Some characterizations of Hom-Leibniz algebras

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Abstract

Some basic properties of Hom-Leibniz algebras are found. These properties are the Hom-analogue of corresponding well-known properties of Leibniz algebras. Considering the Hom-Akivis algebra associated to a given Hom-Leibniz algebra, it is observed that the Hom-Akivis identity leads to an additional property of Hom-Leibniz algebras, which in turn gives a necessary and sufficient condition for Hom-Lie admissibility of Hom-Leibniz algebras. A necessary and sufficient condition for Hom-power associativity of Hom-Leibniz algebras is also found.

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1 Introduction

The theory of Hom-algebras originated from the introduction of the notion of a Hom-Lie algebra by J.T. Hartwig, D. Larsson and S.D. Silvestrov [5] in the study of algebraic structures describing some q-deformations of the Witt and the Virasoro algebras. A Hom-Lie algebra is characterized by a Jacobi-like identity (called the Hom-Jacobi identity) which is seen as the Jacobi identity twisted by an endomorphism of a given algebra. Thus, the class of Hom-Lie algebras contains the one of Lie algebras.

Generalizing the well-known construction of Lie algebras from associative algebras, the notion of a Hom-associative algebra is introduced by A. Makhlouf and S.D. Silvestrov [12] (in fact the commutator algebra of a Hom-associative algebra is a Hom-Lie algebra). The other class of Hom-algebras closely related to Hom-Lie algebras is the one of Hom-Leibniz algebras [12] (see also [8]) which are the Hom-analogue of Leibniz algebras [9]. Roughly, a

Hom-type generalization of a given type of algebras is defined by a twisting of the defining identities with a linear self-map of the given algebra. For various Hom-type algebras one may refer, e.g., to [10], [11], [15], [16], [7]. In [14] D. Yau showed a way of constructing Hom-type algebras starting from their corresponding untwisted algebras and a self-map.

In [9] (see also [3], [4]) the basic properties of Leibniz algebras are given. The main purpose of this note is to point out that the Hom-analogue of some of these properties holds in Hom-Leibniz algebras (section 3). Considering the Hom-Akivis algebra associated to a given Hom-Leibniz algebra, we observe that the property in Proposition 3.3 is the expression of the Hom-Akivis identity. As a consequence we found a necessary and sufficient condition for the Hom-Lie admissibility of Hom-Leibniz algebras (Corollary 3.4). Generalizing power-associativity of rings and algebras [2], the notion of the (right) nth Hom-power associativity x^n of an element x in a Hom-algebra is introduced by D. Yau [17], as well as Hom-power associativity of Hom-algebras. We found that $x^n = 0$, $n \geq 3$, for any x in a left Hom-Leibniz algebra $(L, \cdot, \alpha)$ and that $(L, \cdot, \alpha)$ is Hom-power associative if and only if $\alpha(x)x^2 = 0$, for all x in L (Theorem 3.7). Then we deduce , as a particular case, corresponding characterizations of left Leibniz algebras (Corollary 3.8). Apart of the (right) nth Hom-power of an element of a Hom-algebra [17], we consider in this note the left nth Hom-power of the given element. This allows to prove the Hom-analogue (see Theorem 3.10) of a result of D.W. Barnes ([4], Theorem 1.2 and Corollary 1.3) characterizing left Leibniz algebras. In section 2 we recall some basic notions on Hom-algebras. Modules, algebras, and linearity are meant over a ground field $\mathbb{K}$ of characteristic 0.

2 Preliminaries

In this section we recall some basic notions related to Hom-algebras. These notions are introduced in [5], [10], [12], [14], [7].

Definition 2.1. A *Hom-algebra* is a triple $(A, \cdot, \alpha)$ in which A is a $\mathbb{K}$-vector space, " $\cdot$ " a binary operation on A and $\alpha : A \to A$ is a linear map (the twisting map) such that $\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$ (multiplicativity), for all x, y in A.

Remark 2.2. A more general notion of a Hom-algebra is given (see, e.g., [10], [12]) without the assumption of multiplicativity and A is considered just as a $\mathbb{K}$-module. For convenience, here we assume that a Hom-algebra

$(A, \cdot, \alpha)$ is always multiplicative and that A is a $\mathbb{K}$-vector space.

Definition 2.3. Let $(A, \cdot, \alpha)$ be a Hom-algebra.

(i) The *Hom-associator* of $(A, \cdot, \alpha)$ is the trilinear map $as : A \times A \times A \to A$ defined by

$$as(x, y, z) = xy \cdot \alpha(z) - \alpha(x) \cdot yz,$$

for all x, y, z in A.

(ii) $(A, \cdot, \alpha)$ is said *Hom-associative* if $as(x, y, z) = 0$ (Hom-associativity), for all x, y, z in A.

Remark 2.4. If $\alpha = Id$ (the identity map) in $(A, \cdot, \alpha)$, then its Hom-associator is just the usual associator of the algebra $(A, \cdot)$. In Definition 2.1, the Hom-associativity is not assumed, i.e. $as(x, y, z) \neq 0$ in general. In this case $(A, \cdot, \alpha)$ is said non-hom-associative [7] (or Hom-nonassociative [14]; in [11], $(A, \cdot, \alpha)$ is also called a nonassociative Hom-algebra). This matches the generalization of associative algebras by the nonassociative ones.

Definition 2.5. (i) A *(left) Hom-Leibniz algebra* is a Hom-algebra $(A, \cdot, \alpha)$ such that the identity

$$\alpha(x) \cdot yz = xy \cdot \alpha(z) + \alpha(y) \cdot xz \tag{2.1}$$

holds for all x, y, z in A.

(ii) A *Hom-Lie algebra* is a Hom-algebra $(A, [-, -], \alpha)$ such that the binary operation "$[-, -]''$ is skew-symmetric and the *Hom-Jacobi identity*

$$J_\alpha(x, y, z) = 0 \tag{2.2}$$

holds for all x, y, z in A and $J_\alpha(x, y, z) := [[x, y], \alpha(z)] + [[y, z], \alpha(x)] + [[z, x], \alpha(y)]$ is called the *Hom-Jacobian.*

Remark 2.6. The original definition of a Hom-Leibniz algebra [12] is related to the identity

$$xy \cdot \alpha(z) = xz \cdot \alpha(y) + \alpha(x) \cdot yz \tag{2.3}$$

which is expressed in terms of (right) adjoint homomorphisms $Ad_y x := x \cdot y$ of $(A, \cdot, \alpha)$. This justifies the term of "(right) Hom-Leibniz algebra" that could be used for the Hom-Leibniz algebra defined in [12]. The dual of (2.3) is (2.1) and in this note we consider only left Hom-Leibniz algebras. For $\alpha = Id$ in $(A, \cdot, \alpha)$ (resp. $(A, [-,-], \alpha)$), any Hom-Leibniz algebra (resp. Hom-Lie algebra) is a Leibniz algebra $(A, \cdot)$ [3], [9] (resp. a Lie algebra $(A, [-,-])$). As for Leibniz algebras, if the operation "$\cdot$" of a given Hom-Leibniz algebra $(A, \cdot, \alpha)$ is skew-symmetric, then $(A, \cdot, \alpha)$ is a Hom-Lie algebra (see [12]).

In terms of Hom-associators, the identity (2.1) is written as

$$as(x, y, z) = -\alpha(y) \cdot xz \qquad (2.4)$$

Therefore, from Definition 2.3 and Remark 2.4, we see that Hom-Leibniz algebras are examples of non-Hom-associative algebras.

Definition 2.7. [7] A *Hom-Akivis algebra* is a quadruple $(A, [-,-], [-,-,-], \alpha)$ in which A is a vector space, "$[-,-]$" a skew-symmetric binary operation on A, "$[-,-,-]$" a ternary operation on A and $\alpha : A \to A$ a linear map such that the *Hom-Akivis identity*

$$J_\alpha(x, y, z) = \sigma[x, y, z] - \sigma[y, x, z] \qquad (2.5)$$

holds for all x, y, z in A, where σ denotes the sum over cyclic permutation of x, y, z.

Note that when $\alpha = Id$ in a Hom-Akivis algebra $(A, [-,-], [-,-,-], \alpha)$, then one gets an Akivis algebra $(A, [-,-], [-,-,-])$. Akivis algebras were introduced in [1] (see also references therein), where they were called W-algebras. The term "Akivis algebra" for these objects is introduced in [6].

In [7], it is observed that to each non-Hom-associative algebra is associated a Hom-Akivis algebra (this is the Hom-analogue of a similar relationship between nonassociative algebras and Akivis algebras [1]). In this note we use the specific properties of the Hom-Akivis algebra associated to a given Hom-Leibniz algebra to derive a property characterizing Hom-Leibniz algebras.

3 Characterizations

In this section, Hom-versions of some well-known properties of left Leibniz algebras are displayed. Considering the specific properties of the binary and ternary operations of the Hom-Akivis algebra associated to a given Hom-

Leibniz algebra, we infer a characteristic property of Hom-Leibniz algebras. This property in turn allows to give a necessary and sufficient condition for the Hom-Lie admissibility of these Hom-algebras. The Hom-power associativity of Hom-Leibniz algebras is considered.

Let $(L, \cdot, \alpha)$ be a Hom-Leibniz algebra and consider on $(L, \cdot, \alpha)$ the operations

$$[x, y] := x \cdot y - y \cdot x \tag{3.1}$$

$$[x, y, z] := as(x, y, z). \tag{3.2}$$

Then the operations (3.1) and (3.2) define on L a Hom-Akivis structure [7]. We have the following

Proposition 3.1. *Let $(L, \cdot, \alpha)$ be a Hom-Leibniz algebra. Then*
(i) $(x \cdot y + y \cdot x) \cdot \alpha(z) = 0$,
(ii) $\alpha(x) \cdot [y, z] = [x \cdot y, \alpha(z)] + [\alpha(y), x \cdot z]$,
for all x, y, z in L .

Proof. The identity (2.1) implies that
$xy \cdot \alpha(z) = \alpha(x) \cdot yz - \alpha(y) \cdot xz.$
Likewise, interchanging x and y, we have
$yx \cdot \alpha(z) = \alpha(y) \cdot xz - \alpha(x) \cdot yz.$
Then, adding memberwise these equalities above, we come to the property (i). Next we have

$$\begin{aligned}
[x \cdot y, \alpha(z)] + [\alpha(y), x \cdot z] &= xy \cdot \alpha(z) - \alpha(z) \cdot xy \\
&+ \alpha(y) \cdot xz - xz \cdot \alpha(y) \\
&= \alpha(x) \cdot yz - \alpha(z) \cdot xy - xz \cdot \alpha(y) \text{ (by (2.1))} \\
&= \alpha(x) \cdot yz - zx \cdot \alpha(y) - \alpha(x) \cdot zy \\
&- xz \cdot \alpha(y) \text{ (by (2.1))} \\
&= \alpha(x) \cdot yz - \alpha(x) \cdot zy \text{ (by (i))} \\
&= \alpha(x) \cdot [y, z].
\end{aligned}$$

and so we get (ii). □

Remark 3.2. If set $\alpha = Id$ in Proposition 3.1, then one recovers the well-known properties of Leibniz algebras: $(x \cdot y + y \cdot x) \cdot z = 0$ and

$x \cdot [y,z] = [x \cdot y, z] + [y, x \cdot z]$ (see [3], [9]).

Proposition 3.3. *Let $(L, \cdot, \alpha)$ be a Hom-Leibniz algebra. Then*

$$J_{\alpha}(x,y,z) = \sigma xy \cdot \alpha(z), \tag{3.3}$$

for all x, y, z in L .

Proof. Considering (2.5) and then applying (3.2) and (2.4), we get $J_{\alpha}(x,y,z) = \sigma[-\alpha(y) \cdot xz] - \sigma[-\alpha(x) \cdot yz] = \sigma[\alpha(x) \cdot yz - \alpha(y) \cdot xz] = \sigma xy \cdot \alpha(z)$ (by (2.1)). □

One observes that (3.3) is the specific form of the Hom-Akivis identity (2.5) in case of Hom-Leibniz algebras.

The skew-symmetry of the operation "$\cdot$" of a Hom-Leibniz algebra $(L, \cdot, \alpha)$ is a condition for $(L, \cdot, \alpha)$ to be a Hom-Lie algebra [12]. From Proposition 3.3 one gets the following necessary and sufficient condition for the Hom-Lie admissibility [12] of a given Hom-Lie algebra.

Corollary 3.4. *A Hom-Leibniz algebra $(L, \cdot, \alpha)$ is Hom-Lie admissible if and only if $\sigma xy \cdot \alpha(z) = 0$, for all x, y, z in L .* □

In [17] D. Yau introduced Hom-power associative algebras which are seen as a generalization of power-associative algebras. It is shown that some important properties of power-associative algebras are reported to Hom-power associative algebras.

Let A be a Hom-Leibniz algebra with a twisting linear self-map α and the binary operation on A denoted by juxtaposition. We recall the following

Definition 3.5. [17] Let $x \in A$ and denote by α^m the m-fold composition of m copies of α with $\alpha^0 := Id$.

(1) The nth Hom-power $x^n \in A$ of x is inductively defined by

$$x^1 = x, \quad x^n = x^{n-1}\alpha^{n-2}(x) \tag{3.4}$$

for $n \geq 2$.

(2) The Hom-algebra A is *nth Hom-power associative* if

$$x^n = \alpha^{n-i-1}(x^i)\alpha^{i-1}(x^{n-i}) \tag{3.5}$$

for all $x \in A$ and $i \in \{1, ..., n-1\}$.

(3) The Hom-algebra A is *up to nth Hom-power associative* if A is kth Hom-power associative for all $k \in \{2, ..., n\}$.

(4) The Hom-algebra A is *Hom-power associative* if A is nth Hom-power associative for all $n \geq 2$.

The following result provides a characterization of third Hom-power associativity of Hom-Leibniz algebras.

Lemma 3.6. *Let $(L, \cdot, \alpha)$ be a Hom-Leibniz algebra. Then*
(i) $x^3 = 0$, for all $x \in L$;
(ii) $(L, \cdot, \alpha)$ is third Hom-power associative if and only if $\alpha(x)x^2 = 0$, for all $x \in L$.

Proof. From (3.4) we have $x^3 := x^2\alpha(x)$. Therefore the assertion (i) follows from Proposition 3.1(i) if set $y = x = z$.

Next, from (3.5) we note that the $i = 2$ case of nth Hom-power associativity is automatically satisfied since this case is $x^3 = \alpha^0(x^2)\alpha^1(x^1) = x^2\alpha(x)$, which holds by definition. The $i = 2$ case says that $x^3 = \alpha^1(x)\alpha^0(x^2) = \alpha(x)x^2$. Therefore, since $x^2\alpha(x) = 0$ naturally holds by Proposition 3.1 (i), we conclude that the third Hom-power associativity of $(L, \cdot, \alpha)$ holds if and only if $\alpha(x)x^2 = 0$ for all $x \in L$, which proves the assertion (ii). $\square$

The following result shows that the condition in Lemma 3.6 is also necessary and sufficient for the Hom-power associativity of $(L, \cdot, \alpha)$. To prove this, we rely on the main result of [17] (see Corollary 5.2).

Theorem 3.7. *Let $(L, \cdot, \alpha)$ be a Hom-Leibniz algebra. Then*
(i) $x^n = 0$, $n \geq 3$, for all $x \in L$;
(ii) $(L, \cdot, \alpha)$ is Hom-power associative if and only if $\alpha(x)x^2 = 0$, for all $x \in L$.

Proof. The proof of (i) is by induction on n: the first step $n = 3$ holds by Lemma 3.6(i); now if suppose that $x^n = 0$, then $x^{n+1} := x^{(n+1)-1}\alpha^{(n+1)-2}(x) = x^n\alpha^{n-1}(x) = 0$ so we get (i).

Corollary 5.2 of [17] says that, for a multiplicative Hom-algebra, the Hom-power associativity is equivalent to both of the conditions

$$x^2\alpha(x) = \alpha(x)x^2 \text{ and } x^4 = \alpha(x^2)\alpha(x^2). \tag{3.6}$$

In the situation of multiplicative left Hom-Leibniz algebras, the first equality of (3.6) is satisfied by Lemma 3.6(i) and the hypothesis $\alpha(x)x^2 = 0$. Next we have, from (3.5):

case $i = 1$: $x^4 := \alpha^{4-2}(x)\alpha^0(x^3) = \alpha^2(x)x^3$,

case $i = 2$: $x^4 := \alpha(x^2)\alpha(x^2)$,

case $i = 3$: $x^4 := \alpha^0(x^3)\alpha^2(x) = x^3\alpha^2(x)$.

Because of the assertion (i) above, only the case $i = 2$ is of interest here. From one side we have $x^4 = 0$ (by (i)) and, from the other side we have $\alpha(x^2)\alpha(x^2) = [\alpha(x)]^2\alpha(x^2) = 0$ (by multiplicativity and Proposition 3.1(i)). Therefore, Corollary 5.2 of [17] now applies and we conclude that (3.6) holds (i.e. $(L, \cdot, \alpha)$ is Hom-power associative) if and only if $\alpha(x)x^2 = 0$, which proves (ii). □

Let A be an algebra (over a field of characteristic 0). For an element $x \in A$, the *right powers* are defined by

$$x^1 = x, \text{ and } x^{n+1} = x^n x \tag{3.7}$$

for $n \geq 1$. Then A is power-associative if and only if

$$x^n = x^{n-i}x^i \tag{3.8}$$

for all $x \in A$, $n \geq 2$, and $i \in \{1, ..., n-1\}$. By a theorem of Albert [2], A is power-associative if only if it is third and fourth power-associative, which in turn is equivalent to

$$x^2x = xx^2 \text{ and } x^4 = x^2x^2. \tag{3.9}$$

for all $x \in A$.

Some consequences of the results above are the following simple characterizations of (left) Leibniz algebras.

Corollary 3.8. *Let $(L, \cdot)$ be a left Leibniz algebra. Then*
(i) $x^n = 0$, $n \geq 3$, for all $x \in L$;
(ii) $(L, \cdot)$ is power-associative if and only if $xx^2 = 0$, for all $x \in L$.

Proof. The part (i) of this corollary follows from (3.7) and Theorem 3.7(i) when $\alpha = Id$ (we used here the well-known property $(xy + yx)z = 0$ of left Leibniz algebras). The assertion (ii) is a special case of Theorem 3.7(ii) (when $\alpha = Id$), if keep in mind the assertion (i), (3.8), and (3.9). $\square$

Remark 3.9. Although the condition $xx^2 = 0$ does not always hold in a left Leibniz algebra $(L, \cdot)$, we do have $xx^2 \cdot z = 0$ for all $x, z \in L$ (again, this follows from the property $(xy + yx)z = 0$). In fact, $b \cdot z = 0$, $z \in L$, where $b \neq 0$ is a left mth power of x $(m \geq 2)$, i.e. $b = x(x(...(xx)...))$ ([4], Theorem 1.2 and Corollary 1.3).

Let call the *nth right Hom-power* of $x \in A$ the power defined by (3.4), where A is a Hom-algebra. Then one may consider the *nth left Hom-power* of $a \in A$ defined by

$$a^1 = a, \quad a^n = \alpha^{n-2}(a)a^{n-1} \tag{3.10}$$

for $n \geq 2$. In this setting of left Hom-powers, we have the following

Theorem 3.10. *Let $(L, \cdot, \alpha)$ be a Hom-Leibniz algebra and let $a \in L$. Then $L_{a^n} \circ \alpha = 0$, $n \geq 2$, where L_z denotes the left multiplication by z in $(L, \cdot, \alpha)$, i.e. $L_z x = z \cdot x$, $x \in L$.*

Proof. We proceed by induction on n and the repeated use of Proposition 3.1(i).

From Proposition 3.1(i), we get $a^2\alpha(z) = 0$, $\forall a, z \in L$ and thus the first step $n = 2$ is verified. Now assume that, up to the degree n, we have $a^n\alpha(z) = 0$, $\forall a, z \in L$. Then Proposition 3.1(i) implies that $(a^n\alpha^{n-1}(a) + \alpha^{n-1}(a)a^n)\alpha(z) = 0$, i.e. $(a^n\alpha(\alpha^{n-2}(a)) + \alpha^{n-1}(a)a^n)\alpha(z) = 0$. The application of the induction hypothesis to $a^n\alpha(\alpha^{n-2}(a))$ leads to $(\alpha^{n-1}(a)a^n)\alpha(z) = 0$, i.e. $(\alpha^{(n+1)-2}(a)a^{(n+1)-1})\alpha(z) = 0$ which means (by (3.10)) that $a^{n+1}\alpha(z) = 0$. Therefore we conclude that $a^n\alpha(z) = 0$, $\forall n \geq 2$, i.e. $L_{a^n} \circ \alpha = 0$, $n \geq 2$. $\square$

Remark 3.11. We observed that Theorem 3.10 above is an α-twisted version of a result of D.W. Barnes ([4], Theorem 1.2 and Corollary 1.3), related to left Leibniz algebras. Indeed, setting $\alpha = Id$, Theorem 3.10 reduces to the result of Barnes.

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