COUNTING REAL CUBICS WITH PASSAGE/TANGENCY CONDITIONS

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ABSTRACT. We study the following question: given a set $\mathcal{P}$ of seven points and an immersed curve Γ in the real plane $\mathbb{R}^2$, all in general position, how many real rational nodal plane cubics pass through these points and are tangent to this curve. We count each such cubic with a certain sign, and present an explicit formula for their algebraic number. This number is preserved under small regular homotopies of a pair $(\mathcal{P}, \Gamma)$ but jumps (in a well-controlled way) when in the process of homotopy we pass a certain singular discriminant. We discuss the relation of such enumerative problems with finite type invariants. Our approach is based on maps of configuration spaces and the intersection theory in the spirit of classical algebraic topology.

1. INTRODUCTION

1.1. **History.** A classical problem in enumerative geometry is the study of the number of certain algebraic curves of degree d passing through some number of points in the affine or projective plane. This question is not very interesting if we consider all curves of degree d, due to the fact that the set of such curves forms the projective space of dimension $\frac{1}{2}d(d+3)$, so the question of passing through points is simply a question of solving a system of linear equations. Thus, one usually asks this question about some families of algebraic curves of degree d, e.g., curves of a fixed genus g. The simplest case is $g = 0$, i.e. rational curves of degree d. In the complex (resp. real) case the set of rational curves of degree d forms a smooth algebraic variety of complex (resp. real) dimension $3d - 1$, called the Severi variety. There is an old question of determining the number N_d (resp. $N_d(\mathbb{R})$) of rational curves of degree d passing through $3d - 1$ points in general position in the complex (resp. real) projective plane.

The numbers $N_1 = N_2 = N_1(\mathbb{R}) = N_2(\mathbb{R}) = 1$ go back to antiquity; $N_3 = 12$ was computed by J. Steiner in 1848. The late 19-th century was the golden era for enumerative geometry, and H.G. Zeuthen in 1873 could

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compute the number $N_4 = 620$. By then, the art of resolving enumerative problems had attained a very high degree of sophistication and, in fact, its foundations could no longer support it. Hilbert asked for rigorous foundation of enumerative geometry, including it as the 15-th problem in his list.

The 20-th century witnessed great advances in intersection theory. In the seventies and eighties, a lot of old enumerative problems were solved and many classical results were verified. However, the specific question of determining the numbers N_d turned out to be very difficult. In fact, in the eighties only one more of the numbers was unveiled: the number of quintics $N_5 = 87304$.

The revolution took place around 1994 when a connection between theoretical physics (string theory) and enumerative geometry was discovered. As a corollary, M. Kontsevich and Yu. Manin in [10] (see also [4]) gave a solution in terms of a recursive formula

$$N_d = \sum_{d_1+d_2=d,\ d_1,d_2>0} N_{d_1}N_{d_2}\left(d_1^2d_2^2\binom{3d-4}{3d_1-2} - d_1^3d_2\binom{3d-4}{3d_1-1}\right).$$

But all these advances were done in the complex algebraic geometry. In the real case the situation is different. Until 2000 nothing was known about $N_d(\mathbb{R})$ for $d \geq 3$. In 2000 A. Degtyarev and V. Kharlamov showed that $N_3(\mathbb{R})$ may be $8, 10$ or 12, depending on the configuration of $8 = 3 \cdot 3 - 1$ points in the plane $\mathbb{RP}^2$. This result reflects a general problem of a real enumerative geometry: such numbers are usually not constant, but do depend on a configuration of geometrical objects. A natural way to overcome this difficulty is to try to assign some signs and multiplicities to objects in question so that the corresponding algebraic numbers remain constant. Already in the work of A. Degtyarev and V. Kharlamov one can see that one can assign certain multiplicities (signs) to real cubics through a given 8 points, so that the weighted sum of these cubics is independent on the configuration of points. In 2003 J.-Y. Welschinger found the way to assign multiplicities to real curves of any degree. Welschinger's main theorem is the following: if we assign to each real curve C the multiplicity $(-1)^{m(C)}$, where $m(C)$ is the number of solitary points of C, called the mass of C, then the corresponding weighted sum W_d of all curves through the given points is independent of the choice of points. The number W_d is called the Welschinger's invariant. In particular, $|W_d|$ gives a lower bound for the actual number $N_d(\mathbb{R})$ of real rational plane curves passing through any given set of $3d-1$ generic points. In the case of cubics ($d = 3$) from the Degtyarev-Kharlamov theorem one can see that $W_3 = 8$.

The questions of passing through some number of points is the simplest one. The next step is to ask about the number of algebraic curves passing through some number of points and tangent to some number of another algebraic curves. In the complex case in 1996 L. Caporaso and J. Harris found the recursive formulas in the spirit of M. Kontsevich for such tangency questions. In the real case that kind of questions of tangency is quite new and the serious development is just beginning.

1.2. **Motivation.** We are interested to merge rigid algebro-geometric objects with flexible objects from smooth topology. We consider algebraic curves that pass through some number of points and tangent to a smooth immersed curve.

As a toy model one may consider the case $d = 1$. Let L be a set of lines in $\mathbb{R}^2$ passing through a fixed point p and tangent to a (generic) oriented immersed plane curve Γ. The problem is to introduce a sign ε_l for each such line $l \in L$ so, that the total algebraic number $N = \sum_{l \in L} \varepsilon_l$ of lines does not change under homotopy of Γ in $\mathbb{R}^2 \setminus p$. It is easy to guess such a sign rule. Indeed, under a deformation shown in Figure 1a, two new lines appear, so their contributions to N should cancel out. Thus, their signs should be opposite and we get the sign rule shown in Figure 1b. Note that only the orientation of Γ is used to define it; l is not oriented.

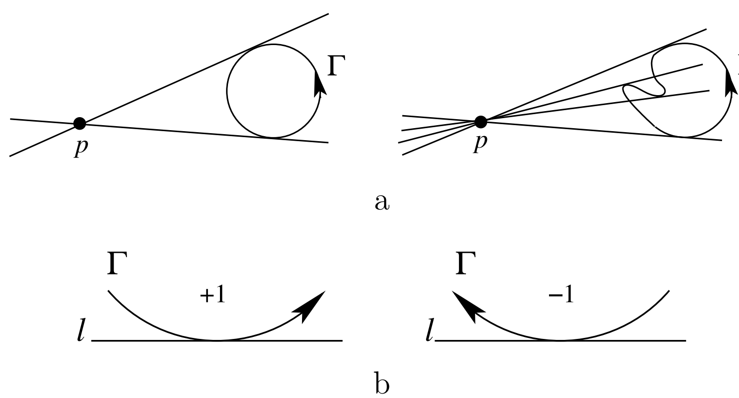
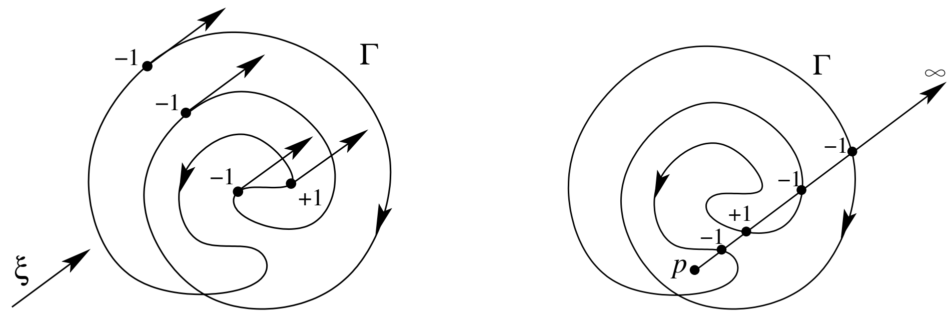


FIGURE 1. Counting lines with signs.

Suppose that p is close to infinity (i.e., lies in the unbounded region of $\mathbb{R}^2 \setminus \Gamma$). In this case we get $N = 2\,\mathrm{ind}(\Gamma)$, where $\mathrm{ind}(\Gamma)$ is the Whitney index (a.k.a. rotation number) of Γ, i.e. the number of turns made by the tangent vector as we pass once along Γ following the orientation. In other words, the Whitney index $\mathrm{ind}(\Gamma)$ equals to the degree of the Gauss map $G_\Gamma : \mathbb{S}^1 \to \mathbb{S}^1$ given by $G_\Gamma(t) = \dfrac{\gamma'(t)}{\|\gamma'(t)\|}$, where $\gamma : \mathbb{S}^1 \looparrowright \mathbb{R}^2$ is a parametrization of Γ. Hence, the Whitney index can be calculated as an algebraic number of preimages of a regular value $\xi \in \mathbb{S}^1$ of the Gauss map G_Γ, see Figure 2a.

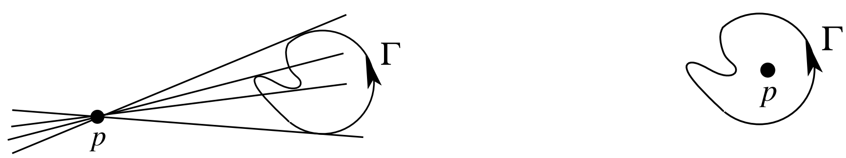


$\mathrm{ind}(\Gamma) = -1 - 1 - 1 + 1 = -2$ $\qquad$ $\mathrm{ind}_p(\Gamma) = -1 + 1 - 1 - 1 = -2$

a $\qquad$ b

FIGURE 2. Whitney index of a curve and an index of a point w.r.t. to a curve.

While N is preserved under homotopies of Γ in $\mathbb{R}^2 \setminus p$, it changes when Γ passes through p, see Figure 3a,b.



$N = +2$ $\qquad$ $N = 0$

a $\qquad$ b

FIGURE 3. Counting lines with signs.

The compensating term is also easy to guess and we finally obtain

$$N = 2\,\mathrm{ind}(\Gamma) - 2\,\mathrm{ind}_p(\Gamma)\ . \tag{1}$$

Here the index $\mathrm{ind}_p(\Gamma)$ of p w.r.t. Γ is the number of turns made by the vector connecting p to a point $x \in \Gamma$, as x passes once along Γ following the orientation. It may be computed as the intersection number $I([p,\infty],\Gamma;\mathbb{R}^2)$ of a 1-chain $[p,\infty]$ (i.e. an interval connecting p with a point near infinity of $\mathbb{R}^2$) with an oriented 1-cycle Γ in $\mathbb{R}^2$. See Figure 2b.

The appearance of $\mathrm{ind}(\Gamma)$ and $\mathrm{ind}_p(\Gamma)$ in the above formula comes as no surprise: in fact, these are the only invariants of the curve Γ under its homotopy in the class of immersions in $\mathbb{R} \setminus p$. These are the simplest finite type invariants of plane curves, see [1].

In this simple example we see two main distinctive differences of real enumerative problems vs. complex problems of a similar passage/tangency type. Firstly, in the real case we are to count algebraic curves under the consideration with signs. Secondly, over $\mathbb{C}$ the answer is a number which does not depend on the relative position of the set of points and the curve Γ. Over $\mathbb{R}$, however, the answer depends on the configuration: it is preserved under small deformations of Γ and the set of points, but experiences certain (well-controlled) jumps when the configuration crosses certain singular discriminant in the process of homotopy. Thus, in the general case for similar enumerative problems we should not expect to get an answer as one number, but rather as a collection of numbers, depending on the relative configuration of points and the smooth curve.

Two main questions in this kind of problems are

1. How to find such sign rules, i.e. how to assign a certain sign to each algebraic curve under consideration, so that the total algebraic number is invariant under small deformations?
2. How does the singular discriminant looks like, and what is the explicit structure of the formula for the algebraic number of curves?

1.3. **Main results and the structure of the paper.** In the present work we study the question of the algebraic number of real plane rational nodal cubics (curves of degree 3) passing through the set $\mathcal{P}$ of seven generic points and tangent to a generic immersed curve Γ in the plane $\mathbb{R}^2$. Degree $d = 3$ is the first case when all general difficulties appear, namely the number $N_3(\mathbb{R})$ is different from one and depends on a configuration of points, and curves may have cuspidal singularities. The general case of real plane rational nodal curves of degree d may be considered in a similar manner, see Remark 2.3.

A mixture of rigid algebro-geometric objects with smooth topology gives to our problem a curious flavor, leading to a nice merging of features and techniques originating in both of these fields. In particular, this type of passage/tangency problems turns out to be intimately related to a theory of finite type invariants of plane curves, similarly to the toy case of $d = 1$ considered in Subsection 1.2 above.

We use the Welschinger's sign and show an easy way to produce new signs suitable for tangency questions. The technique of proves uses the concept of configuration spaces and the intersection theory in the spirit of classical differential topology.

The question of passage/tangency conditions for real rational plane algebraic curves was considered earlier by J.-Y. Welschinger in [14]. He considered projective curves in $\mathbb{RP}^2$ passing through a generic set $\mathcal{P}$ and tangent to a non-oriented smooth simple zero-homologous curve Γ. In [14, Remark 4.3(3)] the author suggested the generalization to the case of a non-oriented smooth immersed curve Γ, which bounds an immersed disk; unfortunately, this formula does not extend to arbitrary immersed curves, e.g. to a figure-eight curve. There is a number of differences between [14] and the present work. Firstly, we consider oriented curves in $\mathbb{R}^2$ (thus adding orientations both to the curve and to the ambient manifold). Secondly, we consider immersed curves. Finally, in contrast with [14], where the author used 4-dimensional symplectic geometry and hard-core techniques from the theory of moduli spaces of pseudo-holomorphic curves, we use usual classical tools of differential topology. In this way we also get a clear geometric interpretation of Welschinger's number w_C as the orientation of a certain surface in $ST^*\mathbb{R}^2$ (i.e. the manifold of oriented contact elements of the plane), which parameterizes real rational algebraic curves passing through $\mathcal{P}$.

The paper is organized in the following way. In Section 2 we introduce objects of our study, define signs of tangency, list the requirements of a general position, and formulate the main theorem. Section 3 is dedicated to the proofs. We interpret the desired number of cubics as a certain intersection number; the main claim follows from different ways of its calculation. In Section 4 we discuss a relation of the real enumerative geometry to finite type invariants.

2. The Statement of the Main Result. Sketch of the Proof.

There are three types of real plane rational cubics: cubics with one crossing point, cubics with one solitary point and cuspidal cubics with one cusp point. The first two types will be called rational nodal cubics. We count the algebraic number of real plane rational nodal cubics passing through seven generic points and tangent to a generic immersed curve. We get a number, which does not depend on a regular homotopy of the curve in a complement of a certain singular discriminant, see Subsection 2.1. As the curve passes through the discriminant, this number changes in a well-controlled way, so that it defines a finite type invariant of degree one, see Section 4. For this we count rational nodal cubics with signs and add certain correction terms, which come from the degenerate cases of reducible and cuspidal cubics.

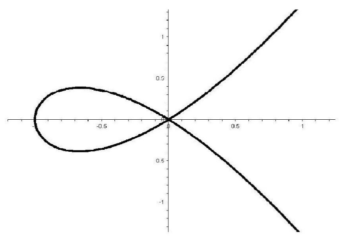
2.1. **Curves and points in general position.** Let $\mathcal{P} = \{p_1, \dots, p_7\}$, $p_i \in \mathbb{R}^2$, $i = 1, \dots, 7$ be a 7-tuple of (distinct) points in $\mathbb{R}^2$. Denote by $\mathcal{D}(\mathcal{P})$ the set of the following degenerate real plane cubics passing through $\mathcal{P}$:

- (i) connected reducible cubics passing through $\mathcal{P}$,
- (ii) cuspidal cubics passing through $\mathcal{P}$,
- (iii) real rational nodal cubics passing through $\mathcal{P}$ and having a crossing point at some point of $\mathcal{P}$.

Denote also by n_p the number of curves in (iii) having a crossing point at $p \in \mathcal{P}$.

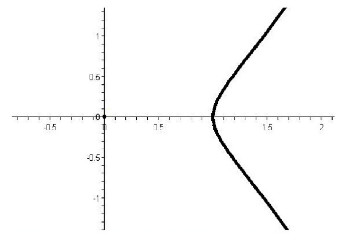
Suppose that the 7-tuple $\mathcal{P}$ is in general position. By a general position we mean the following:

1. No three points from $\mathcal{P}$ lie on one line.



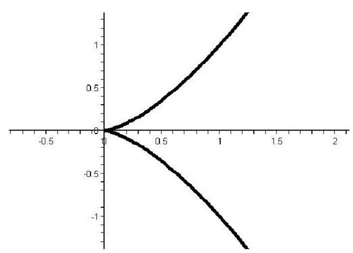
$x^3 + x^2 - y^2 = 0$

FIGURE 4. A crossing point.



$x^3 - x^2 - y^2 = 0$

FIGURE 5. A solitary point.



$x^3 - y^2 = 0$

FIGURE 6. A cuspidal cubic.

2. No six points from $\mathcal{P}$ lie on one irreducible conic.
3. Every connected reducible cubic passing through $\mathcal{P}$ is the union of a unique line through some pair of points $p_i, p_j \in \mathcal{P}, i \neq j$ and a unique non-degenerate conic passing through the remaining five points $\{p_k \in \mathcal{P} | k \neq i, j\}$ that intersect in two different points. Denote by $\mathcal{R}_{\mathcal{P}}$ the set of such intersection points. See Figure 7.[1]
4. There are finitely many cuspidal cubics passing through $\mathcal{P}$. Denote by $\mathcal{C}_{\mathcal{P}}$ the finite set of their cusps.
5. For any $p \in \mathcal{P}$ there are finitely many real rational nodal cubics passing through $\mathcal{P}$ and having a crossing point at p.

[1]The is another type of reducible cubics passing through seven generic points, namely the disjoint union of a line and a conic. We are interested only in the connected reducible cubics. See Remark 3.2.

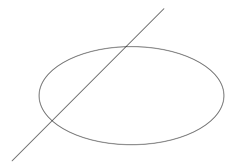


FIGURE 7. Intersection of a line with a conic.

6. All intersection points of curves from $\mathcal{D}(\mathcal{P})$ are transversal points, and all of them but from $\mathcal{P}$ are transversal double points.

Now, let Γ be a generic immersed oriented curve in $\mathbb{R}^2$ in general position w.r.t. $\mathcal{P}$. Let us spell this requirement in more details. By a general position we mean that

7. The curve Γ is generically immersed, i.e., it is a smooth curve with a finite number of double points of transversal self-intersection as the only singularities.
8. The curve Γ intersects each of the cubics from the set $\mathcal{D}(\mathcal{P})$ transversally, in points which do not belong to $\mathcal{P} \cup \mathcal{C}_\mathcal{P} \cup \mathcal{R}_\mathcal{P}$.
9. Every cubic passing through $\mathcal{P}$ is tangent to Γ at most at one point, with the tangency of the first order.

Define the singular discriminant Δ as the set of pairs $(\mathcal{P}, \Gamma)$ that violate the general position requirements listed above.

Remark 2.1. *The number of curves in $\mathcal{D}(\mathcal{P})$ and points in $\mathcal{C}_\mathcal{P}, \mathcal{R}_\mathcal{P}$ are bounded from above as follows. The number of points in $\mathcal{R}_\mathcal{P}$ is no more than $\binom{7}{2} = 21$. Due to* [9], *there are at most* 24 *cuspidal cubics passing through seven points in general position in $\mathbb{CP}^2$, hence $\#\mathcal{C}_\mathcal{P} \leq 24$. From* [12, Theorem 3.2] *one can deduce that $n_p \in \{0, 1\}$.*

2.2. **Signs of points and cubics.** We fix the standard orientation $o_{\mathbb{R}^2}$ on the plane $\mathbb{R}^2$ once and for all. Let us denote $\mathfrak{S} := \mathcal{P} \cup \mathcal{C}_\mathcal{P} \cup \mathcal{R}_\mathcal{P}$. For each $p \in \mathfrak{S}$ we define ι_p by

$$\iota_p = \begin{cases} -8 + 2n_p & \text{if } p \in \mathcal{P}, \\ -1 & \text{if } p \in \mathcal{C}_\mathcal{P}, \\ +1 & \text{if } p \in \mathcal{R}_\mathcal{P}. \end{cases}$$

Denote by $\mathcal{M} := \mathcal{M}^{\Gamma,\mathcal{P}}_{3,\mathrm{rat},\mathrm{nod}}(\mathbb{R})$ the set of real rational plane nodal cubics passing through $\mathcal{P}$ and tangent to Γ. To each $C \in \mathcal{M}$ we assign a sign $\varepsilon_C = w_C \cdot \tau_C$, where

- $w_C = (-1)^{m(C)}$, where $m(C)$ is the mass of C, i.e. the number of solitary points of C. For a cubic C we have $m(C) \in \{0, 1\}$. The number w_C is called the Welschinger's number of C.

- τ_C is a sign of tangency of C with Γ, which is defined similarly to Subsection 1.2 as follows:
  Let p be the point of tangency of Γ with C. For a sufficiently small radius r, C divides the ball $B(p,r)$ into two parts. Since the tangency of Γ and C is of the first order, their quadratic approximations at this point p differ. Hence, $\Gamma \cap B(p,r)$ belongs to the closure of one of the two parts of $B(p,r) \setminus C$. Let n be a normal vector to C at p, which looks into the closure of the part which contains Γ, and let t be the tangent vector t to Γ at p. Set $\tau_C = +1$ if the frame (t,n) defines the positive orientation $o_{\mathbb{R}^2}$ of $\mathbb{R}^2$, and $\tau_C = -1$ otherwise. See Figure 8 (compare also with Figure 1b).

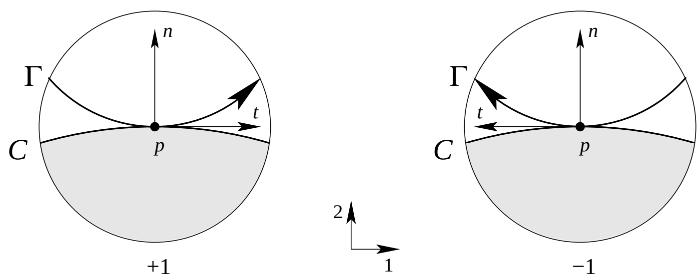


FIGURE 8. Signs of tangency τ_C.

Note that while the immersed curve Γ is oriented, the algebraic curve C is not, and we use just the orientation of Γ in order to define the sign τ_C.

2.3. **The statement of the main result.** Let $N := N^{\Gamma,\mathcal{P}}_{3,\mathrm{rat},\mathrm{nod}}(\mathbb{R})$ be the algebraic number

$$N = \sum_{C \in \mathcal{M}} \varepsilon_C$$

of real rational nodal plane cubics passing through $\mathcal{P}$ and tangent to Γ.

The main result in this work is the following

Theorem 2.2. *Let $\mathcal{P} = \{p_1, \dots p_7\} \subseteq \mathbb{R}^2$ and Γ be an immersed oriented curve in $\mathbb{R}^2$, all in general position. Then*

$$N = 2 \cdot \left(8 \cdot \mathrm{ind}(\Gamma) + \sum_{p \in \mathfrak{S}} \iota_p \cdot \mathrm{ind}_p(\Gamma) \right). \tag{2}$$

The number N is invariant under a regular homotopy of the pair $(\mathcal{P}, \Gamma)$ in (each connected component of) the complement of the singular discriminant Δ.

Remark 2.3. *One can consider the case of algebraic curves of a higher degree $d \geq 4$ in a similar way. The number $8 = W_3$ will be replaced by W_d and the formula becomes*

$$N^{\Gamma,\mathcal{P}}_{d,\mathrm{rat,nod}}(\mathbb{R}) = 2 \cdot \left(W_d \cdot \mathrm{ind}(\Gamma) + \sum_{p \in \mathfrak{S}} \iota_p \cdot \mathrm{ind}_p(\Gamma) \right). \tag{3}$$

The set $\mathfrak{S}$ is again the union $\mathfrak{S} := \mathcal{P} \cup \mathcal{C}_{\mathcal{P}} \cup \mathcal{R}_{\mathcal{P}}$, where $\mathcal{P}$ is the set of $3d-2$ generic (distinct) points in $\mathbb{R}^2$ and $\mathcal{C}_{\mathcal{P}}$ (resp. $\mathcal{R}_{\mathcal{P}}$) is the finite set of singular points of cuspidal (resp. reducible) real plane algebraic curves of degree d passing through $\mathcal{P}$. Weights ι_p are defined as follows

$$\iota_p = \begin{cases} -W_d + 2 \cdot \sum\limits_{m=0}^{\delta} (-1)^m \cdot n_p^d(m) & \text{if } p \in \mathcal{P}, \\ -(-1)^{m(C)} & \text{if } p \in \mathcal{C}_{\mathcal{P}}, \\ (-1)^{m(C)} & \text{if } p \in \mathcal{R}_{\mathcal{P}}, \end{cases}$$

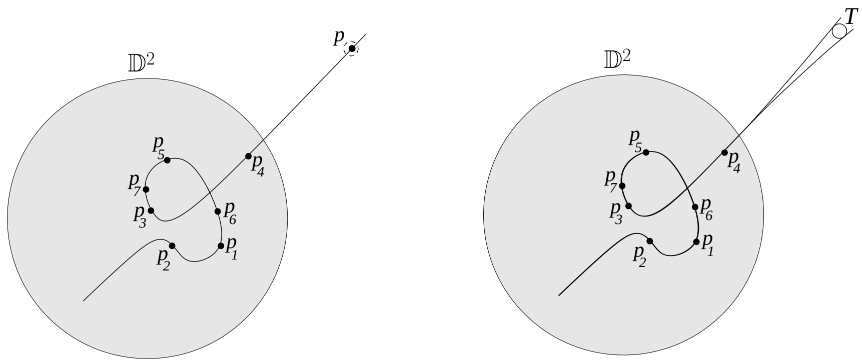
where

- $\delta := \dfrac{(d-1)(d-2)}{2}$ *is the maximal number of nodal points of a rational plane algebraic curve of degree d,*
- $n_p^d(m)$ *is the total number of real rational nodal plane algebraic curves of degree d and of mass m passing through $\mathcal{P}$ and having a crossing point at p,*
- $m(C)$ *is the mass of (the unique) cuspidal or reducible curve of degree d passing through $\{p\} \cup \mathcal{P}$.*

Note that for $d \geq 4$ there are two additional finite sets of singular real rational plane algebraic curves of degree d passing through $\mathcal{P}$. Namely, curves having a unique tacnode (A_3-type singularity), and curves having a unique ordinary triple point (D_4-type singularity). By considering the bifurcation diagrams of tacnodes and triple points, it can be easily verified that these singularities do not contribute to the formula (3). *See e.g. figures in* [12, Pages 225 – 226].

2.4. **The main example.** Firstly, consider $\Gamma = T$, where $T = \partial\mathbb{D}(p, r)$ is a circle of infinitesimally small radius $0 < r << 1$ in $\mathbb{R}^2$, centered at p. Suppose that T is far from $\mathfrak{S}$, i.e, T is in the complement of some closed disk $\mathbb{D}^2$ which contains $\mathfrak{S}$. See Figure 9a.

Viewing T as a point p, i.e. taking the limit $r \to 0$, we get eight generic points $\{p\} \cup \mathcal{P}$ in the plane. We have eight real plane rational nodal cubics passing through $\{p\} \cup \mathcal{P}$, counted with their Welschinger's signs, see [3]. Thus the algebraic number N of real plane rational nodal cubics passing through



a b

FIGURE 9. Changing a point into a circle.

$\mathcal{P}$ and tangent to T counted with the sign ε_C is equal to $2 \cdot 8 = 16$. Indeed, each rational nodal cubic passing through $\{p\} \cup \mathcal{P}$ gives 2 rational nodal cubics passing through $\mathcal{P}$ and tangent to T, see Figure 9b. Moreover, from the definition of the sign τ_C we have that $\tau_C = +1$ for any $C \in \mathcal{M}$. Thus in this case

$$N = \sum_{C \in \mathcal{M}} w_C \cdot \tau_C = \sum_{C \in \mathcal{M}} w_C = 2 \cdot \left(\sum_{C \text{ passes through } \{p\} \cup \mathcal{P}} w_C \right) = 2 \cdot 8 = 16.$$

Reparameterizing the circle T by $\mathbb{S}^1 \to \mathbb{S}^1,\ z \mapsto z^k,\ k \in \mathbb{Z}$ (and deforming it slightly into a general position) we get a curve denoted by $k \cdot T$ for which we have $\operatorname{ind}(k \cdot T) = k$ and

$$N = 2k \cdot 8 = 16 \cdot k.$$

Since every immersed curve Γ is homotopic in the class of immersions in $\mathbb{R}^2$ to $k \cdot T$, where $k = \operatorname{ind}(\Gamma)$, we have that $N = 16 \cdot \operatorname{ind}(\Gamma)$ for a curve Γ lying in the complement of some closed disk of a sufficiently large radius, which contains $\mathfrak{S}$.

2.5. **The idea of the proof.** Consider a solid torus $M = \mathbb{D}^2 \times \mathbb{S}^1$, where $\mathbb{D}^2$ is a sufficiently large closed disk containing $\mathfrak{S}$. We will show that the number N in Theorem 2.2 is the intersection number $I(L, \overline{\Sigma}; M)$ of an oriented smooth curve L with a compactification $\overline{\Sigma}$ of an open two-dimensional surface Σ in M. The surface Σ is constructed as follows:
For each $p \in \mathbb{D}^2 \smallsetminus \mathfrak{S}$, we use a contact element (line) of cubics passing through $\{p\} \cup \mathcal{P}$ to get Σ as a lift of $\mathbb{D}^2 \smallsetminus \mathfrak{S}$ into M. Lifting Γ into M in a similar way we get L. The Welschinger's sign w_C gives rise to the orientation on Σ and the orientation of Γ defines the orientation of L.

In order to define the intersection number, we compactify Σ to get a compact surface $\overline{\Sigma}$ with boundary. This is done by blowing up punctures $\mathfrak{S}$ on $\mathbb{D}^2$, i.e., we cut out a small open disk around each puncture and then we lift the remaining domain into M. Due to generality of a pair $(\Gamma, \mathcal{P})$, L transversally intersects $\overline{\Sigma}$ in a finite number of points (in the interior $int(\Sigma)$ of $\overline{\Sigma}$). Each point $(p, \xi) \in L \pitchfork \overline{\Sigma}$ corresponds to a cubic passing through $\mathcal{P}$ and tangent to Γ. We prove that the local intersection number $I_{(p,\xi)}(L, \overline{\Sigma}; M)$ equals to $\tau_C \cdot w_C$, and thus

$$N = I(L, \overline{\Sigma}; M).$$

Now to get the right hand side of the formula (2) we use the homological interpretation of the intersection number. We take $\Gamma' := \mathrm{ind}(\Gamma) \cdot T$ as in the main example, see Subsection 2.4, so Γ' is homotopic to Γ in the class of immersions. Hence $[\Gamma] - [\Gamma'] = \partial K$ in $C_1(\mathbb{D}^2; \mathbb{Z})$ for some 2-chain $K \in C_2(\mathbb{D}^2; \mathbb{Z})$. Then for the lifts L' and $\mathcal{K}$ of Γ' and K, respectively, into M we have $[L] - [L'] = \partial\mathcal{K}$ in $C_1(M; \mathbb{Z})$, and hence

$$I(L, \overline{\Sigma}; M) = I(L', \overline{\Sigma}; M) + I(\partial\mathcal{K}, \overline{\Sigma}; M).$$

From the main example we obtain

$$I(L', \overline{\Sigma}; M) = 16 \cdot \mathrm{ind}(\Gamma') = 16 \cdot \mathrm{ind}(\Gamma).$$

Finally, we show that

$$I(\mathcal{K}, \partial\overline{\Sigma}; M) = 2 \sum_{p \in \mathfrak{S}} \iota_p \cdot \mathrm{ind}_p(\Gamma).$$

We use the equality

$$I(\partial\mathcal{K}, \overline{\Sigma}; M) = I(\mathcal{K}, \partial\overline{\Sigma}; M)$$

to complete the proof.

Remark 2.4. *A simple way to visualize the surface Σ is to apply the above construction to the model example of Section 1.2. In this case $\mathfrak{S}$ consists of one point p and the contact element of any line passing through p is the line itself (see Figure 18), so the surface is a helicoid, see Figure 13.*

3. The proof of the main result.

The manifold of oriented contact elements (directions) of the plane is $ST^*\mathbb{R}^2$, the spherization of the cotangent bundle of the plane. We fix an orientation $o_{ST^*\mathbb{R}^2} = o_{\mathbb{R}^2} \times o_{\mathbb{S}^1}$ on $ST^*\mathbb{R}^2$, where $o_{\mathbb{S}^1}$ is the standard counterclockwise orientation on $\mathbb{S}^1$.

3.1. **Construction of** $M, \Sigma, \overline{\Sigma}, L$.

Construction of Σ. Consider a 7-tuple $\mathcal{P} = \{p_1, \dots, p_7\}$ of points in $\mathbb{R}^2$ in general position. Recall that $\mathfrak{S} := \mathcal{P} \cup \mathcal{C}_{\mathcal{P}} \cup \mathcal{R}_{\mathcal{P}}$, see Subsections 2.1–2.2. Let $S := \mathbb{R}^2 \smallsetminus \mathfrak{S}$, let S_{sing} be the set of points in S, lying on curves in $\mathcal{D}(\mathcal{P})$, and let $S_{reg} := S \smallsetminus S_{sing}$. Define

$$\Sigma_{reg} := \left\{ (p,\xi) \in ST^*(\mathbb{R}^2 \smallsetminus \mathfrak{S}) \,\middle|\, \begin{array}{l} \text{there is a nodal cubic passing} \\ \text{through } \{p\} \cup \mathcal{P} \text{ and having } \xi \\ \text{as a tangent direction at a point } p \end{array} \right\},$$

$$\Sigma_{sing} := \left\{ (p,\xi) \in ST^*(\mathbb{R}^2 \smallsetminus \mathfrak{S}) \,\middle|\, \begin{array}{l} \text{there is a cubic in } \mathcal{D}(\mathcal{P}) \text{ passing} \\ \text{through } \{p\} \cup \mathcal{P} \text{ and having } \xi \\ \text{as a tangent direction at a point } p \end{array} \right\}.$$

Denote $\Sigma := \Sigma_{reg} \cup \Sigma_{sing}$, and let $\pi : \Sigma \to S$ be the natural projection $\pi((p,\xi)) = p$.

Proposition 3.1. *The set Σ is a non-compact two-dimensional surface in $ST^*\mathbb{R}^2$ with the singular subset Σ_{sing}. The set Σ_{reg} is a non-compact (disconnected) two-dimensional submanifold in $ST^*\mathbb{R}^2$, and the map $\pi\big|_{\Sigma_{reg}} : \Sigma_{reg} \to S_{reg}$ is a smooth covering map. The set Σ_{sing} is an open one-dimensional submanifold in $ST^*\mathbb{R}^2$.*

Proof. Consider an arbitrary $p \in S$.

- (i) If $p \in S_{reg}$, then eight points $\{p\} \cup \mathcal{P}$ are in general position and define a real general pencil of cubics, which contains 8, 10 or 12 rational nodal cubics, see [3]. They all intersect transversally in a finite number of points.
- (ii) If $p \in S_{sing}$, then eight points $\{p\} \cup \mathcal{P}$ are not in general position and define a real pencil of cubics, which (in addition to a certain number of nodal cubics) contains at least one cubic from the set $\mathcal{D}(\mathcal{P})$. They all intersect transversally in a finite number of points.

In both cases, lifting the point p to $PT^*\mathbb{R}^2$ using tangent lines of these, say, k cubics at the point p, we get k points in $PT^*\mathbb{R}^2$, which give $2k$ points in the double covering $ST^*\mathbb{R}^2$ of $PT^*\mathbb{R}^2$.

So $\pi\big|_{\Sigma_{reg}} : \Sigma_{reg} \to S_{reg}$ is a smooth covering. The set Σ_{sing} is a disjoint union of lifts into $ST^*\mathbb{R}^2$ of cubics from $\mathcal{D}(\mathcal{P})$. Since all of them intersect transversally in a finite number of points, each component of Σ_{sing} is a one-dimensional manifold. The structure of Σ in a small neighborhood of $(p,\xi) \in \Sigma_{sing}$ depends on the type of the point p.

Namely, let $C \in \mathcal{D}(\mathcal{P})$ be a degenerate cubic passing through $\{p\} \cup \mathcal{P}$ and having a tangent line ξ at the point p.

If C is reducible or C is nodal c having a node at some $p_i \in \mathcal{P}$, the lift of C

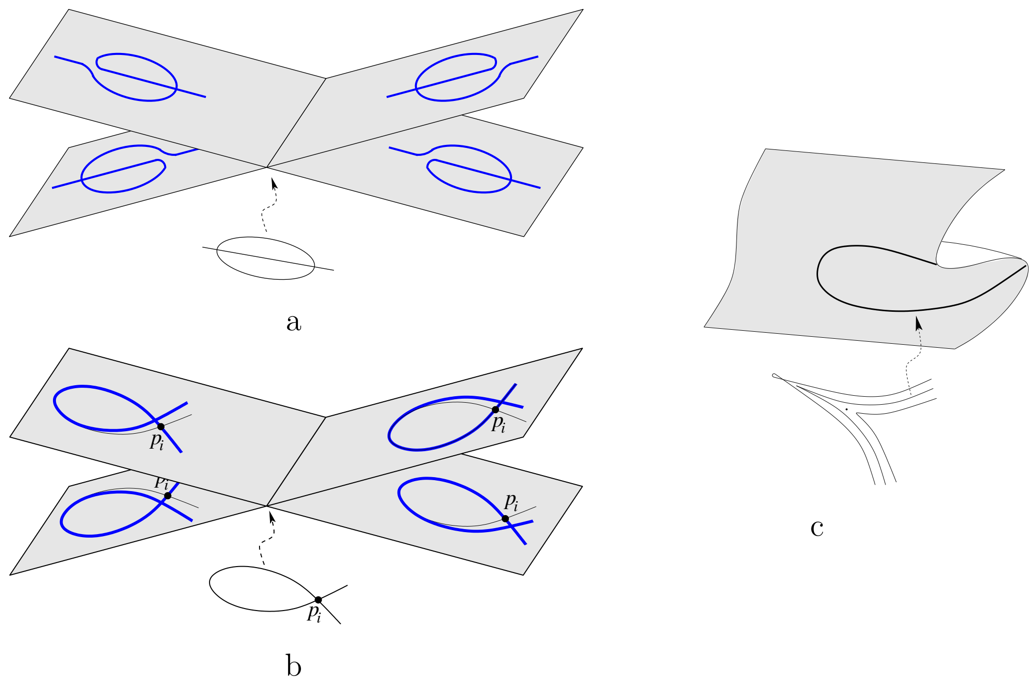


FIGURE 10. Branch points and folds of Σ.

on the surface Σ has the structure of an open book, and sheets of the book come in pairs with each pair forming a smooth surface, see Figure 10a, b. If C is cuspidal, the lift of C on the surface Σ has the structure of a fold, see Figure 10c.

□

Remark 3.2. *The lift of disconnected reducible cubics into $ST^*\mathbb{R}^2$ forms a disjoint union of smooth curves. Since a disconnected reducible cubic is not the limit (in the Gromov-Hausdorff sense) of rational cubics, this lift is disjoint from the surface Σ and hence will not contribute to the intersection number $I(L, \overline{\Sigma}; M)$.*

We define an orientation on Σ as follows.
Every point $(p, \xi) \in \Sigma_{reg}$ corresponds to a nodal cubic $C(p, \xi)$, passing through $\{p\} \cup \mathcal{P}$ with a tangent direction ξ at the point p. To define an orientation on Σ_{reg}, is suffices to indicate a continuous field ν normal to Σ_{reg}. Since $T_{(p,\xi)}\Sigma_{reg} \pitchfork T_{(p,\xi)}\mathbb{S}^1$, such a normal vector $\nu_{(p,\xi)}$ is determined by its projection to $T_{(p,\xi)}\mathbb{S}^1$. Recall, that we have already fixed the orientation $o_{\mathbb{S}^1}$ on the fiber $\mathbb{S}^1$ of $ST^*\mathbb{R}^2$. Then for a nodal cubic $C(p, \xi)$ we set the direction of $T_{(p,\xi)}\mathbb{S}^1$-component of $\nu_{(p,\xi)}$ in the direction of $o_{\mathbb{S}^1}$ if $w_{C(p,\xi)} = +1$, and opposite to this direction if $w_{C(p,\xi)} = -1$. It remains to show that the orientation extends over Σ_{sing}.

In a neighborhood of a branch point of Σ, all four sheets of the open book (which form two smooth leaves) correspond to nodal curves with the positive Welschinger's sign (see Figure 10a,b), so the normal field points in the direction of $o_{\mathbb{S}^1}$ on both smooth leaves of the open book.
In a neighborhood of a fold of Σ, there are two sheets that correspond to nodal curves with opposite Welschinger's signs (see Figure 10c), so the $T_{(p,\xi)}\mathbb{S}^1$-component of $\nu_{(p,\xi)}$ is defined to be zero.
In both cases the orientation extends over Σ_{sing}.
Compactification of $ST^*\mathbb{R}^2$ and Σ. In order to use the intersection theory, we need to compactify both the open manifold $ST^*\mathbb{R}^2$, and the non-compact surface Σ with punctures over $\mathfrak{S}$.

Let $\mathbb{D}^2 := \overline{\mathbb{D}(0,R)}, R > 0$ be a closed disk in the plane $\mathbb{R}^2$ with the center in the origin and of a sufficiently large radius, such that $\mathfrak{S} \subseteq \overline{\mathbb{D}(0,R/2)} \subseteq \mathbb{D}^2$. Define $M := ST^*\mathbb{D}^2 = \mathbb{D}^2 \times \mathbb{S}^1$.

Let us choose $0 < \delta << 1$ sufficiently small, such that

(a) $\overline{\mathbb{D}(p,\delta)} \cap \overline{\mathbb{D}(q,\delta)} = \varnothing$ for all $p \neq q \in \mathfrak{S}$,
(b) $\overline{\mathbb{D}(p,\delta)} \cap \Gamma = \varnothing$ for all $p \in \mathfrak{S}$,
(c) $\overline{\mathbb{D}(p,\delta)} \cap \partial\mathbb{D}^2 = \varnothing$ for all $p \in \mathfrak{S}$,
(d) $\overline{\mathbb{D}(p,\delta)}$ does not contain points but p of mutual intersections of all reducible and all cuspidal cubics passing through $\mathcal{P}$ for all $p \in \mathfrak{S}$,
(e) $\partial\overline{\mathbb{D}(p,\delta)}$ intersects transversally every $C \in \mathcal{D}(\mathcal{P})$ for all $p \in \mathfrak{S}$. These intersections look as shown in Figure 11a if p is a regular point of C, and as in Figure 11b–d if p is a singular point of C.

For each $p \in \mathfrak{S}$ we cut out the disk $\mathbb{D}(p,\delta)$ from $\mathbb{D}^2$ (see Figure 12) and define

$$\bar{S} := \mathbb{D}^2 \setminus \bigcup_{p\in\mathfrak{S}} \mathbb{D}(p,\delta), \qquad \overline{\Sigma} := \pi^{-1}(\bar{S}) = \Sigma \cap \left(\bar{S} \times \mathbb{S}^1\right).$$

For all $p \in \mathfrak{S}$ let $\sigma_p := \Sigma \cap (\partial\mathbb{D}(p,\delta) \times \mathbb{S}^1)$; it is the union of several (16, 20, or 24) smooth closed simple curves σ_p^i on Σ. We equip σ_p with the orientation induced from Σ. Note that σ_p may be also defined as the lift to Σ of the **clockwise-oriented** circle $\partial\mathbb{D}(p,\delta)$. See Figure 13.
Construction of L. Let $\Gamma = f(\mathbb{S}^1)$ be oriented immersed curve, where $f : \mathbb{S}^1 \looparrowright \mathbb{R}^2$ is an immersion. Choosing the unit tangent vector to $f(t)$ as the contact element, we get a lift L of Γ into $ST^*\mathbb{R}^2$:

$$L := F(\mathbb{S}^1), \qquad F : \mathbb{S}^1 \hookrightarrow ST^*\mathbb{R}^2, \qquad t \mapsto \left(f(t), \frac{f'(t)}{\|f'(t)\|}\right).$$

It follows that L is an oriented closed one-dimensional submanifold of $ST^*\mathbb{R}^2$. If Γ is generic in the sense of 2.1 then $L \cap \Sigma_{sing} = \varnothing$ and $L \pitchfork \Sigma_{reg} \neq \varnothing$.

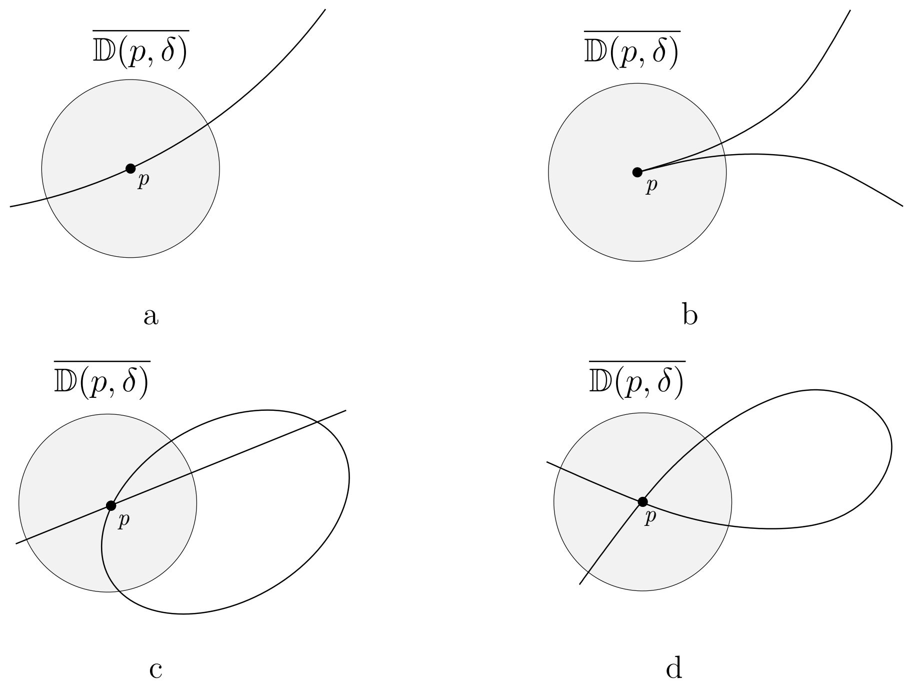


FIGURE 11. The intersection of blowup disks with degenerate cubics.

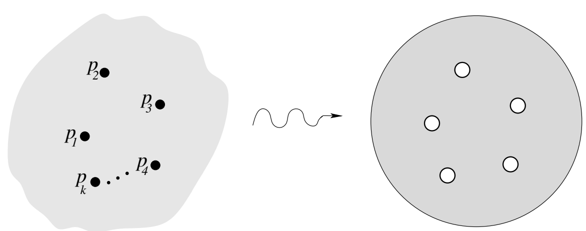


FIGURE 12. A compactification $\overline{S}$ of S.

3.2. **Two ways to calculate the intersection number $I(L, \overline{\Sigma}; M)$.** We will consider two different ways to calculate the intersection number $I(L, \overline{\Sigma}; M)$, which will correspond to the LHS and the RHS of equality (2).

The intersection number $I(L, \overline{\Sigma}; M)$ via the algebraic number N. Every point $(p, \xi) \in \overline{\Sigma} \cap L$ corresponds to a rational nodal cubic $C(p, \xi)$, passing through $\mathcal{P}$ and tangent to Γ at the point p with the tangent direction ξ. Since $\mathcal{P}$ and Γ are in general position, we have that L and $\overline{\Sigma}$ intersect transversally, and since $\dim(L) = 1$, $\dim(\overline{\Sigma}) = 2$ and $\dim(M) = 3$, we have that $\dim(L \pitchfork \overline{\Sigma}) = 0$. So the number of points in $L \pitchfork \overline{\Sigma}$ is finite. Both L and $\overline{\Sigma}$ are oriented as is M, hence the intersection number $I(L, \overline{\Sigma}; M)$ is well

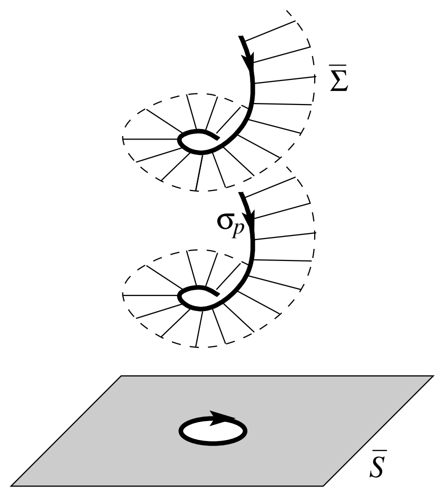


FIGURE 13. A compactification $\overline{\Sigma}$ of Σ.

defined and we have

$$I(L, \overline{\Sigma}; M) = \sum_{(p,\xi) \in L \pitchfork \overline{\Sigma}} I_{(p,\xi)}(L, \overline{\Sigma}; M),$$

where $I_{(p,\xi)}(L, \overline{\Sigma}; M)$ is the local intersection number.

Proposition 3.3. *For every $(p, \xi) \in L \pitchfork \overline{\Sigma}$ we have*

$$I_{(p,\xi)}(L, \overline{\Sigma}; M) = \varepsilon_{C(p,\xi)},$$

where ε_C is the sign of the cubic C, see Subsection 2.2.

Proof. The orientation of $T_{(p,\xi)}\overline{\Sigma}$ is defined by the Welschinger's sign $w_{C(p,\xi)}$. The curve L intersects $\overline{\Sigma}$ in the direction of the oriented fiber F iff $\tau_{C(p,\xi)} = +1$, see Figure 14.

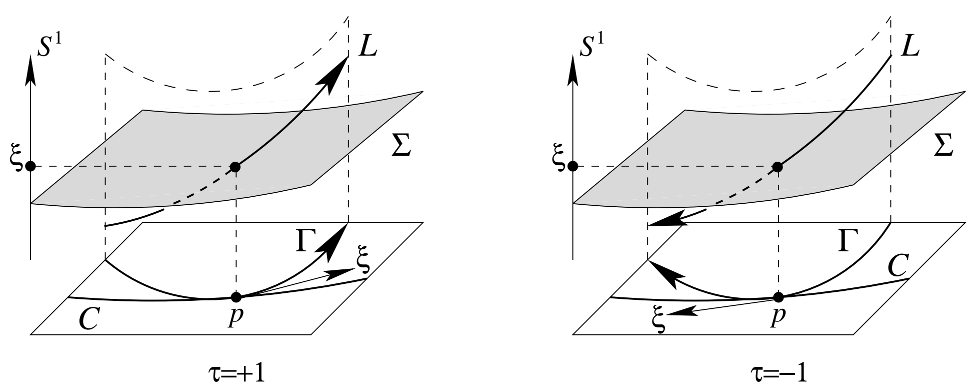


FIGURE 14. Intersection of L with Σ.

Hence the orientation of $T_{(p,\xi)}\overline{\Sigma}\oplus T_{(p,\xi)}L$ differs from the orientation $o_{ST^*\mathbb{R}^2}\big|_M$ of M by the sign $\varepsilon_{C(p,\xi)} = w_{C(p,\xi)}\cdot\tau_{C(p,\xi)}$. So we get $I_{(p,\xi)}(L,\overline{\Sigma}) = \varepsilon_{C(p,\xi)}$. □

Corollary 3.4. *We have*

$$N = \sum_{C\in\mathcal{M}} \varepsilon_C = I(L,\overline{\Sigma};M)$$

The intersection number $I(L,\overline{\Sigma};M)$ via a homological theory. Let us take $k\cdot T,\ k = \mathrm{ind}(\Gamma)$ as in Subsection 2.4 which is regularly homotopic to Γ in $\mathbb{D}^2$, and $h : \mathbb{S}^1\times[0,1]\to\mathbb{D}^2$ be a homotopy between $k\cdot T$ and Γ. Denote $\Gamma_t := h(\mathbb{S}^1\times\{t\}),\ t\in[0,1]$, so $\Gamma_0 = k\cdot T$ and $\Gamma_1 = \Gamma$. Denote by L_t a lift of Γ_t to M. Then $L' = L_0, L = L_1$ and a 2-chain $\mathcal{K} := \{L_t | t\in[0,1]\}\in C_2(M;\mathbb{Z})$ realizes a homotopy between L' and L. We choose an orientation of $\mathcal{K}$ such that $\partial\mathcal{K} = [L]-[L']$. Because of the homotopy invariance of the intersection number, we may choose a special homotopy h as follows. For all $p\in\mathfrak{S}$ pick an open neighborhood U_p of $\overline{\mathbb{D}(p,\delta)}$ and a direction θ_p so that

- ($\mathcal{P}$) if $p\in\mathcal{P}$, θ_p is transversal to any nodal cubic from $\mathcal{D}(\mathcal{P})$ having a node at p.
- ($\mathcal{C}_\mathcal{P}$) if $p\in\mathcal{C}_\mathcal{P}$, θ_p is the tangent direction of the corresponding cuspidal cubic at its cusp p.
- ($\mathcal{R}_\mathcal{P}$) if $p\in\mathcal{R}_\mathcal{P}$, θ_p is the direction of the line component of the corresponding reducible cubic.

See Figure 15a. Now, choose the homotopy h so that for all $t\in[0,1]$ with $\Gamma_t\cap U_p\neq\varnothing$, the fragment $\Gamma_t\cap U_p$ is close to a straight interval in the direction θ_p. For such a homotopy the part $\mathcal{K}\cap(U_p\times\mathbb{S}^1)$ of $\mathcal{K}$ over U_p is almost flat, i.e., lies in a thin cylinder

$$\mathcal{K}\cap(U_p\times\mathbb{S}^1)\subseteq U_p\times(\theta_p-\varepsilon,\theta_p+\varepsilon),$$

for some small $0<\varepsilon<<1$. See Figure 15b.

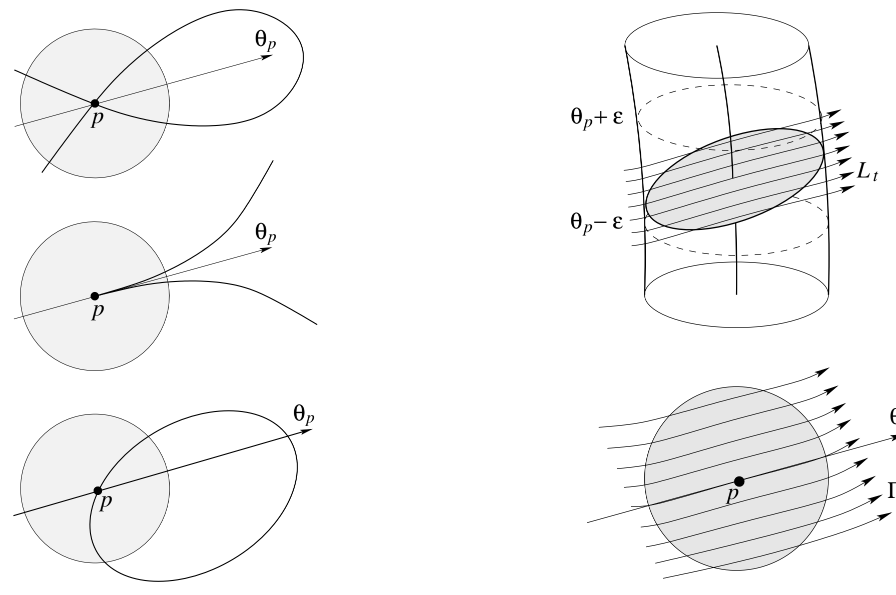
By the additivity of the intersection number and according to the calculations in the Subsection 2.4 we have

$$\begin{aligned} I(L,\overline{\Sigma};M) &= I(L',\overline{\Sigma};M) + I(\partial\mathcal{K},\overline{\Sigma};M) = \\ &= 16\cdot\mathrm{ind}(k\cdot T) + I(\partial\mathcal{K},\overline{\Sigma};M) = 16\cdot\mathrm{ind}(\Gamma) + I(\partial\mathcal{K},\overline{\Sigma};M) \end{aligned}$$

It remains to compute $I(\partial\mathcal{K},\overline{\Sigma};M)$.

Lemma 3.5. *For $\overline{\Sigma},\mathcal{K},\sigma_p$ and L as before we have*

$$I(\partial\mathcal{K},\overline{\Sigma};M) = \sum_{p\in\mathfrak{S}} I(\mathcal{K},\sigma_p;M).$$



a b

FIGURE 15. Flat homotopy of Γ.

Proof. Recall that $I(\partial\mathcal{K},\overline{\Sigma};M) = I(\mathcal{K},\partial\overline{\Sigma};M)$. Now, as a 1-chain in M,

$$\partial\overline{\Sigma} = \partial\overline{\Sigma} \cap (\partial\mathbb{D}^2 \times \mathbb{S}^1)) + \sum_{p\in\mathfrak{S}} \sigma_p.$$

Since $\mathcal{K} \cap (\partial\mathbb{D}^2 \times \mathbb{S}^1) = \varnothing$, we get $I(\mathcal{K},\partial\overline{\Sigma};M) = \sum_{p\in\mathfrak{S}} I(\mathcal{K},\sigma_p;M)$. □

The following proposition completes the proof of the main theorem.

Proposition 3.6. *For every* $p \in \mathfrak{S}$ *we have*

$$I(\mathcal{K},\sigma_p;M) = 2 \cdot \iota_p \cdot \mathrm{ind}_p(\Gamma).$$

Proof. Firstly, recall that $ST^*\mathbb{D}^2 \to PT^*\mathbb{D}^2$ is a 2-fold covering, so for every component of the lift of $\partial\mathbb{D}(p,\delta)$ to $PT^*\mathbb{D}^2$ there are two components in $ST^*\mathbb{D}^2$, which explains the coefficient 2 in the RHS. Secondly, note that $I(\mathcal{K},\sigma_p;M) = I(\mathcal{K}_p,\sigma_p;M)$ for every $p \in \mathfrak{S}$, where $\mathcal{K}_p := \mathcal{K} \cap (U_p \times \mathbb{S}^1)$.
In order to compute $I(\mathcal{K}_p,\sigma_p;M)$ we study the homology class $[\sigma_p^i] \in \mathrm{H}_1(M;\mathbb{Z})$ of each component σ_p^i of σ_p. Since $\mathrm{H}_1(M;\mathbb{Z}) = \mathbb{Z}\langle F\rangle$, where F is the class of the fiber, we conclude that $[\sigma_p^i] = k_p^i \cdot F$ for some $k_p^i \in \mathbb{Z}$. We can compute the number k_p^i as the degree of the corresponding Gauss map $G_p^i : \mathbb{S}^1 \to \mathbb{S}^1$ defined by $G_p^i(t) = \xi$, where $(\varphi(t),\xi) \in \sigma_p^i$ for a parametrization $\varphi : \mathbb{S}^1 \to \partial\mathbb{D}(p,\delta)$ of degree -1. The degree of G_p^i equals to the rotation number of the field ξ of tangent directions of cubics along $\partial\mathbb{D}(p,\delta)$ (with the

clockwise orientation) corresponding to σ_p^i. The behavior of the field ξ on $\partial\mathbb{D}(p,\delta)$ is determined by the standard bifurcation diagrams of the singularity theory.

Fix a point $q \in \partial\mathbb{D}(p,\delta)$, such that $q \cup \mathcal{P}$ are in general position. Then there are $8, 10$ or 12 nodal cubics passing through $q \cup \mathcal{P}$. Consider separately the following three cases.

($p \in \mathcal{P}$) By applying a small deformation to a nodal cubic from $\mathcal{D}(\mathcal{P})$ having a node at p, we get two nodal cubics passing through $q \cup \mathcal{P}$, see Figure 10b and 16. They intersect transversally and hence form four different

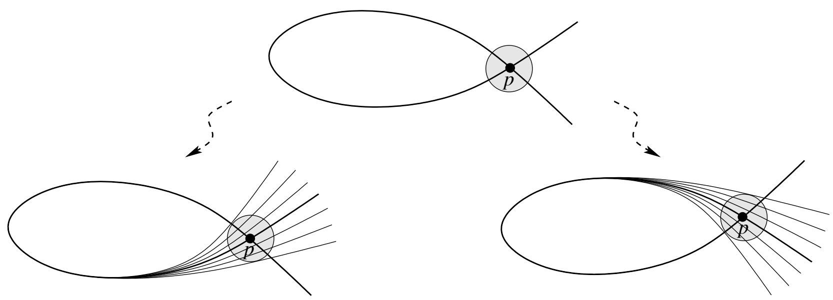


FIGURE 16. Deforming a cubic with a node at $p \in \mathcal{P}$.

components $\sigma_p^1, \sigma_p^2, \sigma_p^3, \sigma_p^4$ of σ_p. The corresponding four fields have the rotation number 0. See Figures 16 and 17. Each of the remaining

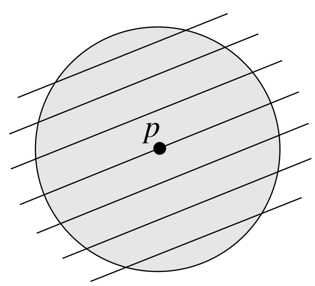


FIGURE 17. A constant field for $p \in \mathfrak{S}$ of the rotation number 0.

$8-2n_p$, or $10-2n_p$, or $12-2n_p$ nodal cubics passing through $q \cup \mathcal{P}$ intersects the boundary $\overline{\partial\mathbb{D}(p,\delta)}$ in exactly two points. They intersect transversally, and hence, form $(2\cdot(8-2n_p), 2\cdot(10-2n_p)$ or $2\cdot(12-2n_p))$ different components of σ_p. The corresponding fields look as shown in Figure 18. As we follow the circle $\partial\mathbb{D}(p,\delta)$ clockwise, this field of tangent directions rotates once clockwise, and thus has a rotation number -1. By the construction of the orientation of Σ and by the

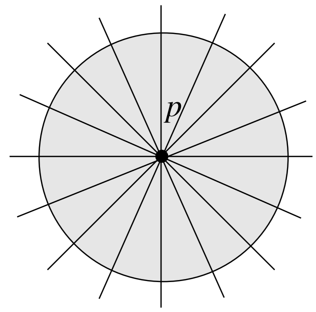


FIGURE 18. The radial field for $p \in \mathcal{P}$ of the rotation number -1.

choice of the homotopy h,

$$I(\mathcal{K}_p, \sigma_p; M) = \sum_{i \neq 1, \dots 4} I(\mathcal{K}_p, \sigma_p^i; M) = -2(8 - 2n_p) \cdot I(\mathcal{K}_p, \{p\} \times \mathbb{S}^1; M).$$

($p \in \mathcal{C}_{\mathcal{P}}$) By applying a small deformation to the cuspidal cubic from $\mathcal{D}(\mathcal{P})$ having a cusp at p, we get a nodal cubic passing through $q \cup \mathcal{P}$, see Figure 10c. It forms two corresponding components σ_p^1, σ_p^2 of σ_p. The corresponding tangent fields look as shown in Figure 19. In this

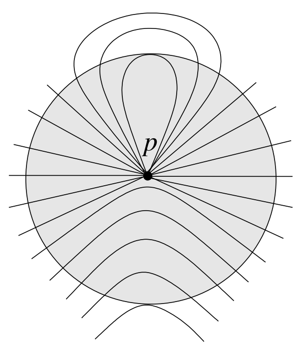


FIGURE 19. The cuspidal field for $p \in \mathcal{C}_{\mathcal{P}}$ of the rotation number -1.

case, as we follow the circle $\partial\mathbb{D}(p, \delta)$ clockwise, the field of tangent directions rotates once clockwise, and thus has the rotation number -1. It follows that $[\sigma_p^1] = [\sigma_p^2] = -F$.
The remaining 9 or 11 nodal cubics passing through $q \cup \mathcal{P}$ intersect transversally, and hence, form 18 or 22 different components of σ_p. The corresponding fields look as shown in Figure 17 and have rotation numbers 0. By the choice of the homotopy h,

$$I(\mathcal{K}_p, \sigma_p; M) = I(\mathcal{K}_p, \sigma_p^1 \cup \sigma_p^2; M) = -2 \cdot I(\mathcal{K}_p, \{p\} \times \mathbb{S}^1; M).$$

($p \in \mathcal{R}_{\mathcal{P}}$) By applying a small deformation to the connected reducible cubic from $\mathcal{D}(\mathcal{P})$ that passes through p, we get a nodal cubic passing through $q \cup \mathcal{P}$, see Figure 10a. It forms two corresponding components σ_p^1, σ_p^2 of σ_p. The corresponding tangent fields look as shown in Figure 20. In this case, as we follow the circle $\partial\mathbb{D}(p, \delta)$ clockwise, the

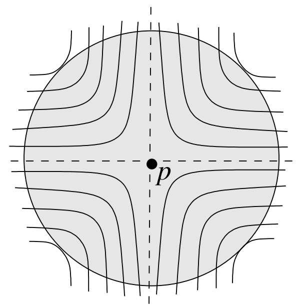


FIGURE 20. The reducible field for $p \in \mathcal{R}_{\mathcal{P}}$ of the rotation number +1.

field of tangent directions rotates once counterclockwise, and thus, has rotation number +1. It follows that $[\sigma_p^1] = [\sigma_p^2] = F$.
The remaining 7, or 9, or 11 nodal cubics passing through $q \cup \mathcal{P}$ intersect transversally and hence form 14, or 18, or 22 different components of σ_p. The corresponding fields look as shown in Figure 17 and have the rotation number 0. By the choice of the homotopy h,

$$I(\mathcal{K}_p, \sigma_p; M) = I(\mathcal{K}_p, \sigma_p^1 \cup \sigma_p^2; M) = 2 \cdot I(\mathcal{K}_p, \{p\} \times \mathbb{S}^1; M).$$

We finish the proof by observing that

$$\begin{aligned} I(\mathcal{K}_p, \{p\} \times \mathbb{S}^1; M) &= I(h(\mathbb{S}^1 \times [0,1]), [p]; \mathbb{R}^2) = \\ &= I(h(\mathbb{S}^1 \times [0,1]), [p] - [\infty]; \mathbb{R}^2) = -I(\partial h(\mathbb{S}^1 \times [0,1]), [p, \infty]; \mathbb{R}^2) = \\ &= \operatorname{ind}_p(\Gamma). \end{aligned}$$

□

4. FINITE TYPE INVARIANTS.

Finite type invariants generalize polynomial functions. This notion is based on the following classical theorem:

Theorem 4.1 (Frechet 1912). *Given $x_0, x_1^{\pm}, \dots, x_n^{\pm} \in \mathbb{R}$ and an n-tuple $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{-1, 1\}^n$, let $x_\varepsilon = x_0 + x_1^{\varepsilon_1} + \dots + x_n^{\varepsilon_n}$ and $|\varepsilon| = \prod_{i=1}^n \varepsilon_i$. Then C^0-function $f : \mathbb{R} \to \mathbb{R}$ is a polynomial of degree less than n, iff*

$$\sum_{\varepsilon \in \{-1,1\}^n} (-1)^{|\varepsilon|} f(x_\varepsilon) = 0$$

for any choice of x_0 *and* $x_1^\pm, \ldots, x_n^\pm$.

Finite type invariants are topological analogues of this definition. Corresponding theories are developed for a variety of objects: knots, 3-manifolds, plane curves, graphs, etc. (see [11] for a general theory of finite type invariants of cubic complexes). Let us briefly recall the main notions in the case of immersed curves in a punctured plane. Let $\mathfrak{S} \subset \mathbb{R}^2$ be a finite set of marked points and Γ_{sing} be an immersed plane curve with n non-generic fragments, contained in n small disks $\mathbb{D}_i$. Fix an arbitrary pair of resolutions for each $\mathbb{D}_i$ and call one of them positive and the other negative (again, arbitrarily). Here by a resolution of Γ_{sing} in a disk $\mathbb{D}_i$ we mean a homotopy of Γ_{sing} inside $\mathbb{D}_i$, fixed on the boundary $\partial\mathbb{D}_i$, so that the resulting curve is a generic immersion inside $\mathbb{D}_i$ and does not pass through $\mathfrak{S} \cap \mathbb{D}_i$. See Figure 21.

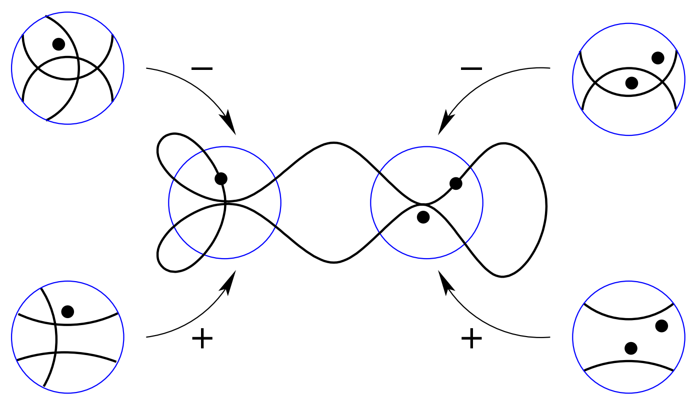


FIGURE 21. A non-generic curve with a pair of resolutions in each disk.

For an n-tuple $\varepsilon \in \{-1, 1\}^n$, resolve all singularities of Γ_{sing} choosing the corresponding ε_i resolution in each disk $\mathbb{D}_i$. Denote by Γ_ε the resulting curve. In this way, as ε runs over $\{-1, 1\}^n$, we obtain 2^n generically immersed curves Γ_ε. See Figure 22.

Denote $|\varepsilon| = \prod_{i=1}^n \varepsilon_i$. A locally-constant function f on the space of generically immersed curves is called an invariant of degree less than n, if

$$\sum_{\varepsilon \in \{-1,1\}^n} (-1)^{|\varepsilon|} f(\Gamma_\varepsilon) = 0,$$

for any choice of the curve Γ_{sing} and its resolutions.

When $\mathfrak{S} = \varnothing$, the only invariant of degree 0 (i.e., a constant function on the space of immersed curves) is the rotation number $\mathrm{ind}(\Gamma)$. Various interesting invariants of degree one for $\mathfrak{S} = \varnothing$ were extensively studied by V. Arnold, see [1]. When $\mathfrak{S}$ consists of one point, we get an additional simple invariant of degree 1, namely $\mathrm{ind}_p(\Gamma)$.

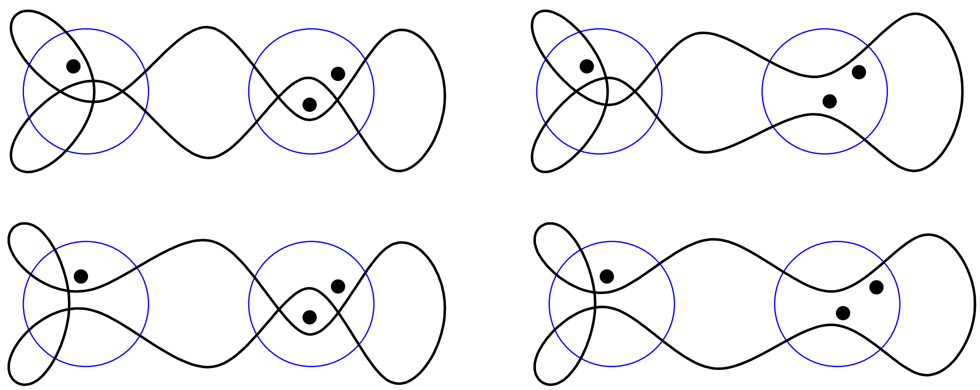


FIGURE 22. Resolved generic curves.

Finite type invariants naturally appear in real enumerative geometry. One of the simplest examples was considered in Section 1.2. Note that in the formula (1), an algebraic number of lines passing through a point p and tangent to a generic immersed curve $\Gamma \subset \mathbb{R}^2 \smallsetminus \{p\}$ is expressed via invariants $\mathrm{ind}(\Gamma)$, $\mathrm{ind}_p(\Gamma)$ of degrees 0 and 1. This fact is easy to explain. Let us show, that if a certain algebraic number of lines satisfying some passage/tangency conditions is a locally constant function f on the space of generic immersed curves, then it is an invariant of degree less than or equal to 2. Indeed, let Γ_{sing} be an immersed curve with three non-generic fragments contained in three discs $\mathbb{D}_i$, $i = 1, 2, 3$. WLOG we may assume that $\mathbb{D}_i$'s are small enough so that they do not lie on one line, i.e., no line passes through all three of them. Suppose that some line l is counted for one of the resolutions Γ_ε of Γ_{sing}. Then l does not pass through at least one of the disks, say, $\mathbb{D}_1$. But then l is counted twice – with opposite signs – for both resolutions of Γ_{sing} inside $\mathbb{D}_1$, hence its contribution to f sums up to 0, and we readily get $f(\Gamma_{sing}) = 0$.

By the same argument (noticing that no rational curves of degree d pass through $3d$ generic points), we immediately obtain the following

Theorem 4.2. *Suppose that a certain algebraic number of real rational algebraic plane curves of degree d, satisfying some passage/tangency conditions, is a locally constant function on the space of generic immersed curves. Then it is an invariant of degree less than or equal to $3d - 1$.*

Moreover, if a curve is required to pass through k fixed points (in general position), then an algebraic number of such curves is an invariant of degree less than or equal to $3d - k - 1$. In particular, for $d = 3$ and $k = 7$ we get the upper bound 1 on the degree of an invariant. This explains the structure of formula (2) of Theorem 2.2.

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