The Annals of Applied Probability
2010, Vol. 20, No. 5, 1801–1830
DOI: 10.1214/09-AAP673

WIENER–HOPF FACTORIZATION AND DISTRIBUTION OF EXTREMA FOR A FAMILY OF LÉVY PROCESSES

By Alexey Kuznetsov[1]

York University

In this paper we introduce a ten-parameter family of Lévy processes for which we obtain Wiener–Hopf factors and distribution of the supremum process in semi-explicit form. This family allows an arbitrary behavior of small jumps and includes processes similar to the generalized tempered stable, KoBoL and CGMY processes. Analytically it is characterized by the property that the characteristic exponent is a meromorphic function, expressed in terms of beta and digamma functions. We prove that the Wiener–Hopf factors can be expressed as infinite products over roots of a certain transcendental equation, and the density of the supremum process can be computed as an exponentially converging infinite series. In several special cases when the roots can be found analytically, we are able to identify the Wiener–Hopf factors and distribution of the supremum in closed form. In the general case we prove that all the roots are real and simple, and we provide localization results and asymptotic formulas which allow an efficient numerical evaluation. We also derive a convergence acceleration algorithm for infinite products and a simple and efficient procedure to compute the Wiener–Hopf factors for complex values of parameters. As a numerical example we discuss computation of the density of the supremum process.

1. Introduction. Wiener–Hopf factorization is a powerful tool in the study of various functionals of a Lévy process, such as extrema of the process, first passage time and the overshoot, the last time the extrema was achieved, etc. These results are very important from the theoretical point of view; for example, they can be used to prove general theorems about short/long time behavior (see [7, 18, 21] and [29]). However, in recent years, there has also

Received January 2009; revised December 2009.
[1]Supported in part by the Natural Sciences and Engineering Research Council of Canada.
AMS 2000 subject classifications. Primary 60G51; secondary 60E10.
Key words and phrases. Lévy process, supremum process, Wiener–Hopf factorization, meromorphic function, infinite product.

been a growing interest in applications of Wiener–Hopf factorization, for example, in Insurance Mathematics and the classical ruin problem (see [2]) and in Mathematical Finance, where the above-mentioned functionals are being used to describe the payoff of a contract and the corresponding probability distribution is used to compute its price (see [3, 10, 26] and [30] and the references therein).

Let us summarize one of the most important results from Wiener–Hopf factorization. Assume that X_t is a one-dimensional real-valued Lévy process started from $X_0 = 0$ and defined by a triple (μ, σ, ν), where $\mu \in \mathbb{R}$ specifies the linear component, $\sigma \geq 0$ is the volatility of the Gaussian component and $\nu(dx)$ is the Lévy measure satisfying $\int_{\mathbb{R}} \min\{1, x^2\}\nu(dx) < \infty$. The characteristic exponent $\Psi(z)$ is defined by

$$\mathbb{E}[e^{izX_t}] = e^{-t\Psi(z)}, \qquad z \in \mathbb{R},$$

and the Lévy–Khintchine representation (see [7]) tells us that $\Psi(z)$ can be expressed in terms of the generating triple (μ, σ, ν) as follows:

$$\Psi(z) = \frac{1}{2}\sigma^2 z^2 - i\mu z - \int_{\mathbb{R}} (e^{izx} - 1 - izh(x))\nu(dx). \tag{1}$$

Here $h(x)$ is the cut-off function, which in general can be taken to be equal to $x\mathbf{I}_{\{|x|<1\}}$; however, in this paper we will use $h(x) \equiv 0$ (Section 2) or $h(x) \equiv x$ (Sections 3 and 4).

We define extrema processes

$$S_t = \sup\{X_s : 0 \leq s \leq t\}, \qquad I_t = \inf\{X_s : 0 \leq s \leq t\}$$

introduce an exponential random variable $\tau = \tau(q)$ with parameter $q > 0$, which is independent of the process X_t, and use the following notation for characteristic functions of S_τ and I_τ:

$$\phi_q^+(z) = \mathbb{E}[e^{izS_{\tau(q)}}], \qquad \phi_q^-(z) = \mathbb{E}[e^{izI_{\tau(q)}}].$$

The Wiener–Hopf factorization states that the random variables S_τ and $X_\tau - S_\tau$ are independent, random variables I_τ and $X_\tau - S_\tau$ have the same distribution; thus for $z \in \mathbb{R}$ we have

$$\begin{aligned} \frac{q}{q + \Psi(z)} &= \mathbb{E}[e^{izX_\tau}] \\ &= \mathbb{E}[e^{izS_\tau}]\mathbb{E}[e^{iz(X_\tau - S_\tau)}] = \phi_q^+(z)\phi_q^-(z). \end{aligned} \tag{2}$$

Moreover, random variable S_τ (I_τ) is infinitely divisible, positive (negative) and has no linear component in the Lévy–Khintchine representation (1). There also exist several integral representations for $\phi_q^\pm$ in terms of $\mathbb{P}(X_t \in dx)$ (see [7, 18, 21] and [29]) or in terms of $\Psi(z)$ [6, 25].

The integral expressions for the Wiener–Hopf factors $\phi_q^{\pm}$ are quite complicated; however, in the case of stable process it is possible to obtain explicit formulas for a dense class of parameters (see [17]). It is remarkable that in some cases we can compute Wiener–Hopf factors explicitly with the help of factorization identity (2). As an example, let us consider the case when the Lévy measure is of phase-type. Phase-type distribution (see [2]) can be defined as the distribution of the first passage time of a finite state continuous time Markov chain. A Lévy process X_t whose jumps are phase-type distributed enjoys the following analytical property: its characteristic function $\Psi(z)$ is a rational function. Thus function $q(q+\Psi(z))^{-1}$ is also a rational function, and therefore it has a finite number of zeros/poles in the complex plane $\mathbb{C}$. And here is the main idea: since the random variable S_τ (I_τ) is positive (negative) and infinitely divisible, its characteristic function must be analytic and have no zeros in $\mathbb{C}^+$ $(\mathbb{C}^-)$, where

$$\mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}, \qquad \mathbb{C}^- = \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}, \qquad \bar{\mathbb{C}}^{\pm} = \mathbb{C}^{\pm} \cup \mathbb{R}.$$

Thus we can *uniquely* identify $\phi_q^+(z)$ $[\phi_q^-(z)]$ as a rational function, which has value one at $z=0$ and whose poles/zeros coincide with poles/zeros of $q(q+\Psi(z))^{-1}$ in $\mathbb{C}^-$ $(\mathbb{C}^+)$.

While Lévy processes with phase-type jumps are very convenient objects to work with and one can implement efficient numerical schemes, there are some unresolved difficulties. One of them is that by definition phase-type distribution has a smooth density on $[0,\infty)$; in particular the density of the Lévy measure cannot have a singularity at zero. This means that if we want to work with a process with infinite activity of jumps, we have to approximate its Lévy measure by a sequence of phase-type measures, but then the degree of rational function $\Psi(z)$ would go to infinity and the above algorithm for computing Wiener–Hopf factors would quickly become unfeasible.

In this paper we address this problem and discuss Wiener–Hopf factorization for processes whose Lévy measure can have a singularity of arbitrary order at zero. The main idea is quite simple: if characteristic exponent $\Psi(z)$ is *meromorphic* in $\mathbb{C}$ and if we have sufficient information about zeros/poles of $q+\Psi(z)$, we can still use factorization identity (2) essentially in the same way as in the case of phase-type distributed jumps, except that all the finite products will be replaced by infinite products, and we have to be careful with the convergence issues. The main analytical tools will be asymptotic expansion of solutions to $q+\Psi(z)=0$ and asymptotic results for infinite products.

The paper is organized as follows: in Section 2 we introduce a simple example of a compound Poisson process, whose Lévy measure has a density given by $\nu(x) = e^{\alpha x}\mathrm{sech}(\mathrm{x})$. We obtain closed form expressions for the

Wiener–Hopf factors and density of S_τ. Also, in this simple case we introduce many ideas and tools which will be used in other sections. In Section 3 we introduce a Lévy process X_t with jumps of infinite variation and the density of the Lévy measure $\nu(x) = e^{\alpha x} \sinh(x/2)^{-2}$. This process is a member of the general β-family defined later in Section 4; however, it is quite unique because its characteristic exponent $\Psi(z)$ is expressed in terms of simpler functions, and thus all the formulas are easier and stronger results can be proved. In this section we derive the localization results and asymptotic expansion for the solutions of $q + \Psi(iz) = 0$, prove that all of them are real and simple, obtain explicit formulas for sums of inverse powers of these solutions and finally obtain semi-explicit formulas for Wiener–Hopf factors and distribution of supremum S_τ. In Section 4 we define the ten-parameter β-family of Lévy processes and derive formulas for characteristic exponent and prove results similar to the ones in Section 3. Section 5 deals with numerical issues: we discuss acceleration of convergence of infinite products and introduce an efficient method to compute roots of $q + \Psi(z)$ for q complex. As an example we compute the distribution of the supremum process S_t.

2. A compound Poisson process. In this section we study a compound Poisson process X_t, defined by a Lévy measure having density

$$\nu(x) = \frac{e^{\alpha x}}{\cosh(x)}.$$

We take the cut-off function $h(x)$ in (1) to be equal to zero, and thus the characteristic exponent of X_t is given by

$$\Psi(z) = -\int_{\mathbb{R}} (e^{ixz} - 1)\nu(x)\, dx = \frac{\pi}{\cos(\pi/2\alpha)} - \frac{\pi}{\cosh(\pi/2(z - i\alpha))}, \tag{3}$$

and the above integral can be computed with the help of formula 3.981.3 in [20]. Our main result in this section is the following theorem, which provides closed-form expressions for the Wiener–Hopf factors and the distribution of S_τ.

Theorem 1. *Assume that $q > 0$. Define*

$$\begin{aligned} \eta &= \frac{2}{\pi} \arccos\left(\frac{\pi}{q + \pi \sec(\pi/2\alpha)}\right), \\ p_0 &= \frac{\Gamma(1/4(1-\alpha))\Gamma(1/4(3-\alpha))}{\Gamma(1/4(\eta-\alpha))\Gamma(1/4(4-\eta-\alpha))}. \end{aligned} \tag{4}$$

Then for $\mathrm{Im}(z) > (\alpha - \eta)$ we have

$$\phi_q^+(z) = p_0 \frac{\Gamma(1/4(\eta - \alpha - iz))\Gamma(1/4(4 - \eta - \alpha - iz))}{\Gamma(1/4(1 - \alpha - iz))\Gamma(1/4(3 - \alpha - iz))}. \tag{5}$$

We have $\mathbb{P}(S_\tau = 0) = p_0$*, and the density of* S_τ *is given by*

$$
\begin{aligned}
&\frac{d}{dx}\mathbb{P}(S_\tau \le x) \\
&\quad = \frac{2p_0}{\pi}\cot\left(\frac{\pi\eta}{2}\right) \\
&\qquad \times \left[\frac{\Gamma(1/4(1+\eta))\Gamma(1/4(3+\eta))}{\Gamma(1/2\eta)}\right. \\
&\qquad\quad \times e^{(\alpha-\eta)x}{}_2F_1\left(\frac{1+\eta}{4}, \frac{3+\eta}{4}; \frac{\eta}{2}; e^{-4x}\right) \\
&\qquad\quad - \frac{\Gamma(1/4(5-\eta))\Gamma(1/4(7-\eta))}{\Gamma(1/2(4-\eta))} \\
&\qquad\qquad \left.\times e^{(\alpha-4+\eta)x}{}_2F_1\left(\frac{5-\eta}{4}, \frac{7-\eta}{4}; \frac{4-\eta}{2}; e^{-4x}\right)\right],
\end{aligned} \tag{6}
$$

where ${}_2F_1(a,b;c;z)$ *is the Gauss hypergeometric function. If* $q=0$ *and* $\alpha<0$*, equation (4) implies* $\eta=|\alpha|$*, and formulas (5) and (6) are still valid. In this case the random variable* $S_{\tau(0)}$ *should be interpreted as* $S_\infty = \sup\{X_s : s \ge 0\}$*.*

First we will state and prove the following lemma, which will be used repeatedly in this paper. It is a variant of the Wiener–Hopf argument, which we have borrowed from the proof of Lemma 45.6 in [29].

LEMMA 2. *Assume we have two functions* $f^+(z)$ *and* $f^-(z)$*, such that* $f^\pm(0)=1$*,* $f^\pm(z)$ *are analytic in* $\mathbb{C}^\pm$*, continuous and have no roots in* $\bar{\mathbb{C}}^\pm$ *and* $z^{-1}\ln(f^\pm(z)) \to 0$ *as* $z\to\infty$*,* $z\in\bar{\mathbb{C}}^\pm$*. If*

$$
\frac{q}{q+\Psi(z)} = f^+(z)f^-(z), \qquad z\in\mathbb{R}, \tag{7}
$$

then $f^\pm(z)\equiv\phi_q^\pm(z)$*.*

PROOF. We define function $F(z)$ as

$$
F(z) = \begin{cases} \dfrac{\phi_q^-(z)}{f^-(z)}, & \text{if } z\in\bar{\mathbb{C}}^-, \\[2ex] \dfrac{f^+(z)}{\phi_q^+(z)}, & \text{if } z\in\bar{\mathbb{C}}^+. \end{cases}
$$

Function $F(z)$ is well defined for z real due to (7) and (2). Using properties of $\phi_q^\pm$ and $f^\pm$ we conclude that $F(z)$ is analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$ and continuous in $\mathbb{C}$, and therefore by analytic continuation (see Theorem 16.8 on page 323 in [28]) it must be analytic in the entire complex plane. Moreover, by construction function $F(z)$ has no zeros in $\mathbb{C}$, thus its logarithm is also an entire function. All that is left to do is to prove that function $\ln(F(z))$ is constant.

Using integration by parts and formula (1) one could prove the following result: if ξ is an infinitely divisible positive random variable with no drift and $\Psi_\xi(z)$ is its characteristic exponent, then $z^{-1}\Psi_\xi(z) \to 0$ as $z \to \infty$, $z \in \bar{\mathbb{C}}^+$ (this statement is similar to Proposition 2 in [7]). Thus

$$z^{-1}\ln(\phi_q^{\pm}(z)) \to 0, \qquad z \to \infty, \qquad z \in \bar{\mathbb{C}}^{\pm}.$$

Since functions $f^{\pm}$ also satisfy the above conditions, we find that $z^{-1}\ln(F(z)) \to 0$ as $|z| \to \infty$ in the entire complex plane. Thus we have an analytic function $\ln(F(z))$ which grows slower than $|z|$ as $z \to \infty$, and therefore we can conclude that this function must be constant (a rigorous way to prove this is to apply Cauchy's estimates, see Proposition 2.14 on page 73 in [16]). The value of this constant is easily seen to be zero, since $f^{\pm}(0) = \phi_q^{\pm}(0) = 1$. $\square$

Proof of Theorem 1. Using expression (3) for $\Psi(z)$ we find that function $q(q+\Psi(z))^{-1}$ has simple zeros at $\{i(1+\alpha+4n), i(3+\alpha+4n)\}$ and simple poles at $\{i(\alpha+\eta+4n), i(\alpha-\eta+4n)\}$, where $n \in \mathbb{Z}$ and η is defined by (4). Next we check that $|\alpha| < \eta < 1$ and define function $f^+(z)$ as product over all zeros/poles lying in $\mathbb{C}^-$

$$f^+(z) = \prod_{n \ge 0} \frac{(1 - iz/(4n+1-\alpha))(1 - iz/(4n+3-\alpha))}{(1 - iz/(4n+\eta-\alpha))(1 - iz/(4n+4-\eta-\alpha))} \tag{8}$$

and similarly $f^-(z)$ as product over zeros/poles in $\mathbb{C}^+$. It is easy to see that the product converges uniformly on compact subsets of $\mathbb{C} \setminus i\mathbb{R}$ since each term is $1 + O(n^{-2})$ (see Corollary 5.6 on page 166 in [16] for sufficient conditions for the absolute convergence of infinite products). The fact that $f^+(z)$ is equal to the right-hand side of formula (5) can be seen by applying the following result from [19]:

$$\prod_{n \ge 0} \frac{1 + x/(n+a)}{1 + x/(n+b)} = \frac{\Gamma(a)\Gamma(b+x)}{\Gamma(b)\Gamma(a+x)}. \tag{9}$$

The formula for $f^-(z)$ is identical to (5) with (z, α) replaced by $(-z, -\alpha)$.

Now we will prove that $f^{\pm}(z) \equiv \phi_q^{\pm}(z)$. First, using the reflection formula for the gamma function (formula 8.334.3 in [20]), one can check that for $z \in \mathbb{R}$ functions $f^{\pm}(z)$ satisfy factorization identity (7). Next, using the following asymptotic expression (formula 6.1.47 in [1]):

$$\frac{\Gamma(a+x)}{\Gamma(b+x)} = x^{a-b} + O(x^{a-b-1}), \tag{10}$$

we conclude that $z^{-1}\ln(f^{\pm}(z)) \to 0$ as $z \to \infty$, $z \in \bar{\mathbb{C}}^{\pm}$, and thus all the conditions of Lemma 2 are satisfied, and we conclude that $f^{\pm}(z) \equiv \phi_q^{\pm}(z)$.

To derive formula (6) for the density of S_τ we use equations (5) and (10) to find that $\mathbb{E}[e^{-\zeta S_\tau}] = \phi_q^+(i\zeta) \to p_0$ as $\zeta \to \infty$, where p_0 is given by (4). This implies that distribution of S_τ has an atom at $x = 0$ (which should not be surprising since X_t is a compound Poisson process), and $\mathbb{P}(S_\tau = 0) = p_0$. The density of S_τ can be computed by the inverse Fourier transform

$$\begin{aligned} &\frac{d}{dx}\mathbb{P}(S_\tau \le x) \\ &\qquad = \frac{1}{2\pi}\int_{\mathbb{R}} [\phi_q^+(z) - p_0] e^{-ixz}\, dz \\ &\qquad = \frac{p_0}{2\pi}\int_{\mathbb{R}} \left[\frac{\Gamma(1/4(\eta-\alpha-iz))\Gamma(1/4(4-\eta-\alpha-iz))}{\Gamma(1/4(1-\alpha-iz))\Gamma(1/4(3-\alpha-iz))} - 1\right] e^{-ixz}\, dz. \end{aligned}$$

Formula (6) is obtained from the above expression by replacing the contour of integration by $ic + \mathbb{R}$, letting $c \to -\infty$ and evaluating the residues at $z \in \{-i(4n+\eta-\alpha), -i(4n+4-\eta-\alpha)\}$ for $n \ge 0$. Evaluating the residues can be made easier by using the reflection formula for the gamma function. □

Remark 1. There are other examples of Lévy measures $\nu(x)\,dx$, which have finite total mass (and thus can define a process with a finite intensity of jumps), and for which the characteristic exponent is a simple meromorphic function. These are two examples based on theta functions (see Section 8.18 in [20] for definition and properties of theta functions):

$$\nu_1(x) = e^{-\alpha x}\theta_2(0, e^{-x}) = e^{-\alpha x}\left[2\sum_{n\ge 0} e^{-(n+1/2)^2 x}\right],$$

$$\nu_2(x) = e^{-\alpha x}\theta_3(0, e^{-x}) = e^{-\alpha x}\left[1 + 2\sum_{n\ge 0} e^{-n^2 x}\right].$$

These two jump densities are defined on $x > 0$, they decay exponentially as $x \to +\infty$ and behave as $x^{-1/2}$ as $x \to 0^+$; thus the total mass is finite. The Fourier transform of these functions can be computed using formulas 6.162 in [20]

$$\int_0^\infty e^{ixz}\nu_1(x)\, dx = \frac{\pi}{\sqrt{\alpha - iz}}\tanh(\pi\sqrt{\alpha - iz}),$$

$$\int_0^\infty e^{ixz}\nu_2(x)\, dx = \frac{\pi}{\sqrt{\alpha - iz}}\coth(\pi\sqrt{\alpha - iz}).$$

Unfortunately equation $q + \Psi(z) = 0$ cannot be solved explicitly which implies that we cannot obtain closed form results as in Theorem 1; however, these processes could be treated using methods presented in the next sections.

3. A process with jumps of infinite variation. In this section we study a Lévy process X_t, defined by a triple (μ, σ, ν), where the density of the Lévy measure is given by

$$\nu(x) = \frac{e^{\alpha x}}{[\sinh(x/2)]^2}$$

with $|\alpha| < 1$ (it is a Lévy measure of a Lamperti-stable process with characteristics $(1, 1+\alpha, 1-\alpha)$, see [22]). The jump part of X_t is similar to the normal inverse Gaussian process (see [4, 15]), as it is also a process of infinite variation, the jump measure decays exponentially as $|x| \to \infty$ and has a $O(x^{-2})$ singularity at $x = 0$. Note that since the Lévy measure has exponential tails we can take the cut-off function $h(x) \equiv x$ in (1).

By definition process X_t has three parameters. However, if we want to achieve greater generality for modeling purposes, we could introduce two additional scaling parameters a and $b > 0$ and define a process $Y_t = aX_{bt}$, thus obtaining a five parameter family of Lévy processes.

PROPOSITION 3. *The characteristic exponent of X_t is given by*

$$\Psi(z) = \tfrac{1}{2}\sigma^2 z^2 + i\rho z + 4\pi(z - i\alpha)\coth(\pi(z - i\alpha)) - 4\gamma, \tag{11}$$

where

$$\gamma = \pi\alpha\cot(\pi\alpha), \qquad \rho = 4\pi^2\alpha + \frac{4\gamma(\gamma-1)}{\alpha} - \mu.$$

PROOF. We start with the series representation valid for $x > 0$,

$$\left[\sinh\left(\frac{x}{2}\right)\right]^{-2} = 4\frac{e^{-x}}{(1-e^{-x})^2} = 4\sum_{n\ge1} n e^{-nx}, \tag{12}$$

which can be easily obtained using binomial series or by taking derivative of a geometric series. The infinite series in (12) converges uniformly on (ε, ∞) for every $\varepsilon > 0$, thus

$$\begin{aligned}
&\int_0^\infty (e^{izx} - 1 - izx)\frac{e^{\alpha x}}{\sinh(x/2)^2}\,dx \\
&\qquad = 4\sum_{n\ge1}\left[\frac{n}{n-\alpha-iz} - \frac{n}{n-\alpha} - \frac{inz}{(n-\alpha)^2}\right] \\
&\qquad = 4\sum_{n\ge1}\left[\frac{\alpha+iz}{n-\alpha-iz} - \frac{\alpha+iz}{n-\alpha} - \frac{i\alpha z}{(n-\alpha)^2}\right].
\end{aligned}$$

The integral in the Lévy–Khintchine representation (1) for $\Psi(z)$ can now be computed as

$$\int_0^{\infty} (e^{izx} - 1 - izx)\frac{e^{\alpha x}}{\sinh(x)^2}\,dx + \int_0^{\infty} (e^{-izx} - 1 + izx)\frac{e^{-\alpha x}}{\sinh(x)^2}\,dx$$
$$= 8(\alpha + iz)^2 \sum_{n\geq 1} \frac{1}{n^2 - (\alpha + iz)^2} - 8(\alpha + iz)\alpha \sum_{n\geq 1} \frac{1}{n^2 - \alpha^2}$$
$$- 4i\alpha z \sum_{n\in\mathbb{Z}} \frac{1}{(n-\alpha)^2} + \frac{4iz}{\alpha}.$$

To complete the proof we need to use the following well-known series expansions (see formulas 1.421.4 and 1.422.4 in [20])

$$\coth(\pi x) = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{n\geq 1} \frac{1}{x^2 + n^2},$$
$$1 + \cot(\pi x)^2 = \operatorname{cosec}(\pi x)^2 = \frac{1}{\pi^2} \sum_{n\in\mathbb{Z}} \frac{1}{(n-x)^2}.$$ □

Note that it is impossible to find solutions to $q + \Psi(z) = 0$ explicitly in the general case, even though the characteristic exponent $\Psi(z)$ is quite simple. It is remarkable that in some special cases, when $\sigma = 0$ and parameters μ, α and q satisfy certain conditions, we can still obtain closed-form results. Below we present just one example of this type.

PROPOSITION 4. *Assume that $\sigma = \alpha = 0$. Define*

$$\eta = \frac{1}{\pi} \operatorname{arccot}\left(\frac{\mu}{4\pi}\right). \tag{13}$$

Then Wiener–Hopf factor $\phi_q^+(z)$ can be computed in closed form when $q = 4$,

$$\phi_4^+(z) = \frac{\Gamma(\eta - iz)}{\Gamma(\eta)\Gamma(1 - iz)}. \tag{14}$$

The density of $S_{\tau(4)}$ is given by

$$\frac{d}{dx}\mathbb{P}(S_{\tau(4)} \leq x) = \frac{\sin(\pi\eta)}{\pi}(e^x - 1)^{-\eta}.$$

The proof of Proposition 4 is identical to the proof of Theorem 1.

REMARK 2. This result is very similar to Proposition 1 in [11], where the authors are able to compute the law of $I_{\tau(q)}$ in closed form only for a single value of q, and this law is essentially identical to the distribution of $S_{\tau(4)}$.

This coincidence seems to be rather surprising, since these propositions study different processes: our Proposition 4 is concerned with a Lamperti-stable process having characteristics $(1,1,1)$ (see [22]) and completely arbitrary drift, while Proposition 1 in [11] studies a Lamperti-stable process with characteristics $(\alpha,1,\alpha)$ but with no freedom in specifying the drift, which must be uniquely expressed in terms of parameters of the Lévy measure.

The following theorem is one of the main results in this section. It describes various properties of solutions to equation $q+\Psi(z)=0$, which will be used later to compute Wiener–Hopf factors and the distribution of the supremum process.

THEOREM 5. *Assume that $q>0$ and that $\Psi(z)$ is given by (11).*

(i) *Equation $q+\Psi(i\zeta)=0$ has infinitely many solutions, all of which are real and simple. They are located as follows:*

$$\begin{aligned}
&\zeta_0^- \in (\alpha-1,0),\\
&\zeta_0^+ \in (0,\alpha+1),\\
&\zeta_n \in (n+\alpha, n+\alpha+1), \qquad n\ge 1\\
&\zeta_n \in (n+\alpha-1, n+\alpha), \qquad n\le -1.
\end{aligned} \tag{15}$$

(ii) *If $\sigma\neq 0$ we have as $n\to\pm\infty$*

$$\begin{aligned}
\zeta_n = (n+\alpha) &+ \frac{8}{\sigma^2}(n+\alpha)^{-1}\\
&-\frac{8}{\sigma^2}\left(\frac{2\rho}{\sigma^2}+\alpha\right)(n+\alpha)^{-2}+O(n^{-3}).
\end{aligned} \tag{16}$$

(iii) *If $\sigma=0$ we have as $n\to\pm\infty$*

$$\begin{aligned}
\zeta_{n+\delta} = (n+\alpha+\omega_0) &+ c_0(n+\alpha+\omega_0)^{-1}\\
&-\frac{c_0}{\rho}(4\gamma-q-4\pi^2 c_0)(n+\alpha+\omega_0)^{-2}+O(n^{-3}),
\end{aligned} \tag{17}$$

where

$$c_0=-\frac{4(4\gamma-q+\alpha\rho)}{16\pi^2+\rho^2}, \qquad \omega_0=\frac{1}{\pi}\operatorname{arccot}\left(\frac{\rho}{4\pi}\right)$$

and $\delta\in\{-1,0,1\}$ depending on the signs of n and ρ.

(iv) *Function $q(q+\Psi(z))^{-1}$ can be factorized as follows:*

$$\frac{q}{q+\Psi(z)}=\frac{1}{(1+iz/\zeta_0^+)(1+iz/\zeta_0^-)}\prod_{|n|\ge 1}\frac{1+iz/(n+\alpha)}{1+iz/\zeta_n}, \tag{18}$$

where the infinite product converges uniformly on the compact subsets of the complex plane excluding zeros/poles of $q+\Psi(z)$.

First we need to prove the following technical result.

LEMMA 6. *Assume that* α *and* β *are not equal to a negative integer, and* $b_n = O(n^{-\varepsilon_1})$ *for some* $\varepsilon_1 > 0$ *as* $n \to \infty$*. Then*

$$\prod_{n\ge 0} \frac{1+z/(n+\alpha)}{1+z/(n+\beta+b_n)} \approx C z^{\beta-\alpha} \tag{19}$$

as $z \to \infty$, $|\arg(z)| < \pi - \varepsilon_2 < \pi$, *where* $C = \frac{\Gamma(\alpha)}{\Gamma(\beta)} \prod_{n\ge 0}(1 + \frac{b_n}{n+\beta})$.

PROOF. First we have to justify absolute convergence of infinite products. A product in the left-hand side of (19) converges since each term is $1+O(n^{-2})$ and the infinite product in the definition of constant C converges since each term is $1+O(n^{-1-\varepsilon_1})$ (see Corollary 5.6 on page 166 in [16] for sufficient conditions for the absolute convergence of infinite products). Thus we can rewrite the left-hand side of (19) as

$$\begin{aligned}\prod_{n\ge 0} \frac{1+z/(n+\alpha)}{1+z/(n+\beta+b_n)} &= \prod_{n\ge 0} \frac{1+z/(n+\alpha)}{1+z/(n+\beta)} \prod_{n\ge 0} \frac{1+z/(n+\beta)}{1+z/(n+\beta+b_n)} \\ &= C\frac{\Gamma(\beta+z)}{\Gamma(\alpha+z)} \prod_{n\ge 0} \frac{z+n+\beta}{z+n+\beta+b_n}.\end{aligned} \tag{20}$$

The ratio of gamma functions gives us the leading asymptotic term $z^{\beta-\alpha}$ due to (10). Now we need to prove that the last infinite product in (20) converges to one as $z \to \infty$, $|\arg(z)| < \pi - \varepsilon_2 < \pi$. We take the logarithm of this product and estimate it as

$$\begin{aligned}\left|\sum_{n\ge 1} \ln\left(\frac{z+n+\beta}{z+n+\beta+b_n}\right)\right| &= \left|\sum_{n\ge 1} \ln\left(1+\frac{b_n}{z+n+\beta}\right)\right| \\ &\le \sum_{n\ge 1} \ln\left(1+\frac{|b_n|}{|z+n+\beta|}\right) \\ &\le \sum_{n\ge 1} \frac{|b_n|}{|z+n+\beta|} \\ &\le A \sum_{n\ge 1} \frac{1}{n^{\varepsilon_1}|z+n+\beta|},\end{aligned}$$

where we have used the fact that $\ln(1+x) < x$ for $x > 0$ and $|b_n| < A n^{-\varepsilon_1}$ for some $A > 0$. Since $|\arg(z)| < \pi - \varepsilon_2 < \pi$ we have for z sufficiently large

$|z+n+\beta| > \max\{1, |n-|z+\beta||\}$. Let $m=[|z+\beta|]$, where $[x]$ denotes the integer part of x. Then

$$\sum_{n\ge 1} \frac{1}{n^{\varepsilon_1}|z+n+\beta|} < \sum_{n=1}^{m} \frac{1}{n^{\varepsilon_1}(m+1-n)} + \sum_{n=m+1}^{\infty} \frac{1}{n^{\varepsilon_1}(n-m)}. \tag{21}$$

The first series in the right-hand side of (21) converges to zero as $m\to\infty$, since

$$\sum_{n=1}^{m} \frac{1}{n^{\varepsilon_1}(m+1-n)} = \sum_{n=1}^{[\sqrt{m}]} \frac{1}{n^{\varepsilon_1}(m+1-n)} + \sum_{n=[\sqrt{m}]+1}^{m} \frac{1}{n^{\varepsilon_1}(m+1-n)}$$
$$< \frac{[\sqrt{m}]}{m+1-[\sqrt{m}]} + m^{-\varepsilon_1/2} \sum_{n=[\sqrt{m}]+1}^{m} \frac{1}{(m+1-n)}$$
$$< \frac{[\sqrt{m}]}{m+1-[\sqrt{m}]} + m^{-\varepsilon_1/2}\ln(m).$$

The second series in the right-hand side of (21) can be rewritten as $\sum_{n=1}^{\infty}(n+m)^{-\varepsilon_1}n^{-1}$, and we see that it is a convergent series of positive terms, where each term converges to zero as $m\to\infty$. By considering its partial sums it is easy to prove that the series itself must converge to zero as $m\to\infty$. □

PROOF OF THEOREM 5. The proof consists of three steps. The first step is to study solutions to equation $q+\Psi(i\zeta)=0$. We will produce a sequence of "obvious" solutions ζ_n and study their asymptotics as $n\to\pm\infty$. Note that this first step requires quite demanding computations, which can be made much more enjoyable if one uses a symbolic computation package. The second step is to represent the function $q(q+\Psi(z))^{-1}$ as a general infinite product, which includes poles of $\Psi(z)$, zeros of $q+\Psi(z)$ (given by "obvious" ones $\{i\zeta_0^{\pm}, i\zeta_n\}$ and possibly some "unaccounted" zeros) and an exponential factor. Our main tool will be Hadamard theorem (Theorem 1, page 26, in [24] or Theorem 3.4, page 289, in [16]). We produce entire functions $P(z)$ and $Q(z)$, such that $P(z)$ has zeros at poles of $\Psi(z)$ and $q(q+\Psi(z))^{-1}=P(z)/Q(z)$. After studying the growth rate of $P(z)$ and $Q(z)$ we apply Hadamard theorem and obtain an infinite product for function $Q(z)$ [function $P(z)$ will have an explicit infinite product]. These results give us an infinite product for $q(q+\Psi(z))^{-1}$. The last step is to prove the absence of exponential factor and "unaccounted" zeros in this infinite product, and here the main tool will be asymptotic relation (19) for infinite products provided by Lemma 6.

First we will prove localization result (15). We use (11) to rewrite equation $q+\Psi(i\zeta)=0$ as

$$4\pi(\zeta-\alpha)\cot(\pi(\zeta-\alpha))-(\rho+\mu)\zeta-4\gamma=\tfrac{1}{2}\sigma^2\zeta^2-\mu\zeta-q. \tag{22}$$

Note that we have separated the jump part of $\Psi(z)$ on the left-hand side and the diffusion part on the right-hand side of (22). See Figure 1, where the jump part is represented by black line and diffusion part by grey dotted line.

The left-hand side of (22) is zero at $\zeta=0$ and goes to $-\infty$ as $\zeta\nearrow\alpha+1$ or $\zeta\searrow\alpha-1$ (see Figure 1). The right-hand side is negative at $\zeta=0$ and continuous everywhere; thus we have at least one solution $\zeta_0^+\in(0,\alpha+1)$ and at least one solution $\zeta_0^-\in(\alpha-1,0)$. In fact it is easy to prove that we have *exactly* one solution on each of these intervals, since $4\pi(\zeta-\alpha)\cot(\pi(\zeta-\alpha))$ is a concave function on $(\alpha-1,\alpha+1)$, while $\frac{1}{2}\sigma^2\zeta^2-\mu\zeta-q$ is convex.

Next, for $n\neq 0$ we have

$$4\pi(\zeta-\alpha)\cot(\pi(\zeta-\alpha))\nearrow+\infty \qquad \text{as } \zeta\nearrow\alpha-n,\zeta\searrow\alpha+n,$$
$$4\pi(\zeta-\alpha)\cot(\pi(\zeta-\alpha))\searrow-\infty \qquad \text{as } \zeta\searrow\alpha-n,\zeta\nearrow\alpha+n,$$

thus there must exist at least one zero ζ_n on each interval $(n+\alpha,n+\alpha+1)$, $(n+\alpha-1,n+\alpha)$.

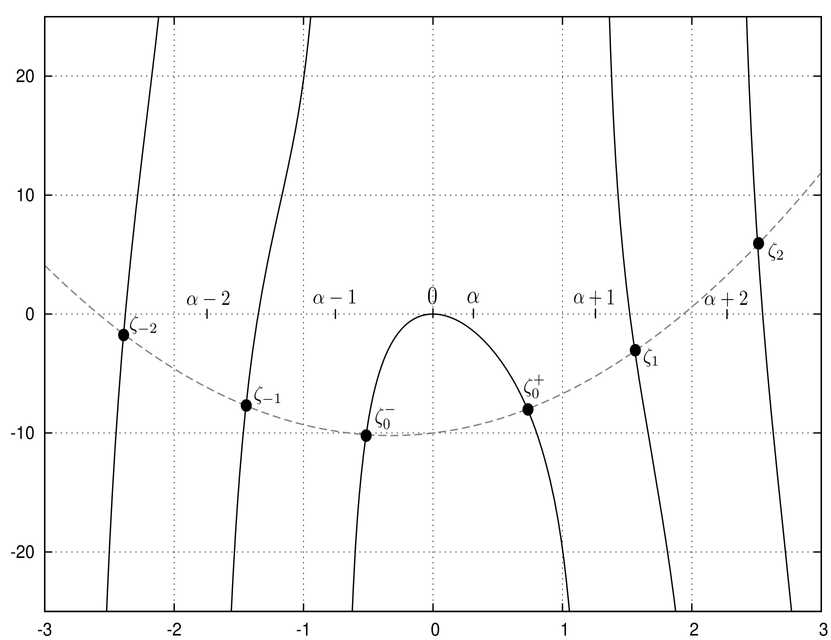

FIG. 1. *Illustration of the proof of Theorem 5.*

Next we will prove the asymptotic expansion (16). Since we have assumed that $\sigma \neq 0$ we can rearrange the terms in (22) to obtain

$$
\begin{aligned}
\frac{1}{\pi}\tan(\pi(\zeta-\alpha)) &= \frac{4(\zeta-\alpha)}{1/2\sigma^2\zeta^2+\rho\zeta+4\gamma-q} \\
&= \frac{8}{\sigma^2}\zeta^{-1}\left[\frac{1-\alpha\zeta^{-1}}{1+2\rho\sigma^{-2}\zeta^{-1}+O(\zeta^{-2})}\right] \\
&= \frac{8}{\sigma^2}\zeta^{-1}-\frac{8}{\sigma^2}\left(\frac{2}{\rho}+\alpha\right)\zeta^{-2}+O(\zeta^{-3}).
\end{aligned}
\tag{23}
$$

The main idea in the above calculation is to expand the rational function in the Taylor series centered at $\zeta=\infty$. Now, the right-hand side of (23) is small when ζ is large, and thus the solution to (23) should be close to the solution of $\tan(\pi(\zeta-\alpha))=0$, which implies

$$
\zeta = n+\alpha+\omega \tag{24}
$$

and $\omega = o(1)$ as $n\to\infty$. Next we expand the right-hand side of (23) in powers of w as

$$
\frac{1}{\pi}\tan(\pi(\zeta-\alpha)) = \frac{1}{\pi}\tan(\pi\omega) = \omega + O(\omega^3)
$$

and, using the first two terms of the Maclaurin series for ζ^{-1} in powers of ω

$$
\zeta^{-1} = (n+\alpha+\omega)^{-1} = (n+\alpha)^{-1} - \omega(n+\alpha)^{-2} + O(\omega^2 n^{-3}),
$$

we are able to rewrite (23) as

$$
\omega + O(\omega^3) = \frac{8}{\sigma^2}((n+\alpha)^{-1}+\omega(n+\alpha)^{-2}) - \frac{8}{\sigma^2}\left(\frac{2}{\rho}+\alpha\right)(n+\alpha)^{-2}+O(n^{-3}).
$$

Asymptotic expansion (16) follows easily from the above formula and (24).

If $\sigma=0$, equation (23) has to be modified as follows:

$$
\begin{aligned}
\frac{1}{\pi}\tan(\pi(\zeta-\alpha)) &= \frac{4(\zeta-\alpha)}{\rho\zeta+4\gamma-q} = \frac{4}{\rho}\left[\frac{1-\alpha\zeta^{-1}}{1+(4\gamma-q)\rho^{-1}\zeta^{-1}}\right] \\
&= \frac{4}{\rho} - \frac{4}{\rho^2}(4\gamma-q+\alpha\rho)\zeta^{-1} \\
&\quad + \frac{4(4\gamma-q)}{\rho^3}(4\gamma-q+\alpha\rho)\zeta^{-2}+O(\zeta^{-3}),
\end{aligned}
\tag{25}
$$

where again we have expanded the rational function in the Taylor series centered at $\zeta=\infty$. As before, when ζ is large the solution of (22) should be close to the solution of

$$
\frac{1}{\pi}\tan(\pi(\zeta-\alpha)) = \frac{4}{\rho},
$$

and thus we should expand both sides of (25) in the Taylor series centered at the solution to the above equation. We define ω as

$$\zeta = n + \alpha + \frac{1}{\pi}\arctan\left(\frac{4\pi}{\rho}\right) + \omega = n + \alpha + \omega_0 + \omega, \tag{26}$$

and again $\omega = o(1)$ as $n \to \infty$. To expand the left-hand side of (25) in power series in ω we use an addition formula for $\tan(\cdot)$ and find that

$$\begin{aligned}\frac{1}{\pi}\tan(\pi(\zeta-\alpha)) &= \frac{1}{\pi}\tan\left(\arctan\left(\frac{4\pi}{\rho}\right) + \pi\omega\right)\\ &= \frac{1}{\pi}\frac{4\pi/\rho + \tan(\pi\omega)}{1 - 4\pi/\rho\tan(\pi\omega)} = \frac{1}{\pi}\frac{4\pi/\rho + \pi\omega + O(\omega^3)}{1 - 4\pi/\rho\pi\omega + O(\omega^3)}\\ &= \frac{4}{\rho} + \frac{1}{\rho^2}(16\pi^2 + \rho^2)\omega\\ &\quad + \frac{4\pi^2}{\rho^3}(16\pi^2+\rho^2)\omega^2 + O(\omega^3).\end{aligned} \tag{27}$$

Again, we use (26) to obtain the Maclaurin series of ζ^{-1} in powers of ω

$$\begin{aligned}\zeta^{-1} &= (n+\alpha+\omega_0+\omega)^{-1}\\ &= (n+\alpha+\omega_0)^{-1} - \omega(n+\alpha+\omega_0)^{-2} + O(\omega^2 n^{-3}).\end{aligned}$$

Using (27) and the above expansion we can rewrite (25) as

$$\begin{aligned}&(16\pi^2+\rho^2)\left[\omega + \frac{4\pi^2}{\rho}\omega^2\right]\\ &\quad = -4(4\gamma - q + \alpha\rho)((n+\alpha+\omega_0)^{-1} - \omega(n+\alpha+\omega_0)^{-2})\\ &\qquad + \frac{4(4\gamma-q)}{\rho}(4\gamma - q + \alpha\rho)(n+\alpha+\omega_0)^{-2} + O(n^{-3}) + O(\omega^3),\end{aligned}$$

and from this equation we obtain the second asymptotic expansion (17).

Now we are ready to prove the factorization identity (18) and the fact that all the zeros of $q + \Psi(i\zeta)$ are real and simple and that there are no other zeros except for the ones described in (15). First we need to find an analytic function $P(z)$ such that $P(0) = 1$ and which has zeros at all poles of $\Psi(z)$ (with the same multiplicity). The choice is rather obvious due to (11):

$$P(z) = \frac{\alpha}{\sin(\pi\alpha)} \times \frac{\sinh(\pi(z - i\alpha))}{z - i\alpha}. \tag{28}$$

By definition, the function

$$Q(z) = q^{-1}(q + \Psi(z))P(z) \tag{29}$$

is also analytic in the entire complex plane.

Next, using the definition of $P(z)$ (28) and $Q(z)$ (29) we check that $Q(z) = 0$ if and only if $q + \Psi(z) = 0$. We have proved already that the zeros of $q + \Psi(i\zeta)$ include $\zeta_n, \zeta_0^{\pm}$; however, some of them might have multiplicity greater than one, and there also might exist other roots (real and/or complex). Let us denote the set of these unaccounted roots (counting with multiplicity) as $\mathfrak{Z}$. Using asymptotic expansions given by equations (23) and (25) one can easily prove that $\mathfrak{Z}$ is a finite set (possibly empty).

Using equations (28), (29) and (11) we obtain an explicit formula for $Q(z)$ from which it easily follows that $Q(z)$ has order equal to one, which means that one is the least lower bound of all $\gamma > 0$ such that $Q(z) = O(\exp(|z|^{\gamma}))$ as $z \to \infty$; the rigorous definition can be found in [24], page 4 or Chapter 11 in [16]. Since $Q(z)$ has order equal to one, we can use the Hadamard theorem (see Theorem 1, page 26, in [24] or Theorem 3.4, page 289, in [16]) to represent it as an infinite product over its zeros

$$Q(z) = \exp(c_1 z)\left(1 + \frac{iz}{\zeta_0^+}\right)\left(1 + \frac{iz}{\zeta_0^-}\right) \times \prod_{z_k \in \mathfrak{Z}} \left(1 + \frac{iz}{z_k}\right) \prod_{|n|\geq 1} \left(1 + \frac{iz}{\zeta_n}\right) \exp\left(-\frac{iz}{\zeta_n}\right)$$

for some constant $c_1 \in \mathbb{C}$. As the next step we rearrange the infinite product in the above formula and obtain

$$Q(z) = \exp(c_2 z)\left(1 + \frac{iz}{\zeta_0^+}\right)\left(1 + \frac{iz}{\zeta_0^-}\right) \times \prod_{z_k \in \mathfrak{Z}} \left(1 + \frac{iz}{z_k}\right) \prod_{n\geq 1} \left(1 + \frac{iz}{\zeta_n}\right)\left(1 + \frac{iz}{\zeta_{-n}}\right) \tag{30}$$

for some other constant $c_2 \in \mathbb{C}$, where the infinite product converges absolutely since each term is $1 + O(n^{-2})$ as $n \to \infty$. Using definition of $P(z)$ (28) and infinite product representation of trigonometric functions (see formulas 1.431 in [20]) we find that

$$P(z) = \prod_{n\geq 1} \left(1 + \frac{iz}{n+\alpha}\right)\left(1 + \frac{iz}{-n+\alpha}\right). \tag{31}$$

Combining equations (28), (29), (30) and (31) we finally conclude that for all $z \in \mathbb{C}$

$$\frac{q}{q + \Psi(z)} = \frac{\exp(c_2 z)}{(1 + iz/\zeta_0^+)(1 + iz/\zeta_0^-)} \prod_{z_k \in \mathfrak{Z}} \frac{1}{1 + iz/z_k} \prod_{|n|\geq 1} \frac{1 + iz/(n+\alpha)}{1 + iz/\zeta_n}. \tag{32}$$

First let us prove that $c_2 = 0$. Denote the left-hand side of (32) as $F_1(z)$ and right-hand side as $F_2(z)$. Since $\Psi(z)$ is a characteristic exponent, it must be $O(z^2)$ as $z \to \infty$, $z \in \mathbb{R}$, thus clearly $z^{-1}\ln(F_1(z)) \to 0$ as $z \to \infty$, $z \in \mathbb{R}$. Using Lemma 6 we find that $z^{-1}\ln(F_2(z)) \to c_2$ as $z \to \infty$ $z \in \mathbb{R}$, which implies that $c_2 = 0$.

All that is left to do it to prove that $\mathfrak{Z}$ is an empty set. The main tool is again Lemma 6. Assuming that $\sigma \neq 0$ and using asymptotic expansion (16) and Lemma 6 we find that the infinite product in (32) converges to a constant as $z \to \infty$, $z \in \mathbb{R}$. Thus function $F_2(z) \approx A_2 z^{-2-M}$ where M is equal to the number of elements in the set $\mathfrak{Z}$. However, function $F_1(z) \approx A_1 z^{-2}$ as $z \to \infty$, $z \in \mathbb{R}$, thus $M = 0$ and the set $\mathfrak{Z}$ must be empty. In the case $\sigma \neq 0$ the proof is identical, except that both $F_1(z)$ and $F_2(z)$ behave like Az^{-1}, which can be established by the asymptotic expression for ζ_n given in (17) and Lemma 6. □

Theorem 5 provides us with all the information about the zeros of $q + \Psi(z)$ that we will need later to prove results about Wiener–Hopf factors and perform numerical computations. However, we can also compute explicitly the sums of inverse powers of zeros. These results can be useful for checking the accuracy, but more importantly, for approximating the smallest solutions $\zeta_0^{\pm}$. We assume that $\alpha \neq 0$ and define for $m \geq 0$

$$\Omega_m = \alpha^{-m-1} + (\zeta_0^-)^{-m-1} + (\zeta_0^+)^{-m-1} + \sum_{n \geq 1} [\zeta_n^{-m-1} + \zeta_{-n}^{-m-1}]. \tag{33}$$

Asymptotic expansions (16) and (17) guarantee that the series converges absolutely for $m \geq 0$, thus the sequence $\{\Omega_m\}_{m \geq 0}$ is correctly defined.

Lemma 7. *The sequence $\{\Omega_m\}_{m \geq 0}$ can be computed using the following recurrence relation:*

$$\Omega_m = -\frac{1}{b_0}\left[(m+1)b_{m+1} + \sum_{n=0}^{m-1} \Omega_n b_{m-n}\right], \qquad m \geq 0,$$

where coefficients $\{b_n\}_{n \geq 0}$ are defined as

$$\begin{aligned} b_{2n} &= \frac{(-1)^{n-1}\pi^{2n-1}}{(2n)!}[n(2n-1)\alpha\sigma^2 + \pi^2\alpha(q+8n) - 2n\gamma\rho] \\ b_{2n+1} &= \frac{(-1)^n \pi^{2n}}{(2n+1)!}\left[n(2n+1)\frac{\gamma\sigma^2}{\pi} - \pi(4\pi^2\alpha^2 + 4\gamma^2 - \gamma q) \right. \\ &\qquad \left. + \pi(2n+1)(4\gamma + \alpha\rho)\right]. \end{aligned} \tag{34}$$

Proof. This statement is just an application of the following general result. Assume that we have an entire function $H(z)$ which can be expressed as an infinite product over the set of its zeros $\mathfrak{Z}$

$$H(z) = \prod_{z_k \in \mathfrak{Z}} \left(1 - \frac{z}{z_k}\right).$$

Taking derivative of $\ln(H(z))$ we find

$$H'(z) = -H(z) \sum_{z_k \in \mathfrak{Z}} (z_k - z)^{-1} = -h(z) \sum_{m \geq 0} \left[\sum_{z_k \in \mathfrak{Z}} z_k^{-m-1}\right] z^m,$$

and the recurrence relation for $\sum_{z_k \in \mathfrak{Z}} z_k^{-m-1}$ is obtained by expanding $H(z)$ and $H'(z)$ as a Maclaurin series, multiplying two series in the right-hand side and comparing the coefficients in front of z^m. The statement of Lemma 7 follows by considering an entire function

$$H(z) = q\pi(z - \alpha)Q(iz),$$

where $Q(z)$ is defined by (29). Function $H(z)$ has zeros at $\{\alpha, \zeta_0^{\pm}, \zeta_n\}$, and one can check that the Maclaurin expansion is given by $H(z) = \sum_{n \geq 0} b_n z^n$ where coefficients b_n are defined in (34). □

Finally we can state and prove our main results: expressions for Wiener–Hopf factors and density of S_τ.

Theorem 8. *For $q > 0$*

(35)
$$\phi_q^-(z) = \frac{1}{1 + iz/\zeta_0^+} \prod_{n \geq 1} \frac{1 + iz/(n + \alpha)}{1 + iz/\zeta_n},$$
$$\phi_q^+(z) = \frac{1}{1 + iz/\zeta_0^-} \prod_{n \leq -1} \frac{1 + iz/(n + \alpha)}{1 + iz/\zeta_n}.$$

Infinite products converge uniformly on compact subsets of $\mathbb{C} \setminus i\mathbb{R}$. The density of S_τ is given by

(36)
$$\frac{d}{dx}\mathbb{P}(S_\tau \leq x) = -c_0^- \zeta_0^- e^{\zeta_0^- x} - \sum_{k \leq -1} c_k^- \zeta_k e^{\zeta_k x},$$

where

(37)
$$c_0^- = \prod_{n \leq -1} \frac{1 - \zeta_0^-/(n + \alpha)}{1 - \zeta_0^-/\zeta_n},$$
$$c_k^- = \frac{1 - \zeta_k/(k + \alpha)}{1 - \zeta_k/\zeta_0^-} \prod_{n \leq -1, n \neq k} \frac{1 - \zeta_k/(n + \alpha)}{1 - \zeta_k/\zeta_n}.$$

PROOF. Expressions (35) for Wiener–Hopf factors are obtained using factorization identity (18) and Lemmas 2 and 6. Expression (36) for the density of S_τ is derived by computing the inverse Fourier transform via residues. □

REMARK 3. Theorem 8 remains true for $q = 0$ if $\mu < 0$. In this case $\mathbb{E}X_1 < 0$ and $S_\tau \to S_\infty$ and $I_\tau \to -\infty$ as $q \to 0^+$. From the analytical point of view we have $\zeta_0^- < 0$ and $\zeta_0^+ = 0$ [see Figure 1 and (22)]. If $\mu = 0$, then $\mathbb{E}X_1 = 0$ and the process X_t oscillates; thus $S_\infty = I_\infty = \infty$, which is expressed analytically by the fact that $\zeta_0^+ = \zeta_0^- = 0$.

4. A family of Lévy processes.

DEFINITION 4. We define a β-family of Lévy processes by the generating triple (μ, σ, ν), where the density of the Lévy measure is defined as

$$\nu(x) = c_1 \frac{e^{-\alpha_1\beta_1 x}}{(1 - e^{-\beta_1 x})^{\lambda_1}} \mathbf{I}_{\{x>0\}} + c_2 \frac{e^{\alpha_2\beta_2 x}}{(1 - e^{\beta_2 x})^{\lambda_2}} \mathbf{I}_{\{x<0\}} \tag{38}$$

and parameters satisfy $\alpha_i > 0$, $\beta_i > 0$, $c_i \ge 0$ and $\lambda_i \in (0, 3)$. This Lévy measure has exponential tails; thus we will use the cut-off function $h(x) \equiv x$ in (1).

The β-family is quite rich: in particular, by controlling parameters λ_i, we can obtain an arbitrary behavior of small jumps, and parameters α_i and β_i are responsible for the tails of the Lévy measure (which are always exponential). Parameters c_i control the total "intensity" of positive/negative jumps. The processes in β-family are similar to the generalized tempered stable processes (see [15]) which were also named KoBoL processes in [9] and [10]

$$\nu(x) = c_+ \frac{e^{-\alpha_+ x}}{x^{\lambda_+}} \mathbf{I}_{\{x>0\}} + c_- \frac{e^{\alpha_- x}}{|x|^{\lambda_-}} \mathbf{I}_{\{x<0\}}.$$

In fact we can obtain the above measure as the limit of Lévy measures in β-family. If we set $c_1 = c_+\beta^{\lambda_+}$, $c_2 = c_-\beta^{\lambda_-}$, $\alpha_1 = \alpha_+\beta^{-1}$, $\alpha_2 = \alpha_-\beta^{-1}$, $\beta_1 = \beta_2 = \beta$ and let $\beta \to 0^+$ we see that the Lévy measure defined in (38) will converge to the Lévy measure of the generalized tempered stable process. Next, when $\lambda_1 = \lambda_2$, the processes in β-family are similar to the tempered stable processes (see [5]). If we restrict the parameters even further, $c_1 = c_2$, $\lambda_1 = \lambda_2$ and $\beta_1 = \beta_2$ so that the small positive/negative jumps have the same behavior, while large jumps which are controlled by α_i may be different, and we obtain a process very similar to the CGMY family defined in [13]. Finally, if $c_i = 4$, $\beta_i = 1/2$, $\lambda_i = 2$ and $\alpha_1 = 1 - \alpha$ and $\alpha_2 = 1 + \alpha$, we obtain the process X_t discussed in Section 3.

If we restrict parameters as $\sigma = 0$, $\beta_1 = \beta_2$, $\lambda_1 = \lambda_2$ (and μ uniquely specified in terms of these parameters), then β-family reduces to a family of Lamperti-stable processes, which can be obtained by Lamperti transformation from the stable processes conditioned to stay positive (see the original paper [23] by Lamperti for the definition of this transformation and its various properties). Spectrally one-sided Lamperti-stable processes appeared in [8] and [27], and two-sided processes were studied in [11, 12, 14] and [22]. Lamperti-stable processes are a particularly interesting subclass of the β-family since they offer many examples of fluctuation identities related to Wiener–Hopf factorization which can be computed in closed form (see [11, 14] and [22]).

In the following proposition we derive a formula for the characteristic exponent $\Psi(z)$ for processes in the β-family. As we will see, the characteristic exponent can be expressed in terms of beta and digamma functions (see Chapter 6 in [1] or Section 8.3 in [20])

$$\mathrm{B}(x;y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad \psi(x) = \frac{d}{dx}\ln(\Gamma(x)), \tag{39}$$

which justifies the name of the family.

Proposition 9. *If $\lambda_i \in (0,3) \setminus \{1,2\}$, then*

$$\begin{aligned} \Psi(z) = \frac{\sigma^2 z^2}{2} + i\rho z - \frac{c_1}{\beta_1}\mathrm{B}\left(\alpha_1 - \frac{iz}{\beta_1}; 1-\lambda_1\right) \\ - \frac{c_2}{\beta_2}\mathrm{B}\left(\alpha_2 + \frac{iz}{\beta_2}; 1-\lambda_2\right) + \gamma, \end{aligned} \tag{40}$$

where

$$\begin{aligned} \gamma &= \frac{c_1}{\beta_1}\mathrm{B}(\alpha_1; 1-\lambda_1) + \frac{c_2}{\beta_2}\mathrm{B}(\alpha_2; 1-\lambda_2), \\ \rho &= \frac{c_1}{\beta_1^2}\mathrm{B}(\alpha_1; 1-\lambda_1)(\psi(1+\alpha_1-\lambda_1) - \psi(\alpha_1)) \\ &\quad - \frac{c_2}{\beta_2^2}\mathrm{B}(\alpha_2; 1-\lambda_2)(\psi(1+\alpha_2-\lambda_2) - \psi(\alpha_2)) - \mu. \end{aligned}$$

If λ_1 or $\lambda_2 \in \{1,2\}$ the characteristic exponent can be computed using the following two integrals:

$$\begin{aligned} &\int_0^\infty (e^{ixy} - 1 - ixy)\frac{e^{-\alpha\beta x}}{1-e^{-\beta x}}\,dx \\ &\quad = -\frac{1}{\beta}\left[\psi\left(\alpha - \frac{iy}{\beta}\right) - \psi(\alpha)\right] - \frac{iy}{\beta^2}\psi'(\alpha) \end{aligned} \tag{41}$$

$$\begin{aligned}&\int_0^{\infty}(e^{ixy}-1-ixy)\frac{e^{-\alpha\beta x}}{(1-e^{-\beta x})^2}\,dx\\ &\quad=-\frac{1}{\beta}\left(1-\alpha+\frac{iy}{\beta}\right)\left[\psi\left(\alpha-\frac{iy}{\beta}\right)-\psi(\alpha)\right]-\frac{iy(1-\alpha)}{\beta^2}\psi'(\alpha).\end{aligned}\tag{42}$$

Proof. First we assume that $\lambda \in (0,1)$. Performing change of variables $u = \exp(-\beta x)$ and using integral representation for beta function (formula 8.380.1 in [20]) we find that

$$\begin{aligned}&\beta\int_0^{\infty}(e^{ixz}-1-ixz)\frac{e^{-\alpha\beta x}}{(1-e^{-\beta x})^{\lambda}}\,dx\\ &\quad=\mathrm{B}\left(\alpha-\frac{iz}{\beta};1-\lambda\right)-\mathrm{B}(\alpha;1-\lambda)-z\left[\frac{d}{dz}\mathrm{B}\left(\alpha-\frac{iz}{\beta};1-\lambda\right)\right]_{z=0},\end{aligned}$$

and we obtain the desired result (40). The left-hand side of the above equation is analytic in λ for $\mathrm{Re}(\lambda)<3$, and the right-hand side is analytic and well defined for $\mathrm{Re}(\lambda)<3$, $\lambda \neq \{1,2\}$; thus by analytic continuation they should be equal for $\lambda \in (0,3)\setminus\{1,2\}$.

Assume that $\lambda = 2$. Then using binomial series we can expand

$$(1-\exp(-x))^{-2}=\sum_{n\ge 0}(n+1)\exp(-nx),$$

which converges uniformly on (ε,∞) and obtain

$$\begin{aligned}&\beta\int_0^{\infty}(e^{ixz}-1-ixz)\frac{e^{-\alpha\beta x}}{(1-e^{-\beta x})^2}\,dx\\ &\quad=\sum_{n\ge 0}\left[\frac{n+1}{n+\alpha-iy/\beta}-\frac{n+1}{n+\alpha}-\frac{iy}{\beta}\frac{n+1}{(n+\alpha)^2}\right]\\ &\quad=\left(1-\alpha+\frac{iy}{\beta}\right)\sum_{n\ge 0}\left[\frac{1}{n+\alpha-iy/\beta}-\frac{1}{n+\alpha}\right]\\ &\qquad-\frac{iy}{\beta}(1-\alpha)\sum_{n\ge 0}\frac{1}{(n+\alpha)^2},\end{aligned}$$

and using the series representation for digamma function (formula 8.362.1 in [20]) we obtain (42). Derivation of formula (41) corresponding to the case $\lambda = 1$ is identical. □

The following theorem is the analogue of Theorem 5, and it is the main result in this section.

Theorem 10. *Assume that $q>0$ and that $\Psi(z)$ is given by (40).*

(i) *Equation $q+\Psi(i\zeta)=0$ has infinitely many solutions, all of which are real and simple. They are located as follows:*

$$\begin{aligned}
&\zeta_0^- \in (-\beta_1\alpha_1, 0),\\
&\zeta_0^+ \in (0, \beta_2\alpha_2),\\
&\zeta_n \in (\beta_2(\alpha_2+n-1), \beta_2(\alpha_2+n)), \qquad n\ge 1,\\
&\zeta_n \in (\beta_1(-\alpha_1+n), \beta_1(-\alpha_1+n+1)), \qquad n\le -1.
\end{aligned} \tag{43}$$

(ii) *If $\sigma\neq 0$ we have*

$$\begin{aligned}
\zeta_{n+1} &= \beta_2(n+\alpha_2) + \frac{2c_2}{\sigma^2\beta_2^2\Gamma(\lambda_2)}(n+\alpha_2)^{\lambda_2-3}\\
&\quad + O(n^{\lambda_2-3-\varepsilon}), \qquad n\to+\infty,\\
\zeta_{n-1} &= \beta_1(n-\alpha_1) - \frac{2c_1}{\sigma^2\beta_1^2\Gamma(\lambda_1)}(-n+\alpha_1)^{\lambda_1-3}\\
&\quad + O(n^{\lambda_1-3-\varepsilon}), \qquad n\to-\infty.
\end{aligned} \tag{44}$$

(iii) *If $\sigma=0$ we have*

$$\begin{aligned}
\zeta_{n+\delta} &= \beta_2(n+\alpha_2+\omega_0) + A(n+\alpha_2+\omega_0)^{\lambda}\\
&\quad + O(n^{\lambda-\varepsilon}), \qquad n\to+\infty,
\end{aligned} \tag{45}$$

where coefficients w_0 and A are presented in Table 1, $\delta\in\{0,1\}$ depending on the signs of w_0 and A and

$$x_0 = \frac{1}{\pi}\arctan\left(\sin(\pi\lambda_2)\left(\frac{c_1\beta_2^{\lambda_2}\Gamma(1-\lambda_1)}{c_2\beta_1^{\lambda_1}\Gamma(1-\lambda_2)} - \cos(\pi\lambda_2)\right)^{-1}\right).$$

The corresponding results for $n\to-\infty$ can be obtained by symmetry considerations.

(iv) *Function $q(q+\Psi(z))^{-1}$ can be factorized as follows:*

$$\begin{aligned}
\frac{q}{q+\Psi(z)} &= \frac{1}{(1+iz/\zeta_0^+)(1+iz/\zeta_0^-)}\prod_{n\ge1}\frac{1+iz/(\beta_2(n-1+\alpha_2))}{1+iz/\zeta_n}\\
&\quad\times\prod_{n\le -1}\frac{1+iz/(\beta_1(n+1-\alpha_1))}{1+iz/\zeta_n},
\end{aligned} \tag{46}$$

where the infinite products converge uniformly on the compact subsets of the complex plane excluding zeros/poles of $q+\Psi(z)$.

Remark 5. When $\sigma=0$ the remaining cases $\lambda_1<2$, $\lambda_2=2$ and $\lambda_1=2$, $0<\lambda_2<3$ are not covered by Theorem 10. The interested reader can derive these asymptotic expansions by using formulas (41), (42) and the following results for the digamma function (see formulas 6.3.7 and 6.3.18 in [1]):

$$\psi(1-z)=\psi(z)+\pi\cot(\pi z), \qquad \psi(z)=\ln(z)-\frac{1}{2z}+O(z^{-2}), \qquad z\to\infty.$$

Proof of Theorem 10. The proof of (i) is very similar to the corresponding part of the proof of Theorem 5. We separate equation $q+\Psi(i\zeta)=0$ into a jump part and a diffusion part, find points where the jump part goes to infinity and by analyzing the signs we conclude that on every interval between these points there should exist a solution.

The proof of (ii) and (iii) is based on the following two asymptotic formulas as $\zeta\to+\infty$:

$$\mathrm{B}(\alpha+\zeta;\gamma)=\Gamma(\gamma)\zeta^{-\gamma}\left[1-\frac{\gamma(2\alpha+\gamma-1)}{2\zeta}+O(\zeta^{-2})\right],$$

$$\mathrm{B}(\alpha-\zeta;\gamma)=\Gamma(\gamma)\frac{\sin(\pi(\zeta-\alpha-\gamma))}{\sin(\pi(\zeta-\alpha))}\zeta^{-\gamma}\left[1+\frac{\gamma(2\alpha+\gamma-1)}{2\zeta}+O(\zeta^{-2})\right].$$

The first asymptotic expansion follows from the definition of beta function (39) and formula 6.1.47 in [1], while the second formula can be reduced to the first one by applying a reflection formula for the gamma function.

If $\sigma\neq 0$ and $\zeta\to+\infty$ we use (40) and the above formulas and rewrite equation $q+\Psi(i\zeta)=0$ as

$$\frac{\sin(\pi(\zeta/\beta_2-\alpha_2+\lambda_2))}{\sin(\pi(\zeta/\beta_2-\alpha_2))}=\frac{\sigma^2\beta_2^{\lambda_2}}{2c_2\Gamma(1-\lambda_2)}\zeta^{3-\lambda_2}+O(\zeta^{1-\lambda_2})+O(\zeta^{\lambda_1-\lambda_2}),$$

while if $\sigma=0$ and $\zeta\to+\infty$ we have

$$\frac{\sin(\pi(\zeta/\beta_2-\alpha_2+\lambda_2))}{\sin(\pi(\zeta/\beta_2-\alpha_2))}=\frac{\beta_2^{\lambda_2}\rho}{c_2\Gamma(1-\lambda_2)}\zeta^{2-\lambda_2}+\frac{c_1\beta_2^{\lambda_2}\Gamma(1-\lambda_1)}{c_2\beta_1^{\lambda_1}\Gamma(1-\lambda_2)}\zeta^{\lambda_1-\lambda_2}$$

Table 1

Coefficients for asymptotic expansion of ζ_n when $\sigma=0$

	$\boldsymbol{\omega_0}$	$\boldsymbol{A}$	$\boldsymbol{\lambda}$
$\lambda_1<2,\ \lambda_2<2$	0	$\frac{c_2}{\rho\beta_2\Gamma(\lambda_2)}$	λ_2-2
$\lambda_1<2,\ \lambda_2>2$	$2-\lambda_2$	$-\frac{\sin(\pi\lambda_2)\beta_2^3\rho}{\pi c_2\Gamma(1-\lambda_2)}$	$2-\lambda_2$
$\lambda_1>2,\ \lambda_2<\lambda_1$	0	$\frac{c_2\beta_1^{\lambda_1}}{c_1\beta_2^{\lambda_1-1}\Gamma(1-\lambda_1)\Gamma(\lambda_2)}$	$\lambda_2-\lambda_1$
$\lambda_1>2,\ \lambda_2>\lambda_1$	$2-\lambda_2$	$-\frac{\sin(\pi\lambda_2)}{\pi}\frac{c_1\beta_2^{\lambda_1+1}\Gamma(1-\lambda_1)}{c_2\beta_1^{\lambda_1}\Gamma(1-\lambda_2)}$	$\lambda_1-\lambda_2$
$\lambda_1>2,\ \lambda_2=\lambda_1$	x_0	$-\rho\frac{\sin(\pi x_0)^2}{\pi^2}\frac{\beta_2^3}{c_2}\Gamma(\lambda_2)$	$2-\lambda_2$

$$+O(\zeta^{1-\lambda_2})+O(\zeta^{\lambda_1-\lambda_2-1}).$$

Asymptotic expansions (44) and (45) can be derived from the above formulas using the same method as in the proof of Theorem 5.

In order to prove factorization identity (46) and the fact that there are no other roots, we use exactly the same approach as in the proof of Theorem 5. Again we choose an entire function $P(z)$ which has zeros at the poles of $\Psi(z)$ with the same multiplicity, and the choice is obvious due to (40):

$$P(z)=\left[\Gamma\left(\alpha_1-\frac{iz}{\beta_1}\right)\Gamma\left(\alpha_2+\frac{iz}{\beta_2}\right)\right]^{-1}, \tag{47}$$

and function $Q(z)$ is defined by as $q^{-1}(1+\Psi(z))P(z)$. Function $P(z)$ can be expanded in infinite product using Euler's formula (see formula 6.1.3 in [1]). Using (40) and (47) and asymptotics for gamma function we find that function $Q(z)$ has order equal to one, and thus again we can use Hadamard's theorem to expand it as infinite product, and finally we use asymptotics for infinite products supplied by Lemma 6 and asymptotics for ζ_n given by (44) and (44) to prove factorization identity (46). □

We can also derive a result similar to Lemma 7 using the entire function $Q(z)$ defined by (47). While there is no closed form expression for derivatives of gamma function, they can be easily computed numerically. Our final result in this section is the analogue of Theorem 8, and the proof is identical.

THEOREM 11. *For $q>0$*

$$\phi_q^-(z)=\frac{1}{1+iz/\zeta_0^+}\prod_{n\ge 1}\frac{1+iz/(\beta_2(n-1+\alpha_2))}{1+iz/\zeta_n},$$

$$\phi_q^+(z)=\frac{1}{1+iz/\zeta_0^-}\prod_{n\le -1}\frac{1+iz/(\beta_1(n+1-\alpha_1))}{1+iz/\zeta_n}.$$

Infinite products converge uniformly on compact subsets of $\mathbb{C}\setminus i\mathbb{R}$. The density of S_τ is given by

$$\frac{d}{dx}\mathbb{P}(S_\tau\le x)=-c_0^-\zeta_0^-e^{\zeta_0^-x}-\sum_{k\le -1}c_k^-\zeta_k e^{\zeta_k x},$$

where

$$c_0^-=\prod_{n\le -1}\frac{1-\zeta_0^-/(\beta_1(n+1-\alpha_1))}{1-\zeta_0^-/\zeta_n},$$

$$c_k^-=\frac{1-\zeta_k/(\beta_1(k+1-\alpha_1))}{1-\zeta_k/\zeta_0^-}\prod_{n\le -1,n\ne k}\frac{1-\zeta_k/(\beta_1(n+1-\alpha_1))}{1-\zeta_k/\zeta_n}.$$

REMARK 6. In the case $\sigma=0$, $\lambda_i<2$, and $\rho>0$ we have a process of bounded variation and negative drift, and thus the distribution of S_τ will have an atom at zero, which can be computed using the following formula:

$$\mathbb{P}(S_\tau=0)=\lim_{z\to+\infty}\mathbb{E}[e^{-zS_\tau}]=\lim_{z\to+\infty}\phi_q^+(iz)$$
$$=\frac{-\zeta_0^-}{\alpha_1\beta_1}\prod_{n\le -1}\frac{\zeta_n}{\beta_1(n-\alpha_1)}.$$

Using asymptotic relation (45) one can see that the above infinite product converges to a number between zero and one.

5. Implementation and numerical results. In this section we discuss implementation details for computing the probability density function of $S_{\tau(q)}$ and S_t. In order to illustrate the main ideas we will use the process X_t defined in Section 3; however, the implementation for a general X_t from the β-family would be quite similar. Our main tools are Theorem 8 and asymptotic expansion for ζ_n given in Theorem 5.

First let us discuss the computation of density of S_τ. The first step would be to compute solutions to equation $q+\Psi(i\zeta)=0$, and for q real this is a simple task: for n large we use Newton's method which is started from the point given by asymptotic expansion (16) or (17). To compute $\zeta_0^\pm$ or ζ_n with n small we use localization result (15) and the secant (or bisection) method to get the starting point for Newton's iteration. Overall this part of the algorithm is very computationally efficient and can be made even faster if we compute different ζ_n in parallel.

The second step is to compute coefficient c_k^- which are given by (37). Each term in the infinite product is $1+O(n^{-2})$; however, as we show in Proposition 12, we can considerably improve convergence by using our knowledge of the asymptotic expansion for ζ_n. The final step is to compute the density of S_τ using formula (36). Note that the series converges exponentially for $x>0$. When x is small the convergence is slow, and the asymptotic behavior as $x\to 0^+$ would depend on the decay rate of coefficient c_k^-; however, we were unable to prove any results in this direction.

PROPOSITION 12. *Assume that* $\zeta_n=n+\beta+\frac{A_1}{(n+\beta)}+\frac{A_2}{(n+\beta)^2}+O(n^{-3})$ *as* $n\to+\infty$. *Then as* $N\to+\infty$ *we have*

$$\prod_{n\ge N}\frac{1+z/(n+\alpha)}{1+z/\zeta_n}$$
$$=\frac{\Gamma(N+\alpha)\Gamma(N+\beta+z)}{\Gamma(N+\beta)\Gamma(N+\alpha+z)} \tag{48}$$

$$\times \exp[A_1(f_{1,1}(\beta,\beta;N) - f_{1,1}(z+\beta,\beta;N)) + A_2(f_{1,2}(\beta,\beta;N) - f_{1,2}(z+\beta,\beta;N)) + O(N^{-3})],$$

where $f_{\alpha_1,\alpha_2}(z_1,z_2;N)$ *can be computed as follows:*

$$\begin{aligned} f_{\alpha_1,\alpha_2}(z_1,z_2;N) &= \sum_{k\ge 0} \frac{\binom{-\alpha_2}{k}(z_2-z_1)^k}{\alpha_1+\alpha_2+k-1}(z_1+N)^{1-\alpha_1-\alpha_2-k} \\ &+ (z_1+N)^{-\alpha_1}(z_2+N)^{-\alpha_2}\left[\frac{1}{2} + \frac{\alpha_1}{12(z_1+N)} + \frac{\alpha_2}{12(z_2+N)}\right] \\ &+ O(N^{-\alpha_1-\alpha_2-3}). \end{aligned} \tag{49}$$

Proof. First we define for $\alpha_1+\alpha_2>1$

$$f_{\alpha_1,\alpha_2}(z_1,z_2;N) = \sum_{n\ge N}(n+z_1)^{-\alpha_1}(n+z_2)^{-\alpha_2}.$$

The proof of the asymptotic expansion (49) is based on the Euler–Maclaurin formula

$$\sum_{n\ge N} f(n) = \int_N^\infty f(x)\,dx + \frac{f(N)}{2} - \frac{f'(N)}{12} + O(f^{(3)}(N)),$$

where we take $f(x) = (x+z_1)^{-\alpha_1}(x+z_2)^{-\alpha_2}$. To obtain (49) we compute the integral by changing variables $y=(x+z_1)^{-1}$, expanding the resulting integrand in Taylor's series at $y=0$ and integrating term by term.

In order to obtain formula (48) we follow the steps of the proof of Lemma 6

$$\begin{aligned} \prod_{n\ge N}\frac{1+z/(n+\alpha)}{1+z/\zeta_n} &= \prod_{n\ge N}\frac{1+z/(n+\alpha)}{1+z/(n+\beta)}\prod_{n\ge N}\frac{1+z/(n+\beta)}{1+z/\zeta_n} \\ &= \frac{\Gamma(N+\alpha)\Gamma(N+\beta+z)}{\Gamma(N+\beta)\Gamma(N+\alpha+z)}\prod_{n\ge N}\frac{1+z/(n+\beta)}{1+z/\zeta_n}. \end{aligned}$$

Next, we approximate $\ln(1+\omega) = \omega + O(\omega^2)$ as $\omega\to 0$ and obtain

$$\begin{aligned} &\sum_{n\ge N}\ln\left(\frac{1+z/(n+\beta)}{1+z/\zeta_n}\right) \\ &\quad = \sum_{n\ge N}\ln\left(\frac{\zeta_n}{n+\beta}\right) - \sum_{n\ge N}\ln\left(\frac{z+\zeta_n}{z+n+\beta}\right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n \ge N} \left(\frac{A_1}{(n+\beta)^2} + \frac{A_2}{(n+\beta)^3} + O(n^{-4}) \right) \\
&\quad - \sum_{n \ge N} \left(\frac{A_1}{(z+n+\beta)(n+\beta)} + \frac{A_2}{(z+n+\beta)(n+\beta)^2} + O(n^{-4}) \right),
\end{aligned}
$$

which completes the proof. □

Computing the density $p_t(x) = \frac{d}{dx}\mathbb{P}(S_t \le x)$ of S_t at a deterministic time $t > 0$ requires more work. Our starting point is the the fact that the density of S_τ [which we denote by $p^S(q,x)$] is the Laplace transform of $q \times p_t(x)$

$$
\begin{aligned}
p^S(q,x) &= \frac{d}{dx}\mathbb{P}(S_{\tau(q)} \le x) = \frac{d}{dx}\int_0^\infty q e^{-qt}\mathbb{P}(S_t \le x)\,dt \\
&= q\int_0^\infty e^{-qt} p_t(x)\,dt.
\end{aligned}
$$

Thus $p_t(x)$ can be recovered as the following cosine transform:

$$
p_t(x) = \frac{2}{\pi} e^{q_0 t} \int_0^\infty \mathrm{Re}\left[\frac{p^S(q_0+iu,x)}{q_0+iu}\right]\cos(tu)\,du, \qquad q_0 > 0.
$$

We see that to compute this Fourier integral numerically we need to be able to compute $p^S(q,x)$ for q lying in some interval $q \in [q_0, q_0 + iu_0]$ in the complex plane. The main problem is that we need to solve many equations (22) with complex q. While the asymptotic expansions for ζ_n presented in Theorem 5 are still true, we do not have any localization results in the complex plane. It is certainly possible to compute the roots using the argument principle, and originally all the computations were done by the author using this method. However as we will see, there is a much more efficient algorithm.

We need to compute the solutions of equation $q + \Psi(i\zeta) = 0$ for all $q \in [q_0, q_0 + iu_0]$. First we compute the initial values: the roots $\zeta_0^{\pm}$, ζ_n for real value of $q = q_0$ using the method discussed above. Next we consider each root as an implicit function of $u : \zeta_n(u)$ is defined as

$$
\Psi(i\zeta_n(u)) + (q_0 + iu) = 0, \qquad \zeta_n(0) = \zeta_n.
$$

Using implicit differentiation we obtain a first order differential equation

$$
\frac{d\zeta_n(u)}{du} = -\frac{1}{\Psi'(i\zeta_n(u))}
$$

with initial condition $\zeta_n(0) = \zeta_n$. We compute the solution to this ODE using a numerical scheme, for example, an adaptive Runge–Kutta method, and at each step we correct the solution by applying several iterations of Newton's method. Again, for different n we can compute $\zeta_n(u)$ in parallel.

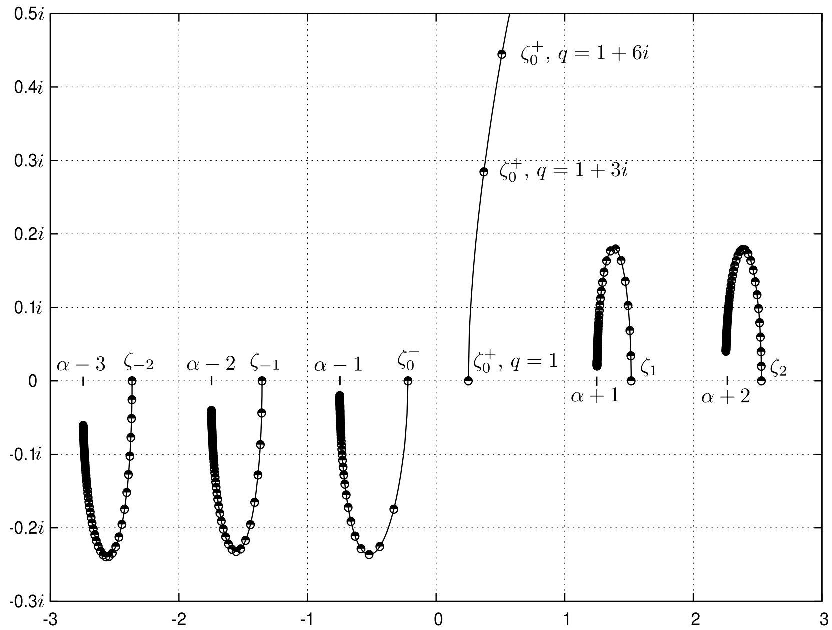

FIG. 2. *The values of* $\zeta_0^{\pm}$ *and* ζ_n *for* $q \in [1, 1+200i]$.

Figure 2 shows the result of this procedure. We have used the following values of parameters: $\sigma = 1$, $\mu = -0.1$ and $\alpha = 0.25$ and computed zeros $\zeta_0^{\pm}$ and ζ_n for $q \in [1, 1+200i]$. The graph shows interesting qualitative behavior: all zeros except ζ_0^+ converge to the closest pole of $\Psi(i\zeta)$ at $\alpha + n$, while ζ_0^+ has no pole nearby [since $\Psi(i\zeta)$ is regular at $\zeta = \alpha$] and it converges to ∞ while always staying in $\mathbb{C}^+$. If $\alpha < 0$ the situation is exactly the same, except that now ζ_0^- escapes to ∞ while always staying in $\mathbb{C}^-$. We have repeated this procedure for many different values of parameters, and from this numerical evidence we can make some observations/conjectures. It appears to be true that the roots never collide which means that we have no higher order solutions to $q + \Psi(z) = 0$ for all $q \in \mathbb{C}$. It also seems that the roots never cross the real line. All these observations are based on numerical evidence, and we did not pursue this any further to obtain rigorous proofs. However there is one fact that we can prove rigorously: there are not going to appear any new, unaccounted zeros. This could be proved by an argument that we have used in the proof of Theorem 5 to show that there are no extra zeros.

The results of our computations are presented in Figure 3. The parameters are $\sigma = 1$, $\mu = -0.1$ and $\alpha = 0.25$; the surface $p^S(q, x)$ is on the left and $p_t(x)$ on the right.

6. Conclusion. In this paper we have introduced a ten-parameter family of Lévy processes characterized by the fact that the characteristic exponent

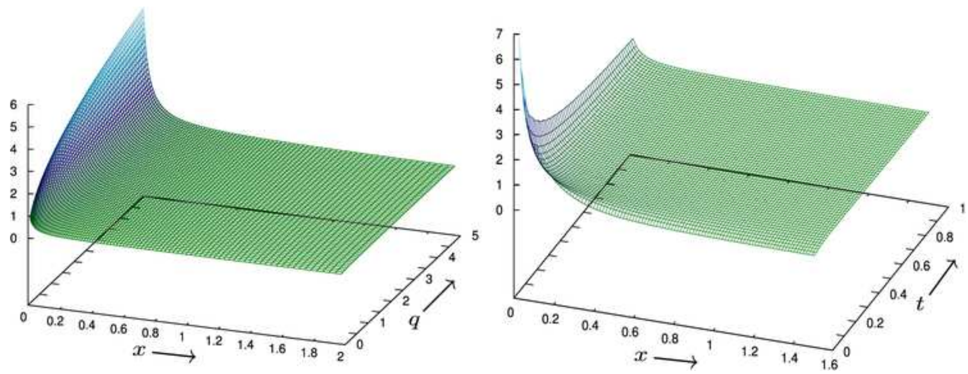

FIG. 3. *Surface plot of* $p^S(q, x)$ *(left) and* $p_t(x)$ *(right).*

is a meromorphic function expressed in terms of beta and digamma functions. This family is quite rich, and, in particular, it includes processes with the complete range of behavior of small positive/negative jumps. We have presented results of the Wiener–Hopf factorization for these processes, including semi-explicit formulas for Wiener–Hopf factors and the density of the supremum process S_τ.

These Lévy processes might be used for modeling purposes whenever one needs to compute distributions related to such functionals as the first passage time, overshoot, extrema, last time before achieving extrema, etc. Some possible applications in Mathematical Finance and Insurance Mathematics include pricing barrier, lookback and perpetual American options, building structural models with jumps in Credit Risk, computing ruin probabilities, etc.

Finally, we would like to mention that one can use the methods presented in this paper, and, in particular, the technique to solve $q + \Psi(z) = 0$ for complex values of q discussed in Section 5, to compute Wiener–Hopf factors arising from a more general factorization identity (see Theorem 6.16 in [21])

$$\frac{q}{q - iw + \Psi(z)} = \Psi_q^+(w, z)\Psi_q^-(w, z), \qquad w, z \in \mathbb{R},$$

where

$$\Psi_q^+(w, z) = E[e^{iw\overline{G}_\tau + izS_\tau}], \qquad \Psi_q^-(w, z) = E[e^{iw\underline{G}_\tau + izI_\tau}],$$

and $\overline{G}_t$ ($\underline{G}_t$) are defined as the last time before t when maximum (minimum) was achieved

$$\overline{G}_t = \sup\{0 \le u \le t : X_u = S_u\}, \qquad \underline{G}_t = \sup\{0 \le u \le t : X_u = I_u\}.$$

Acknowledgment. The author would like to thank Andreas Kyprianou and Geert Van Damme for many detailed comments and constructive suggestions.

REFERENCES

[1] Abramowitz, M. and Stegun, I. A., eds. (1970). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover, New York. MR1225604

[2] Asmussen, S. (2000). *Ruin Probabilities. Advanced Series on Statistical Science and Applied Probability* **2**. World Scientific, River Edge, NJ. MR1794582

[3] Asmussen, S., Avram, F. and Pistorius, M. R. (2004). Russian and American put options under exponential phase-type Lévy models. *Stochastic Process. Appl.* **109** 79–111. MR2024845

[4] Barndorff-Nielsen, O. E. (1997). Normal inverse Gaussian distributions and stochastic volatility modelling. *Scand. J. Statist.* **24** 1–13. MR1436619

[5] Barndorff-Nielsen, O. E. and Levendorskii, S. Z. (2001). Feller processes of normal inverse Gaussian type. *Quant. Finance* **1** 318–331. MR1830659

[6] Baxter, G. and Donsker, M. D. (1957). On the distribution of the supremum functional for processes with stationary independent increments. *Trans. Amer. Math. Soc.* **85** 73–87. MR0084900

[7] Bertoin, J. (1996). *Lévy Processes. Cambridge Tracts in Mathematics* **121**. Cambridge Univ. Press, Cambridge. MR1406564

[8] Bertoin, J. and Yor, M. (2001). On subordinators, self-similar Markov processes and some factorizations of the exponential variable. *Electron. Comm. Probab.* **6** 95–106 (electronic). MR1871698

[9] Boyarchenko, S. and Levendorskii, S. (2000). Option pricing for truncated Lévy processes. *Int. J. Theor. Appl. Finance* **3** 549–552.

[10] Boyarchenko, S. and Levendorskii, S. (2002). Barrier options and touch-and-out options under regular Lévy processes of exponential type. *Ann. Appl. Probab.* **12** 1261–1298. MR1936593

[11] Caballero, M. E. and Chaumont, L. (2006). Conditioned stable Lévy processes and the Lamperti representation. *J. Appl. Probab.* **43** 967–983. MR2274630

[12] Caballero, M. E., Pardo, J. C. and Perez, J. L. (2008). On the Lamperti stable processes. Preprint. Available at `http://arxiv.org/abs/0802.0851`.

[13] Carr, P., Geman, H., Madan, D. and Yor, M. (2002). The fine structure of asset returns: An empirical investigation. *J. Bus.* **75** 305–332.

[14] Chaumont, L., Kyprianou, A. E. and Pardo, J. C. (2009). Some explicit identities associated with positive self-similar Markov processes. *Stochastic Process. Appl.* **119** 980–1000. MR2499867

[15] Cont, R. and Tankov, P. (2004). *Financial Modelling with Jump Processes*. Chapman and Hall/CRC, Boca Raton, FL. MR2042661

[16] Conway, J. B. (1978). *Functions of One Complex Variable*, 2nd ed. *Graduate Texts in Mathematics* **11**. Springer, New York. MR503901

[17] Doney, R. A. (1987). On Wiener–Hopf factorisation and the distribution of extrema for certain stable processes. *Ann. Probab.* **15** 1352–1362. MR905336

[18] Doney, R. A. (2007). *Fluctuation Theory for Lévy Processes*. Springer.

[19] Erdélyi, A., ed. (1955). *Higher Transcendental Functions* **1**. McGraw-Hill, New York.

[20] Jeffrey, A., ed. (2007). *Table of Integrals, Series, and Products*, 7th ed. Elsevier, Amsterdam.

[21] Kyprianou, A. E. (2006). *Introductory Lectures on Fluctuations of Lévy Processes with Applications*. Springer, Berlin. MR2250061

[22] Kyprianou, A. E., Pardo, J. C. and Rivero, V. (2010). Exact and asymptotic n-tuple laws at first and last passage. *Ann. Appl. Probab.* **20** 522–564.

[23] Lamperti, J. (1972). Semi-stable Markov processes. I. *Z. Wahrsch. Verw. Gebiete* **22** 205–225. MR0307358

[24] Levin, B. Y. (1996). *Lectures on Entire Functions. Translations of Mathematical Monographs* **150**. Amer. Math. Soc., Providence, RI. MR1400006

[25] Lewis, A. L. and Mordecki, E. (2008). Wiener–Hopf factorization for Lévy processes having positive jumps with rational transforms. *J. Appl. Probab.* **45** 118–134. MR2409315

[26] Mordecki, E. (2002). Optimal stopping and perpetual options for Lévy processes. *Finance Stoch.* **6** 473–493. MR1932381

[27] Patie, P. (2009). Exponential functional of a new family of Lévy processes and self-similar continuous state branching processes with immigration. *Bull. Sci. Math.* **133** 355–382. MR2532690

[28] Rudin, W. (1986). *Real and Complex Analysis*. McGraw-Hill, New York.

[29] Sato, K.-i. (1999). *Lévy Processes and Infinitely Divisible Distributions. Cambridge Studies in Advanced Mathematics* **68**. Cambridge Univ. Press, Cambridge. MR1739520

[30] Schoutens, W. (2006). Exotic options under Lévy models: An overview. *J. Comput. Appl. Math.* **189** 526–538. MR2202995

Department of Mathematics and Statistics
York University
Toronto, Ontario, M3J 1P3
Canada
E-mail: kuznetsov@mathstat.yorku.ca