Restriction maps on 1-cohomology of (algebraic) groups

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Abstract. Our first theorem is a version of the five lemma in a certain situation of pointed sets arising from non-abelian 1-cohomology. For our second, we show that a connected, unipotent algebraic group Q acted on by a reductive algebraic group G admits a filtration $Q = Q(1) \geq Q(2) \geq \cdots \geq Q(n) = 1$ with successive quotients having the structure of G-modules. From these two theorems we deduce our third theorem: if G is a reductive algebraic group, with B a Borel subgroup and Q a unipotent algebraic G-group, then the restriction map $H^1(G, Q) \to H^1(B, Q)$ is an isomorphism. This is a generalisation in the case $n = 1$ of Cline, Parshall, Scott and van der Kallen's result that $H^n(G, V) \cong H^n(B, V)$ for any rational G-module V. We also prove the easy generalisation when $n = 0$. In the case $n = 1$ we use our result to get a corollary about complete reducibility and subgroup structure.

Contents

1 Introduction

Let G be a group. The low degree cohomology of G with coefficients in another group Q on which G acts is often very important in group theory. For instance in degree 0, $H^0(G, Q)$ is the number of fixed points of the action of G on Q, $H^1(G, Q)$ measures the number of complements to Q in the semidirect product QG, and $H^2(G, Q)$ (where defined) measures the number of non-equivalent group extensions

$$1 \to Q \to E \to G \to 1.$$

For instance, a classical result of Schur-Zassenhaus says that if G is a finite group and Q is a normal Hall subgroup (i.e. $Q \lhd G$ and $(|G|, |G/Q|) = 1$), then Q has a complement K such that $G = QK$ is a semidirect product. When Q is abelian, this amounts to saying $H^2(G, Q) = 0$.

In this paper we consider the situation where Q is in general, a non-abelian G-group. First off, we give the definitions of the 0th, 1st and 2nd cohomology groups of G with coefficients in Q (where defined). When Q fits into a short exact sequence

$$\{1\} \to R \to Q \to S \to \{1\}$$

and where the image of R in Q is contained in the centre $Z(Q)$ of Q, we get an exact sequence of non-abelian cohomology (Proposition 2.2). We consider the restriction of this exact sequence to a subgroup B of G and show how conditions on the restriction maps of cohomology $H^n(G, R) \to H^n(B, R)$ and $H^n(G, S) \to H^n(B, S)$ can be used to deduce conditions on the restriction $H^1(G, Q) \to H^1(B, Q)$ and $H^0(G, Q) \to H^0(B, Q)$ (Theorems 3.1 and 3.2). For abelian Q this would come down to a simple application of the equally simple 'five lemma' but in the non-abelian case, we must use the technique of twisting from Galois cohomology, which we explain in §2.3.

We apply this in the special situation where G is a reductive algebraic group acting (morphically) on a unipotent algebraic group Q. Let B be a Borel subgroup of G.

A result due to Cline, Parshall, Scott and van der Kallen says that $H^n(G,V) \cong H^n(B,V)$ for any finite dimensional rational G-module V. We show that Q has a central filtration by G-modules (Proposition 3.9) and then use this fact to extend the Cline, Parshall, Scott and van der Kallen result by applying our Theorems 3.1 and 3.2 inductively to show that $H^n(G,Q) \cong H^n(B,Q)$ for $n=0$ and $n=1$.

In the case $n=1$ this has the Corollary 3.12 on subgroup structure related to Serre's notion of G-complete reducibility via a result of Bate, Martin, Roerhle and Tange: Let P be a parabolic subgroup of G with Levi subgroup L, let H be a closed, reductive subgroup of P and let B be a Borel subgroup of H. Then H is conjugate to a subgroup of L if and only if B is.

2 Preliminaries

2.1 Definitions

We start with some standard material. Let G be a group and let Q be a G-set, that is, a set with an action of G.

Then we define $H^0(G,Q)$ as the set Q^G of fixed points of G acting on Q, i.e. $H^0(G,Q) = \{q \in Q : q^g = q, \forall g \in G\}$.

If further Q is a G-group, then we may define the 1-cohomology $H^1(G,Q)$ as follows.

Firstly, a 1-cocycle is a map $\gamma : G \to Q$ satisfying the cocycle condition, i.e. $\gamma(gh) = \gamma(g)^h\gamma(h)$. The set of all 1-cocycles is denotes $Z^1(G,Q)$. We say that two 1-cocycles γ, δ are cohomologous and write $\gamma \sim \delta$ if $\gamma(g) = q^{-g}\gamma(g)q$. Then we define $H^1(G,Q)$ as the set of equivalence classes of $Z^1(G,Q)$ under $\sim$; i.e. $H^1(G,Q) = Z^1(G,Q)/\sim$.

There is a distinguished point in $Z^1(G,Q)$ given by the trivial cocycle $\mathbf{1} : G \to Q$; $\mathbf{1}(g) = 1$. The equivalence class of the trivial cocycle, $[\mathbf{1}] \subseteq Z^1(G,Q)$ is the set of coboundaries and we denote this distinguished point in $H^1(G,Q)$ by $B^1(G,Q)$.

If R is an abelian G-group we can go further and define the second cohomology.

We define a 2-cocycle to be a map $\gamma : G \times G \to R$ satisfying the 2-cocycle condition, i.e. $\gamma(gh,k)\gamma(g,h)^k = \gamma(g,hk)\gamma(g,k)$. The set of all 2-cocycles is denoted $Z^2(G,R)$. Two 2-cocycles γ and δ are cohomologous and write $\gamma \sim \delta$ if there is a map (morphism) $\phi : G \to R$ with $\delta(g,h) = \gamma(g,h)\phi(g)^h\phi(h)\phi(gh)^{-1}$.

We then define $H^2(G,R)$ to be the set of equivalence classes of $Z^2(G,R)$ modulo $\sim$, i.e. $H^2(G,R) := Z^2(G,R)/\sim$.

Again there is a distinguished point in $Z^2(G,R)$ given by the trivial 2-cocycle $\mathbf{1}$ and its class in $H^2(G,R)$ is denoted by $B^2(G,R)$.

2.2 Exact sequences

Where one has a G-homomorphism of G-groups $\rho : R \to Q$, it is clear that one gets natural maps of the sets of cocycles $Z^i(G,R) \to Z^i(G,Q)$ by post-composing with ρ, for each i that we have defined above. It is then easy to see that cohomologous cocycles map to cohomologous cocycles, and so this induces maps of cohomology $H^i(G,R) \to H^i(G,S)$ for $0 \leq i \leq 2$ (again, where defined).

If $\{1\} \to R \to Q \to S \to \{1\}$ is a short exact sequence of G-groups, then as usual:

Proposition 2.1 ([Ser94, Prop. 38]). *Let $R \triangleleft Q$ be G-groups and let $S := Q/R$ be the corresponding G-group obtained from the natural action of G on the quotient. Then there is an exact sequence of cohomology*

$$\begin{aligned}\{1\} \to H^0(G,R) \to H^0(G,Q) \to H^0(G,S)& \\ \xrightarrow{\delta} H^1(G,R) \to H^1(G,Q) \to H^1(G,S)&\end{aligned}$$

where if qR represents an element in $H^0(G,S) \cong (Q/R)^G$ the map δ is given by $\delta(qR) = [q^{-g}q]$ and all other maps are natural.

Proof. One checks easily that δ does indeed give a map $H^0(G,S) \to H^1(G,R)$ as stated and that it is well-defined; i.e. if $q'R = qR$ then $[q'^{-g}q'] = [q^{-g}q]$; to prove exactness at each point is equally straightforward. $\square$

If R is a central subgroup of Q then one can say slightly more:

Proposition 2.2 ([Ser94, Prop. 43]). *Let $R \leq Z(Q)$ and $S := Q/R$ having the natural G-action on the quotient. Then there is an exact sequence of cohomology*

$$\begin{aligned}\{1\} \to H^0(G,R) \to H^0(G,Q) \to H^0(G,S)& \\ \xrightarrow{\delta} H^1(G,R) \to H^1(G,Q) \to H^1(G,S) \xrightarrow{\Delta} H^2(G,R),&\end{aligned}$$

where the map Δ is given by $\Delta([\gamma]) = [\alpha]$ and α is formed by $\alpha(g,h) = \gamma(g)^h\gamma(h)\gamma(gh)^{-1}$.

Proof. Again it is straightforward to check that Δ is well-defined and that the sequence is exact at each point. $\square$

2.3 Twisting

Let R be a G-set and let Q be G-group with an action on R which commutes with the action of G. i.e.

$$(r^q)^g = (r^g)^{q^g} \qquad \text{for all } r \in R,\ q \in Q,\ g \in G$$

This happens for instance if R and Q are both G-groups with $R \leq Q$.

Now fix an arbitrary 1-cocycle $\gamma \in Z^1(G, Q)$ and define $r * g = r^{g\gamma(g)}$.

This is a new action of G on R as

$$r * (gh) = r^{gh\gamma(gh)} = r^{gh\gamma(g)^h\gamma(h)} = r^{g\gamma(g)hh\gamma(h)} = r * g * h.$$

We call this the $*$-action of G on R with respect to γ. The set R with the $*$-action is again a G-set, denoted R_γ and is called a *twisted form* of R. We say that R_γ is obtained by twisting R by γ.

Observe that if $\rho : Q \to S$ is a homomorphism of G-groups with the image of γ under $\rho \circ \_$ being $\beta \in Z^1(G, S)$ then we get a map $\rho_\beta : Q_\gamma \to S_\beta$.

In particular, if R is a G-stable normal subgroup of Q with Q/R given the natural G-action then for any cocycle $\gamma \in Z^1(G, Q)$, we have a well-defined twisted form $(Q/R)_\gamma$ of Q/R. (Here we denote the image of γ in $Z^1(G, Q/R)$ by γ, also.)

Now we note further that if $R \leq Z(Q)$ then the $*$-action of G on R coincides with the usual action of G on R. So in this case, if $\{1\} \to R \to Q \to S \to \{1\}$ is a short exact sequence of G-groups, then it is clear that so is $\{1\} \to R \to Q_\gamma \xrightarrow{\rho} S_\gamma \to \{1\}$. (The map ρ obviously commutes with the action of G.)

Thus for any γ, we get from Proposition 2.2, a new exact sequence of cohomology:

$$1 \to R^G \to Q_\gamma^G \to S_\gamma^G \to H^1(G, R) \to H^1(G, Q_\gamma) \to H^1(G, S_\gamma) \to H^2(G, R_\gamma),$$

where by the proposition below, $H^1(G, Q) \cong H^1(G, Q_\gamma)$ and $H^1(G, S_\gamma) \cong H^1(G, S)$.

Proposition 2.3 ([Ser94, Prop. 35 bis]). *Let R be a G-group and $\gamma \in Z^1(G, R)$. Then the map*

$$\theta_\gamma : H^1(G, R_\gamma) \to H^1(G, R);\ [\delta] \mapsto [\delta\gamma],$$

where $\delta\gamma$ denotes the map $g \mapsto \gamma(g)\delta(g)$, is a well-defined bijection, taking the trivial class in $H^1(G, R_\gamma)$ to the class of γ in $H^1(G, R)$

2.4 Commutative diagrams

We now give some commutative diagrams that we shall need in the proof of the main result. Note that if $B \leq G$ then for each $0 \leq i \leq 2$ we get maps $H^i(G, Q) \to H^i(B, Q)$ by restriction.

Proposition 2.4. *Let B be a subgroup of G acting on a short exact sequence $\{1\} \to R \to Q \to S \to \{1\}$, such that the image of R in Q is central. Then:*

(i) Restriction to B of the exact sequence of cohomology from Proposition 2.2 gives rise to the following commutative diagram, where the rows are exact and the vertical arrows are restrictions:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & R^G & \longrightarrow & Q^G & \longrightarrow & S^G & \xrightarrow{\delta_G} & H^1(G,R) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & R^B & \longrightarrow & Q^B & \longrightarrow & S^B & \xrightarrow{\delta_B} & H^1(B,R)
\end{array}$$

$$\begin{array}{ccccccc}
\longrightarrow & H^1(G,Q) & \longrightarrow & H^1(G,S) & \longrightarrow & H^2(G,R) \\
& \downarrow & & \downarrow & & \downarrow \\
\longrightarrow & H^1(B,Q) & \longrightarrow & H^1(B,S) & \longrightarrow & H^2(B,R).
\end{array}$$

(ii) Let $\gamma \in Z^1(G, Q)$ be a 1-cocycle, and let $\{1\} \to R \to Q_\gamma \to S_\gamma \to \{1\}$ be the short exact sequence from §2.3. If $\beta \in Z^1(B, Q)$ denotes the restriction of γ to B, then we have the following commutative diagram

$$\begin{array}{ccccccccc}
\{1\} & \longrightarrow & R & \longrightarrow & Q_\gamma & \longrightarrow & S_\gamma & \longrightarrow & \{1\} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\{1\} & \longrightarrow & R & \longrightarrow & Q_\beta & \longrightarrow & S_\beta & \longrightarrow & \{1\}.
\end{array},$$

where the vertical arrows are restrictions. Moreover,

(iii) restriction from G to B gives rise to the following commutative diagram,

where the rows are exact and the vertical arrows are restrictions:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & R^G & \longrightarrow & Q_\gamma^G & \longrightarrow & S_\gamma^G & \xrightarrow{\delta_G} & H^1(G,R) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & R^B & \longrightarrow & Q_\beta^B & \longrightarrow & S_\beta^B & \xrightarrow{\delta_B} & H^1(B,R)
\end{array}$$

$$\begin{array}{ccccccc}
\longrightarrow & H^1(G,Q_\gamma) & \longrightarrow & H^1(G,S_\gamma) & \longrightarrow & H^2(G,R) \\
 & \downarrow & & \downarrow & & \downarrow \\
\longrightarrow & H^1(B,Q_\beta) & \longrightarrow & H^1(B,S_\beta) & \longrightarrow & H^2(B,R).
\end{array}$$

Proof. A moment's thought is required to see that (i) is true. For instance, if $qR \in Q/R \cong S$ with $(qR)^g = qR$ for all g then $(\mathrm{res}_B^G \circ \delta_G)(qR)(b) = [q^{-\bullet}q](b) = q^{-b}q = \delta_B(qR) = \delta_B(\mathrm{res}_B^G qR)$ for all $b \in B$.

(ii) is clear.

(iii) follows immediately from (i) and (ii). □

We also need to relate some of maps of (i) and (iii) together.

Proposition 2.5 ([Ser94, p47]). *Let G be a group and $\rho : Q \to S$ any G-homomorphism of G-groups with $\gamma \in Z^1(G,Q)$ and β its image in $Z^1(G,S)$. Then we have the following commutative diagram:*

$$\begin{array}{ccc}
H^1(G,Q) & \xrightarrow{\rho\circ_} & H^1(G,S) \\
{\scriptstyle\theta_\gamma}\uparrow & & {\scriptstyle\theta_\beta}\uparrow \\
H^1(G,Q_\gamma) & \xrightarrow{\rho_\gamma\circ_} & H^1(G,S_\beta),
\end{array}$$

where the vertical maps are the bijections of Proposition 2.3.

Proof. Again this is clear: take $[\delta] \in H^1(G,Q_\gamma)$. Then $\theta_\gamma[\delta] = [\delta\gamma] \in H^1(G,Q)$. In turn

$$\rho \circ_{\_} ([\delta\gamma]) = [(\rho\circ\delta)(\rho\circ\gamma)] = [(\rho\circ\delta)\beta]$$

and the latter is clearly equal to $\theta_\beta(\rho_\gamma([\delta]))$. □

Proposition 2.6. *Let B be a subgroup of G and Q a G-group, with $\gamma \in Z^1(G,Q)$ and $\beta \in Z^1(B,Q)$ its restriction to B. Then we have the following commutative diagram*

$$
\begin{tikzcd}
H^1(G, Q_\gamma) \arrow[r, "\theta_\gamma"] \arrow[d] & H^1(G, Q) \arrow[d] \\
H^1(B, Q_\beta) \arrow[r, "\theta_\beta"] & H^1(B, Q)
\end{tikzcd}
$$

where the vertical maps are restriction.

Proof. On elements

$$
\begin{tikzcd}
{[\delta]} \arrow[r] \arrow[d] & {[\delta\gamma]} \arrow[d] \\
{[\delta|_B]} \arrow[r] & {[\delta|_B \beta] = [\delta|_B \gamma|_B]}
\end{tikzcd}
$$

□

Putting together the last three propositions:

Proposition 2.7. *With the hypotheses of Proposition 2.4, we have the following commutative partial cuboid*

$$
\begin{tikzcd}
& H^1(G, R) \arrow[rr] \arrow[dd] & & H^1(G, Q) \arrow[rr] \arrow[dd] & & H^1(G, S) \arrow[dd] \\
H^1(G, R) \arrow[rr] \arrow[dd] & & H^1(G, Q_\gamma) \arrow[rr] \arrow[dd] \arrow[ur] & & H^1(G, S_\gamma) \arrow[dd] \arrow[ur] & \\
& H^1(B, R) \arrow[rr] & & H^1(B, Q) \arrow[rr] & & H^1(B, S) \\
H^1(B, R) \arrow[rr] & & H^1(B, Q_\beta) \arrow[rr] \arrow[ur] & & H^1(B, S_\beta) \arrow[ur] &
\end{tikzcd}
$$

where rightward arrows are part of four exact sequences running through the central vertical square.

Proof. The front and back faces are subdiagrams of Proposition 2.4(i),(iii); the top and bottom faces commute by Proposition 2.5; the vertical squares commute by Proposition 2.6. □

2.5 Group theoretic interpretations

The following material is completely standard but we include it so as to put our main results into a group theoretic context.

Let G be a group and Q a G-group. Then we can define the semidirect product of G and Q, $Q \rtimes G$ to be $Q \times G$ as a set, and with a multiplication $(q_1, g_1).(q_2, g_2) = (q_1.q_2^{g_1}, g_1.g_2)$. It is easy to check that this makes $Q \rtimes G$ into a group with identity element 1 and inverse $(q, g)^{-1} = (q^{-g^{-1}}, g^{-1})$.

One has a short exact sequence of groups

$$\{1\} \to Q \to Q \rtimes G \to G \xrightarrow{\pi} \{1\}$$

where the last map is simply projection to the second factor. This sequence splits since $\iota : G \to Q \rtimes G;\ g \mapsto (1, g)$ gives $\pi \circ \iota$ the identity map on G. We have $\{(q, 1) : q \in Q\}$ identifying Q with its image in $Q \rtimes G$. Thus we usually just write $Q \rtimes G = QG$ and $(q, g) = qg$.

Now take a cocycle $\gamma \in Z^1(G, Q)$. Then in the semidirect product $Q \rtimes G$ the group $G' := \{\gamma(g)g \in QG : g \in G\}$ is isomorphic to G, $QG' = QG$ and G' has trivial intersection with the image of Q in $Q \rtimes G$. In other words, G' is a *complement* to Q in QG'. Similarly if G' is a complement to Q, projection to G defines a cocycle $\gamma : G \to Q$. Thus one shows that $Z^1(G, Q)$ is in bijection with the set of complements to Q in QG. In fact, two complements G_1 and G_2 are conjugate in QG by an element q of Q if an only if the two cocycles γ_1 and γ_2 defined by them are cohomologous via q. Thus the Q-conjugacy classes of complements to Q in QG are in bijection with the pointed set $H^1(G, Q)$.

One then has $H^1(G, Q) \cong H^1(B, Q)$ if and only if the Q-conjugacy classes of complements to Q in QG are in bijection with the conjugacy classes of complements to Q in QB.

3 Main results

3.1 Versions of the five lemma

Theorem 3.1. *Let $\{1\} \to R \to Q \to S \to \{1\}$ be a short exact sequence of G-groups, such that the image of R is central in Q. Let $B \leq G$. Then in the following diagram,*

$$\begin{array}{ccccccccc}
S^G & \xrightarrow{\delta_G} & H^1(G,R) & \xrightarrow{\iota_G} & H^1(G,Q) & \xrightarrow{\pi_G} & H^1(G,S) & \xrightarrow{\Delta_G} & H^2(G,R) \\
{\scriptstyle h_1}\downarrow & & {\scriptstyle h_2}\downarrow & & {\scriptstyle h_3}\downarrow & & {\scriptstyle h_4}\downarrow & & {\scriptstyle h_5}\downarrow \\
S^B & \xrightarrow{\delta_B} & H^1(B,R) & \xrightarrow{\iota_B} & H^1(B,Q) & \xrightarrow{\pi_B} & H^1(B,S) & \xrightarrow{\Delta_B} & H^2(B,R).
\end{array}$$

the following hold:

(i) If h_2 and h_4 are surjective and h_5 is injective, then h_3 is surjective.

(ii) If h_2 and h_4 are injective and the restriction maps $S^G_\gamma \to S^B_\beta$ are surjective for any $\gamma \in Z^1(G,S)$ with $\gamma|^G_B = \beta$, then h_3 is injective.

(iii) If the hypotheses of (i) and (ii) hold, then h_3 is an isomorphism.

Proof. Assume the hypotheses of (i) and take $\gamma \in H^1(B,Q)$. We need to produce a pre-image β, say, such that $h_3(\beta) = \gamma$. We get started by the diagram chase used to prove the five lemma for diagrams of abelian groups.

Let $\delta := \pi_B(\gamma)$. As the bottom row is exact, $\Delta_B(\delta) = 1$. As h_4 is surjective, we have a pre-image ϵ with $h_4(\epsilon) = \delta$.

Since the last square commutes, $h_5(\Delta_G(\epsilon)) = 1$ and since h_5 is injective, $\Delta_G(\epsilon) = 1$.

Now, the top row is exact, so $\epsilon \in \ker \Delta_G$ and hence $\epsilon \in \mathrm{im}\pi_G$; say, $\epsilon = \pi_G(\eta)$, say. Let $h_3(\eta) = \theta$.

Since the penultimate square commutes, we have $\sigma(\theta) = \delta$.

The picture is now as follows:

$$\begin{array}{ccccccccc}
* & \xrightarrow{\delta_G} & ?,* & \xrightarrow{\iota_G} & ?,\eta & \xrightarrow{\pi_G} & ?,\epsilon & \xrightarrow{\Delta_G} & 1 \\
{\scriptstyle h_1}\downarrow & & {\scriptstyle h_2}\downarrow & & {\scriptstyle h_3}\downarrow & & {\scriptstyle h_4}\downarrow & & {\scriptstyle h_5}\downarrow \\
* & \xrightarrow{\delta_B} & ?,* & \xrightarrow{\iota_B} & \gamma,\theta & \xrightarrow{\pi_B} & \delta & \xrightarrow{\Delta_B} & 1,
\end{array}$$

where we would like to establish the $?$s and are not interested in $*$s.

In the case of abelian groups, one would continue the proof of the five lemma by taking the difference $\gamma - \theta$; observing that this maps under π_B to 1 and continuing the diagram chase. Since we cannot do this in the case of pointed sets we use twisting by η and continue into the partial cuboid of Propostion 2.7.

$$
\begin{tikzcd}
& * \arrow[rr, "\iota_G"] \arrow[dd, "h_2"] & & ?, \eta \arrow[rr, "\pi_G"] \arrow[dd, "h_3"] & & \epsilon \, , \arrow[dd, "h_4"] \\
?_2 \arrow[rr, "\iota'_G"] \arrow[dd, "h_2"'] & & ?_3, 1 \arrow[rr, "\pi'_G"] \arrow[dd, "h'_3"] \arrow[ur, "\theta_\eta"] & & 1 \arrow[dd, "h'_4"] \arrow[ur, "\theta_\epsilon"] & \\
& * \arrow[rr, "\iota_B"] & & \gamma, \theta \arrow[rr, "\pi_B"] & & \delta \\
?_1, 1 \arrow[rr, "\iota'_B"] & & \gamma', 1 \arrow[rr, "\pi'_B"] \arrow[ur, "\theta_\theta"] & & 1 \arrow[ur, "\theta_\delta"] &
\end{tikzcd}
$$

Here we are using the fact that the bijection in Proposition 2.3 takes the neutral element in $H^1(G, Q_\eta)$ to the element η in $H^1(G, Q)$ and the fact that the partial cuboid is commutative.

Now in the front bottom row, as $\gamma' \in \ker \pi'_B$, we have $\gamma' \in \mathrm{im}\ \iota'_B$. Thus we may put κ in place of $?_1$. Since h_2 is a surjection, we may put λ in place of $?_2$. Then we replace $?_3$ with $\mu = \iota'_G(\lambda)$ and by the fact that the front left square commutes, $h'_3(\mu) = \gamma'$. Finally if we replace $?$ with $\nu := \theta_\eta(\mu)$ the commutativity of the central vertical square gives us our preimage of γ.

For (ii) the picture in the partial cuboid is as follows:

$$
\begin{tikzcd}
& & * \arrow[rr, "\iota_G"] \arrow[dd, "h_2"] & & \gamma, \delta \arrow[rr, "\pi_G"] \arrow[dd, "h_3"] & & \zeta \arrow[dd, "h_4"] \\
1, \nu \arrow[r, "\delta'_G"] \arrow[dd, "h'_1"'] & 1, \kappa \arrow[rr, "\iota'_G"] \arrow[dd, "h_2"] & & 1, \delta' \arrow[rr, "\pi'_G"] \arrow[dd, "h'_3"] \arrow[ur, "\theta_\gamma"] & & 1 \arrow[dd, "h'_4"] \arrow[ur, "\theta_\zeta"] & \\
& & * \arrow[rr, "\iota_B"] & & \epsilon \arrow[rr, "\pi_B"] & & \eta \\
1, \mu \arrow[r, "\delta'_B"] & 1, \lambda \arrow[rr, "\iota'_B"] & & 1 \arrow[rr, "\pi'_B"] \arrow[ur, "\theta_\epsilon"] & & 1 \arrow[ur, "\theta_\eta"] &
\end{tikzcd}
$$

where we have twisted by γ. Here a preimage κ of δ' under i'_G exists since δ' is in the kernel of π'_G. Simlarly μ is a preimage of λ under δ'_B; and ν is a preimage of μ under h'_1.

This shows that $\delta' = \iota'_G(\delta'_G(\nu))$ and hence is equal to 1 since the composition of these two maps is trivial. Thus $\gamma = \delta$ since θ_γ is a bijection.

(iii) is now obvious. $\square$

Proposition 3.2. *Let $1 \to R \to Q \to S \to 1$ be a short exact sequence of G-groups, (with R not necessarily central in Q) and let B be a subgroup of G. Then in the following diagram:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & R^G & \longrightarrow & Q^G & \longrightarrow & S^G & \xrightarrow{\delta_G} & H^1(G,R) \\ {\scriptstyle h_1}\big\downarrow & & {\scriptstyle h_2}\big\downarrow & & {\scriptstyle h_3}\big\downarrow & & {\scriptstyle h_4}\big\downarrow & & {\scriptstyle h_5}\big\downarrow \\ 1 & \longrightarrow & R^B & \longrightarrow & Q^B & \longrightarrow & S^B & \xrightarrow{\delta_B} & H^1(B,R). \end{array},$$

(i) If h_2 and h_4 are surjective and h_5 is injective, then h_3 is surjective.

(ii) If h_2 and h_4 are injective then h_3 is injective.

(iii) If h_2 and h_4 are isomorphisms and h_5 is an injection then h_3 is an isomorphism.

Proof. The usual proof of the five lemma goes through in this case. Where one would take the 'difference' of two elements g and h in an abelian group one uses the element gh^{-1}. The proof then works as normal. $\square$

Corollary 3.3. *Suppose $H^i(G,R) \cong H^i(B,R)$ for $0 \le i \le 2$ and all abelian G-groups R. Let Q be any G-group with a finite central filtration, i.e. $Q = Q_1 \ge Q_2 \ge Q_3 \ge \cdots \ge Q_n = \{1\}$ with $Q_i \triangleleft Q$ and $Q_i/Q_{i+1} \le Z(Q/Q_{i+1})$ for all i; then we have $Q^G \cong Q^B$ and $H^1(G,Q) \cong H^1(B,Q)$.*

Proof. This is a simple induction using the previous two results. The case $n = 2$ is given by the hypotheses as Q is then abelian. If we know the result up to $n-1$ then set $R = Q_n$, and $S = Q/Q_n$. Then the hypotheses of Theorem 3.1(iii) and Proposition 3.2(iii) hold and we are done. $\square$

3.2 Algebraic G-groups

Let G be a reductive algebraic group defined over a field k and let B be a Borel subgroup of G. A parabolic subgroup P of G is any closed subgroup of G containing a Borel subgroup. Any parabolic subgroup has a Levi decomposition $P = LR_u(P)$ where L is a Levi subgroup of P, i.e. any closed, maximal reductive subgroup of P; and $R_u(P)$ is the unipotent radical of P, i.e. $R_u(P)$ is the maximal, closed, unipotent, normal subgroup of P. This decomposition is a semidirect product. Thus there is a short exact sequence

$$\{1\} \to R_u(P) \to P \xrightarrow{\pi} L \to \{1\}$$

with a splitting $\iota : L \to P$ such that $\pi \circ \iota$ is the identity map on L.

Let Q be another algebraic group defined over k. Then we say Q *is a G-group* if there is a homomorphism $\phi : G \to \mathrm{Aut}_k(Q)$. One checks that all the work of the previous sections goes through in the category of linear algebraic groups provided one adds 'closed' in the right places; for instance $H^1(G, Q)$ is now in bijection with the set of closed complements to Q in GQ up to Q-conjugacy.

We need some further definitions.

Definition 3.4. Let G be a group and Q a G-group. Then we say that Q *has a central filtration by G-modules* if there is a sequence

$$Q = Q(1) \geq Q(2) \geq \cdots \geq Q(n) = 1$$

of G-groups with $Q(i) \lhd Q$ and where for each $1 \leq i \leq n$, $Q(i)/Q(i+1)$ has the structure of a G-module and is central in $Q/Q(i+1)$; i.e. $Q(i)/Q(i+1) \leq Z(Q/Q(i+1))$.

Recall finally that an algebraic group G comes equipped with an algebra of regular function $k[G]$. If $\mathbb{A}_n$ denotes n-dimensional affine k-space, then $k[\mathbb{A}_n] = k[T_1, \ldots, T_n]$ is a polynomial ring in the n co-ordinate functions T_i.

Any variety X can be seen as a closed subvariety of $\mathbb{A}_n$ for some n; i.e it is the vanishing set $V(I(X))$ of some (radical) ideal $I(X) \leq k[\mathbb{A}_n]$. Moreover, we have $k[X] \cong k[A_n]/I(X)$.

3.3 Unipotent algebraic G-groups

In [Fau75] it is proved that any normal subgroup N of a unipotent algebraic group Q defined over a perfect field is the zeros of $\mathrm{codim}_G(N)$ p-polynomials in the co-ordinate functions of the ambient group. We do not need the full weight of that result. Nonetheless, we make the

Definition 3.5. A polynomial $f \in k[T_1, \ldots, T_n]$ is a *p-polynomial*, if it is a linear combination of terms $T_i^{p^j}$ for $j \geq 0$.

and the

Proposition 3.6. *Let V be a vector group over k (i.e. $V \cong \mathbb{A}_n$) with co-ordinate functions $T_1, \ldots T_n$. If W is any closed subgroup of V then $I(W)$ is generated by codim$_V W$ p-polynomials in $T_1, \ldots, T_n$.*

Proof. This is [Fau75, Proposition 2(i)]. □

Note that when these p-polynomials are not linear, one can have subgroups which are not subspaces:

Example 3.7. Let char $k = p > 0$ and take the one dimensional additive group $\mathbb{G}_a$ embedded in $\mathbb{A}_2$ by $x \mapsto (x, x^p)$. Then $\mathbb{G}_a$ is a closed, connected subgroup of $\mathbb{A}_2$ but not a subspace. It is determined by the vanishing of the single p-polynomial $T_1^p - T_2$

However, if we can find a k^*-action on W as multiplication by scalars, we really do have a subspace.

Proposition 3.8. *With the hypotheses of 3.6, if W is stable under the action of the multiplicative group $\mathbb{G}_m$ on V determined by $v \mapsto \lambda v$ for each $\lambda \in \mathbb{G}_m(k) \cong k^*$, then W is generated by codim$_V W$ linear polynomials in $T_1, \ldots, T_n$ and is hence a subspace.*

Proof. By 3.6 we have that $I(W) = \langle f_1, \ldots f_m \rangle$ with each f_i a p-polynomial, with no dependence $k[\mathbb{A}_n]$-dependence amongst the f_i. More specifically,

$$\begin{aligned} f_1 = &a_{01}T_1 + a_{02}T_2 + \cdots + a_{0n}T_n \\ &a_{11}T_1^p + a_{12}T_2^p + \cdots + a_{1n}T_n^p \\ &\vdots \\ &a_{r1}T_1^{p^r} + a_{r2}T_2^{p^r} + \cdots + a_{rn}T_n^{p^r}. \end{aligned}$$

Call the linear part f_1' of f_1 the linear polynomial $a_{01}T_1 + \cdots + a_{0n}T_n$.

Observe that since the map $k \to k$ defined by $x \mapsto x^p$ is a field automorphism, the zeros of f_1 are the same as the series of $F^*(f_1)$, where F^* is the comorphism of the Frobenius morphism acting as $F^*(T_i) = T_i^p$. Thus if $f_i' = 0$ for any i we may replace f_i with $F^{*-1}(f_i)$ without changing $V(I(W))$ and hence the hypotheses of the proposition.

Now we claim we can arrange that the f_i have no linear relation between their linear parts.

To prove the claim, suppose we have $k_1f_1'+k_2f_2'+\cdots+k_mf_m' = 0$ with $0 \neq k_i \in k$ for some i. By reordering the f_i, we may assume this is k_m. Then setting

$$\hat{f}_m = f_m - \frac{k_1f_1 + k_2f_2 + \cdots + k_{m-1}f_{m-1}}{k_m}$$

we have $I(W) = \langle f_1, \ldots, f_{m-1}, \hat{f}_m \rangle$ with $\hat{f}_m' = 0$. Now we can replace $\hat{f}_m$ with $F^{*-1}(\hat{f}_m)$ and repeat. As the highest power of T_1 amongst all the f_i is finite, this process must terminate in a set of f_i with no linear relationship in the f_i'.

Now if f_1 is a linear polynomial then we get W as a subgroup of a lower dimensional vector space $\mathbb{A}_{n-1} \leq \mathbb{A}_n$ defined by the vanishing of f_1 and then we have the claim by induction.

So assume, looking for a contradiction that (*) $a_{ij} > 0$ for some $i > 0$.

Choose $\lambda \in k \backslash \mathbb{F}_{p^i}$. The morphism $V \to V; v \to \lambda v$ correponds to a comorphism λ^* where on coordinate functions $\lambda^*(T_j) = \lambda T_j$ (so $\lambda^*(T_j^p) = \lambda^p T_j^p$).

Now consider $\lambda^*(f_1)$. If $\lambda^*(f_1)$ is contained in the ideal $I(W)$ then as its linear part $\lambda^*(f_1)' = \lambda f_1'$ we must have $\lambda^*(f_1) \leq \langle f_1 \rangle$ as there are no linear relationships between any of the linear parts of the f_i by the above claim. Now $\lambda^*(f_1) = \lambda f_1$ since the linear parts must agree; thus the coefficient of $T_j^{p^i}$ in λf_1 is λa_{ij} whereas in $\lambda^*(f_1)$ it is $\lambda^{p^i} a_{ij}$. But since we chose $\lambda \notin \mathbb{F}_{p^i}$, $\lambda^{p^i} \neq \lambda$, we must have $a_{ij} = 0$. This is a contradiction to (*). $\square$

We are now in a position to prove our second main result.

Theorem 3.9. *Let G be a reductive group over an algebraically closed field k and Q a G-group. Then Q has a central filtration by G-modules.*

Proof. Form the semidirect product $H := Q \rtimes G$ of Q and G; as G is reductive and Q is unipotent, we note that $H = QG$ is a Levi decomposition of H. Since this is again a linear algebraic group we may take $Q \leq GL(V)$ for some V; note also that $Q = R_u(H)$. Take H in a parabolic subgroup P of $GL(V)$ with $P = R_u(P)L$ chosen minimal subject to containing H.

Suppose, looking for a contradiction, that the intersection $Q' := Q \cap L$ is non-trivial. Then under the projection $\pi : P \to L$, the image $H' := \pi(H) \leq L$ is a subgroup of the reductive subgroup L with a non-trivial unipotent radical $Q' := R_u(H')$ contained in L. Thus H' is in a non-trivial parabolic subgroup $R = MR_u(R)$ of L, with $\dim M < \dim L$. But now $P' = MR_u(R)R_u(P)$ contains a Borel subgroup of G and thus is a parabolic subgroup of G and is strictly contained in P (since the maximal reductive subgroup M of P' is strictly contained in L). However, $H \leq \pi(H)R_u(P) \leq MR_u(R)R_u(P) = P' < P$. This is a contraction as P was chosen minimal subject to containing H; thus we conclude that $Q \cap L = \{1\}$.

Now $G \cap Q = \{1\}$ as G is reductive, thus we have $G \cong G' = \pi(G)$ and so $GQ = G'Q$ with $G' \leq L$.

We apply the main result of [ABS90] which states that $R_u(P)$ has a central filtration by modules for L. That means a sequence of subgroups $R_u(P) = Z(1) \geq Z(2) \geq Z(3) \geq \cdots \geq Z(n)$ with $Z(i)/Z(i+1)$ central in $Z/Z(i)$ and each $Z(i)/Z(i+1)$ has the structure of an L-module. Restriction to G' gives each $Z(i)/Z(i+1)$ the structure of an G' module. Finally, we get a filtration of Q by intersection with the $Z(i)$; that is, letting $Q(i) = Z(i) \cap Q$ we get a central filtration $Q(i)$ of the G'-group Q by G'-stable subgroups of G'-modules.

We wish now to find an action of $\mathbb{G}_m$ on each $Q(i)/Q(i+1)$ so that we may invoke 3.8.

To do this, observe that as $Z(L)$ centralises G', $Z(L)$ also stabilises each $Q(i)/Q(i+1) \leq Z(i)/Z(i+1)$. Indeed it is easy to see that one can choose a one-dimensional torus $\mathbb{G}_m \cong S \leq Z(L)$ acting on each $Z(i)/Z(i+1)$ as scalars.

(For the interested reader, one can do this by choosing standard parabolic subgroups such that a level $Z(i)/Z(i+1)$ is generated by root groups corresponds to roots

$$0 \dots 01 \dots 10 \dots 0 * \cdots * 0 \dots 01 \dots 10 \dots 0 * \cdots * \dots \dots 0 \dots 0$$

where there are $i > 0$ 1s and $\text{rank}_{ss}L$ $*$s taking values 0 or 1 according to whether the resulting string represents a root. Then if the first time a 1 appears is in position j not next to a $*$ we have that the torus $\{h_{\alpha_j}(t) : t \in k^*\}$ acts as scalars on $Q(i)/Q(i+1)$ and is in $Z(L)$ whereas if the first time a 1 appears is position j next to a single star we can use $\{h_{\alpha_{j-1}}(t)^{-1} : t \in k^*\}$ and if the first time j is in between two stars we can use $\{h_{\alpha_{j-1}}(t)h_{\alpha_j}(t^2)h_{\alpha_{j+1}}(t) : t \in k^*\}$.)

As we have our action of $\mathbb{G}_m$ as hypothesised in 3.8 we conclude that each $Q(i)/Q(i+1)$ is a G'-stable subspace of $Z(i)/Z(i+1)$ and hence a G'-submodule. Hence we have a central filtration of Q by G'-modules.

Finally through the isomorphism $G \cong G'$ we then get a central filtration $Q(i)$ of the G-group Q by G-modules. $\square$

3.4 Cohomology of G with coefficients in a unipotent group

In [CPSvdK77], Cline, Parshall, Scott and van der Kallen proved a result which has the following generalisation

Theorem 3.10 ([Jan03, II.4.7]). *Let G be a reductive algebraic group over k and let P be a parabolic subgroup of G. If V is a rational G-module, then $H^n(G, V) \cong H^n(P, V)$ for all $n \geq 0$.*

Using the results of the previous sections we will generalise this result in the cases $n = 0$ and $n = 1$ replacing V by an arbitrary, connected unipotent G-group Q.

Theorem 3.11. *Let G be a reductive group over an algebraically closed field k and let Q be a G-group. Let P be a parabolic subgroup of G. Then the restriction maps*

(i) $Q^G = H^0(G, Q) \to H^0(P, Q) = Q^P$*; and*

(ii) $H^1(G, Q) \to H^1(P, Q)$

are isomorphisms of pointed sets.

Proof. Mimicking the proof of 3.3, we see that induction on the length of the filtration in 3.9, together with 3.2 gives part (i) of the theorem.

Now since we have proved $H^0(G, Q) \cong H^0(P, Q)$ for all G-actions on all connected unipotent groups Q (in particular, for all twists $(Q/R)_\gamma$ in the situation of 3.1) we have the hypotheses for 3.1(iii) and so we conclude, again inductively, that $H^1(G, Q) \cong H^1(P, Q)$. □

Finally, we give a corollary on subgroup structure.

We recall Serre's notion of G-complete reducibility from [Ser98]. A subgroup H of G is said to be G-completely reducible (or G-cr) if whenever H is contained in a parabolic subgroup P of G, it is contained in some Levi subgroup of that parabolic.

Using the above theorem we can show that a closed reductive subgroup H of G is G-cr if and only if whenever H is in a parabolic subgroup of G, one of its Borel subgroups is in a Levi subgroup of that parabolic; in other words

Corollary 3.12. *Let H be a closed reductive subgroup of G contained in a parabolic $P = R_u(P)L$ of G and let B be a Borel subgroup of H. Then H is G-conjugate to a subgroup of L if and only if B is.*

Proof. One direction is trivial, so assume that B is G-conjugate to a subgroup of L.

Since H is reductive, we have as usual that if $\bar{H} := \pi(H)$ denotes the projection of H to L then H is a complement to $R_u(P)$ in $R_u(P)\bar{H}$; hence H corresponds to some cocycle $\gamma \in Z^1(\bar{H}, R_u(P))$. Now consider $\bar{B} := \pi(B) \leq \bar{H}$. It is clear that if β denotes the restriction $\gamma|^{\bar{H}}_{\bar{B}}$ then B corresponds to the cocycle $\beta \in Z^1(\bar{B}, R_u(P))$.

The hypothesis that B is G-conjugate to L implies that B is $R_u(P)$-conjugate to L by [BMRT09, 5.9(ii)]. Thus β must be in the trivial cocycle class in $H^1(\bar{B}, R_u(P))$. But by Theorem 3.11, $H^1(\bar{B}, R_u(P)) \cong H^1(\bar{H}, R_u(P))$ and so γ is in the trivial cocycle class in $H^1(\bar{H}, R_u(P))$. Thus H is $R_u(P)$-conjugate to a subgroup of L and so clearly it is G-conjugate to a subgroup of L. $\square$

Remark 3.13. One has of course the same result with B replaced by any parabolic subgroup Q of H.

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