Analytical Solution of Covariance Evolution for Irregular LDPC Codes

Takayuki Nozaki, *Student Member, IEEE,* Kenta Kasai, *Member, IEEE*, and Kohichi Sakaniwa, *Member, IEEE*

Abstract

A scaling law developed by Amraoui et al. is a powerful technique to estimate the block error probability of finite length low-density parity-check (LDPC) codes. Solving a system of differential equations called covariance evolution is a method to obtain the scaling parameter. However, the covariance evolution has not been analytically solved. In this paper, we present the analytical solution of the covariance evolution for irregular LDPC code ensembles.

Index Terms

LDPC codes, scaling law, covariance evolution, binary erasure channel

I. Introduction

Gallager invented low-density parity-check (LDPC) codes [1]. LDPC codes are linear codes defined by sparse bipartite graphs, called *Tanner graphs*. *Peeling algorithm* (PA) [3], [7] introduced by Luby et al. is a sequential iterative decoding algorithm for the binary erasure channel (BEC). As PA proceeds, edges and nodes are progressively removed from the original Tanner graph and the so-called *residual graph* is left at each iteration. The residual graph at each iteration consists of variable nodes that are still unknown and the check nodes and the edges connecting to those variable nodes. The decoding successfully halts if and only if the residual graph vanishes. It is known that PA and brief propagation (BP) decoder have the same decoding result.

The *scaling law* developed by Amraoui et al. [6] is a powerful technique to estimate the block and bit error probability of finite length LDPC codes. Let r_i and l_j be random variables representing the number of edges connecting to the check nodes of degree i and the variable nodes of degree j, respectively, in the residual graph. Then, the *scaling parameter* is obtained from the mean and the variance of r_1. The means of r_i and l_j are determined from a system of differential equations which was derived and analytically solved by Luby et al. [3]. The covariances of r_i and l_j also satisfy a system of differential equations called *covariance evolution* which was derived by Amraoui et al. [6]. However, the analytical solution of the covariance evolution has not been known. Therefore, one had to resort to numerical computation to solve the covariance evolution.

In [5], Amraoui et al. proposed an alternative way to determine the variance of r_1, though only at the decoding threshold. Thereby they have given the analytic expression for the scaling parameters without using covariance evolution. They used BP decoding instead of PA. This method was applied to irregular repeat-accumulate codes in [9], [10] and to turbo-like codes in [11] and was extended to binary memoryless symmetric channels in [8].

Denote by ξ the total number of edges in the Tanner graph. Let μ_i be the random variable which is 1 if the edge i conveys an erasure message from a variable node to a check node, and 0 otherwise, in the BP decoding. The method in [5] analyzed the random variable $M := \sum_{i=1}^{\xi} \mu_i$ in the BP decoding and derived the analytical expression for the variance of M. Finally, they did make an unproved assumption that the random variable $r_1 - E[r_1]$ in PA is proportional to the random variable $M - E[M]$ in BP and under this assumption they have given the analytical solution for the variance of r_1.

However, no such assumption is needed if the covariance evolution is solved analytically. Moreover, we can obtain the variance of r_1 at any channel erasure probability. In this paper, we present the analytical solution of the covariance evolution for irregular LDPC code ensembles.

II. Preliminaries

In this section, we recall some basic facts on the finite length analysis of LDPC codes under iterative decoding. We also introduce some notations used throughout this paper.

The authors are with Department of Communication and Integrated Systems, Tokyo Institute of Technology, Tokyo 152-8550, Japan e-mail: {nozaki, kenta, sakaniwa}@comm.ss.titech.ac.jp .

Manuscript received January 1, 20**; revised January 1, 20**. The material in this paper was presented in part at IEEE International Symposium on Information Theory (ISIT'09), Seoul, Korea, July 2009.

A. Ensemble and Channel Model

In this paper, we consider irregular LDPC code ensembles [2]. An irregular LDPC code ensemble is defined by the set of bipartite graphs with variable nodes and check nodes. Let $\mathcal{L}$ and $\mathcal{R}$ be the sets of degrees of variable nodes and check nodes, respectively. Irregular LDPC code ensembles are characterized with the block length n and two polynomials, $\lambda(x) = \sum_{i\in\mathcal{L}} \lambda_i x^{i-1}$ and $\rho(x) = \sum_{i\in\mathcal{R}} \rho_i x^{i-1}$, where λ_i and ρ_i are the fractions of edges connected to variable nodes and check nodes of degree i, respectively. The derivatives of $\lambda(x)$ and $\rho(x)$ are $\lambda'(x) = \sum_{i\in\mathcal{L}}(i-1)\lambda_i x^{i-2}$ and $\rho'(x) = \sum_{i\in\mathcal{R}}(i-1)\rho_i x^{i-2}$, respectively.

We assume the transmission over the binary erasure channel (BEC) with channel erasure probability ϵ.

B. Peeling Algorithm

The peeling algorithm (PA) [3] is a sequential iterative decoding algorithm for BEC. It is know that PA and brief propagation (BP) decoder have the same decoding result. A *residual graph* at each iteration consists of variable nodes that are still unknown and the check nodes and the edges connecting to those variable nodes. The decoder proceeds as follows.

a) Initialization: Variable nodes receive the channel outputs. The variable nodes receiving the known values send their values to the check nodes connected to them. Then the variable nodes sending their values and edges connecting to those variable nodes are removed from the graph.

b) Iteration: The decoder uniformly chooses a check node of degree one in the residual graph. The chosen check node sends the value computed from the received values to the adjacent variable node. The variable node propagates this value to all adjacent check nodes. The variable node is removed together with its edges.

c) Decision: If the decoder does not find any check nodes of degree one in the residual graph, then the decoding halts. If the residual graph is empty, then the decoding succeeds, otherwise it fails.

C. Analysis of Residual Graph

Let t denote the iteration round of PA and ξ be the total number of edges in the original graph. We define

$$\tau := \frac{t}{\xi}. \tag{1}$$

Define a parameter y such that $dy/d\tau = -1/(\epsilon\lambda(y))$ and $y = 1$ at $\tau = 0$. Let $l_{k,t}$ and $r_{i,t}$ denote random variables representing the number of edges connecting to the variable nodes of degree k and the check nodes of degree i, respectively, in the residual graph at the iteration round t. Let d_c be the maximum degree of check nodes. We define $\bar{\mathcal{R}} := \{1, 2, \dots, d_c - 1\}$. We also define a set of random variables

$$\mathcal{D}_t := \{l_{k,t} \mid k \in \mathcal{L}\} \cup \{r_{k,t} \mid k \in \bar{\mathcal{R}}\}.$$

To simplify the notation, we drop the subscript t. For $X \in \mathcal{D} \cup \{r_{d_c}\}$, we define $\bar{X}(y)$ by

$$\bar{X}(y) := \frac{\mathbb{E}[X]}{\xi}.$$

For $i \in \mathcal{L}$ and $j \in \{2, \dots, d_c\}$ as the block length tends to infinity, Luby et al. [3] showed that $\bar{X}(y)$ is given by

$$\begin{aligned}\bar{l}_i(y) &= \epsilon\lambda_i y^i,\\ \bar{r}_j(y) &= \sum_{i\in\mathcal{R}} \rho_i \binom{i-1}{j-1} x^j \tilde{x}^{i-j},\\ \bar{r}_1(y) &= x(y - 1 + \rho(\tilde{x})),\end{aligned}$$

where $x := \epsilon\lambda(y)$ and $\tilde{x} := 1 - x$. We define $\delta^{(X,Y)}(y)$ by

$$\delta^{(X,Y)}(y) := \frac{\mathrm{Cov}[X,Y]}{\xi}, \qquad (X, Y \in \mathcal{D}),$$

where $\mathrm{Cov}[X,Y]$ is the covariance of X and Y. To simplify the notation, we drop y. In [4], [6], Amraoui et al. showed that $\delta^{(X,Y)}$ satisfy the following system of differential equations for irregular LDPC code ensembles as the block length tends to infinity.

$$\begin{aligned}\frac{d\delta^{(X,Y)}}{dy} = -\frac{e}{y}\Big[\sum_{Z\in\mathcal{D}}\Big(\frac{\partial \hat{f}^{(X)}}{\partial \bar{Z}}\delta^{(Y,Z)} + \frac{\partial \hat{f}^{(Y)}}{\partial \bar{Z}}\delta^{(X,Z)}\Big)\\ + \hat{f}^{(X,Y)}\Big],\end{aligned} \tag{2}$$

and this system of differential equation is referred to as covariance evolution. Let $I_{\{.\}}$ be the indicator function which is 1 if the condition inside the braces is fulfilled and 0 otherwise. Define $e(y) := \sum_{i\in\mathcal{L}} \bar{l}_i = xy,\ \ x' := \frac{dx}{dy},\ \ a := \sum_{i\in\mathcal{L}} \frac{i\bar{l}_i}{e} = \frac{x'y+x}{x}$ and $G_j(y) := \frac{j(\bar{r}_{j+1}-\bar{r}_j)}{x}$. The terms in the covariance evolution are given by the following for $k,s \in \mathcal{L}$, $i \in \bar{\mathcal{R}}$ and $j \in \{1,2,\ldots,d_c-2\}$

$$\begin{aligned}
\frac{\partial \hat{f}^{(l_k)}}{\partial \bar{l}_s} &= \frac{k\bar{l}_k}{e^2} - I_{\{k=s\}}\frac{k}{e},\\
\frac{\partial \hat{f}^{(l_k)}}{\partial \bar{r}_i} &= 0,\\
\frac{\partial \hat{f}^{(r_j)}}{\partial \bar{l}_k} &= -\frac{2a-k-1}{e}\frac{G_j}{y},\\
\frac{\partial \hat{f}^{(r_j)}}{\partial \bar{r}_i} &= j\frac{a-1}{e}(I_{\{i=j+1\}} - I_{\{i=j\}}),\\
\frac{\partial \hat{f}^{(r_{d_c-1})}}{\partial \bar{l}_k} &= (d_c-1)\frac{a-1}{e} - \frac{2a-k-1}{e}\frac{G_{d_c-1}}{y},\\
\frac{\partial \hat{f}^{(r_{d_c-1})}}{\partial \bar{r}_i} &= -(d_c-1)\frac{a-1}{e}(1+I_{\{i=d_c-1\}}),
\end{aligned}$$

and for $k,s \in \mathcal{L}$ and $i,j \in \bar{\mathcal{R}}$

$$\begin{aligned}
\hat{f}^{(l_k,l_s)} &= ks\frac{\bar{l}_k}{e}(I_{\{k=s\}} - \frac{\bar{l}_s}{e}),\\
\hat{f}^{(l_k,r_i)} &= (a-k)\frac{k\bar{l}_k}{e}\frac{G_i}{y},\\
\hat{f}^{(r_i,r_j)} &= \frac{x''x-(x')^2}{x^2}G_iG_j\\
&\quad + ij\frac{x'}{x^2}\big[I_{\{i=j\}}(\bar{r}_{j+1}+\bar{r}_j) - I_{\{i=j+1\}}\bar{r}_i - I_{\{j=i+1\}}\bar{r}_j\big].
\end{aligned}$$

Initial conditions of the covariance evolution are also given by Amraoui et al. [4], [6]. For $i,j \in \bar{\mathcal{R}} \cup \{d_c\}$ and $k,s \in \mathcal{L}$, the initial conditions of the covariance evolution are derived as follows:

$$\begin{aligned}
\delta^{(l_k,l_s)}(1) &= I_{\{k=s\}}k\lambda_k\epsilon\tilde{\epsilon},\\
\delta^{(l_k,r_i)}(1) &= -k\lambda_k\epsilon\tilde{\epsilon}G_i(1),\\
\delta^{(r_i,r_j)}(1) &= I_{\{i=j\}}i\bar{r}_i(1) - V_{i,j}(1) + \lambda'(1)\epsilon\tilde{\epsilon}G_i(1)G_j(1),
\end{aligned}$$

where $\tilde{\epsilon} := 1-\epsilon$ and

$$V_{i,j}(y) := \sum_{s\in\mathcal{R}} s\rho_s\binom{s-1}{i-1}\binom{s-1}{j-1}x^{i+j}\tilde{x}^{2s-i-j}.$$

D. Scaling Law

Let $P_B(\epsilon,n)$ be the block error probability under BP decoding for channel erasure probability ϵ and block length n. *Threshold* is defined by

$$\epsilon^* := \sup\{\epsilon \in [0,1] \mid \lim_{n\to\infty} P_B(\epsilon,n) = 0\},$$

and characterized via *density evolution* as follows:

$$\epsilon^* = \sup\{\epsilon \in [0,1] \mid y > 1-\rho(1-\epsilon\lambda(y)), \forall y \in (0,1]\}.$$

The curve of the block error probability for finite length LDPC codes is divided two regions which called *waterfall region* and *error floor region*. In the waterfall region, the block error probability drops off steeply as the function of channel erasure probability. In the error floor region, the block error probability has a gentle slope. A *scaling law* is a technique to estimate the waterfall region. The scaling law is based on the analysis of the residual graphs.

In [6], the block error probability $P_B(n,\epsilon)$ is given by

$$P_B(n,\epsilon) = Q\big(\frac{\sqrt{n}(\epsilon^*-\epsilon)}{\alpha}\big) + o(1),$$

where α is *slope scaling parameter* depending on the ensemble and the Q-function is defined by

$$Q(z) := \frac{1}{\sqrt{2\pi}} \int_{z}^{\infty} e^{-\frac{x^2}{2}} dx.$$

In [6], the slope scaling parameter is derived as

$$\alpha = -\sqrt{\frac{n}{\xi}} \sqrt{\delta^{(r_1,r_1)}\Big|_{\epsilon^*;y^*}} \left(\frac{\partial \bar{r}_1}{\partial \epsilon}\Big|_{\epsilon^*;y^*} \right)^{-1} \tag{3}$$

where y^* is the non-zero solution of $\bar{r}_1(y) = 0$ at the threshold (i.e. define y^* such that $y^* = 1 - \rho(1 - \epsilon^* \lambda(y^*))$) and ξ is the total number of edges in the original graph.

III. Main Results

We show, in the following theorem, the analytical solution of the covariance evolution, for irregular LDPC code ensembles. The proof shall be given in Section IV.

Theorem 1. Consider transmission over the BEC(ϵ). Let τ be the normalized iteration round of PA as defined in (1). A parameter y is defined by $dy/d\tau = -1/(\epsilon\lambda(y))$. For an irregular LDPC code ensemble, $i, j \in \bar{\mathcal{R}}$ and $k, s \in \mathcal{L}$, in the limit of the code length, we obtain the following.

$$\begin{aligned}
\delta^{(l_k,l_s)} &= -\frac{ks\bar{l}_k\bar{l}_s}{e^2} F + \frac{\epsilon \bar{l}_k \bar{l}_s}{e} \big[k(y^s - 1) + s(y^k - 1)\big] \\
&\quad + I_{\{k=s\}} k\bar{l}_k (1 - \epsilon y^k), && (4)\\
\delta^{(l_s,r_j)} &= \big[F\frac{s\bar{l}_s}{e} - \epsilon\bar{l}_s(y^s - 1)\big]\big(\frac{x'}{x} G_j - I_{\{j=1\}}\big) \\
&\quad - \frac{s\bar{l}_s}{e} G_j \big(\frac{F' + x}{2} - \epsilon x y^s\big), && (5)\\
\delta^{(r_i,r_j)} &= -F\big(\frac{x'}{x} G_i - I_{\{i=1\}}\big)\big(\frac{x'}{x} G_j - I_{\{j=1\}}\big) \\
&\quad + G_i G_j \big(F' \frac{x'}{x} - \sum_{s \in \mathcal{L}} \epsilon^2 s \lambda_s y^{2s-2} + x^2\big) - V_{i,j} \\
&\quad + \big(I_{\{j=1\}} G_i + I_{\{i=1\}} G_j\big)\big[x(e - x) - \frac{F' - x}{2}\big] \\
&\quad + I_{\{i=j\}} i \bar{r}_i + I_{\{i=j=1\}} (e - x)^2, && (6)
\end{aligned}$$

where $F := \sum_i \frac{\lambda_i}{i}[\epsilon^2(y^i - 1)^2 + \epsilon(y^i - 1)]$ and $F' = \frac{dF}{dy} = 2\sum_i \epsilon^2 \lambda_i y^{2i-1} - (\epsilon - \tilde{\epsilon})x$.

Using Theorem 1, we can obtain the following corollary.

Corollary 1. Let ϵ^* be the threshold of the ensemble under BP decoding, n be the block length and ξ be the total number of edges in the original graph. For irregular LDPC codes, the slope scaling parameter α is given by

$$\begin{aligned}
\alpha = \Big[&\frac{\rho(\tilde{x}^*)^2 - \rho(\tilde{x}^{*2}) - \tilde{x}^{*2}\rho'(\tilde{x}^{*2})}{\rho'(\tilde{x}^*)^2} + \frac{1 - 2x^*\rho(\tilde{x}^*)}{\rho'(\tilde{x}^*)} \\
&+ x^{*2} - \epsilon^{*2}\lambda(y^{*2}) - \epsilon^{*2} y^{*2} \lambda'(y^{*2})\Big]^{\frac{1}{2}} \sqrt{\frac{n}{\xi}} \frac{1}{\lambda(y^*)}, && (7)
\end{aligned}$$

where $x^* := \epsilon^* \lambda(y^*)$ and $\tilde{x}^* := 1 - x^*$.

Proof: Since $\bar{r}_1|_{\epsilon^*;y^*} = 0$ and $\frac{\partial \bar{r}_1}{\partial y}\Big|_{\epsilon^*;y^*} = 0$, we see that $1 - y^* = \rho(\tilde{x}^*)$ and $\rho'(\tilde{x}^*)\epsilon^*\lambda'(y^*) = 1$. Using those equations, we have from (6),

$$\begin{aligned}
&\delta^{(r_1,r_1)}\Big|_{\epsilon^*;y^*} \\
&= x^{*2}[\rho(\tilde{x}^*)^2 - \tilde{x}^{*2}\rho'(\tilde{x}^{*2}) - \rho(\tilde{x}^{*2})] \\
&\quad + x^{*2}\rho'(\tilde{x}^*)[1 - 2x^*\rho(\tilde{x}^*)] \\
&\quad + (x^*\rho'(\tilde{x}^*))^2[x^{*2} - \epsilon^{*2}\lambda'(y^{*2})y^{*2} - \epsilon^{*2}\lambda(y^{2*})].
\end{aligned}$$

Recall that $\bar{r}_1 = x(y - 1 + \rho(\tilde{x}))$. We see that

$$\frac{\partial \bar{r}_1}{\partial \epsilon}\Big|_{\epsilon^*;y^*} = -\lambda(y^*) x^* \rho'(\tilde{x}^*).$$

From (3), we can obtain (7). ■

Remark 1. The result of Corollary 1 is the same as the result in [5] for irregular LDPC code ensembles. In particular, for (d_v, d_c)-regular LDPC code ensembles we can write

$$\alpha = \epsilon^* \sqrt{\frac{d_v - 1}{d_v}(\frac{1}{x^*} - \frac{1}{y^*})}.$$

IV. LEMMAS AND PROOFS

In this section, we state three lemmas and prove Theorem 1. Section IV-A, IV-B and IV-C give (4), (5) and (6), respectively.

A. Lemma and Proof of (4)

In this section, we give a lemma to prove (4) and we prove (4).

1) Lemma to Prove (4):

Lemma 1. Define $U^{(l_k;l_s)} := \frac{\delta^{(l_k,l_k)}}{(k\bar{l}_k)^2} - \frac{\delta^{(l_s,l_s)}}{(s\bar{l}_s)^2}$. For $k, s \in \mathcal{L}$, we have the following equations.

$$\sum_{k,s\in\mathcal{L}} \frac{\delta^{(l_k,l_s)}}{ks} = \epsilon\tilde{\epsilon} \sum_{i\in\mathcal{L}} \frac{\lambda_i}{i}, \tag{8}$$

$$2\frac{\delta^{(l_k,l_s)}}{ks\bar{l}_k\bar{l}_s} - \frac{\delta^{(l_k,l_k)}}{(k\bar{l}_k)^2} - \frac{\delta^{(l_s,l_s)}}{(s\bar{l}_s)^2} = (\frac{\epsilon y^k - 1}{k\bar{l}_k} + \frac{\epsilon y^s - 1}{s\bar{l}_s}) I_{\{k\neq s\}}, \tag{9}$$

$$U^{(l_k;l_s)} = -\frac{\epsilon y^k - 1}{k\bar{l}_k} + \frac{\epsilon y^s - 1}{s\bar{l}_s} + \frac{2\epsilon}{e}(\frac{y^k - 1}{k} - \frac{y^s - 1}{s}). \tag{10}$$

Proof: Define $\delta^{(l_k,l_\Sigma)} = \sum_{s\in\mathcal{L}} \delta^{(l_k,l_s)}$. From the covariance evolution, we have

$$\frac{d\delta^{(l_k,l_s)}}{dy} = -x\Big(\frac{s\bar{l}_s}{e^2}\delta^{(l_k,l_\Sigma)} + \frac{k\bar{l}_k}{e^2}\delta^{(l_s,l_\Sigma)} - \frac{k+s}{e}\delta^{(l_k,l_s)}\Big) - x\hat{f}^{(l_k,l_s)}. \tag{11}$$

a) Proof of (8): From (11), we have the following equation:

$$\sum_{k,s\in\mathcal{L}} \frac{1}{ks}\frac{d\delta^{(l_k,l_s)}}{dy} = 0.$$

From initial conditions, we have

$$\sum_{k,s\in\mathcal{L}} \frac{1}{ks}\delta^{(l_k,l_s)} = \epsilon\tilde{\epsilon}\sum_{i\in\mathcal{L}} \frac{\lambda_i}{i}.$$

This leads to (8).

b) Proof of (9): Obviously we can get (9) for $k = s$. From (11), we have

$$\frac{d}{dy}(\frac{\delta^{(l_k,l_s)}}{ks\bar{l}_k\bar{l}_s}) = \frac{1}{ks\bar{l}_k\bar{l}_s}\frac{d\delta^{(l_k,l_s)}}{dy} - \frac{k+s}{ks\bar{l}_k\bar{l}_s y}\delta^{(l_k,l_s)} = -x\Big(\frac{\hat{f}^{(l_k,l_s)}}{ks\bar{l}_k\bar{l}_s} + \frac{\delta^{(l_k,l_\Sigma)}}{k\bar{l}_k e^2} + \frac{\delta^{(l_s,l_\Sigma)}}{s\bar{l}_s e^2}\Big). \tag{12}$$

From those equations, we have

$$\frac{d}{dy}\Big(2\frac{\delta^{(l_k,l_s)}}{ks\bar{l}_k\bar{l}_s} - \frac{\delta^{(l_k,l_k)}}{(k\bar{l}_k)^2} - \frac{\delta^{(l_s,l_s)}}{(s\bar{l}_s)^2}\Big) = -2\frac{x\hat{f}^{(l_k,l_s)}}{ks\bar{l}_k\bar{l}_s} + \frac{x\hat{f}^{(l_k,l_k)}}{k^2\bar{l}_k^2} + \frac{x\hat{f}^{(l_s,l_s)}}{s^2\bar{l}_s^2} = \frac{1}{y}(\frac{1}{\bar{l}_k} + \frac{1}{\bar{l}_s}),$$

for $j \neq k$. This differential equation can be solve as follow:

$$2\frac{\delta^{(l_k,l_s)}}{ks\bar{l}_k\bar{l}_s} - \frac{\delta^{(l_k,l_k)}}{(k\bar{l}_k)^2} - \frac{\delta^{(l_s,l_s)}}{(s\bar{l}_s)^2} = -\frac{1}{k\bar{l}_k} - \frac{1}{s\bar{l}_s} + C,$$

with a constant C which can be determined from initial conditions. From initial conditions, we get

$$C = \frac{1}{k\lambda_k} + \frac{1}{s\lambda_s}.$$

Thus we have for $k \neq s$

$$2\frac{\delta^{(l_k,l_s)}}{ks\bar{l}_k\bar{l}_s} - \frac{\delta^{(l_k,l_k)}}{(k\bar{l}_k)^2} - \frac{\delta^{(l_s,l_s)}}{(s\bar{l}_s)^2} = \frac{\epsilon y^k - 1}{k\bar{l}_k} + \frac{\epsilon y^s - 1}{s\bar{l}_s}.$$

This leads to (9).

c) Proof of (10): From (9), we have for all $k, s \in \mathcal{L}$

$$\delta^{(l_k,l_s)} = \big[\frac{s\bar{l}_s}{2}(\epsilon y^k - 1) + \frac{k\bar{l}_k}{2}(\epsilon y^s - 1)\big]I_{\{k\neq s\}} + \frac{s\bar{l}_s}{2k\bar{l}_k}\delta^{(l_k,l_k)} + \frac{k\bar{l}_k}{2s\bar{l}_s}\delta^{(l_s,l_s)}.$$

The sum of this equation for $s \in \mathcal{L}$ is written as follows

$$\delta^{(l_k,l_\Sigma)} = \frac{ae}{2}(\epsilon y^k - 1) + \frac{k\bar{l}_k}{2}\sum_{s\in\mathcal{L}}(\epsilon y^s - 1) - k\bar{l}_k(\epsilon y^k - 1) + \frac{ae}{2k\bar{l}_k}\delta^{(l_k,l_k)} + k\bar{l}_k\sum_{s\in\mathcal{L}}\frac{\delta^{(l_s,l_s)}}{2s\bar{l}_s}.$$

Combining (12) with this equation, we have

$$\frac{d}{dy}\Big(\frac{\delta^{(l_k,l_k)}}{(k\bar{l}_k)^2}\Big) = K^{(l_k,l_k)} - \frac{x}{e^2}\sum_{s\in\mathcal{L}}(\epsilon y^s - 1) - \frac{x}{e^2}\sum_{s\in\mathcal{L}}\frac{\delta^{(l_s,l_s)}}{s\bar{l}_s} - \frac{a}{y}\frac{\delta^{(l_k,l_k)}}{(k\bar{l}_k)^2},$$

where

$$K^{(l_k,l_k)} := -x\frac{\hat{f}^{(l_k,l_k)}}{(k\bar{l}_k)^2} - \frac{a}{k\bar{l}_k y}(\epsilon y^k - 1) + \frac{2}{ey}(\epsilon y^k - 1).$$

From this equation, we have

$$\frac{dU^{(l_k;l_s)}}{dy} = K^{(l_k,l_k)} - K^{(l_s,l_s)} - \frac{a}{y}U^{(l_k;l_s)}. \tag{13}$$

Note that

$$\int \frac{a}{y}dy = \log xy.$$

Since (13) is a first order differential equation, it can be solved as follows:

$$U^{(l_k;l_s)} = \frac{1}{e}\int e\big(K^{(l_k,l_k)} - K^{(l_s,l_s)}\big)dy + \frac{1}{e}C,$$

with a constant C which is determined from initial conditions. Note that

$$\begin{aligned}&\int eK^{(l_k,l_k)}dy\\ &= \int\Big[-\frac{x'y + x}{k\lambda_k} + \frac{x'y - (k-1)x}{k\bar{l}_k} + 2\epsilon y^{k-1} - \frac{1}{y}\Big]dy\\ &= -\frac{e}{k\lambda_k} + \sum_{i\in\mathcal{L}}\frac{\bar{l}_i}{k\bar{l}_k}I_{\{i\neq k\}} + \frac{2\epsilon}{k}y^k - \log y.\end{aligned}$$

We get

$$\begin{aligned} U^{(l_k;l_s)} =& -\frac{\epsilon y^k - 1}{k\bar{l}_k} + \frac{\epsilon y^s - 1}{s\bar{l}_s} \\ &+ \frac{1}{e}\Big(\frac{2\epsilon y^k - 1}{k} - \frac{2\epsilon y^s - 1}{s} + C\Big). \end{aligned}$$

From initial conditions, we have $U^{(l_k;l_s)}(1) = \frac{\tilde{\epsilon}}{\epsilon}\big(\frac{1}{k\lambda_k} - \frac{1}{s\lambda_s}\big)$ and $C = \frac{1-2\epsilon}{k} - \frac{1-2\epsilon}{s}$. Therefore we have

$$U^{(l_k;l_s)} = \frac{1-\epsilon y^k}{k\bar{l}_k} - \frac{1-\epsilon y^s}{s\bar{l}_s} + \frac{2\epsilon}{e}\Big(\frac{y^k - 1}{k} - \frac{y^s - 1}{s}\Big).$$

This leads (10). ■

2) Proof of (4): By definition of $U^{(l_k;l_s)}$, we have

$$\bar{l}_k\delta^{(l_s,l_s)} = (s\bar{l}_s)^2\big(\frac{\delta^{(l_k,l_k)}}{k^2\bar{l}_k} - \bar{l}_k U^{(l_k;l_s)}\big).$$

The sum of this equation for $k \in \mathcal{L}$ is written as follows:

$$e\delta^{(l_s,l_s)} = (s\bar{l}_s)^2 \sum_{k\in\mathcal{L}}\big(\frac{\delta^{(l_k,l_k)}}{k^2\bar{l}_k} - \bar{l}_k U^{(l_k;l_s)}\big). \tag{14}$$

From (9), we see that for all $k, s \in \mathcal{L}$

$$\begin{aligned} &\frac{1}{2}\frac{\bar{l}_s}{k^2\bar{l}_k}\delta^{(l_k,l_k)} + \frac{1}{2}\frac{\bar{l}_k}{s^2\bar{l}_s}\delta^{(l_s,l_s)} \\ &= \frac{1}{ks}\delta^{(l_k,l_s)} - \frac{1}{2}\bar{l}_s\frac{\epsilon y^k - 1}{k}I_{\{k\neq s\}} - \frac{1}{2}\bar{l}_k\frac{\epsilon y^s - 1}{s}I_{\{k\neq s\}}. \end{aligned}$$

The sum over this equation for $k, s \in \mathcal{L}$ is written as follows:

$$e\sum_{k\in\mathcal{L}}\frac{\delta^{(l_k,l_k)}}{k^2\bar{l}_k} = \sum_{k,s\in\mathcal{L}}\frac{\delta^{(l_k,l_s)}}{ks} + \sum_{k\in\mathcal{L}}(\bar{l}_k - e)\frac{\epsilon y^k - 1}{k}. \tag{15}$$

Combining (15) with (8), we have

$$\sum_{k\in\mathcal{L}}\frac{\delta^{(l_k,l_k)}}{k^2\bar{l}_k} = \frac{\epsilon\tilde{\epsilon}}{e}\sum_{k\in\mathcal{L}}\frac{\lambda_k}{k} + \sum_{k\in\mathcal{L}}\frac{\bar{l}_k - e}{e}\frac{\epsilon y^k - 1}{k}. \tag{16}$$

From (10), we have

$$\begin{aligned} \sum_{k\in\mathcal{L}}\bar{l}_k U^{(l_k,l_s)} =& \frac{e}{s\bar{l}_s}(\epsilon y^s - 1) - 2\epsilon\frac{y^s - 1}{s} \\ &- \sum_{k\in\mathcal{L}}\frac{\epsilon y^k - 1}{k} + \frac{2\epsilon}{e}\sum_{k\in\mathcal{L}}\frac{\bar{l}_k(y^k - 1)}{k}. \end{aligned} \tag{17}$$

Combining (14) with (16) and (17), we obtain

$$\delta^{(l_s,l_s)} = -\frac{(s\bar{l}_s)^2}{e^2}F + 2\epsilon\frac{s\bar{l}_s^2}{e}(y^s - 1) + s\bar{l}_s(1 - \epsilon y^s).$$

From this equation and (9), we can obtain (4) for $k, s \in \mathcal{L}$.

B. Lemma and Proof of (5)

In this section, we introduce a lemma to prove (5) and we prove (5).

1) Lemma to Prove (5):

Lemma 2. Define $A^{(l_\Sigma,r_j)} := \sum_{i\in\mathcal{L}} \frac{1}{i}\delta^{(l_i,r_j)}$, $A^{(l_\Sigma,r_\Sigma)} := \sum_{j\in\bar{\mathcal{R}}} A^{(l_\Sigma,r_j)}$, $S^{(l_i,l_s;r_j)} := \frac{1}{i\bar{l}_i}\delta^{(l_i,r_j)} - \frac{1}{s\bar{l}_s}\delta^{(l_s,r_j)}$, $S^{(l_i,l_s;r_\Sigma)} := \sum_{j\in\bar{\mathcal{R}}} S^{(l_i,l_s;r_j)}$ and $G_\Sigma := \sum_{j\in\bar{\mathcal{R}}} G_j = \frac{d_c\bar{r}_{d_c}-e}{x}$. For $j\in\bar{\mathcal{R}}$ and $k,s\in\mathcal{L}$, we have the following equations.

$$A^{(l_\Sigma,r_\Sigma)} = \epsilon\tilde{\epsilon}\sum_{i\in\mathcal{L}}\frac{\lambda_i}{i}(y^i-1)(G_\Sigma\frac{x'}{x}-1) - \tilde{\epsilon}xG_\Sigma, \tag{18}$$

$$A^{(l_\Sigma,r_j)} = \epsilon\tilde{\epsilon}\sum_{i\in\mathcal{L}}\frac{\lambda_i}{i}(y^i-1)(G_j\frac{x'}{x}-I_{\{j=1\}}) - \tilde{\epsilon}xG_j, \tag{19}$$

$$\begin{aligned} S^{(l_k,l_s;r_\Sigma)} &= -\epsilon\big(G_\Sigma\frac{x'}{x}-1\big)\big(\frac{y^k-1}{k}-\frac{y^s-1}{s}\big) \\ &\quad + \epsilon G_\Sigma(y^{k-1}-y^{s-1}), \end{aligned} \tag{20}$$

$$\begin{aligned} S^{(l_k,l_s;r_j)} &= -\epsilon\big(G_j\frac{x'}{x}-I_{\{j=1\}}\big)\big(\frac{y^k-1}{k}-\frac{y^s-1}{s}\big) \\ &\quad + \epsilon G_j(y^{k-1}-y^{s-1}). \end{aligned} \tag{21}$$

We use (18) and (20) to prove the basis of the mathematical induction in proof of (19) and (21), respectively.

Proof: First, we will derive differential equations. We define $\delta^{(l_\Sigma,r_j)} := \sum_{k\in\mathcal{L}}\delta^{(l_k,r_j)}$, $\delta^{(l_k,r_\Sigma)} := \sum_{j\in\bar{\mathcal{R}}}\delta^{(l_k,r_j)}$ and $\delta^{(l_k,r_{d_c})} := \delta^{(l_k,l_\Sigma)} - \delta^{(l_k,r_\Sigma)}$, respectively. From the covariance evolution (2), we can write for $j\in\bar{\mathcal{R}}$ and $k\in\mathcal{L}$

$$\begin{aligned} \frac{d\delta^{(l_k,r_j)}}{dy} &= D^{(l_k,r_j)} - \frac{k\bar{l}_k}{ey}\delta^{(l_\Sigma,r_j)} + \frac{k}{y}\delta^{(l_k,r_j)} \\ &\quad - j\frac{x'}{x}(\delta^{(l_k,r_{j+1})} - \delta^{(l_k,r_j)}), \end{aligned} \tag{22}$$

where

$$\begin{aligned} D^{(l_k,r_j)} &:= 2\frac{x'}{e}G_j\delta^{(l_k,l_\Sigma)} - \frac{G_j}{y^2}\sum_{i\in\mathcal{L}}(i-1)\delta^{(l_k,l_i)} \\ &\quad - x\hat{f}^{(l_k,r_j)}. \end{aligned}$$

We define $A^{(l_\Sigma,r_j)} := \sum_{k\in\mathcal{L}}\frac{1}{k}\delta^{(l_k,r_j)}$, $A^{(l_\Sigma,r_\Sigma)} := \sum_{j\in\bar{\mathcal{R}}} A^{(l_\Sigma,r_j)}$ and $D^{(l_k,r_\Sigma)} := \sum_{j\in\bar{\mathcal{R}}} D^{(l_k,r_j)}$. From (22), we have for $k\in\bar{\mathcal{R}}$

$$\frac{dA^{(l_\Sigma,r_j)}}{dy} = \sum_{k\in\mathcal{L}}\frac{D^{(l_k,r_j)}}{k} - j\frac{x'}{x}\big(A^{(l_\Sigma,r_{j+1})} - A^{(l_\Sigma,r_j)}\big). \tag{23}$$

The sum over this equation for $j\in\bar{\mathcal{R}}$ is written as the follows:

$$\begin{aligned} \frac{dA^{(l_\Sigma,r_\Sigma)}}{dy} &= \sum_{k\in\mathcal{L}}\frac{D^{(l_k,r_\Sigma)}}{k} - (d_c-1)\frac{x'}{x}\sum_{k\in\mathcal{L}}\frac{1}{k}\delta^{(l_k,l_\Sigma)} \\ &\quad + d_c\frac{x'}{x}A^{(l_\Sigma,r_\Sigma)}. \end{aligned} \tag{24}$$

From (22), we see that

$$\begin{aligned} \frac{d}{dy}\Big(\frac{\delta^{(l_k,r_j)}}{k\bar{l}_k}\Big) &= \frac{D^{(l_k,r_j)}}{k\bar{l}_k} - \frac{1}{ey}\delta^{(l_\Sigma,r_j)} \\ &\quad - j\frac{x'}{x}\frac{\delta^{(l_k,r_{j+1})} - \delta^{(l_k,r_j)}}{k\bar{l}_k}. \end{aligned} \tag{25}$$

Define $S^{(l_k,l_s;r_j)} := \frac{\delta^{(l_k,r_j)}}{k\bar{l}_k} - \frac{\delta^{(l_s,r_j)}}{s\bar{l}_s}$, $S^{(l_k,l_s;l_i)} := \frac{\delta^{(l_k,l_i)}}{k\bar{l}_k} - \frac{\delta^{(l_s,l_i)}}{s\bar{l}_s}$ and $S^{(l_k,l_s;l_\Sigma)} := \frac{\delta^{(l_k,l_\Sigma)}}{k\bar{l}_k} - \frac{\delta^{(l_s,l_\Sigma)}}{s\bar{l}_s}$. From (25), we have

$$\begin{aligned} \frac{dS^{(l_k,l_s;r_j)}}{dy} &= \frac{D^{(l_k,r_j)}}{k\bar{l}_k} - \frac{D^{(l_s,r_j)}}{s\bar{l}_s} \\ &\quad - j\frac{x'}{x}\big(S^{(l_k,l_s;r_{j+1})} - S^{(l_k,l_s;r_j)}\big), \end{aligned} \tag{26}$$

for $k,s\in\mathcal{L}$ and $j\in\bar{\mathcal{R}}$. The sum over this equation for $j\in\bar{\mathcal{R}}$ is written as the follows:

$$\begin{aligned} \frac{dS^{(l_k,l_s;r_\Sigma)}}{dy} &= \frac{D^{(l_k,r_\Sigma)}}{k\bar{l}_k} - \frac{D^{(l_s,r_\Sigma)}}{s\bar{l}_s} - (d_c-1)\frac{x'}{x}S^{(l_k,l_s;l_\Sigma)} \\ &\quad + d_c\frac{x'}{x}S^{(l_k,l_s;r_\Sigma)}. \end{aligned} \tag{27}$$

a) Proof of (18): Since (24) is a first order differential equation, it can be solve as follows [1]:

$$\begin{aligned} &A^{(l_\Sigma,r_\Sigma)} \\ &= x^{d_c}\int \frac{1}{x^{d_c}}\Big[\sum_{k\in\mathcal{L}}\frac{D^{(l_k,r_\Sigma)}}{k} - (d_c-1)\frac{x'}{x}\sum_{k\in\mathcal{L}}\frac{\delta^{(l_k,l_\Sigma)}}{k}\Big]dy \\ &\quad + Cx^{d_c} \\ &= \epsilon\tilde{\epsilon}\sum_{k\in\mathcal{L}}\frac{\lambda_k}{k}(y^k-1)\big(G_\Sigma\frac{x'}{x}-1\big) + \tilde{\epsilon}xy + Cx^{d_c}, \end{aligned}$$

with a constant C which is determined initial conditions. From initial conditions, we see that

$$A^{(l_\Sigma,r_\Sigma)}(1) = \epsilon\tilde{\epsilon}(1-d_c\rho_{d_c}\epsilon^{d_c-1}).$$

From this equation, we can determine $C=-d_c\rho_{d_c}\tilde{\epsilon}$. Thus, we get

$$A^{(l_\Sigma,r_\Sigma)} = \epsilon\tilde{\epsilon}\sum_{k\in\mathcal{L}}\frac{\lambda_k}{k}(y^k-1)\big(G_\Sigma\frac{x'}{x}-1\big) - \tilde{\epsilon}xG_\Sigma.$$

Hence, we have (18).

b) Proof of (19): Since (23) is a first order differential equation, it can be solve as follows:

$$\begin{aligned} A^{(l_\Sigma,r_j)} = x^j\int\frac{1}{x^j}\Big(\sum_{k\in\mathcal{L}}\frac{D^{(l_k,r_j)}}{k} - j\frac{x'}{x}A^{(l_\Sigma,r_{j+1})}\Big)dy \\ + C_{l_\Sigma,r_j}x^j, \end{aligned} \tag{28}$$

with a constant C_{l_Σ,r_j} which can be determined from initial conditions.

We solve (28) by mathematical induction for $j\in\{2,3,\dots,d_c-1\}$. From (18), we have

$$A^{(l_\Sigma,r_{d_c})} = \epsilon\tilde{\epsilon}\sum_{i\in\mathcal{L}}\frac{\lambda_i}{i}(y^i-1)\frac{x'}{x}G_{d_c} - \tilde{\epsilon}xG_{d_c}$$

where $G_{d_c} = -\frac{d_c\bar{r}_{d_c}}{x}$. Using the same method in the induction step, we can show that $A^{(l_\Sigma,r_{d_c-1})}$ fulfill (19).

We show that if $A^{(l_\Sigma,r_{j+1})}$ fulfill (19), then also $A^{(l_\Sigma,r_j)}$ fulfill (19). Using the induction hypothesis, we have

$$\begin{aligned} &\sum_{k\in\mathcal{L}}\frac{D^{(l_k,r_j)}}{k} - j\frac{x'}{x}A^{(l_\Sigma,r_{j+1})} \\ &= -j\frac{(x')^2}{x^2}\epsilon\tilde{\epsilon}\sum_{i\in\mathcal{L}}\frac{\lambda_i}{i}(y^i-1)G_{j+1} + jx'G_{j+1}\tilde{\epsilon} \\ &\quad + x'G_j\tilde{\epsilon} + G_j\epsilon\tilde{\epsilon}\sum_{i\in\mathcal{L}}\frac{\lambda_i}{i}(y^i-1)\Big(\frac{x''}{x}-2\frac{(x')^2}{x^2}\Big). \end{aligned}$$

Using integration by parts, we have

$$\begin{aligned} &\int\frac{1}{x^j}G_j\epsilon\tilde{\epsilon}\sum_{i\in\mathcal{L}}\frac{\lambda_i}{i}(y^i-1)\frac{x''}{x}dy \\ &= \epsilon\tilde{\epsilon}\frac{G_j}{x^{j+1}}\sum_{i\in\mathcal{L}}\frac{\lambda_i}{i}(y^i-1)x' \\ &\quad - \epsilon\tilde{\epsilon}\int\Big(\frac{G_j}{x^{j+1}}\sum_{i\in\mathcal{L}}\frac{\lambda_i}{i}(y^i-1)\Big)'x'dy. \end{aligned} \tag{29}$$

Note that $G'_j = -j\frac{x'}{x}G_{j+1} + (j-1)\frac{x'}{x}G_j$ for $j\in\{2,\dots,d_c-1\}$. From (29), we have

$$\begin{aligned} &\int\frac{1}{x^j}\Big[-\epsilon\tilde{\epsilon}j\frac{(x')^2}{x^2}G_{j+1}\sum_{i\in\mathcal{L}}\frac{\lambda_i}{i}(y^i-1) + \sum_{i\in\mathcal{L}}\frac{D^{(l_i,r_j)}}{i}\Big]dy \\ &= \epsilon\tilde{\epsilon}G_j\sum_{i\in\mathcal{L}}\frac{\lambda_i}{i}(y^i-1)\frac{x'}{x^{j+1}}. \end{aligned} \tag{30}$$

[1]In a way similar to Section IV-B1b, we perform this calculation.

We get

$$\int \frac{1}{x^j} j G_{j+1} x' \tilde{\epsilon} dy = -\frac{\tilde{\epsilon}}{x^{j-1}} G_j. \tag{31}$$

From the sum over (30) and (31), we have

$$\begin{aligned} x^j \int \frac{1}{x^j} \big(\sum_{i\in\mathcal{L}} \frac{D^{(l_i,r_j)}}{i} - j\frac{x'}{x} A^{(l_\Sigma, r_{j+1})}\big) dy \\ = \epsilon\tilde{\epsilon} \sum_{i\in\mathcal{L}} \frac{\lambda_i}{i}(y^i - 1) G_j \frac{x'}{x} - \tilde{\epsilon} G_j x. \end{aligned}$$

Thus, we have

$$A^{(l_\Sigma, r_j)} = \epsilon\tilde{\epsilon} \sum_{i\in\mathcal{L}} \frac{\lambda_i}{i}(y^i - 1) G_j \frac{x'}{x} - \tilde{\epsilon} G_j x + C_{l_\Sigma, r_j} x^j.$$

From initial conditions, we have $A^{(l_\Sigma, r_j)}(1) = -\epsilon\tilde{\epsilon} G_j(1)$ and $C_{l_\Sigma, r_j} = 0$. Hence we obtain

$$A^{(l_\Sigma, r_j)} = \epsilon\tilde{\epsilon} \sum_{i\in\mathcal{L}} \frac{\lambda_i}{i}(y^i - 1) G_j \frac{x'}{x} - \tilde{\epsilon} G_j x.$$

This leads to (19) for $j \in \{2, 3, \ldots, d_c - 1\}$.

Note that $A^{(l_\Sigma, r_1)} = A^{(l_\Sigma, r_\Sigma)} - \sum_{j=2}^{d_c-1} A^{(l_\Sigma, r_j)}$. We have

$$A^{(l_\Sigma, r_1)} = \epsilon\tilde{\epsilon} \sum_{i\in\mathcal{L}} \frac{\lambda_i}{i}(y^i - 1)\big(G_1 \frac{x'}{x} - 1\big) - \tilde{\epsilon} G_1 x.$$

Hence we obtain (19).

c) Proof of (20): Since (27) is a first order differential equation, it can be solve as follows:

$$\begin{aligned} S^{(l_k, l_s; r_\Sigma)} = x^{d_c} \int \frac{1}{x^{d_c}} \big[\frac{D^{(l_k, r_\Sigma)}}{k\bar{l}_k} - \frac{D^{(l_s, r_\Sigma)}}{s\bar{l}_s} \\ - (d_c - 1)\frac{x'}{x} S^{(l_k, l_s; l_\Sigma)}\big] dy + C x^{d_c}. \end{aligned}$$

Note that

$$\begin{aligned} \frac{D^{(l_k, r_\Sigma)}}{k\bar{l}_k} - \frac{D^{(l_s, r_\Sigma)}}{s\bar{l}_s} - (d_c - 1)\frac{x'}{x} S^{(l_k, l_s; l_\Sigma)} \\ = K^{(l_k, r_\Sigma)} - K^{(l_s, r_\Sigma)}, \end{aligned}$$

where

$$\begin{aligned} & K^{(l_k, r_\Sigma)} \\ & := \epsilon G_\Sigma \big[-2\frac{x'}{e} y^k + (k-1) y^{k-2} + \frac{2(x')^2 - x'' x}{x^2} \frac{y^k - 1}{k}\big] \\ & + (d_c - 1)\frac{x'}{x} \epsilon \big(y^k - \frac{x' y + x}{x} \frac{y^k - 1}{k}\big). \end{aligned}$$

Note that

$$x^{d_c} \int \frac{1}{x^{d_c}} K^{(l_k, r_\Sigma)} dy = -\epsilon\big(G_\Sigma \frac{x'}{x} - 1\big) \frac{y^k - 1}{k} + \epsilon G_\Sigma y^{k-1}.$$

Thus we have

$$\begin{aligned} S^{(l_k, l_s; r_\Sigma)} = & -\epsilon\big(G_\Sigma \frac{x'}{x} - 1\big)\big(\frac{y^k - 1}{k} - \frac{y^s - 1}{s}\big) \\ & + \epsilon G_\Sigma (y^{k-1} - y^{s-1}) + C x^{d_c}. \end{aligned}$$

From the initial covariance, we have $S^{(l_i, l_s; r_\Sigma)}(1) = 0$ and $C = 0$. This leads to (20).

d) Proof of (21): In a way similar to Section IV-B1b, we can obtain (21). ■

2) Proof of (5): From definitions of $S^{(l_k,l_s;r_j)}$ and $A^{(l_\Sigma,r_j)}$, we see that

$$\begin{aligned}\delta^{(l_s,r_j)} &= \frac{s\bar{l}_s}{e}\Big(A^{(l_\Sigma,r_j)} - \sum_{k\in\mathcal{L}}\bar{l}_k S^{(l_k,l_s;r_j)}\Big)\\ &= \big[F\frac{s\bar{l}_s}{e} - \epsilon\bar{l}_s(y^s-1)\big]\big(\frac{x'}{x}G_j - I_{\{j=1\}}\big)\\ &\quad - \frac{s\bar{l}_s}{e}G_j\big(\frac{F'+x}{2} - \epsilon x y^s\big).\end{aligned}$$

Thus, we obtain (5).

C. Lemma and Proof of (6)

In this section, we introduce a lemma to prove (6) and we prove (6).

1) Lemma to Prove (6):

Lemma 3. We define $\delta^{(r_j,r_\Sigma)} := \sum_{k\in\bar{\mathcal{R}}}\delta^{(r_j,r_k)}$ and $\delta^{(r_\Sigma,r_\Sigma)} := \sum_{j\in\bar{\mathcal{R}}}\delta^{(r_j,r_\Sigma)}$. For $j\in\bar{\mathcal{R}}$, we have the following equations.

$$\begin{aligned}\delta^{(r_\Sigma,r_\Sigma)} =&\ -F\big(\frac{x'}{x}G_\Sigma - 1\big)^2 + F'G_\Sigma\big(\frac{x'}{x}G_\Sigma - 1\big)\\ &- G_\Sigma^2\sum_{i\in\mathcal{L}} i\epsilon^2\lambda_i y^{2i-2} + d_c^2\bar{r}_{d_c}^2 - V_{d_c,d_c},\end{aligned} \tag{32}$$

$$\begin{aligned}\delta^{(r_j,r_\Sigma)} =&\ -F\big(\frac{x'}{x}G_\Sigma - 1\big)\big(\frac{x'}{x}G_j - I_{\{j=1\}}\big)\\ &+ F'G_j\big(\frac{x'}{x}G_\Sigma - 1\big) - G_\Sigma G_j\sum_{i\in\mathcal{L}}\epsilon^2 i\lambda_i y^{2i-2}\\ &+ d_c\bar{r}_{d_c}xG_j + V_{j,d_c} + \frac{F'-x}{2}\big(G_j - I_{\{j=1\}}G_\Sigma\big)\\ &+ I_{\{j=1\}}d_c\bar{r}_{d_c}(e-x).\end{aligned} \tag{33}$$

We use (32) to prove of the basis for the mathematical induction in proof of (33). Similarly, we use (33) to prove of the basis for the mathematical induction in proof of (6).

Proof: First, we derive differential equations. We define $\delta^{(r_i,r_\Sigma)} := \sum_{j\in\bar{\mathcal{R}}}\delta^{(r_i,r_j)}$, $\delta^{(r_\Sigma,r_\Sigma)} := \sum_{i\in\bar{\mathcal{R}}}\delta^{(r_i,r_\Sigma)}$ and $\delta^{(r_{d_c},r_j)} := \delta^{(l_\Sigma,r_j)} - \delta^{(r_\Sigma,r_j)}$. From covariance evolution (2), we get

$$\begin{aligned}\frac{d\delta^{(r_i,r_j)}}{dy} =&\ -\frac{x'}{x}\big[i\delta^{(r_{i+1},r_j)} + j\delta^{(r_{j+1},r_i)} - (i+j)\delta^{(r_j,r_i)}\big]\\ &+ D^{(r_i,r_j)},\end{aligned} \tag{34}$$

where

$$\begin{aligned}D^{(r_i,r_j)} :=& \sum_{k\in\mathcal{L}}\frac{2a-k-1}{y^2}\big(\delta^{(l_k,r_j)}G_i + \delta^{(l_k,r_i)}G_j\big)\\ &- x\hat{f}^{(r_i,r_j)}.\end{aligned}$$

Define $D^{(r_i,r_\Sigma)} := \sum_{j\in\mathcal{L}} D^{(r_i,r_j)}$ and $D^{(r_\Sigma,r_\Sigma)} := \sum_{i\in\mathcal{L}} D^{(r_i,r_\Sigma)}$. For $\delta^{(r_i,r_\Sigma)}$, we have

$$\begin{aligned}\frac{d\delta^{(r_i,r_\Sigma)}}{dy} =&\ -\frac{x'}{x}\big[i\delta^{(r_{i+1},r_\Sigma)} - (d_c+i)\delta^{(r_i,r_\Sigma)}\big]\\ &- \frac{x'}{x}(d_c-1)\delta^{(l_\Sigma,r_i)} + D^{(r_i,r_\Sigma)}.\end{aligned}$$

The sum over this equation for $i\in\bar{\mathcal{R}}$ is written as follows:

$$\begin{aligned}\frac{d\delta^{(r_\Sigma,r_\Sigma)}}{dy} =&\ -2\frac{x'}{x}\big[(d_c-1)\delta^{(l_\Sigma,r_\Sigma)} - d_c\delta^{(r_\Sigma,r_\Sigma)}\big]\\ &+ D^{(r_\Sigma,r_\Sigma)}.\end{aligned} \tag{35}$$

a) Proof of (32): Since (35) is a first order differential equation, it can be solve as follows:

$$\begin{aligned}\delta^{(r_\Sigma,r_\Sigma)} &= x^{2d_c}\int \frac{1}{x^{2d_c}}\big[D^{(r_\Sigma,r_\Sigma)} - 2(d_c-1)\frac{x'}{x}\delta^{(l_\Sigma,r_\Sigma)}\big]dy \\ &\quad + x^{2d_c}C_{r_\Sigma,r_\Sigma} \\ &= -F\big(\frac{x'}{x}G_\Sigma - 1\big)^2 + G_\Sigma\big(\frac{x'}{x}G_\Sigma - 1\big)F' \\ &\quad - G_\Sigma^2\sum_{i\in\mathcal{L}} i\epsilon^2\lambda_i y^{2i-2} + C_{r_\Sigma,r_\Sigma}x^{2d_c},\end{aligned}$$

with a constant C_{r_Σ,r_Σ} which can be determined from initial conditions. From initial conditions, we have

$$\begin{aligned}\delta^{(r_\Sigma,r_\Sigma)}(1) &= \lambda'(1)\epsilon\tilde{\epsilon}(d_c\rho_{d_c}\epsilon^{d_c-1}-1)^2 + \epsilon\tilde{\epsilon} \\ &\quad - 2\epsilon\tilde{\epsilon}d_c\rho_{d_c}\epsilon^{d_c-1} + d_c\rho_{d_c}\epsilon^{d_c} - d_c\rho_{d_c}\epsilon^{2d_c}\end{aligned}$$

and $C_{r_\Sigma,r_\Sigma} = d_c^2\rho_{d_c}^2 - d_c\rho_{d_c}$. Thus we have

$$\begin{aligned}\delta^{(r_\Sigma,r_\Sigma)} &= -F\big(\frac{x'}{x}G_\Sigma - 1\big)^2 + G_\Sigma\big(\frac{x'}{x}G_\Sigma - 1\big)F' \\ &\quad - G_\Sigma^2\sum_{i\in\mathcal{L}} i\epsilon^2\lambda_i y^{2i-2} + d_c^2 r_{d_c}^2 - V_{d_c,d_c}.\end{aligned}$$

This leads to (32).

b) Proof of (33): In a way similar to Section IV-B1b, we can obtain (33). ■

2) Proof of (6): (34) can be solve as follows:

$$\begin{aligned}&\delta^{(r_i,r_j)} \\ &= x^{i+j}\int\frac{1}{x^{i+j}}\big(D^{(r_i,r_j)} - \frac{x'}{x}i\delta^{(r_{i+1},r_j)} - \frac{x'}{x}j\delta^{(r_i,r_{j+1})}\big)dy \\ &\quad + C_{r_i,r_j}x^{i+j}.\end{aligned} \tag{36}$$

This equation can be solved by mathematical induction for $i,j\in\{2,3,\dots,d_c-1\}$. Note that from (33)

$$\begin{aligned}&\delta^{(r_j,r_{d_c})} \\ &\quad = G_jG_{d_c}\big[-F\big(\frac{x'}{x}\big)^2 + F'\frac{x'}{x} - \sum_{s\in\mathcal{L}}\epsilon^2 s\lambda_s y^{2s-2} + x^2\big] \\ &\qquad - V_{j,d_c},\end{aligned}$$

for $j\in\{2,3,\dots,d_c-1\}$. Using the same method in the induction step, we see that $\delta^{(r_{d_c-1},r_{d_c-1})}$ fulfill (6).

We show that if $\{\delta^{(r_i,r_j)} \mid i,j\in\{2,3,\dots,d_c-1\}, i+j=k+1\}$ fulfill (6), then $\{\delta^{(r_i,r_j)} \mid i,j\in\{2,3,\dots,d_c-1\}, i+j=k\}$ fulfill (6). Using the induction hypothesis, we can solve (36)

$$\begin{aligned}&\delta^{(r_i,r_j)} \\ &= S_iS_j\big[-F\big(\frac{x'}{x}\big)^2 + F'\frac{x'}{x} + x^2 - \sum_s\epsilon^2 s\lambda_s y^{2s-2}\big] - V_{i,j} \\ &\quad + I_{\{i=j\}}i\sum_s\rho_s\binom{s-1}{i-1}\big[x^i\tilde{x}^{s-i} - \binom{s-i}{i}x^i(-x)^i\big] \\ &\quad + C_{r_i,r_j}x^{i+j}.\end{aligned}$$

From the initial condition, we get

$$C_{r_i,r_j} = I_{\{i=j\}}i\sum_s\rho_s\binom{s-1}{i-1}\binom{s-i}{i}(-1)^i.$$

Thus, we have (6) for $i,j\in\{2,\dots,d_c-1\}$.

Note that $\delta^{(r_i,r_1)} = \delta^{(r_i,r_\Sigma)} - \sum_{j=2}^{d_c-1}\delta^{(r_i,r_j)}$. We show that $\delta^{(r_i,r_1)}$ fulfill (6) for $i\in\bar{\mathcal{R}}$. Hence we obtain (6).

V. Conclusion

In this paper, we have analytically solved the covariance evolution for irregular LDPC code ensembles. We have also obtained the slope scaling parameter.

REFERENCES

[1] R. G. Gallager, *Low-density parity-check codes*, MIT Press, 1963.
[2] M. Luby, M. Mitzenmacher, A. Shokrollahi, and D. Spielman, "Improved low-density parity-check codes using irregular graphs," *IEEE Trans. Inform. Theory*, vol. 47, no. 2, pp. 585–598, Feb. 2001.
[3] M. Luby, M. Mitzenmacher, A. Shokrollahi, D. Spielman, and V. Stemann. "Practical loss-resilient codes," in *Proc. the 29th annual ACM Symposium on Theory of Computing*, 1997, pp. 150-159.
[4] A. Amraoui, A. Montanari, and R. Urbanke, "Finite-length scaling of irregular LDPC code ensembles," in *Proc. IEEE Inform. Theory Workshop*, Rotorua, New-Zealand, Aug./Sep. 2005.
[5] A. Amraoui, A. Montanari, and R. Urbanke, "How to find good finite-length codes: From art towards science," *Europ. Trans. Telecomm.* , vol. 18, pp. 491–508, Aug. 2007.
[6] A. Amraoui, A. Montanari, T. Richardson, and R. Urbanke, "Finite-length scaling for iteratively decoded LDPC ensembles," *IEEE Trans. Inform. Theory*, vol. 55, no. 2, pp. 473–498, Feb. 2009.
[7] T. Richardson and R. Urbanke, *Modern Coding Theory*, Cambridge University Press, 2008.
[8] J. Ezri, A. Montanari, S. Oh, and R. Urbanke, "The Slope Scaling Parameter for General Channels, Decoders, and Ensembles," in *Proc. IEEE Int. Symp. on Inform. Theory,* Toronto, Canada, pp. 1443–1447, Jul. 2008.
[9] H. D. Pfister, "Finite-length analysis of a capacity-achieving ensemble for the binary erasure channel," in *Proc. IEEE Inform. Theory Workshop*, Rotorua, New Zealand, pp. 166–170, Sep. 2005.
[10] I. Andriyanova, "Finite-length scaling of repeat-accumulate codes on the BEC," in *Proc. 2008 IEEE Int. Zurich Seminar on Communications,* Zurich, Switzerland, pp. 64–67, Mar. 2008.
[11] I. Andriyanova, "Finite-length scaling of turbo-like code ensembles on the binary erasure channel," *IEEE Journal on Selected Areas in Communications*, vol. 27, no. 6, pp. 918–927, Aug. 2009.