

# A non-standard approach to introduce simple harmonic motion

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**Abstract.** We'll be presenting an approach to solve the equation of simple harmonic motion (SHM) which is non-standard as compared with the usual way of solution presented in textbooks. In addition to help students avoid the unnecessary memorization of formulas to solve physics problems, this approach could help instructors to present the subject in a teaching framework which integrates conceptual and mathematical reasoning, in a systemic way of thinking that will help students to reinforce their quantitative reasoning skills by using mathematical knowledge already familiar to students in a first calculus-based introductory physics course, such as the chain rule for derivatives, inverse trigonometric functions, and integration methods.

**Keywords:** Physics Education Research; Students Performance.

## 1 Introduction

An interesting account on the teaching and learning of math concepts suggest the following: "Teaching advanced algebra in middle school pushes concepts on students that are beyond normal development at that age." [1] Possibly, this observation also applies to the teaching of introductory university calculus-based physics courses.

In fact, consider, for example, how the ideas of simple harmonic motion (SHM) are presented in widely used textbooks [2, 3, 4, 5]. SHM is introduced in mathematical terms as solving a second order ordinary non-homogeneous differential equation, an alien topic to most students attending their first introductory calculus-based physics. To further increase anxiety on students, the standard procedure to find the solution (which for most students would be an unimaginable guess that might be justified afterward if they guessed correctly) is taught in such a way that could cause students to "see mathematical reasoning as a mysterious process that only experts with advanced degrees consulting books filled with incomprehensible hieroglyphics can fathom." [1]

Accordingly, in what follows will be presenting a non-standard straightforward approach to introduce SHM as a completely solvable example of Newton's second law of motion, and which only requires knowledge of the chain rule for derivatives and of standard integration techniques. In addition to helping students to fully understand the concepts of Newtonian physics, since along the solution process students also have the opportunity to practice inverse trigonometric functions, such a straightforward technique also offers the advantage of helping them to find more sense on what they are learning in their math courses and how to apply that knowledge.

After consulting several calculus-based introductory physics textbooks (we performed a particular thorough search of the well known textbook by Halliday and Resnick, and the

historical archives of Journals devoted to the teaching of Physics) we were surprised to find no mention of our alternative approach for instructors to teach the subject of SHM.

## 2 The Simple Harmonic Motion Equation

The usual example for presenting SHM is the spring-mass system driven by a elastic force which is given by Hooke's Law,  $\vec{F} = -k\vec{x}$ , where  $k$  is the spring constant and  $\vec{x}$  is the displacement from the spring's equilibrium position. For vertical oscillations, the equation of motion via Newton's second law applied to a mass  $m$  that could oscillate attached at one end of the spring has the form:

$$m \frac{d^2 z}{dt^2} + k(z - L) = mg, \quad (1)$$

where  $L$  is the length of the spring,  $g$  is the magnitude of the local gravitational constant, and  $z$  is the vertical position, with  $z = 0$  at the top of the unstretched spring and positive downward.

To further motivate students, one could mention that equation (1) can also be found in the study of the small oscillations of a pendulum and in the description of the oscillations of partially submerged objects.

## 3 Solving the Simple Harmonic Motion equation

As mentioned earlier, without much explanation, calculus-based introductory physics textbooks introduce the solution of equation (1) as if students were already familiar with the subject of non-homogeneous ordinary differential equations. This is hardly true, and such an approach causes many troubles not only for students, who might be trying to think hard and quick enough to truly understand the significance of the fancy term ordinary differential equation, but also for instructors, who might be having a hard time talking about the subject due to timetable constrain.

Thus, our alternative approach to find the solution of equation (1) can be started via the chain rule for derivative by considering the one dimensional speed  $v$  as a function of the vertical position  $z$ , which allow us to rewrite the second derivative in equation (1) in the form

$$v = \frac{dz}{dt} = v(z(t)) \implies \frac{d^2 z}{dt^2} = \frac{d}{dt} \left( \frac{dz}{dt} \right) = \frac{dv}{dt} = \frac{dv}{dz} \frac{dz}{dt} = v \frac{dv}{dz}. \quad (2)$$

Correspondingly, using this relation in equation (1) one obtains,

$$v \frac{dv}{dz} = \left( \frac{k}{m} \right) \left( \left( \frac{m}{k} \right) C - z \right) \implies \int_{v=v_0}^v v dv = \int_{z=z_0}^z \left( \frac{k}{m} \right) \left( \left( \frac{m}{k} \right) C - z \right) dz, \quad (3)$$

where  $C = g + (k/m)L$ . Carrying out the integral yields,

$$v^2 = v_0^2 + 2C(z - z_0) - \left( \frac{k}{m} \right) (z^2 - z_0^2), \quad (4)$$

which can be casted in the forms,

$$v^2 = v_0^2 + \left(\frac{k}{m}\right) \left( \left(z_0 - \frac{m}{k}C\right)^2 - \left(z - \frac{m}{k}C\right)^2 \right) \quad (5a)$$

$$v^2 + \left(\frac{k}{m}\right) \left(z - \frac{m}{k}C\right)^2 = v_0^2 + \left(\frac{k}{m}\right) \left(z_0 - \frac{m}{k}C\right)^2. \quad (5b)$$

Arriving at this point, one could take advantage of the arrangements of terms given by equation (5b) and explain students about a conserved quantity that holds at every stage of the motion, which later on will be introduced in the context of energy conservation for this type of problems. In addition, one could also explore some particular cases for  $v_0$  and  $z_0$  in order to help students to get a better grasp of the evolution  $v$  versus the position  $z$ . Moreover, instructors could introduce the idea of the frequency of the oscillatory motion by considering the units of  $k/m = \omega^2$ .

To continue, without loss of generality equation (5a) can be written in the form,

$$v = \frac{dz}{dt} = \sqrt{\frac{k}{m}} \left[ \left(\frac{m}{k}\right)v_0^2 + \left(z_0 - \frac{m}{k}C\right)^2 - \left(z - \frac{m}{k}C\right)^2 \right]^{\frac{1}{2}}, \quad (6)$$

from which one has,

$$\int_{z=0}^z \frac{dz}{\left[b^2 - \left(z - \frac{m}{k}C\right)^2\right]^{\frac{1}{2}}} = \int_{t=0}^t \sqrt{\frac{k}{m}} dt \quad (7)$$

on which  $b = \sqrt{\left(\frac{m}{k}\right)v_0^2 + \left(z_0 - \frac{m}{k}C\right)^2}$ . The solution of equation (7) has the form,

$$\arcsin \left[ \frac{z - \frac{m}{k}C}{b} \right] - \arcsin \left[ \frac{z_0 - \frac{m}{k}C}{b} \right] = \sqrt{\frac{k}{m}} t. \quad (8)$$

Now, from this last equation one can obtain,

$$\frac{z - \frac{m}{k}C}{b} = \sin \left[ \sqrt{\frac{k}{m}} t \right] \cos \left[ \arcsin \left[ \frac{z_0 - \frac{m}{k}C}{b} \right] \right] + \cos \left[ \sqrt{\frac{k}{m}} t \right] \sin \left[ \arcsin \left[ \frac{z_0 - \frac{m}{k}C}{b} \right] \right]. \quad (9)$$

To write equation (9) in a familiar form, one needs to use the following relations,

$$\sin [\arcsin (x)] = x \quad (10a)$$

$$\cos [\arcsin (x)] = \sqrt{1 - x^2}. \quad (10b)$$

While the identity given by equation (10a) could be already familiar to students, equation (10b) might not be so. Nevertheless, it can be introduced by means of the already known trigonometric expression  $\cos(\theta) = \sqrt{1 - \sin^2(\theta)}$ . If one lets  $x = \sin(\theta)$ , then  $\theta = \arcsin(x)$  and equation (10b) follows immediately. Consequently, using equation (10) to rearrange equation (9) one obtains the familiar solution,

$$z = \left(L + \frac{m}{k}g\right) + \left[v_0 \sqrt{\frac{m}{k}}\right] \sin \left[ \sqrt{\frac{k}{m}} t \right] + \left[z_0 - \left(L + \frac{m}{k}g\right)\right] \cos \left[ \sqrt{\frac{k}{m}} t \right], \quad (11)$$

where we used that  $b^2 - (z_0 - \frac{m}{k}C)^2 = \frac{m}{k}v_0$  and that  $C = g + \frac{k}{m}L$ .

Once the solution of equation (1) is obtained as equation (11), instructors have the freedom to briefly talk about the subject of non-homogeneous ordinary differential equations, followed by a further exploration of the obtained solution for the oscillations of a mass attached to a spring from the physical point of view.

## 4 Concluding Remarks

A non-standard approach for solving the equation of motion leading to SHM has been presented. Our interest is to help students to avoid solving physics problems merely by applying memorized formulas, a practice encouraged by physics textbook writers through their typical “formulae summary” found at the end of each chapter, a bad habit which unfortunately can be found even in classroom teaching [6].

Rather than memorizing an ansatz to solve a non-homogeneous ordinary differential equation, a subject with which generally students in their first calculus-based introductory physics course are not familiar, we make use of the chain rule for derivatives and of first steps of integration techniques to yield the solution of the involved equation for SHM. Moreover, while solving the equation, in addition to practicing the use of inverse trigonometric functions, students are also introduced to a non-familiar inverse trigonometry identity. In this way we could encourage them to be more proactive in applying what they are learning of their math courses.

We also note that research in *Physics Education Research* [7, 8, 9, 10, 11, 12] shows clear evidence that when a problem solving methodology [12, 13] is applied via active teaching and learning strategies [14, 15, 16, 17], students’ abilities to solve physics problems quantitatively is strengthened, and their conceptual understanding of physics is enhanced.

To paraphrase Heron and Meltzer, learning to approach problems in a systematic way starts from teaching and learning the interrelationships among conceptual knowledge, mathematical skills and logical reasoning.[11] In physics, this necessarily requires the teaching of a good deal of mathematical computations. In such context, the problem solving methodology proposed here for the SHM equation could help instructors to present the subject in a teaching framework which integrates conceptual and mathematical reasoning, which will help students to reinforce their quantitative reasoning skills by using mathematical knowledge already familiar to them, as could be the chain rule for derivatives, inverse trigonometric functions, and integration methods.

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