, are invariant under time translation. Indeed, due to the conventional constraint $\nabla_\mu e^a_\nu + w_\mu{}^a{}_b e^b_\nu = 0$, it is true that

$$e^a_\nu \nabla_\mu e^\rho_a = -e^\rho_a \nabla_\mu e^a_\nu = -e_{\nu a} w_\mu{}^a{}_b e^{\rho b}\,, \tag{B.5}$$

which, using the commutator $[\gamma_{ab}, \gamma^\mu] = 4 e^{\mu c}\gamma_{[a}\eta_{b]c}$, implies

$$\frac{1}{4}w_\mu{}^{ab}[\gamma_{ab}, \gamma^\mu] = \gamma_a w_\mu{}^{ab} e^\mu_b = -(\nabla_\mu e^\mu_c)\gamma^c = -\nabla_\mu \gamma^\mu\,. \tag{B.6}$$

The cancellation follows from the identity

$$\partial_i \gamma^i + \gamma^i \partial_i \ln\sqrt{-g} = \nabla_i \gamma^i = \nabla_\mu \gamma^\mu\,, \tag{B.7}$$

as $\partial_\mu \ln\sqrt{-g} = \Gamma^\nu_{\mu\nu}$ and $\partial_t \gamma^t = \gamma^t \partial_t \ln\sqrt{-g} = 0$ if and only if $\partial_t g_{\mu\nu} = 0$.

At this point, we can use expression (B.3) for $H^\dagger$ to get a Hamiltonian symmetrical in ψ and $\psi^\dagger$. By writing $H \equiv \frac{1}{2}(H + H^\dagger)$ we find

$$H = \frac{i}{2}\int \sqrt{-g} \left((\partial_i \bar{\psi}\gamma^i\psi - \bar{\psi}\gamma^i\partial_i\psi) - \frac{1}{4}\bar{\psi}w_\mu{}^{ab}\{\gamma_{ab}, \gamma^\mu\}\psi \right)\,. \tag{B.8}$$

Let us characterize the term $\Theta \equiv -\frac{1}{4}\bar{\psi}w_\mu{}^{ab}\{\gamma_{ab}, \gamma^\mu\}$, which can be regarded as the obstruction to write the lattice Hamiltonian simply as

$$H = \frac{i}{2}\sum J_{mn} \left(\psi^\dagger_{m+1,n}\sigma_x\psi_{m,n} + \psi^\dagger_{m,n+1}\sigma_y\psi_{m,n} \right) + h.c.\,, \tag{B.9}$$

as found for the special cases discussed in the previous sections. Using the relation between the spin-connection and the driebein and the anticommutator $\{\gamma_{ab}, \gamma^\mu\} = 2e^{\mu c}\gamma_{abc}$ –in 2+1 dimensions it reduces to $\{\gamma_{ab}, \gamma^\mu\} = -\frac{1}{3}e^{\mu c}\epsilon_{abc}$, it follows that

$$\Theta = \frac{1}{12}e^{\mu a}e^{\nu b}\partial_\mu e^c_\nu \, \epsilon_{abc} = \frac{1}{12}e^{ia}e^{\nu b}\partial_i e^c_\nu \, \epsilon_{abc} \,. \tag{B.10}$$

Hence, we conclude that Θ is identically zero when the metric is diagonal, as for the Rindler metric (9) and the spatially deformed metric (A.1). In order to be as general as possible, we observe that any 2+1 spacetime admitting a timelike Killing vector can be always reduced, at least locally, to the form

$$ds^2 = -f^2(x,y)(dt + \lambda(x,y))^2 + f^{-2}(x,y)e^{2\Phi(x,y)}(dx^2 + dy^2)\,, \tag{B.11}$$

where $\lambda(x,y) = \lambda_x(x,y)dx + \lambda_y(x,y)dy$ is one-form independent of time.

It follows that:

$$e^0 = f(dt + \lambda)\,, \quad e^i = f^{-1}e^\Phi dx_i\,, \quad x_i = x, y\,, \quad i = 1, 2\,,$$

and that the inverse driebein are

$$e_0 = f^{-1}\partial_t\,, \quad e_1 = -fe^{-\Phi}\lambda_x\partial_t + fe^{-\Phi}\partial_x\,, \quad e_2 = -fe^{-\Phi}\lambda_y\partial_t + fe^{-\Phi}\partial_y$$

The spin-connection can be chosen to be:

$$w^{01} = fe^{-\Phi}\partial_x f dt + \frac{fe^{-\Phi}}{2}\left(\partial_x(f\lambda_y) - \partial_y(f\lambda_x)\right)dy$$

$$w^{02} = fe^{-\Phi}\partial_y f dt - \frac{fe^{-\Phi}}{2}\left(\partial_x(f\lambda_y) - \partial_y(f\lambda_x)\right)dx$$

$$w^{12} = \frac{f^{-1}e^{-\Phi}}{2}\frac{fe^{-\Phi}}{2}\left(\partial_x(f\lambda_y) - \partial_y(f\lambda_x)\right)dt$$
$$+ \left(\partial_y(\Phi - \ln f) - \frac{f^2 e^{-\Phi}}{2}\lambda_x\left(\partial_x(f\lambda_y) - \partial_y(f\lambda_x)\right)\right)dx$$
$$- \left(\partial_x(\Phi - \ln f) + \frac{f^2 e^{-\Phi}}{2}\lambda_y\left(\partial_x(f\lambda_y) - \partial_y(f\lambda_x)\right)\right)dy\,. \tag{B.12}$$

This implies the following relation for the coefficients:

$$w_y{}^{01} = -w_x{}^{02} = f^{-2}e^\Phi w_t{}^{12}\,,$$

and that $w_y{}^{12}$ can be obtained from $w_x{}^{12}$ by exchanging x with y and 1 with 2 (which amounts for the minus sign), accordingly to the symmetry of the metric (B.11).

We are ready to compute Θ. After some algebra we get

$$\Theta = \frac{f^3 e^{-2\Phi}}{12}\left(\partial_x\lambda_y - \partial_y\lambda_x\right)\,. \tag{B.13}$$

The above expression means that the spin-connection will cancel in the symmetric form of the Hamiltonian if and only the metric is static, i.e. the off-diagonal terms due to λ can be reabsorbed by a change of coordinates. To prove this we note that the condition $\Theta = 0$ implies that the form λ is exact, i.e. there exists a function $F = F(x,y)$ such that $dF = \lambda$. Indeed, this is the case as as can be seen by redefining the time coordinate as $T = t + F$, as the final metric is diagonal.

With the choice of the metric (B.11), the overall hopping rate J_{mn} is

$$J_{mn} = e^{\Phi(x_m, y_n)}\,.$$

Appendix C. The density of states

This section is devoted to deriving the relation between the density of states as a function of the energy $\rho(E)$ and the propagator in a spacetime of dimensions $(1, d)$. In order to do so we will follow a constructive procedure. By definition we have

$$\int \mathrm{d}E \, \rho(E) = \# \text{ of eigenstates of the Hamiltonian } H \, . \tag{C.1}$$

We will express the number of states as the number of poles in the propagator when averaged over the set of eigenstates of H, $\{|n\rangle\}$, which we demand be complete and normalizable (as is the case for any sound Hamiltonian operator). Furthermore, we assume that the position operator eigenstates $|\mathbf{r}\rangle$ can be completed to give an orthonormal basis $|\mathbf{r}\rangle|i\rangle$ where i encodes all internal degrees of freedom, such as the spin. Using the theorem of residues and integrating on a rectangular contour around the real axis of width $2\epsilon > 0$, we can write:

$$
\begin{aligned}
\# \text{ e.s. of } H &= \frac{1}{-2\pi i} \int \mathrm{d}E \sum_n \left(\frac{1}{E + i\epsilon - E_n} - \frac{1}{E - i\epsilon - E_n} \right) \\
&= -\frac{1}{\pi} \mathrm{Im} \int \mathrm{d}E \sum_n \langle n| \frac{1}{E - H + i\epsilon} |n\rangle \\
&= -\frac{1}{\pi} \mathrm{Im} \int \mathrm{d}E \int \mathrm{d}^d r \mathrm{d}^d r' \sum_{n,i,i'} \langle n||\mathbf{r}\rangle|i\rangle\langle\mathbf{r}|\langle i| \frac{1}{E - H + i\epsilon} |\mathbf{r}'\rangle|i'\rangle\langle\mathbf{r}'|\langle i'||n\rangle \\
&= -\frac{1}{\pi} \mathrm{Im} \int \mathrm{d}E \int \mathrm{d}^d r \langle \mathbf{r}|\langle i| \frac{1}{E - H + i\epsilon} |\mathbf{r}\rangle|i\rangle \, . \tag{C.2}
\end{aligned}
$$

Form the above equation we conclude that

$$\rho(E) = \frac{1}{\pi} \mathrm{Im} \int \mathrm{d}^d r \langle \mathbf{r}|\langle i| \frac{1}{H - E - i\epsilon} |\mathbf{r}\rangle|i\rangle \, , \tag{C.3}$$

that is the equation 34 given in [104]. To warm up we compute the density of state in the free case. As the Hamiltonian is diagonal in momentum space, it is convenient to write $\rho(E)$ in this basis:

$$
\begin{aligned}
\rho(E) &= \frac{1}{\pi} \mathrm{Im} \int \mathrm{d}_2\Omega_p \left(\langle \mathbf{p}, +| \frac{1}{E - H + i\epsilon} |\mathbf{p}, +\rangle + \langle \mathbf{p}, -| \frac{1}{E - H + i\epsilon} |\mathbf{p}, -\rangle \right) \\
&= \frac{1}{\pi} \mathrm{Im} \int \mathrm{d}^2 r \int \mathrm{d}_2\Omega_p \left(\langle \mathbf{p}, +||\mathbf{r}\rangle \frac{|\mathbf{p}| + E}{|\mathbf{p}|^2 - E^2 - i\mathrm{sign}(E)\epsilon} \langle \mathbf{r}||\mathbf{p}, +\rangle \right. \\
&\qquad\qquad\qquad\qquad \left. + \langle \mathbf{p}, -||\mathbf{r}\rangle \frac{-|\mathbf{p}| + E}{|\mathbf{p}|^2 - E^2 - i\mathrm{sign}(E)\epsilon} \langle \mathbf{r}||\mathbf{p}, -\rangle \right) \\
&= \frac{1}{\pi} \mathrm{Im} \int \mathrm{d}^2 r \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{2E}{|\mathbf{p}|^2 - E^2 - i\mathrm{sign}(E)\epsilon} \, , \tag{C.4}
\end{aligned}
$$

where $\mathrm{d}_2\Omega_p = \frac{\mathrm{d}^2 p}{(2\pi)^2 2|\mathbf{p}|}$ is the Lorentz invariant measure and the normalization of the momentum states $|\mathbf{p}, \pm\rangle$ is fixed accordingly. By taking to be $S = \int \mathrm{d}^2 r$ the volume of the system and by going to polar coordinates one finds

$$\rho(E) = \frac{S}{\pi} \mathrm{Im} \int_0^{+\infty} \frac{\mathrm{d}p}{2\pi} \frac{2pE}{p^2 - E^2 - i\mathrm{sign}(E)\epsilon} \, . \tag{C.5}$$

The last integral can be solved in many ways, for example by changing variable to $z = p^2$ and regularizing the integral with cut-off

$$\int_0^{\Lambda^2} \frac{\mathrm{d}z}{2\pi} \frac{E}{z - E^2 - i\mathrm{sign}(E)\epsilon} = \frac{E}{2\pi}\left(\log(\frac{\Lambda^2 - E^2}{E^2}) + \log\left(\frac{1 - i\mathrm{sign}(E)\epsilon}{1 + i\mathrm{sign}(E)\epsilon}\right)\right)$$
$$= \frac{E}{2\pi}\left(\log(\frac{\Lambda^2 - E^2}{E^2}) + 2\pi i(\Theta(E) - \Theta(-E))\right) \text{(C.6)}$$

As the imaginary part is independent of the cut-off, the final result, as expected, is

$$\rho(E) = S\frac{|E|}{\pi}, \tag{C.7}$$

which is in agreement with the result quoted in [104].

It is worth-while to note that $\rho(E)$ can be written in terms of the Feynman propagator. This fact can be derived in a more general setting. By definition, $\frac{1}{E-H+i\epsilon}$ is the Fourier transformation in time of the retarded propagator defined by the equation

$$(i\partial_t - H)\mathcal{G}_+(t - t') = \delta(t - t') \otimes \mathbb{I}_{\mathrm{spin}} \tag{C.8}$$

with boundary conditions $\mathcal{G}_+(t - t') = 0$ for $t < t'$. On the right-hand side of (C.8), the identity in spin-space has been written explicitly to remind the reader that $\mathcal{G}_+(t-t')$ in general acts as a matrix on the internal degrees of freedom. After multiplying on the left by γ^t and on the right by γ_t and taking the expectation value with eigenstates of the position operator the above equation gives

$$-i\gamma^\mu \partial_\mu \mathcal{G}_+(t - t', \mathbf{r}, \mathbf{r}')\gamma_\mathbf{t} = \delta(\mathbf{t} - \mathbf{t}')\delta^\mathbf{d}(\mathbf{r} - \mathbf{r}') \otimes \mathbb{I}_{\mathrm{spin}}, \tag{C.9}$$

where we use $\gamma^t\gamma_t = 1$. Hence, we conclude that $\mathcal{G}_+(t - t', \mathbf{r}, \mathbf{r}')\gamma_\mathbf{t}$ is related to the Feynman propagator as it solves the same equation. The precise relation can be derived by taking the boundary conditions into account. This can be explicitly checked by Fourier-transforming to momentum space. Indeed,

$$\frac{1}{E - H + i\epsilon}\gamma_t = \frac{E - \gamma_t\gamma^i p_i}{E^2 - |\mathbf{p}|^2 + i\mathrm{sign}(E)\epsilon}\gamma_t = \frac{E\gamma_t - \gamma^i p_i}{E^2 - |\mathbf{p}|^2 + i\mathrm{sign}(E)\epsilon}$$
$$= \frac{1}{\gamma^\mu p_\mu - i\mathrm{sign}(E)\epsilon}. \tag{C.10}$$

The above relation between the retarded propagator and Feynman propagator can be extended using perturbation theory to the interactive second quantized formalism. To conclude, let us remark that it is better to use the local definition of density (17) because it is easy to make it generally covariant in order to apply it in a curved gravitational background. In this way, the generalize notion of inner product is properly taken into account due to (19).

Appendix D. The computation of Γ

In order to compute $\mathrm{Im}\,\Gamma(w, \mathbf{p})$ we note that the integral of (26) can be split in

$$\Gamma(w, \mathbf{p}) = 4w\int\frac{\mathrm{d}^2 k}{(2\pi)^2}\left(\frac{1}{\mathbf{k}^2 - w^2 - i\epsilon} + \frac{w^2 - \frac{3}{4}\mathbf{p}^2 + \mathbf{p}\cdot\mathbf{k}}{(\mathbf{k}^2 - w^2 - i\epsilon)((\mathbf{k} - \mathbf{p})^2 - w^2 - i\epsilon)}\right). \tag{D.1}$$

The first integral has been computed above for the free case and its imaginary part gives

$$\mathrm{Im}\int\frac{\mathrm{d}^2 k}{(2\pi)^2}\frac{4w}{\mathbf{k}^2 - w^2 - i\epsilon} = 2w. \tag{D.2}$$

The second integral is convergent and can be computed using Feynman parameter. After some algebra one gets

$$4w \operatorname{Im} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{w^2 - \frac{3}{4}\mathbf{p}^2 + \mathbf{p} \cdot \mathbf{k}}{(\mathbf{k}^2 - w^2 - i\epsilon)((\mathbf{k} - \mathbf{p})^2 - w^2 - i\epsilon)} =$$

$$= -\frac{w}{\pi p} \operatorname{Im} \left(\sqrt{4w^2 + i\epsilon - p^2} \arctan \left[\frac{p}{\sqrt{4w^2 + i\epsilon - p^2}} \right] \right)$$

$$= -\frac{w}{\pi p} \sqrt{|4w^2 - p^2|} \left(\Theta(4w^2 - p^2) \operatorname{Im} \arctan \left[\frac{p}{\sqrt{4w^2 + i\epsilon - p^2}} \right] + \right.$$

$$\left. -\Theta(-4w^2 + p^2) \operatorname{Re} \arctan \left[\frac{p}{\sqrt{4w^2 + i\epsilon - p^2}} \right] \right), \quad \text{(D.3)}$$

where $\sqrt{4w^2 + i\epsilon - p^2} = \sqrt{|4w^2 - p^2|} \left(\Theta(4w^2 - p^2) - i\Theta(-4w^2 + p^2) \right)$ and p is the modulus of $\mathbf{p}$, $p = |\mathbf{p}|$.

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