AN EXAMPLE OF MELKERSSON SUBCATEGORY WHICH IS NOT CLOSED UNDER INJECTIVE HULLS

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Abstract. The Melkersson subcategory is a special Serre subcategory which satisfies useful conditions C_I defined in [1]. It was proved that a Serre subcategory which is closed under injective hulls is a Melkersson subcategory. However, it has been an open question whether the contrary implication holds. In this paper, we shall show that this question has a negative answer in general.

1. Introduction

Throughout this paper, all rings are commutative noetherian ring, all modules are unitary and R denotes a ring. We assume that all full subcategories $\mathcal{S}$ of the modules category R-Mod and the finitely generated R-modules category R-mod are closed under isomorphisms, that is if M is in $\mathcal{S}$ and R-module N is isomorphic to M then N is in $\mathcal{S}$.

In [1], M. Aghapournahr and L. Melkersson gave a useful condition C_I on the Serre subcategory $\mathcal{S}$ of R-Mod where I is an ideal of R. It is said that $\mathcal{S}$ satisfies the condition C_I if the following condition holds: if $M = \Gamma_I(M)$ and $(0 :_M I)$ is in $\mathcal{S}$, then M is in $\mathcal{S}$. They showed that local cohomology modules and Serre subcategories which satisfy such a condition have affinity for each other. After of this, the Serre subcategory which satisfies the condition C_I for all ideals I of R was named Melkersson subcategory by M. Aghapournahr, A. J. Taherizadeh and A. Vahidi in [2]. For example, all Serre subcategories which are closed under injective hulls are Melkersson subcategory. So it is natural to ask the following question which was given in [1]:

Question. Is Melkersson subcategory closed under injective hulls?

In this paper, we shall show that this question has a negative answer in general. To be more precise, we denote by $\mathcal{S}_{f.g.}$ the Serre subcategory of all finitely generated R-modules and by $\mathcal{M}_{f.s.}$ the Serre subcategory of all R-modules with finite support. We shall see that a class

$$(\mathcal{S}_{f.g.}, \mathcal{M}_{f.s.}) = \left\{ X \in R\text{-Mod} \;\middle|\; \begin{array}{c} \text{there are } S \in \mathcal{S}_{f.g.} \text{ and } M \in \mathcal{M}_{f.s.} \text{ such that} \\ 0 \to S \to X \to M \to 0 \text{ is exact.} \end{array} \right\}$$

is Melkersson subcategory which is not closed under injective hulls on the ring of formal power series $R = k[[x, y]]$ in the indeterminate x and y with the coefficients in a field k.

The organization of this paper is as follows.

In section 2, we shall recall definitions of Melkersson subcategory and classes $(\mathcal{S}_1, \mathcal{S}_2)$ of extension modules of a Serre subcategory $\mathcal{S}_1$ by another Serre subcategory $\mathcal{S}_2$. In section 3, we shall give a proof of main result. In Section 4, we shall see several remarks on Melkersson subcategory.

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2. Preliminaries

In this section, we shall recall several definitions which are necessary to prove the main result of this paper.

A class $\mathcal{S}$ of R-modules is called a Serre subcategory of R-Mod if it is closed under submodules, quotients and extensions. We also say that a Serre subcategory $\mathcal{S}$ of R-Mod is a Serre subcategory of R-mod if $\mathcal{S}$ consists of finitely generated R-modules.

In [1], M. Aghapournahr and L. Melkersson gave the following condition on Serre subcategories of R-Mod.

Definition 2.1. Let $\mathcal{S}$ be a Serre subcategory of R-Mod and I be an ideal of R. We say that $\mathcal{S}$ satisfies the condition C_I if the following condition satisfied:

$$(C_I) \quad \text{If } M = \Gamma_I(M) \text{ and } (0 :_M I) \text{ is in } \mathcal{S}, \text{ then } M \text{ is in } \mathcal{S}.$$

The following special Serre subcategory was named Melkersson subcategory by M. Aghapournahr, A. J. Taherizadeh and A. Vahidi in [2].

Definition 2.2. Let $\mathcal{M}$ be a Serre subcategory of R-Mod.

(1) $\mathcal{M}$ is called a Melkersson subcategory with respect to an ideal I of R if $\mathcal{M}$ satisfies the condition C_I.
(2) $\mathcal{M}$ is called a Melkersson subcategory if $\mathcal{M}$ satisfies the condition C_I for all ideals I of R.

It has already shown that any Serre subcategory which is closed under injective hulls is the Melkersson subcategory with respect to all ideals I of R, so that it is a Melkersson subcategory. (See [1, Lemma 2.2].)

Next, we consider classes of extension modules of Serre subcategory by another one.

Definition 2.3. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be Serre subcategories of R-Mod. We denote by $(\mathcal{S}_1, \mathcal{S}_2)$ the class of all R-modules M with some R-modules $S_1 \in \mathcal{S}_1$ and $S_2 \in \mathcal{S}_2$ such that a sequence $0 \to S_1 \to M \to S_2 \to 0$ is exact, that is

$$(\mathcal{S}_1, \mathcal{S}_2) = \left\{ M \in R\text{-Mod} \;\middle|\; \begin{array}{c} \text{there are } S_1 \in \mathcal{S}_1 \text{ and } S_2 \in \mathcal{S}_2 \text{ such that} \\ 0 \to S_1 \to M \to S_2 \to 0 \text{ is exact.} \end{array} \right\}.$$

We shall refer to $(\mathcal{S}_1, \mathcal{S}_2)$ as a class of extension modules of $\mathcal{S}_1$ by $\mathcal{S}_2$.

For example, a class $(\mathcal{S}_{f.g.}, \mathcal{S}_{Artin})$ is the set of all Minimax R-modules where $\mathcal{S}_{f.g.}$ denotes the Serre subcategory consists of all finitely generated R-modules and $\mathcal{S}_{Artin}$ denotes the Serre subcategory consists of all Artinian R-modules. We note that a class $(\mathcal{S}_1, \mathcal{S}_2)$ is not necessarily Serre subcategory. (For more detail, see [7].)

3. Main result

In this section, we shall give an example of Melkersson subcategory which is not closed under injective hulls. We denote by $\mathcal{M}_{f.s.}$ the class of R-modules with finite support. A class $\mathcal{M}_{f.s.}$ is Serre subcategory of R-Mod which is closed under injective hulls, so that $\mathcal{M}_{f.s.}$ is a Melkersson subcategory. (See [1, Example 2.4].) Furthermore, a class $(\mathcal{S}_{f.g.}, \mathcal{M}_{f.s.})$ is a Serre subcategory of R-Mod by [7, Corollary 4.3 or 4.5].

The main result in this paper is as follows.

Theorem 3.1. *Let $(R, \mathfrak{m})$ be a local ring with a maximal ideal $\mathfrak{m}$. Then the following assertions hold.*

(1) *If R has infinite many prime ideals, then $(\mathcal{S}_{f.g.}, \mathcal{M}_{f.s.})$ is not closed under injective hulls.*

(2) *If R is a 2-dimensional local domain, then $(\mathcal{S}_{f.g.}, \mathcal{M}_{f.s.})$ is a Melkersson subcategory.*

In particular, if R is a 2-dimensional local domain with infinite many prime ideals, then $(\mathcal{S}_{f.g.}, \mathcal{M}_{f.s.})$ is a Melkersson subcategory which is not closed under injective hulls.

Proof. (1) We assume that R has infinite many prime ideals. (We note that the dimension of R must be at least two.) Since the set $\mathrm{Min}(R)$ of all minimal prime ideals of R is finite set, there exists a prime ideal $\mathfrak{p} \in \mathrm{Min}(R)$ such that $V(\mathfrak{p}) = \{\mathfrak{q} \in \mathrm{Spec}(R) \mid \mathfrak{p} \subseteq \mathfrak{q}\}$ is infinite set. We fix this prime ideal $\mathfrak{p}$.

We assume that $(\mathcal{S}_{f.g.}, \mathcal{M}_{f.s.})$ is closed under injective hulls and shall derived a contradiction. Since $R/\mathfrak{p}$ is in $(\mathcal{S}_{f.g.}, \mathcal{M}_{f.s.})$, the injective hull $E_R(R/\mathfrak{p})$ of $R/\mathfrak{p}$ is also in $(\mathcal{S}_{f.g.}, \mathcal{M}_{f.s.})$ by assumption. Therefore, there exists a short exact sequence

$$0 \to F \to E_R(R/\mathfrak{p}) \to M \to 0$$

with $F \in \mathcal{S}_{f.g.}$ and $M \in \mathcal{M}_{f.s.}$. Since $V(\mathfrak{p})$ is infinite set and $\mathrm{Supp}(M)$ is finite set, we can choose a prime ideal $\mathfrak{n} \in V(\mathfrak{p}) \setminus \big(\mathrm{Supp}(M) \cup \{\mathfrak{p}\}\big)$. Here, we set $T = R_\mathfrak{n}$ and $\mathfrak{q} = \mathfrak{p}R_\mathfrak{n} = \mathfrak{p}T$. We note that T is local ring with at least dimension one and $\mathfrak{q}$ is a minimal prime ideal of T.

Now here, we claim that $E_{T/\mathfrak{q}}(T/\mathfrak{q})$ is a finitely generated as $T/\mathfrak{q}$-module and shall show this. By applying the exact functor $(-) \otimes_R T$ to the above short exact sequence, we see that it holds

$$F_\mathfrak{n} \cong E_R(R/\mathfrak{p}) \otimes_R T \cong E_T(T/\mathfrak{q}).$$

(Also see [4, Lemma 3.2.5].) Furthermore, it holds

$$E_{T/\mathfrak{q}}(T/\mathfrak{q}) \cong \big(0 :_{E_T(T/\mathfrak{q})} \mathfrak{q}\big) \cong (0 :_{F_\mathfrak{n}} \mathfrak{q})$$

by the above isomorphisms. (Also see [3, 10.1.15 Lemma].) Since F is a finitely generated R-module, $F_\mathfrak{n}$ is so as T-module. Thus $E_{T/\mathfrak{q}}(T/\mathfrak{q})$ is a finitely generated T-module. Consequently, we see that $E_{T/\mathfrak{q}}(T/\mathfrak{q})$ is a finitely generated as $T/\mathfrak{q}$-module.

A local domain $T/\mathfrak{q}$ is $\dim T/\mathfrak{q} \geq 1$ and has a finitely generated injective $T/\mathfrak{q}$-module $E_{T/\mathfrak{q}}(T/\mathfrak{q})$. So it follows from the Bass formula that it holds

$$0 < \mathrm{depth}_{T/\mathfrak{q}}\, T/\mathfrak{q} = \mathrm{inj}\dim_{T/\mathfrak{q}} E_{T/\mathfrak{q}}(T/\mathfrak{q}) = 0.$$

This is a contradiction.

(2) We note that any minimal element in $\mathrm{Supp}(M)$ is in $\mathrm{Ass}(M)$ for any (not necessarily finitely generated) R-module M. (e.g. see [5, Theorem 2.4.12].)

We assume that R is a 2-dimensional local domain and have to show that a Serre subcategory $(\mathcal{S}_{f.g.}, \mathcal{M}_{f.s.})$ satisfies the condition C_I for all ideals I of R. We fix an ideal I of R. We suppose that X is an R-module such that $X = \varGamma_I(X)$ and $(0 :_X I)$ is in $(\mathcal{S}_{f.g.}, \mathcal{M}_{f.s.})$, and shall show that X is in $(\mathcal{S}_{f.g.}, \mathcal{M}_{f.s.})$. There exists a short exact sequence

$$0 \to F \to (0 :_X I) \to M \to 0$$

with $F \in \mathcal{S}_{f.g.}$ and $M \in \mathcal{M}_{f.s.}$.

In the case of $\dim (0 :_X I) \leq 1$. Then it holds $\mathrm{Supp}(X) = \mathrm{Ass}(X) \cup \{\mathfrak{m}\}$. Indeed, since it holds $\mathrm{Ass}(X) = \mathrm{Ass}\big((0 :_X I)\big)$, it is easy to see that the zero ideal (0) of R does not belong to $\mathrm{Supp}(X)$. Therefore, if there exists a prime ideal $\mathfrak{p} \in \mathrm{Supp}(X) \setminus \{\mathfrak{m}\}$, $\mathfrak{p}$ is minimal in $\mathrm{Supp}(X)$.

Thus $\mathfrak{p}$ is in $\mathrm{Ass}(X)$, so we see that the above equality holds. On the other hand, it holds

$$\begin{aligned}\mathrm{Ass}(X) &= \mathrm{Ass}\big((0:_X I)\big)\\ &\subseteq \mathrm{Ass}(F)\cup \mathrm{Ass}(M)\\ &\subseteq \mathrm{Ass}(F)\cup \mathrm{Supp}(M).\end{aligned}$$

Since F is a finitely generated R-module and M is in $\mathcal{M}_{f.s.}$, $\mathrm{Ass}(X)$ is finite set. Consequently, $\mathrm{Supp}(X)$ is also finite set, so we see that X is in $\mathcal{M}_{f.s.} \subseteq (\mathcal{S}_{f.g.}, \mathcal{M}_{f.s.})$.

In the case of $\dim(0:_X I) = 2$. Since R is a 2-dimensional domain, the zero ideal (0) of R must be in $\mathrm{Supp}\big((0:_X I)\big)$ and this is a minimal in $\mathrm{Supp}\big((0:_X I)\big)$. It follows that

$$(0) \in \mathrm{Ass}\big((0:_X I)\big) = V(I) \cap \mathrm{Ass}(X) \subseteq V(I).$$

Therefore, it holds $I = (0)$. Consequently, $X = (0:_X I)$ is in $(\mathcal{S}_{f.g.}, \mathcal{M}_{f.s.})$.

The proof is completed. □

Remark 3.2. If $(R, \mathfrak{m})$ is a local ring with at most one dimension, then $\mathrm{Spec}(R)$ is finite set. Thus, any support of R-module is finite set, so we see $(\mathcal{S}_{f.g.}, \mathcal{M}_{f.s.}) = R$-Mod. Therefore, in this case, $(\mathcal{S}_{f.g.}, \mathcal{M}_{f.s.})$ is a Melkersson subcategory and is closed under injective hulls.

Example 3.3. Let R be the ring of formal power series $k[[x, y]]$ in the indeterminate x and y with the coefficients in a field k. Then R is a 2-dimensional local domain and has infinite many prime ideals $(x + y^n)$ for each non-negative integer n. Thus, in this case, $(\mathcal{S}_{f.g.}, \mathcal{M}_{f.s.})$ is a Melkersson subcategory which is not closed under injective hulls by Theorem 3.1.

4. Several remarks on Melkersson subcategories

In this section, we assume that any full subcategory contains a non-zero R-module.

In a local ring R, it is clear that any Serre subcategory of R-Mod contains all finite length modules. On the other hand, we can see the following assertion holds.

Proposition 4.1. *Let $(R, \mathfrak{m})$ be a local ring and $\mathcal{M}$ be a Melkersson subcategory with respect to $\mathfrak{m}$. Then any Artinian module is in $\mathcal{M}$. In particular, Melkersson subcategory contains all Artinian modules.*

Proof. Let $\mathcal{M}$ be a Melkersson subcategory with respect to $\mathfrak{m}$. Since all finite length R-modules belong to any Serre subcategory, we can see that the injective hull $E_R(R/\mathfrak{m})$ of $R/\mathfrak{m}$ belongs to $\mathcal{M}$. Indeed, since it holds

$$\begin{cases} E_R(R/\mathfrak{m}) = \Gamma_{\mathfrak{m}}(E_R(R/\mathfrak{m})) \ \text{ and} \\ (0:_{E_R(R/\mathfrak{m})} \mathfrak{m}) \cong \mathrm{Hom}_R(R/\mathfrak{m}, E_R(R/\mathfrak{m})) = R/\mathfrak{m} \ \text{ is in } \ \mathcal{M},\end{cases}$$

it follows from the condition $C_{\mathfrak{m}}$ that $E_R(R/\mathfrak{m})$ is in $\mathcal{M}$.

Let M be an Artinian module. Then M is embedded in $\oplus^n E_R(R/\mathfrak{m})$ for some integer n. Therefore, since Melkersson subcategory is closed under finite direct sums and submodules, we see that M is in $\mathcal{M}$. □

To see whether Serre subcategory is Melkersson subcategory, we have only to check that it satisfies the condition C_I for all radical ideals I of R.

Proposition 4.2. *Let $\mathcal{M}$ be a Serre subcategory. Then following conditions are equivalent:*

(1) *$\mathcal{M}$ is a Melkersson subcategory;*

(2) *$\mathcal{M}$ is a Melkersson subcategory with respect to $\sqrt{I}$ for all ideals I of R.*

Proof. We assume that $\mathcal{M}$ is a Melkersson subcategory with respect to $\sqrt{I}$ for all ideals I of R. Let I be an ideal of R and shall show that $\mathcal{M}$ satisfies condition C_I. We suppose that M is an R-module such that $M = \varGamma_I(M)$ and $(0 :_M I)$ is in $\mathcal{M}$. Then it holds $\varGamma_{\sqrt{I}}(M) = \varGamma_I(M) = M$. Furthermore, since $\mathcal{M}$ is closed under submodules and $(0 :_M \sqrt{I}) \subseteq (0 :_M I)$, we see $(0 :_M \sqrt{I})$ is in $\mathcal{M}$. If follows from the condition $C_{\sqrt{I}}$ that M is in $\mathcal{M}$. □

Serre subcategory is defined not only in the category R-Mod but also in the category R-mod. Therefore, it stands to reason that we consider the Melkersson subcategory of R-mod which is defined by considering the condition C_I for only finitely generated R-modules as follows: the Serre subcategory $\mathcal{M}$ of R-mod is Melkersson subcategory of R-mod if it satisfies the condition

$$(C_I) \quad \text{If } M = \varGamma_I(M) \in R\text{-mod and } (0 :_M I) \text{ is in } \mathcal{M}, \text{ then } M \text{ is in } \mathcal{M}$$

for all ideal I of R. However, by the following proposition, we can see that it is not necessary to treat Serre subcategory which satisfies such a condition specially.

Proposition 4.3. *Any Serre subcategory $\mathcal{S}$ of R-mod is a Melkersson subcategory of R-mod in the above sense.*

Proof. By [6, Theorem 4.1], there exists a specialization closed subset W of $\mathrm{Spec}(R)$ corresponding to the Serre subcategory $\mathcal{S}$. In particular, we can denote

$$\mathcal{S} = \{M \in R\text{-mod} \mid \mathrm{Supp}(M) \subseteq W\} \text{ and } W = \bigcup_{M \in \mathcal{S}} \mathrm{Supp}(M).$$

Let I be an ideal of R. We suppose that M is a finitely generated R-module such that $M = \varGamma_I(M)$ and $(0 :_M I)$ is in $\mathcal{S}$. Since $(0 :_M I)$ is in $\mathcal{S}$, it holds $\mathrm{Ass}(M) = \mathrm{Ass}((0 :_M I)) \subseteq \mathrm{Supp}((0 :_M I)) \subseteq W$, and so we have $\mathrm{Supp}(M) \subseteq W$. Consequently, M is in $\mathcal{S}$. □

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